# The limits of coefficients of the species diffusion and the rate of reactant to one-dimensional compressible Navier-Stokes equations for a reacting mixture 

Mingyu Zhang ${ }^{1,2^{*}}$ ©

Correspondence:
myuzhang@126.com
${ }^{1}$ School of Mathematics \& Information Sciences, Weifang University, Weifang, P.R. China
${ }^{2}$ School of Mathematical Sciences, Xiamen University, Xiamen, P.R. China


#### Abstract

In this paper, we consider the initial-boundary value problem of the one-dimensional compressible viscous and heat-conductive Navier-Stokes equations with a reacting mixture. This model is used to describe the dynamic combustion. Respectively, we obtain the vanishing species diffusion limit, the rate of reactant limit, and the convergence rates.


MSC: 35K57; 35Q60; 76N15; 80A25
Keywords: Reacting mixture; Vanishing species diffusion limit; Vanishing rate of reactant limit; Convergence rate

## 1 Introduction

The equations of one-dimensional compressible viscous and heat-conductive NavierStokes equations for a reacting mixture in the Lagrange coordinates are of the following form (see [1-3]):

$$
\left\{\begin{array}{l}
v_{t}-u_{x}=0,  \tag{1.1}\\
u_{t}+\left(\frac{a \theta}{v}\right)_{x}=\left(\frac{\mu u_{x}}{v}\right)_{x}, \\
\left(\theta+\frac{u^{2}}{2}\right)_{t}+\left(\frac{a u \theta}{v}\right)_{x}=\left(\frac{\mu u u_{x}}{v}+\frac{v \theta_{x}}{v}\right)_{x}+q k \phi z, \\
z_{t}+k \phi z=\left(\frac{\lambda z_{x}}{v^{2}}\right)_{x},
\end{array}\right.
$$

where $x \in \Omega:=(0,1)$ denotes the Lagrange mass coordinate, $t>0$ is time, the unknown functions $v>0, u, \theta>0, z$ are the specific volume, the fluid velocity, the absolute temperature, and the mass fraction of the reactant; the constants $\mu, \nu, q, \lambda$, and $k$ are the coefficients of bulk viscosity, the heat conduction, the difference in the heats of formation of the reactant and the product, the species diffusion, and the rate of reactant, respectively.

The total specific energy has the form

$$
\widetilde{E}:=e+\frac{u^{2}}{2}+q z
$$

where $e$ is the specific internal energy.

For a perfect gas mixture with the same $\gamma$-gas laws, the pressure $p=p(\nu, \theta)$ and the internal energy $e=e(\nu, \theta)$ are related with the specific volume and the absolute temperature which have the following form:

$$
p(v, \theta)=\frac{a \theta}{v}, \quad e(v, \theta)=\frac{p v}{\gamma-1},
$$

where $a=R M>0, R$ is Boltzmann's gas constant and $M$ is the molecular weight.
The rate function $\phi(\theta)$, which describes the intensity of a chemical reaction, is typically determined by the Arrhenius law (see [1, 4, 5]):

$$
\phi(\theta)=\theta^{\alpha} e^{-A / \theta},
$$

where the positive constant $A$ is the activation energy, $\alpha \geq 0$ is a physical number.
When species diffusion $\lambda>0$, the initial boundary value problems for (1.1) with the initial data are as follows:

$$
\left\{\begin{array}{l}
(v(x, 0), u(x, 0), \theta(x, 0), z(x, 0))=\left(v_{0}(x), u_{0}(x), \theta_{0}(x), z_{0}(x)\right), \quad x \in[0,1]  \tag{1.2}\\
0<m_{0} \leq v_{0}(x), \quad \theta_{0}(x) \leq M_{0}<\infty, \quad 0 \leq z_{0}(x) \leq 1, \\
\int_{0}^{1} v_{0}(x) d x=1,
\end{array}\right.
$$

and the impermeably insulated boundary conditions

$$
\left\{\begin{array}{l}
u(0, t)=u(1, t)=0  \tag{1.3}\\
\theta_{x}(0, t)=\theta_{x}(1, t)=0 \\
z_{x}(0, t)=Z_{x}(1, t)=0
\end{array}\right.
$$

with the compatibility conditions

$$
\left\{\begin{array}{l}
\left.\left(u_{0}, \theta_{0 x}, z_{0 x}\right)\right|_{x=0,1}=0,  \tag{1.4}\\
\left.\left(\frac{a \theta_{0}}{v_{0}}-\left(\frac{\mu u_{0 x}}{v_{0}}\right)_{x}\right)\right|_{x=0,1}=0 .
\end{array}\right.
$$

The existence and behavior of steady plane wave solutions to the compressible NavierStokes equations for a reacting gas have been investigated by Gardner (see [6]) and Wagner (see [7]), and they confirmed some phenomena observed in numerical calculations and predicted by the ZND theory, which has been developed independently by Zeldovich, von Neumann, and Döring (see [8]). In [9] and the references cited therein, lots of theoretical and computational properties regarding the structure and stability of reacting shock waves of (1.1) are analyzed. For recent developments and strategies, see [10, 11] and [12, 13], the authors also gave the mathematical theory of combustion.
The existence of global solutions to the one-dimensional nonsteady equations of a viscous compressible gas was first studied in [14, 15]. The global existence and large-time behavior of solutions for the one-dimensional models of compressible, viscous, and heatconductive fluids have been studied by many researchers. In particular, the case $\lambda>0$ was treated in $[1,2,16,17]$ and the references therein. For the binary-mixture case $\lambda=0$, we can find in [18, 19]. In [1], when $\phi(\theta)$ is discontinuous, existence theorems are established
for global generalized solutions to the compressible Navier-Stokes equations for a reacting mixture. So, it is of great importance to understand how the model changes when $\lambda \rightarrow 0$ and $\lambda, k \rightarrow 0$. The convergence rates need some careful analysis, based on the elementary energy methods and the application of Sobolev's inequality.

In this paper, the initial-boundary value problems (1.1) with the vanishing species diffusion and rate of reactant limits are considered. With the help of global- $\lambda$ and global $-\lambda, k$ independent estimates, we obtain the convergence rates.

Formally, if the species diffusion $\lambda=0$, then system (1.1) turns into the binary-mixture form

$$
\left\{\begin{array}{l}
v_{t}-u_{x}=0  \tag{1.5}\\
u_{t}+\left(\frac{a \theta}{v}\right)_{x}=\left(\frac{\mu u_{x}}{v}\right)_{x}, \\
\left(\theta+\frac{u^{2}}{2}\right)_{t}+\left(\frac{a u \theta}{v}\right)_{x}=\left(\frac{\mu u u_{x}}{v}+\frac{v \theta_{x}}{v}\right)_{x}+q k \phi z, \\
z_{t}+k \phi z=0
\end{array}\right.
$$

which is equipped with the following initial data:

$$
\left\{\begin{array}{l}
(v(x, 0), u(x, 0), \theta(x, 0), z(x, 0))=\left(v_{0}(x), u_{0}(x), \theta_{0}(x), z_{0}(x)\right), \quad x \in[0,1]  \tag{1.6}\\
0<m_{0} \leq v_{0}(x), \quad \theta_{0}(x) \leq M_{0}<\infty, \quad 0 \leq z_{0}(x) \leq 1, \\
\int_{0}^{1} v_{0}(x) d x=1,
\end{array}\right.
$$

and boundary conditions

$$
\left\{\begin{array}{l}
u(0, t)=u(1, t)=0  \tag{1.7}\\
\theta_{x}(0, t)=\theta_{x}(1, t)=0
\end{array}\right.
$$

with the compatibility conditions

$$
\left\{\begin{array}{l}
\left.\left(u_{0}, \theta_{0 x}\right)\right|_{x=0,1}=0,  \tag{1.8}\\
\left.\left(\frac{a \theta_{0}}{v_{0}}-\left(\frac{\mu u_{0 x}}{v_{0}}\right)_{x}\right)\right|_{x=0,1}=0 .
\end{array}\right.
$$

Compared with the large literature body for compressible reacting mixture equations, for system (1.1) with initial-boundary conditions (1.2), (1.3), (1.4) and system (1.5) with initial-boundary conditions (1.6), (1.7), (1.8), we also assume that the reacting rate function $\phi(\theta)$ is a smooth function.

In the other case, if the coefficients of the species diffusion $\lambda=0$ and the rate of reactant $k=0$, then system (1.1) turns into the following form:

$$
\left\{\begin{array}{l}
v_{t}-u_{x}=0  \tag{1.9}\\
u_{t}+\left(\frac{a \theta}{v}\right)_{x}=\left(\frac{\mu u_{x}}{v}\right)_{x}, \\
\left(\theta+\frac{u^{2}}{2}\right)_{t}+\left(\frac{a u \theta}{v}\right)_{x}=\left(\frac{\mu u u_{x}}{v}+\frac{v \theta_{x}}{v}\right)_{x}, \\
z_{t}=0,
\end{array}\right.
$$

which is equipped with the following initial data:

$$
\left\{\begin{array}{l}
(v(x, 0), u(x, 0), \theta(x, 0), z(x, 0))=\left(v_{0}(x), u_{0}(x), \theta_{0}(x), z_{0}(x)\right), \quad x \in[0,1]  \tag{1.10}\\
0<m_{0} \leq v_{0}(x), \quad \theta_{0}(x) \leq M_{0}<\infty, \quad 0 \leq z_{0}(x) \leq 1 \\
\int_{0}^{1} v_{0}(x) d x=1
\end{array}\right.
$$

and boundary conditions

$$
\left\{\begin{array}{l}
u(0, t)=u(1, t)=0  \tag{1.11}\\
\theta_{x}(0, t)=\theta_{x}(1, t)=0
\end{array}\right.
$$

with the compatibility conditions

$$
\left\{\begin{array}{l}
\left.\left(u_{0}, \theta_{0 x}\right)\right|_{x=0,1}=0,  \tag{1.12}\\
\left.\left(\frac{a \theta_{0}}{v_{0}}-\left(\frac{\mu u_{0 x}}{v_{0}}\right)_{x}\right)\right|_{x=0,1}=0 .
\end{array}\right.
$$

Our main results are as follows.

Theorem 1.1 (i) Suppose that

$$
\begin{equation*}
0<v_{0}, 0<\theta_{0}, \quad\left(v_{0}, u_{0}, \theta_{0}, z_{0}\right)(x) \in H^{1} . \tag{1.13}
\end{equation*}
$$

Then, for each fixed $\lambda>0$, there exists a unique global solution $(v, u, \theta, z)$ to the initialboundary value problem (1.1)-(1.4) on $(0,1) \times[0, \infty)$ such that

$$
\begin{align*}
& M^{-1} \leq v(x, t), \quad \theta(x, t) \leq M \quad \text { for all } x \in[0,1], t \in[0, \infty),  \tag{1.14}\\
& \sup _{t \in[0, \infty)}\left(\|(v-\tilde{v}, u)\|_{H^{2}}^{2}+\|\theta-\tilde{\theta}\|_{H^{1}}^{2}+\left\|\left(v_{t}, u_{t}\right)\right\|_{L^{2}}^{2}\right) \\
& \quad+\int_{0}^{\infty}\left(\left\|\left(v_{x}, u_{x}, \theta_{x}\right)\right\|_{H^{1}}^{2}+\left\|\left(v_{x t}, u_{x t}\right)\right\|_{L^{2}}^{2}\right) d t+\int_{0}^{\infty}\left(\left\|\left(v_{t}, u_{t}, \theta_{t}, z_{t}\right)\right\|_{L^{2}}^{2}\right) d t \leq M,  \tag{1.15}\\
& \sup _{t \in[0, \infty)}\|z(t)\|_{H^{2}}^{2}+\lambda \int_{0}^{\infty}\left\|z_{x}\right\|_{H^{2}}^{2} d t+\int_{0}^{\infty} \int_{0}^{1} k \phi(\theta)\left(z^{2}+z_{x}^{2}+z_{x x}^{2}\right) d x d t \leq M, \tag{1.16}
\end{align*}
$$

where $\tilde{v}=\int_{0}^{1} v(x, t) d x$, and the constant $\tilde{\theta}$ is determined by

$$
e(\tilde{v}, \tilde{\theta})=\widetilde{E}_{0}:=\int_{0}^{1}\left(e\left(v_{0}, \theta_{0}\right)+\frac{1}{2} u_{0}^{2}\right) d x
$$

(ii) Assume that $\left(v_{0}, u_{0}, \theta_{0}, z_{0}\right)$ satisfies (1.6). Then there exists a unique global solution ( $v, u, \theta, z$ ) to problem (1.5)-(1.8) on $(0,1) \times[0, \infty)$ such that

$$
\begin{align*}
& M^{-1} \leq v(x, t), \quad \theta(x, t) \leq M \quad \text { for all } x \in[0,1], t \in[0, \infty),  \tag{1.17}\\
& \sup _{t \in[0, \infty)}\left(\|(v-\tilde{v}, u, z)(t)\|_{H^{2}}^{2}+\|\theta-\tilde{\theta}\|_{H^{1}}^{2}+\left\|\left(v_{t}, u_{t}\right)\right\|_{L^{2}}^{2}\right) \\
& \quad+\int_{0}^{\infty}\left(\left\|\left(v_{t}, u_{t}, \theta_{t}, z_{t}\right)\right\|_{L^{2}}^{2}\right) d t+\int_{0}^{\infty}\left(\left\|\left(v_{x}, u_{x}, \theta_{x}\right)\right\|_{H^{1}}^{2}+\left\|\left(v_{x t}, u_{x t}\right)\right\|_{L^{2}}^{2}\right) d t
\end{align*}
$$

$$
\begin{equation*}
+\int_{0}^{\infty} \int_{0}^{1} k \phi(\theta)\left(z^{2}+z_{x}^{2}+z_{x x}^{2}\right) d x d t \leq M \tag{1.18}
\end{equation*}
$$

Here, the letter $M$ denotes the generic positive constant which depends on a, $\mu, v, q, k,\|\phi\|_{L^{\infty}}$, but not on $\lambda$ and $t$.

In terms of (1.1)-(1.4) and (1.9)-(1.12), the second important theorem is as follows.

Theorem 1.2 (i) Under the conditions of Theorem 1.1, for each fixed $\lambda, k>0$, there exists a unique global solution $(v, u, \theta, z)$ to the initial-boundary value problem (1.1)-(1.4) on $(0,1) \times[0, \infty)$ such that

$$
\begin{aligned}
& M^{*-1} \leq v(x, t), \quad \theta(x, t) \leq M^{*} \quad \text { for all }(x, t) \in[0,1] \times[0, \infty), \\
& \sup _{t \in[0, \infty)}\left(\|(v-\tilde{v}, u)\|_{H^{2}}^{2}+\|\theta-\tilde{\theta}\|_{H^{1}}^{2}+\left\|\left(v_{t}, u_{t}\right)\right\|_{L^{2}}^{2}\right) \\
& \quad+\int_{0}^{\infty}\left(\left\|\left(v_{x}, u_{x}, \theta_{x}\right)\right\|_{H^{1}}^{2}+\left\|\left(v_{x t}, u_{x t}\right)\right\|_{L^{2}}^{2}\right) d t+\int_{0}^{\infty}\left(\left\|\left(v_{t}, u_{t}, \theta_{t}, z_{t}\right)\right\|_{L^{2}}^{2}\right) d t \leq M^{*}, \\
& \sup _{t \in[0, \infty)}\|z(t)\|_{H^{2}}^{2}+\lambda \int_{0}^{\infty}\left\|z_{x}\right\|_{H^{2}}^{2} d t+k \int_{0}^{\infty} \int_{0}^{1} \phi(\theta)\left(z^{2}+z_{x}^{2}+z_{x x}^{2}\right) d x d t \leq M^{*} .
\end{aligned}
$$

(ii) Assume that $\left(v_{0}, u_{0}, \theta_{0}, z_{0}\right)$ satisfies (1.10). Then there exists a unique global solution ( $v, u, \theta, z$ ) to problem (1.9)-(1.12) on $(0,1) \times[0, \infty)$ such that

$$
\begin{aligned}
& M^{*-1} \leq v(x, t), \quad \theta(x, t) \leq M^{*}, \quad z(x, t)=z_{0}(x), \quad \text { for all } x \in[0,1], t \in[0, \infty), \\
& \sup _{t \in[0, \infty)}\left(\|(v-\tilde{v}, u)(t)\|_{H^{2}}^{2}+\|\theta-\tilde{v}\|_{H^{1}}^{2}+\left\|\left(v_{t}, u_{t}\right)\right\|_{L^{2}}^{2}\right) \\
& \quad+\int_{0}^{\infty}\left(\left\|\left(v_{x}, u_{x}, \theta_{x}\right)\right\|_{H^{1}}^{2}+\left\|\left(v_{x t}, u_{x t}\right)\right\|_{L^{2}}^{2}\right) d t+\int_{0}^{\infty}\left(\left\|\left(v_{t}, u_{t}, \theta_{t}\right)\right\|_{L^{2}}^{2}\right) d t \leq M^{*} .
\end{aligned}
$$

Here, the letter $M^{*}$ denotes the generic positive constant which may depend on $a, \mu, v, q,\|\phi\|_{L^{\infty}}$, but does not depend on $\lambda, k$, and $t$.
It was pointed out in $[1,18]$ that the justification of $(1.1)$ with vanishing species diffusion limit is still open. Indeed, the study of the vanishing species diffusion and rate of reactant limits relies on the global uniform-in- $\lambda$ estimates and the global uniform-in- $\lambda, k$ estimates of the solutions respectively of problem (1.1)-(1.4), which are more difficult to achieve than those for problem (1.5)-(1.8) and (1.9), (1.10), (1.11), and (1.12) due to the presence of reacting-diffusion equation. Our third and fourth results of this paper are concerned with the vanishing species diffusion and rate of reactant limit, which are shown by making a full use of some strong condition of the heat-conductive Navier-Stokes equations for a reacting mixture.

Theorem 1.3 Under the conditions of Theorem 1.1, for any fixed $0<T<\infty$, let $\left(v^{\lambda}, u^{\lambda}, \theta^{\lambda}\right.$, $\left.z^{\lambda}\right)$ and $(v, u, \theta, z)$, defined on $(0,1) \times[0, T)$, be the solutions of problems (1.1)-(1.4) and
(1.5)-(1.8), respectively. Then

$$
\begin{aligned}
& \sup _{t \in[0, T)}\left(\left\|\left(v^{\lambda}-v, u^{\lambda}-u, \theta^{\lambda}-\theta, z^{\lambda}-z\right)(t)\right\|_{H^{1}}^{2}\right)+\int_{0}^{T}\left(\left\|u^{\lambda}-u\right\|_{H^{2}}^{2}+\left\|\left(\theta^{\lambda}-\theta\right)\right\|_{H^{1}}^{2}\right) d t \\
& \quad+\int_{0}^{T}\left(\left\|\left(u^{\lambda}-u, \theta^{\lambda}-\theta\right)_{t}\right\|_{L^{2}}^{2}\right) d t \leq N \lambda^{1 / 2},
\end{aligned}
$$

where $N$ is a generic positive constant independent of $\lambda$.
Theorem 1.4 Under the condition of Theorem 1.1 , for any fixed $0<T<\infty$, let ( $v^{\lambda, k}, u^{\lambda, k}$, $\left.\theta^{\lambda, k}, z^{\lambda, k}\right)$ and $(v, u, \theta, z)$, defined on $(0,1) \times[0, T)$, be the solutions of problems (1.1)-(1.4) and (1.9)-(1.12), respectively. Then

$$
\begin{aligned}
& \sup _{t \in[0, T)}\left(\left\|\left(v^{\lambda, k}-v, u^{\lambda, k}-u, \theta^{\lambda, k}-\theta, z^{\lambda, k}-z\right)(t)\right\|_{H^{1}}^{2}\right) \\
& \quad+\int_{0}^{T}\left(\left\|u^{\lambda, k}-u\right\|_{H^{2}}^{2}+\left\|\left(\theta^{\lambda, k}-\theta\right)\right\|_{H^{1}}^{2}\right) d t \\
& \quad+\int_{0}^{T}\left(\left\|\left(u^{\lambda, k}-u, \theta^{\lambda, k}-\theta\right)_{t}\right\|_{L^{2}}^{2}\right) d t \leq N^{*}\left(\lambda^{1 / 2}+k^{1 / 2}\right)
\end{aligned}
$$

where $N^{*}$ is a generic positive constant independent of $\lambda$ and $k$.

The rest of this paper is organized as follows. In Sect. 2, we establish the global $\lambda$ independent estimates of the solutions $\left(v^{\lambda}, u^{\lambda}, \theta^{\lambda}, z^{\lambda}\right)$ to problem (1.1)-(1.4), the global estimates of the solutions $(v, u, \theta, z)$ to problem (1.5)-(1.8), respectively. With the help of global (uniform) estimates at hand, we justify the vanishing species diffusion limit and obtain the convergence rates. In Sect. 3, we establish the global $\lambda, k$-independent estimates of the solutions $\left(\nu^{\lambda, k}, u^{\lambda, k}, \theta^{\lambda, k}, z^{\lambda, k}\right)$ to problem (1.1)-(1.4), the global estimates of the solutions ( $v, u, \theta, z$ ) to problem (1.9), (1.10), (1.11), and (1.12), respectively. With the help of global (uniform) estimates, we justify the vanishing species diffusion and rate of reactant limit and obtain the convergence rates.

## 2 The vanishing species diffusion limit

### 2.1 Global $\lambda$-independent estimates of (1.1)-(1.4)

Based on the standard local existence results and the global a priori estimates, the global well-posedness of the solutions to (1.1)-(1.4) can be shown in the same way as that in [ $1,18,20$ ]. Our main purpose is to obtain the global $\lambda$-independent estimates of solutions, which are used to justify the vanishing species diffusion limit. For simplicity, in this section, we use ( $v, u, \theta, z$ ) to denote the solutions of (1.1)-(1.4), the letter $M$ denotes the generic positive constant which depends on $a, \mu, \nu, q, k,\|\phi\|_{L^{\infty}}$, but not on $\lambda$ and $t$.
We begin with the following elementary estimates.

Lemma 2.1 Under the conditions of Theorem 1.1,

$$
\begin{aligned}
& \int_{0}^{1} v(x, t) d x=\int_{0}^{1} v_{0}(x) d x=1, \quad \forall t \in[0, \infty) \\
& \int_{0}^{1} z(x, t) d x+\int_{0}^{\infty} \int_{0}^{1} k \phi(\theta) z(x, t) d x d \tau=\int_{0}^{1} z_{0}(x) d x
\end{aligned}
$$

$$
\int_{0}^{1}\left(\theta+\frac{u^{2}}{2}+q z\right) d x=\int_{0}^{1}\left(\theta_{0}+\frac{u_{0}^{2}}{2}+q z_{0}\right) d x
$$

and

$$
E(v, u, \theta)+\int_{0}^{\infty} \int_{0}^{1}\left(\frac{\mu u_{x}^{2}}{v \theta}+\frac{v \theta_{x}^{2}}{v \theta^{2}}\right) d x d \tau \leq M
$$

where $E(v, u, \theta)$ is defined as

$$
E(v, u, \theta) \triangleq \int_{0}^{1}\left[a(v-\ln v-1)+\frac{u^{2}}{2}+(\theta-\ln \theta-1)\right] d x .
$$

The proof of Lemma 2.1 is the same as in [1, Lemma 1]; here, we omit it for simplicity. With the help of Lemma 2.1, similar to the proof in [1, 20], it is easy to establish the following lemma, we omit its proof as well.

Lemma 2.2 The following inequalities hold:

$$
\begin{equation*}
0 \leq z(x, t) \leq 1, \quad \alpha_{0} \leq \int_{0}^{1} \theta(x, t) d x \leq \beta_{0}, \quad \forall(x, t) \in[0,1] \times[0, \infty) \tag{2.1}
\end{equation*}
$$

Here, the positive constants $\alpha_{0}, \beta_{0}$ are the roots of

$$
y-\ln y-1=E_{1}:=\frac{1}{\min \{1, a\}}\left(E_{0}+q \int_{0}^{1} z_{0}(x) d x\right)
$$

where

$$
E_{0}=E\left(v_{0}, u_{0}, \theta_{0}\right) \triangleq \int_{0}^{1}\left[a\left(v_{0}-\ln v_{0}-1\right)+\frac{u_{0}^{2}}{2}+\left(\theta_{0}-\ln \theta_{0}-1\right)\right] d x
$$

Next, we adapt and modify an idea of Kazhikhov [3] (also cf. the survey article [21]) for the polytropic ideal gas to give a representation of solutions of (1.1)-(1.4). In order to do this, we define

$$
\sigma(x, t) \triangleq-p(v, \theta)+\frac{\mu u_{x}}{v}, \quad \psi(x, t) \triangleq \int_{0}^{t} \sigma(x, t) d \tau+\int_{0}^{x} u_{0} d x .
$$

Then we have $\psi_{x}=u$ and $\psi_{t}=\sigma$. Thus $\psi$ satisfies

$$
\begin{equation*}
\psi_{t}=\frac{\mu}{v} \psi_{x x}-p(v, \theta) \tag{2.2}
\end{equation*}
$$

Multiplying (2.2) by $v$ and using (1.1) ${ }_{1}$, we can see that

$$
\begin{equation*}
(v \psi)_{t}-(u \psi)_{x}=\mu \psi_{x x}-v p-u^{2} \tag{2.3}
\end{equation*}
$$

Keeping in mind that $\psi_{x}=u$ vanishes on the boundary and integrating (2.3) over $(0,1) \times$ $[0, t]$, one has

$$
\begin{equation*}
\int_{0}^{1}(\nu \psi)(x, t) d x=\int_{0}^{1}(\nu \psi)(x, 0) d x-\int_{0}^{t} \int_{0}^{1}\left(u^{2}+v p\right)(x, \tau) d x d \tau:=\Psi(t) \tag{2.4}
\end{equation*}
$$

Applying the mean value theorem to (2.4), by Lemma 2.1, $v>0$, we get that for each $t \geq 0$ there exists $x_{1}(t) \in[0,1]$ such that

$$
\psi\left(x_{1}(t), t\right)=\int_{0}^{1} \psi(x, t) v(x, t) d x=\Psi(t)
$$

Therefore, by the definition of $\psi(x, t)$ and (2.4), we have

$$
\begin{align*}
\int_{0}^{1} \psi\left(x_{1}(t), \tau\right) d \tau= & \psi\left(x_{1}(t), t\right)-\int_{0}^{x_{1}(t)} u_{0}(\xi) d \xi=\Psi(t)-\int_{0}^{x_{1}(t)} u_{0}(\xi) d \xi \\
= & -\int_{0}^{t} \int_{0}^{1}\left(u^{2}+v p\right) d x d \tau+\int_{0}^{1} v_{0}(x) \int_{0}^{x} u_{0}(\xi) d \xi \\
& -\int_{0}^{x_{1}(t)} u_{0}(\xi) d \xi \tag{2.5}
\end{align*}
$$

and $t \geq 0$. Thanks to (2.5), one can establish the following lemma.

Lemma 2.3 For system (1.1)-(1.4), we have the following representations:
(i) For any $t \geq 0$, there exists $x_{1}(t) \in[0,1]$ such that

$$
\begin{equation*}
v(x, t)=B(x, t) D(x, t) \exp \left(-\frac{1}{\mu} \int_{0}^{t} \int_{0}^{1}\left(u^{2}+a \theta\right) d x d \tau\right) \tag{2.6}
\end{equation*}
$$

where

$$
\begin{align*}
B(x, t)= & v_{0}(x) \exp \left\{\frac { 1 } { \mu } \left(\int_{x_{1}(t)}^{x}\left(u-u_{0}\right)(\xi) d \xi-\int_{0}^{x_{1}(t)} u_{0}(\xi) d \xi\right.\right. \\
& \left.\left.+\int_{0}^{1} v_{0}(x) \int_{0}^{x} u_{0}(\xi) d \xi d x\right)\right\}  \tag{2.7}\\
D(x, t)= & 1+\frac{a}{\mu} \int_{0}^{t} \frac{\theta(x, \tau)}{B(x, \tau)} \exp \left(\frac{1}{\mu} \int_{0}^{\tau} \int_{0}^{1}\left(u^{2}+a \theta\right)(x, s) d x d s\right) d \tau
\end{align*}
$$

Proof In view of the definition of $\sigma$, we rewrite $(1.1)_{2}$ as follows:

$$
\begin{equation*}
u_{t}+p(v, \theta)_{x}=(\mu \log v)_{x t} . \tag{2.8}
\end{equation*}
$$

Integrating (2.8) over $[0, t] \times\left[x_{1}(t), x\right]$ and by (2.5), we obtain

$$
\begin{aligned}
& \log v(x, t)-\frac{1}{\mu} \int_{0}^{1} v_{0}(x) \int_{0}^{x} u_{0}(\xi) d \xi d x+\frac{1}{\mu} \int_{0}^{x_{1}(t)} u_{0}(\xi) d \xi-\frac{1}{\mu} \int_{x_{1}(t)}^{x}\left(u-u_{0}\right)(\xi) d \xi \\
& \quad=\log v_{0}(x)-\frac{1}{\mu} \int_{0}^{t} \int_{0}^{1}\left(u^{2}+a \theta\right)(x, \tau) d x d \tau+\frac{1}{\mu} \int_{0}^{t} \frac{a \theta}{v}(x, \tau) d \tau
\end{aligned}
$$

which, upon taking the exponential, turns into

$$
\begin{aligned}
& v(x, t) \exp \left\{-\frac{1}{\mu}\left(\int_{x_{1}(t)}^{x}\left(u-u_{0}\right)(\xi) d \xi-\int_{0}^{x_{1}(t)} u_{0}(\xi) d \xi+\int_{0}^{1} v_{0}(x) \int_{0}^{x} u_{0}(\xi) d \xi d x\right)\right\} \\
& \quad=v_{0}(x) \exp \left\{-\frac{1}{\mu} \int_{0}^{t} \int_{0}^{1}\left(u^{2}+a \theta\right)(x, \tau) d x d \tau\right\} \exp \left\{\frac{1}{\mu} \int_{0}^{t} \frac{a \theta}{v}(x, \tau) d \tau\right\}
\end{aligned}
$$

that is,

$$
\begin{equation*}
v(x, t)=B(x, t) \exp \left\{-\frac{1}{\mu} \int_{0}^{t} \int_{0}^{1}\left(u^{2}+a \theta\right)(x, \tau) d x d \tau\right\} \exp \left\{\frac{1}{\mu} \int_{0}^{t} \frac{a \theta}{v}(x, \tau) d \tau\right\} . \tag{2.9}
\end{equation*}
$$

It follows from (2.7) and (2.9) that

$$
\frac{a \theta}{\mu \nu} \exp \left\{\frac{1}{\mu} \int_{0}^{t} \frac{a \theta}{v}(x, \tau) d \tau\right\}=\frac{v(x, t) A(t)}{B(x, t)} \frac{a \theta}{\mu v}
$$

where

$$
A(t)=\exp \left\{\frac{1}{\mu} \int_{0}^{t} \int_{0}^{1}\left(u^{2}+a \theta\right)(x, \tau) d x d \tau\right\}
$$

Integrating the above identity over $(0, t)$, one has

$$
\begin{equation*}
\exp \left\{\frac{1}{\mu} \int_{0}^{t} \frac{a \theta}{v}(x, \tau) d \tau\right\}=1+\frac{a}{\mu} \int_{0}^{t} \frac{\theta(x, \tau) A(\tau)}{D(x, \tau)} d \tau \tag{2.10}
\end{equation*}
$$

Inserting (2.10) into (2.9), we get (2.6).

With the aid of Lemma 2.3, we can obtain the following important lemma about the uniform upper and lower bounds of $v$.

Lemma 2.4 For any $t \geq 0$, we have

$$
M^{-1} \leq v(x, t) \leq M, \quad \frac{1}{M(1+t)} \leq \theta(x, t), \quad \forall(x, t) \in[0,1] \times[0, \infty)
$$

The proofs of Lemma 2.4 and the below lemma are in the same way as in the paper [1, 20], here we omit them.

Lemma 2.5 The following estimates hold:

$$
\begin{align*}
& \sup _{t \in[0, \infty)}\left(\left\|\left(v_{x}, v_{t}\right)\right\|_{L^{2}}^{2}+\|(u, \theta-\bar{\theta})\|_{H^{1}}^{2}\right) \\
& \quad+\int_{0}^{\infty}\left(\left\|\left(v_{x}, v_{x t}\right)\right\|_{L^{2}}^{2}+\left\|\left(u_{x}, \theta_{x}\right)\right\|_{H^{1}}^{2}+\left\|\left(u_{t}, \theta_{t}\right)\right\|_{L^{2}}^{2}\right) d t \leq M  \tag{2.11}\\
& \sup _{t \in[0, \infty)}\|z\|_{H^{1}}^{2}+\lambda \int_{0}^{\infty}\left\|z_{x}\right\|_{H^{1}}^{2} d t+\int_{0}^{\infty} \int_{0}^{1} k \phi(\theta)\left(z^{2}+z_{x}^{2}\right) d x d t \leq M .
\end{align*}
$$

By means of Lemma 2.5, we next establish the following estimates.

Lemma 2.6 For $t \geq 0$, we have

$$
\begin{aligned}
& |u(x, t)| \leq M, \quad \theta(x, t) \leq M, \quad \forall(x, t) \in[0,1] \times[0, \infty) \\
& \int_{0}^{\infty}\left(\left\|u_{x}\right\|_{L^{\infty}}^{2}+\left\|\theta_{x}\right\|_{L^{\infty}}^{2}\right) d t \leq M
\end{aligned}
$$

Proof By Hölder's inequality, we obtain

$$
u^{2} \leq \int_{0}^{x}\left(u^{2}\right)_{x} d x \leq 2\left(\int_{0}^{1} u^{2} d x\right)^{1 / 2}\left(\int_{0}^{1} u_{x}^{2} d x\right)^{1 / 2} \leq M
$$

With the aid of (2.1), by the mean value theorem, there exists $x^{*} \in(0,1)$ such that

$$
\alpha_{0} \leq \int_{0}^{1} \theta(x, t)=\theta\left(x^{*}, t\right) \triangleq \beta \leq \beta_{0}
$$

Then we define an auxiliary function $g$ by

$$
g(\theta)=\int_{0}^{\theta} \sqrt{\theta-1-\ln \theta} d \theta
$$

Thus we have

$$
\frac{d g(\theta)}{d x}=g^{\prime}(\theta) \theta_{x}, \quad g^{\prime}(\theta)=\sqrt{\theta-1-\ln \theta}
$$

Integrating the above identity over $\left(x^{*}, x\right)$, we obtain

$$
g(\theta)(x, t)=\int_{x^{*}}^{x} g^{\prime}(\theta) \theta_{x} d x+g(\theta)\left(x^{*}, t\right)
$$

Next, we prove that $g(\theta)\left(x^{*}, t\right)$ is a bounded function. In fact

$$
g(\theta)\left(x^{*}, t\right)=\int_{0}^{\theta\left(x^{*}, t\right)} \sqrt{s-1-\ln s} d s .
$$

Notice that when $\theta \geq 1, f(\theta) \triangleq \theta-1-\ln \theta$ is a monotone increasing function, when $0 \leq$ $\theta \leq 1, f(\theta)$ is a monotone decreasing function. If $\theta\left(x^{*}, t\right) \geq 1\left(\theta\left(x^{*}, t\right) \leq 1\right.$ is similar $)$

$$
\begin{aligned}
g(\theta)\left(x^{*}, t\right) & =\int_{0}^{1} \sqrt{s-1-\ln s} d s+\int_{0}^{\theta(x *, t)} \sqrt{s-1-\ln s} d s \\
& \leq \frac{1}{2} \int_{0}^{1} s-1-\ln s d s+\frac{1}{2}+M \leq M .
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
g(\theta)(x, t) & =\int_{0}^{x} \sqrt{\theta-1-\ln \theta}\left|\theta_{x}\right| d x+M \\
& \leq \frac{1}{2} \int_{0}^{1} \theta-1-\ln \theta d x+\frac{1}{2}+M \int_{0}^{1} \theta_{x}^{2} d x+M \\
& \leq M
\end{aligned}
$$

By the definition of $g(\theta)$, we know that when $\theta \rightarrow \infty, g(\theta) \rightarrow \infty$, this together with the above inequality, there exists a constant $M>0$ such that

$$
\theta(x, t) \leq M
$$

By virtue of the interpolation inequality, we have

$$
\begin{aligned}
\left\|u_{x}\right\|_{L^{\infty}}^{2} & \leq M\left\|u_{x}\right\|_{L^{2}}\left\|u_{x}\right\|_{H^{1}} \\
& \leq M\left(\left\|u_{x}\right\|_{L^{2}}^{2}+\left\|u_{x x}\right\|_{L^{2}}^{2}\right) .
\end{aligned}
$$

With the help of (2.11), we obtain

$$
\int_{0}^{\infty}\left\|u_{x}\right\|_{L^{\infty}}^{2} d \tau \leq M
$$

By the same way, we get

$$
\int_{0}^{\infty}\left\|\theta_{x}\right\|_{L^{\infty}}^{2} d \tau \leq M
$$

This completes the proof of Lemma 2.6.

Next the large-time behavior of global generalized solutions is obtained.

## Lemma 2.7 It holds that

$$
\lim _{t \rightarrow \infty}\left(\|v-\tilde{v}, u, \theta-\tilde{\theta}\|_{L^{s}}(t)+\left\|v_{x}, u_{x}, \theta_{x}\right\|_{L^{2}}(t)\right)=0, \quad \forall s \in[2, \infty], x \in \Omega
$$

where $\widetilde{v}$ and $\widetilde{\theta}$ are defined in Theorem 1.1.

Proof By Lemma 2.5, we have

$$
\begin{equation*}
\int_{0}^{\infty}\left(\left\|u_{x}\right\|_{L^{2}}^{2}+\left\|\theta_{x}\right\|_{L^{2}}^{2}\right) d t \leq M \tag{2.12}
\end{equation*}
$$

We rewrite (1.1) ${ }_{3}$ as follows:

$$
\begin{equation*}
\theta_{t}+\frac{a \theta}{v} u_{x}=\left(\frac{\nu \theta_{x}}{v}\right)_{x}+\frac{\mu u_{x}^{2}}{v}+q k \phi(\theta) z . \tag{2.13}
\end{equation*}
$$

Multiplying both sides of (2.13) by $\theta_{x x}$ and using Lemma 2.1, we obtain

$$
\begin{aligned}
\left|\frac{d}{d t}\left\|\theta_{x}\right\|_{L^{2}(\Omega)}^{2}\right|= & \left|\int_{0}^{1} \frac{a \theta}{v} u_{x} \theta_{x x}+\frac{v}{v^{2}} \theta_{x} v_{x} \theta_{x x}-\frac{v}{v^{2}} \theta_{x x}^{2}-\frac{\mu}{v} u_{x}^{2} \theta_{x x}-q k \phi(\theta) z \theta_{x x}\right| \\
\leq & M\left\|\theta_{x x}\right\|_{L^{2}}^{2}+M\left(\|\theta\|_{L^{\infty}}^{2}\left\|u_{x}\right\|_{L^{2}}^{2}+\left\|\theta_{x}\right\|_{L^{\infty}}^{2}\left\|v_{x}\right\|_{L^{2}}^{2}+\left\|u_{x}\right\|_{L^{\infty}}^{2}\left\|u_{x}\right\|_{L^{2}}^{2}\right) \\
& +M \int_{0}^{1} k^{2} \phi(\theta)^{2} z^{2} d x .
\end{aligned}
$$

Integrating it over $(0, \infty)$, with the aid of Lemma 2.5 , we obtain

$$
\begin{aligned}
& \int_{0}^{\infty}\left|\frac{d}{d t}\left\|\theta_{x}\right\|_{L^{2}(\Omega)}^{2}\right| d t \\
& \quad \leq M \int_{0}^{\infty}\left\|\theta_{x x}\right\|_{L^{2}}^{2} d t+M \int_{0}^{\infty}\left(\left\|u_{x}\right\|_{L^{2}}^{2}+\left\|\theta_{x}\right\|_{L^{\infty}}^{2}\|+\| u_{x} \|_{L^{\infty}}^{2}\right) d t
\end{aligned}
$$

$$
\begin{align*}
& +M \int_{0}^{1} k \phi(\theta) z^{2} d x \\
\leq & M . \tag{2.14}
\end{align*}
$$

Multiplying both sides of $(1.1)_{2}$ by $u_{x x}$ and using Lemma 2.1, we derive

$$
\begin{aligned}
\left|\frac{d}{d t}\left\|u_{x}\right\|_{L^{2}(\Omega)}^{2}\right|+\mu \int_{0}^{1} \frac{u_{x x}^{2}}{v} d x & =\left|\int_{0}^{1} a\left(\frac{\theta}{v}\right)_{x} u_{x x}+\mu \frac{u_{x}}{v^{2}} v_{x} u_{x x} d x\right| \\
& \leq \frac{\mu}{4} \int_{0}^{1} \frac{u_{x x}^{2}}{v} d x+M \int_{0}^{1}\left(\theta^{2} v_{x}^{2}+\theta_{x}^{2}+u_{x}^{2} v_{x}^{2}\right) d x
\end{aligned}
$$

Integrating the above inequality over $(0, \infty)$, with the aid of Lemma 2.5 and Lemma 2.6, we obtain

$$
\begin{align*}
& \int_{0}^{\infty}\left|\frac{d}{d t}\left\|u_{x}\right\|_{L^{2}(\Omega)}^{2}\right| d t+\int_{0}^{\infty} \int_{0}^{1} \frac{u_{x x}^{2}}{v} d x d t \\
& \quad \leq \frac{\mu}{4} \int_{0}^{\infty} \int_{0}^{1} \frac{u_{x x}^{2}}{v} d x d t+M \int_{0}^{\infty} \int_{0}^{1}\left(\theta^{2} v_{x}^{2}+\theta_{x}^{2}+u_{x}^{2} v_{x}^{2}\right) d x d t \\
& \quad \leq M+\frac{\mu}{4} \int_{0}^{\infty} \int_{0}^{1} \frac{u_{x x}^{2}}{v} d x d t+M\left\|u_{x}(\cdot, t)\right\|_{L^{\infty}}^{2}+M \underset{[0,1] \times[0, \infty]}{\max } \theta \int_{0}^{\infty} \int_{0}^{1} \theta^{2} v_{x}^{2} d x d t \\
& \quad \leq M+\frac{\mu}{2} \int_{0}^{\infty} \int_{0}^{1} \frac{u_{x x}^{2}}{v} d x d t \tag{2.15}
\end{align*}
$$

With the aid of (2.12), (2.14), and (2.15), one has

$$
\lim _{t \rightarrow \infty}\left(\left\|u_{x}(\cdot, t)\right\|_{L^{2}(\Omega)}+\left\|\theta_{x}(\cdot, t)\right\|_{L^{2}(\Omega)}\right)=0
$$

By virtue of Lemma 2.5, we deduce

$$
\int_{0}^{\infty}\left(\left\|v_{x}(\cdot, t)\right\|_{L^{2}(\Omega)}^{2}+\left|\frac{d}{d t}\left\|v_{x}(\cdot, t)\right\|_{L^{2}(\Omega)}^{2}\right|\right) d t \leq M
$$

then

$$
\lim _{t \rightarrow \infty}\left\|v_{x}(\cdot, t)\right\|_{L^{2}(\Omega)}=0 .
$$

Using the interpolation inequality, we have

$$
\lim _{t \rightarrow \infty}\|(v-\bar{v}, u, \theta-\bar{\theta})(\cdot, t)\|_{L^{2}(\Omega)}=0
$$

The proof of Lemma 2.7 is thus complete.
Thanks to Lemma 2.6, one can deduce the uniform lower bounds of temperature $\theta(x, t)$.

## Lemma 2.8 It holds that

$$
M^{-1} \leq \theta(x, t) \leq M, \quad \forall(x, t) \in[0,1] \times[0, \infty)
$$

Proof With the aid of Lemma 2.7, and by using the interpolation inequality, we have

$$
\|\theta-\tilde{\theta}\|_{C(\bar{\Omega})}^{2} \leq M\left\|\theta_{x}\right\|_{L^{2}}^{2} .
$$

Taking the limit on both sides of the above inequality, we have

$$
\lim _{t \rightarrow \infty}\|(\theta-\tilde{\theta})(\cdot, t)\|_{C(\bar{\Omega})}=0
$$

Hence, there exists some $T_{0}$ such that

$$
M^{-1} \leq \theta(x, t) \leq M, \quad \forall(x, t) \in[0,1] \times\left[T_{0}, \infty\right) .
$$

On the other hand, from Lemma 2.5, we get

$$
\theta(x, t) \geq \frac{1}{M\left(1+T_{0}\right)}, \quad \forall(x, t) \in[0,1] \times\left[0, T_{0}\right] .
$$

This completes the proof of Lemma 2.8.

Lemma 2.9 For any $t \geq 0$, it holds that

$$
\begin{equation*}
\sup _{t \in[0, \infty)}\left(\left\|u_{t}\right\|_{L^{2}}^{2}+\left\|u_{x x}\right\|_{L^{2}}^{2}\right)+\int_{0}^{\infty} \int_{0}^{1} u_{x t}^{2}(x, t) d x d t \leq M \tag{2.16}
\end{equation*}
$$

Proof Differentiating $(1.1)_{2}$ with respect to $t$, multiplying it by $u_{t}$, and integrating the resulting equation over $[0,1]$, by using $(1.1)_{1}$ and $\left.u_{t}\right|_{x=0,1}=0$, we get

$$
\begin{aligned}
\frac{1}{2} \frac{d}{d t} \int_{0}^{1} u_{t}^{2} d x & =\int_{0}^{1}\left(\frac{\mu u_{x}}{v}\right)_{t x} u_{t} d x-\int_{0}^{1}\left(\frac{a \theta}{v}\right)_{t x} u_{t} d x \\
& =-\int_{0}^{1}\left(\frac{\mu u_{x}}{v}\right)_{t} u_{x t} d x+\int_{0}^{1}\left(\frac{a \theta}{v}\right)_{t} u_{x t} d x \\
& =\int_{0}^{1} \frac{\mu}{v^{2}} u_{x}^{2} u_{x t} d x-\int_{0}^{1} \frac{\mu}{v} u_{x t}^{2} d x+\int_{0}^{1}\left(\frac{a \theta}{v}\right)_{t} u_{x t} d x .
\end{aligned}
$$

By the Cauchy-Schwarz inequality, we obtain

$$
\begin{aligned}
\frac{1}{2} & \frac{d}{d t} \int_{0}^{1} u_{t}^{2} d x+\int_{0}^{1} \frac{\mu}{v} u_{x t}^{2} d x \\
& =\int_{0}^{1} \frac{\mu}{v^{2}} u_{x}^{2} u_{x t} d x+\int_{0}^{1}\left(\frac{a \theta}{v}\right)_{t} u_{x t} d x \\
& \leq \epsilon \int_{0}^{1} u_{x t}^{2} d x+C_{\epsilon}\left\|u_{x}\right\|_{L^{\infty}}^{2} \int_{0}^{1} u_{x}^{2} d x+C_{\epsilon} \int_{0}^{1}\left(u_{x}^{2}+\theta_{t}^{2}\right) d x
\end{aligned}
$$

Taking $\epsilon>0$ suitably small and applying Gronwall's inequality, we get

$$
\begin{equation*}
\int_{0}^{1} u_{t}^{2} d x+\int_{0}^{\infty} \int_{0}^{1} u_{x t}^{2} d x \leq M \int_{0}^{T}\left(\left\|u_{x}\right\|_{L^{\infty}}^{2}+\left\|u_{x}\right\|_{L^{2}}^{2}+\left\|\theta_{t}\right\|_{L^{\infty}}^{2}\right) d t \leq M \tag{2.17}
\end{equation*}
$$

Next, we show the boundedness of $\left\|u_{x x}(t)\right\|_{L^{2}} .(1.1)_{2}$ can be rewritten as follows:

$$
\frac{\mu}{v} u_{x x}=u_{t}+\left(\frac{a \theta}{v}\right)_{x}+\frac{\mu}{v^{2}} v_{x} u_{x}
$$

then

$$
\begin{equation*}
\int_{0}^{1} u_{x x}^{2} d x \leq M \int_{0}^{1}\left(u_{t}^{2}+\theta_{t}^{2}+v_{x}^{2}\right) d x+M\left\|u_{x}\right\|_{L^{\infty}}^{2} \int_{0}^{1} v_{x}^{2} d x \leq M \tag{2.18}
\end{equation*}
$$

where the Cauchy-Schwarz inequality and the following interpolation inequality have been used:

$$
\left\|u_{x}\right\|_{L^{\infty}}^{2} \leq C\left\|u_{x}\right\|_{H^{1}}\left\|u_{x}\right\|_{L^{2}} \leq \epsilon\left\|u_{x x}\right\|_{L^{2}}^{2}+C_{\epsilon}\left\|u_{x}\right\|_{L^{2}}^{2}
$$

From inequalities (2.17) and (2.18), we obtain (2.16).

## Lemma 2.10 It holds that

$$
\begin{equation*}
\sup _{t \in[0, \infty)}\left\|v_{x x}\right\|_{L^{2}}^{2}+\int_{0}^{\infty} \int_{0}^{1} v_{x x}^{2}(x, t) d x d t \leq M \tag{2.19}
\end{equation*}
$$

Proof It follows from $(1.1)_{1}$ and $(1.1)_{2}$ that

$$
\begin{equation*}
\frac{\mu}{v} v_{x t}+\frac{a}{v^{2}} \theta v_{x}=u_{t}+\frac{a}{v} \theta_{x}+\frac{\mu}{v^{2}} v_{x} u_{x} . \tag{2.20}
\end{equation*}
$$

Differentiating (2.20) with respect to $x$, multiplying it by $v_{x x}$, and integrating the resulting equation over [ 0,1 ], by $(1.1)_{1}$, we have

$$
\begin{align*}
\frac{1}{2} \frac{d}{d t} & \int_{0}^{1} v_{x x}^{2} d x+\int_{0}^{1} v_{x x}^{2} d x \\
\leq & M \int_{0}^{1}\left(\left|u_{x} v_{x x}^{2}\right|+\left|v_{x} v_{x t} v_{x x}\right|+\left|v_{x}^{2} \theta v_{x x}\right|+\left|\theta_{x} v_{x} v_{x x}\right|\right) d x \\
& +M \int_{0}^{1}\left(\left|u_{x t} v_{x x}\right|+\left|\theta_{x x} v_{x x}\right|+\left|v_{x} v_{x x} u_{x x}\right|+\left|v_{x}^{2} u_{x} v_{x x}\right| d x\right) d x \\
\leq & \epsilon \int_{0}^{1} v_{x x}^{2} d x+C_{\epsilon}\left\|u_{x}\right\|_{L^{\infty}}^{2} \int_{0}^{1} v_{x x}^{2} d x+C_{\epsilon}\left\|v_{x}\right\|_{L^{\infty}}^{2} \int_{0}^{1}\left(v_{x t}^{2}+v_{x}^{2}+u_{x x}^{2}+u_{x}^{2}+v_{x x}^{2}\right) d x \\
& +C_{\epsilon}\left\|\theta_{x}\right\|_{L^{\infty}}^{2} \int_{0}^{1} v_{x}^{2} d x+C_{\epsilon} \int_{0}^{1}\left(\left|u_{x t}\right|^{2}+\theta_{x x}^{2}\right) d x \tag{2.21}
\end{align*}
$$

Notice that

$$
\begin{equation*}
\left\|v_{x}\right\|_{L^{\infty}}^{2} \leq \epsilon\left\|v_{x x}\right\|_{L^{2}}^{2}+C_{\epsilon}\left\|v_{x}\right\|_{L^{2}}^{2} \tag{2.22}
\end{equation*}
$$

Taking $\epsilon>0$ appropriately small, inserting (2.22) into (2.21), with the help of Gronwall's inequality and Lemma 2.9, we obtain (2.19).

Lemma 2.11 For any $t \geq 0$, it holds that

$$
\begin{equation*}
\sup _{t \in[0, \infty)}\left\|z_{x x}\right\|_{L^{2}}^{2}+\int_{0}^{\infty} \int_{0}^{1} k \phi(\theta) z_{x x}^{2}(x, t) d x d t+\lambda \int_{0}^{\infty} \int_{0}^{1} z_{x x x}^{2}(x, t) d x d t \leq M . \tag{2.23}
\end{equation*}
$$

Proof Differentiating (1.1) $)_{4}$ with respect to $x$ and setting $h=z_{x}$, we obtain

$$
\begin{equation*}
h_{t}+k \phi(\theta) h+k \phi^{\prime}(\theta) \theta_{x} z h=\left(\frac{\lambda}{v^{2}} h\right)_{x x} . \tag{2.24}
\end{equation*}
$$

Differentiating (2.24) with respect to $x$, multiplying it by $h_{x}$, and integrating the resulting equation over $[0,1]$, we have

$$
\begin{align*}
& \frac{1}{2} \frac{d}{d t} \int_{0}^{1} h_{x}^{2} d x+\int_{0}^{1} k \phi(\theta) h_{x}^{2} d x+\int_{0}^{1} \frac{\lambda}{v^{2}} h_{x x}^{2} d x \\
& \leq \int_{0}^{1}\left|\left(\left(\frac{\lambda}{v^{2}}\right)_{x x} h d x+2\left(\frac{\lambda}{v^{2}}\right)_{x} h_{x}\right) h_{x x}\right| d x+\int_{0}^{1}\left|k \phi^{\prime}(\theta) \theta_{x} h h_{x}\right| d x \\
& \quad+\int_{0}^{1}\left|\left(k \phi^{\prime}(\theta) \theta_{x} z h\right)_{x} h_{x}\right| d x \\
& =\sum_{i=1}^{3} I_{i} \tag{2.25}
\end{align*}
$$

With the help of the Cauchy-Schwarz inequality and Sobolev's imbedding theorem, we have

$$
\begin{align*}
I_{1} & \leq \epsilon \lambda \int_{0}^{1} h_{x x}^{2} d x+C_{\epsilon} \int_{0}^{1}\left(v_{x}^{4}+v_{x x}^{2}\right) h^{2}+v_{x}^{2} h_{x}^{2} d x \\
& \leq \epsilon \lambda \int_{0}^{1} h_{x x}^{2} d x+C_{\epsilon}\|h\|_{L^{\infty}}^{2}\left(\left\|v_{x}\right\|_{L^{\infty}}^{2} \int_{0}^{1} v_{x}^{2} d x+\int_{0}^{1} v_{x x}^{2} d x\right)+C_{\epsilon}\left\|v_{x}\right\|_{L^{\infty}}^{2} \int_{0}^{1} h_{x}^{2} d x \\
& \leq \epsilon \lambda \int_{0}^{1} h_{x x}^{2} d x+C_{\epsilon}\left(\left\|v_{x x}\right\|_{L^{2}}^{2}+\left\|v_{x}\right\|_{L^{2}}^{2}\right) \int_{0}^{1} h_{x}^{2} d x \tag{2.26}
\end{align*}
$$

and

$$
I_{2} \leq \int_{0}^{1}\left|k \phi^{\prime}(\theta) \theta_{x} h h_{x}\right| d x \leq M \int_{0}^{1} h_{x}^{2} d x+M\left\|\theta_{x}\right\|_{L^{\infty}}^{2} \int_{0}^{1} h^{2} d x .
$$

By $\left.h\right|_{x=0,1}=0, z_{x}=h$, the interpolation inequality and the Cauchy-Schwarz inequality, we have

$$
\begin{align*}
I_{3} & \leq \int_{0}^{1}\left|\left(\left(k \phi^{\prime}(\theta) \theta_{x}\right)_{x} z h+k \phi^{\prime}(\theta) \theta_{x} h^{2}+k \phi^{\prime}(\theta) \theta_{x} z h_{x}\right) h_{x}\right| d x \\
& \leq M\left(1+\left\|\theta_{x}\right\|_{L^{\infty}}^{2}\right) \int_{0}^{1} h_{x}^{2} d x+M\|h\|_{L^{\infty}}^{2}\left(\left\|\theta_{x}\right\|_{L^{\infty}}^{2} \int_{0}^{1}\left(\theta_{x}^{2}+h^{2}\right) d x+\int_{0}^{1} \theta_{x x}^{2} d x\right) \\
& \leq M\left(1+\left\|\theta_{x}\right\|_{L^{\infty}}^{2}+\int_{0}^{1} \theta_{x x}^{2} d x\right) \int_{0}^{1} h_{x}^{2} d x . \tag{2.27}
\end{align*}
$$

Thanks to (2.26), (2.27), and (2.25), take $\epsilon>0$ suitably small, by Gronwall's inequality and $z_{x}=h$, we can get (2.23).

### 2.2 Global estimates of (1.5)-(1.8)

For simplicity, in this section, we still use $(v, u, \theta, z)$ to denote the solution of (1.5)-(1.8). The following elementary estimates of the solution of (1.5)-(1.8) can be deduced by the same way as the above section. Here we do not repeat it.

Lemma 2.12 Under the conditions of Theorem 1.1, assume that $(v, u, \theta, z)$ is the solution of (1.5)-(1.8) defined on $[0,1] \times[0, \infty)$. Then

$$
\begin{aligned}
& M^{-1} \leq v(x, t), \quad \theta(x, t) \leq M \quad \text { for all } x \in[0,1], t \in[0, \infty), \\
& \sup _{t \in[0, \infty)}\left(\|(v-\tilde{v}, u, z)(t)\|_{H^{2}}^{2}+\|\theta-\tilde{\theta}\|_{H^{1}}^{2}+\left\|\left(v_{t}, u_{t}\right)\right\|_{L^{2}}^{2}\right) \\
& \quad+\int_{0}^{\infty}\left(\left\|\left(v_{t}, u_{t}, \theta_{t}, z_{t}\right)\right\|_{L^{2}}^{2}\right) d t+\int_{0}^{\infty}\left(\left\|\left(v_{x}, u_{x}, \theta_{x}\right)\right\|_{H^{1}}^{2}+\left\|\left(v_{x t}, u_{x t}\right)\right\|_{L^{2}}^{2}\right) d t \\
& \quad+\int_{0}^{\infty} \int_{0}^{1} k \phi(\theta)\left(z^{2}+z_{x}^{2}+z_{x x}^{2}\right) d x d t \leq M .
\end{aligned}
$$

### 2.3 Species diffusion limit and convergence rates

In this section, we use the previous estimates to prove the species diffusion limit and convergence rates. Assume that $(v, u, \theta, z)$ and $(\bar{v}, \bar{u}, \bar{\theta}, \bar{z})$ are the solutions of problems (1.1)(1.4) and (1.5)-(1.8) defined on $[0,1] \times[0, \infty)$, respectively. Let

$$
\hat{v}=v-\bar{v}, \quad \hat{u}=u-\bar{u}, \quad \hat{\theta}=\theta-\bar{\theta}, \quad \hat{z}=z-\bar{z}
$$

Thus, by (1.1) and (1.5), one can derive

$$
\left\{\begin{array}{l}
\hat{v}_{t}-\hat{u}_{x}=0,  \tag{2.28}\\
\hat{u}_{t}+\left(\frac{a \hat{\theta}}{v}\right)_{x}+\left(\frac{\mu \bar{u}_{x}}{\bar{v}} \frac{\hat{v}}{v}\right)_{x}=\left(\frac{\mu \hat{u}_{x}}{v}\right)_{x}+\left(\frac{a \bar{\theta}}{\bar{v}} \frac{\hat{v}}{v}\right)_{x}, \\
\hat{\theta}_{t}+\frac{a \hat{\theta} u_{x}}{v}+\frac{a \overline{\hat{\theta}} \hat{u}_{x}}{v}+\left(\frac{v \bar{\theta}_{x}}{\bar{v}} \frac{\hat{v}}{v}\right)_{x}+\frac{\mu \bar{u}_{x}^{2}}{\bar{v}} \frac{\hat{v}}{v}=\frac{a \overline{\bar{u}} \bar{u}_{x}}{\bar{v}} \frac{\hat{v}}{v} \\
\quad+\left(\frac{v \hat{\theta}_{x}}{v}\right)_{x}+\frac{\mu \hat{u}_{x}^{2}}{v}+q k(\phi(\theta) z-\phi(\bar{\theta}) \bar{z}) \\
\hat{z}_{t}+k(\phi(\theta) z-\phi(\bar{\theta}) \bar{z})=\left(\frac{\lambda z_{x}}{v^{2}}\right)_{x} .
\end{array}\right.
$$

Next, we use the following four lemmas to show species diffusion limit and convergence rates with $L^{2}$-norm and $H^{1}$-norm, respectively, and this can be illustrated by Theorem 1.3.

Lemma 2.13 Under the conditions of Theorem 1.3, for any fixed $0<T<\infty$, let $(\hat{v}, \hat{u}, \hat{\theta}, \hat{z})$, which is defined on $(0,1) \times[0, T)$, be the solution of problem (2.28). Then

$$
\begin{equation*}
\sup _{t \in[0, T)}\|(\hat{v}, \hat{u}, \hat{\theta}, \hat{z})(\cdot, t)\|_{L^{2}}^{2}+\int_{0}^{T}\left(\|\left(\hat{u}_{x}\left\|_{L^{2}}^{2}+\right\| \hat{\theta}_{x} \|_{L^{2}}^{2}\right) d t \leq N \lambda^{1 / 2}\right. \tag{2.29}
\end{equation*}
$$

where $N$ is a constant independent of $\lambda$.
Proof Multiplying $(2.28)_{1},(2.28)_{2},(2.28)_{3}$, and $(2.28)_{4}$ by $\hat{v}, \hat{u}, \hat{\theta}$, and $\hat{z}$, respectively, and integrating over $[0,1]$, by the Cauchy-Schwarz inequality, we can deduce

$$
\frac{1}{2} \frac{d}{d t}\left(\|(\hat{v}, \hat{u}, \hat{\theta}, \hat{z})\|_{L^{2}}^{2}\right)+\mu\left\|\hat{u}_{x}\right\|_{L^{2}}^{2}+v\left\|\hat{\theta}_{x}\right\|_{L^{2}}^{2}
$$

$$
\begin{aligned}
\leq & \epsilon\left(\left\|\hat{u}_{x}\right\|_{L^{2}}^{2}+\left\|\hat{\theta}_{x}\right\|_{L^{2}}^{2}\right)+C_{\epsilon}\left(1+\left\|\bar{u}_{x}\right\|_{L^{\infty}}^{2}+\left\|\bar{\theta}_{x}\right\|_{L^{\infty}}^{2}\right)\|\hat{v}\|_{L^{2}}^{2} \\
& +C_{\epsilon}\left(1+\left\|u_{x}\right\|_{L^{\infty}}^{2}+\left\|\bar{u}_{x}\right\|_{L^{\infty}}^{2}\right)\|\hat{\theta}\|_{L^{2}}^{2}+C \epsilon\|\hat{z}\|_{L^{2}}^{2}+\lambda^{1 / 2} C_{\epsilon}\left(\lambda\left\|z_{x}\right\|_{L^{2}}^{2}+\left\|\bar{z}_{x}\right\|_{L^{2}}^{2} d x\right) .
\end{aligned}
$$

By virtue of $\hat{v}_{0}(x)=\hat{u}_{0}(x)=\hat{\theta}_{0}(x)=\hat{z}_{0}(x)=0$, the previous estimates in the above sections and Gronwall's inequality, we can verify inequality (2.29). That is, we arrive at the species diffusion limit and convergence rate with $L^{2}$-norm.

Next, by Lemma 2.13, one can establish the following lemma which gives the species diffusion limit and convergence rate with $H^{1}$-norm.

Lemma 2.14 Under the conditions of Theorem 1.3, for any fixed $0<T<\infty$, let $(\hat{v}, \hat{u}, \hat{\theta}, \hat{z})$, which is defined on $(0,1) \times[0, T)$, be the solution of problem (2.28). Then

$$
\sup _{t \in[0, T)}\left(\left\|\hat{v}_{x}\right\|_{L^{2}}^{2}+\left\|\hat{u}_{x}\right\|_{L^{2}}^{2}+\left\|\hat{\theta}_{x}\right\|_{L^{2}}^{2}\right)+\int_{0}^{T}\left(\left\|\hat{u}_{x x}\right\|_{L^{2}}^{2}+\left\|\hat{u}_{t}\right\|_{L^{2}}^{2}+\left\|\hat{\theta}_{t}\right\|_{L^{2}}^{2}\right) \leq N \lambda^{1 / 2}
$$

where $N$ is a constant independent of $\lambda$.

Proof Striving for equation $(2.28)_{1}$ about the derivative of $x$, multiplying by $\hat{v}_{x}$, and integrating over $[0,1]$, by the Cauchy-Schwarz inequality, we obtain

$$
\begin{equation*}
\frac{1}{2} \frac{d}{d t} \int_{0}^{1} \hat{v}_{x}^{2} d x=\int_{0}^{1} \hat{u}_{x x} \hat{v}_{x} d x \leq \epsilon \int_{0}^{1} \hat{u}_{x x}^{2} d x+C_{\epsilon} \int_{0}^{1} \hat{v}_{x}^{2} d x \tag{2.30}
\end{equation*}
$$

Multiplying $(2.28)_{2}$ by $\hat{u}_{t}$, integrating over $[0,1]$ on $x$, using the Cauchy-Schwarz inequality, we have

$$
\begin{align*}
& \frac{1}{2} \frac{d}{d t}\left\|\hat{u}_{x}\right\|_{L^{2}}^{2} d x+\left\|\hat{u}_{t}\right\|_{L^{2}}^{d} x \\
& \quad \leq \epsilon\left\|\hat{u}_{t}\right\|_{L^{2}}^{2}+C_{\epsilon}\left(1+\left\|u_{x}\right\|_{L^{\infty}}^{2}\right)\left\|\hat{u}_{x}\right\|_{L^{2}}^{2}+C_{\epsilon}\left(\left\|\hat{\theta}_{x}\right\|_{L^{2}}^{2}+\left(\left\|\hat{\theta}_{x}\right\|_{L^{2}}^{2}+\|\hat{\theta}\|_{L^{2}}^{2}\right)\left\|v_{x}\right\|_{L^{2}}^{2}\right) \\
& \quad+C_{\epsilon}\left(\left\|\bar{\theta}_{x}\right\|_{L^{\infty}}^{2}+\left\|\bar{x}_{x}\right\|_{L^{2}}^{2}+\left\|v_{x}\right\|_{L^{2}}^{2}+\left\|\bar{u}_{x x}\right\|_{L^{2}}^{2}+\left\|\bar{u}_{x}\right\|_{L^{\infty}}^{2}\left\|\bar{v}_{x}\right\|_{L^{2}}^{2}\right)\|\hat{v}\|_{L^{2}}^{2} \\
& \quad+C_{\epsilon}\left(1+\left\|\bar{v}_{x}\right\|_{L^{2}}^{2}+\left\|v_{x}\right\|_{L^{2}}^{2}+\left\|\bar{u}_{x x}\right\|_{L^{2}}^{2}+\left\|u_{x}\right\|_{L^{\infty}}^{2}+\left\|\bar{u}_{x}\right\|_{L^{\infty}}^{2}\left\|\bar{v}_{x}\right\|_{L^{2}}^{2}\right. \\
& \left.\quad+\left\|\bar{u}_{x}\right\|_{L^{\infty}}^{2}\left\|v_{x}\right\|_{L^{2}}^{2}\right)\left\|\hat{v}_{x}\right\|_{L^{2}}^{2} . \tag{2.31}
\end{align*}
$$

On the other hand, it follows from $(2.28)_{2}$ that

$$
\begin{align*}
& \left\|\hat{u}_{x x}\right\|_{L^{2}}^{2} d x \\
& \leq \\
& \leq M\left(\left\|\hat{u}_{t}\right\|_{L^{2}}^{2}+\left\|\hat{\theta}_{x}\right\|_{L^{2}}^{2}+\|\hat{\theta}\|_{L^{\infty}}^{2}\left\|v_{x}\right\|_{L^{2}}^{2}+\left\|\bar{u}_{x}\right\|_{L^{\infty}}^{2}\|\hat{\hat{v}}\|_{L^{\infty}}^{2}\left(\left\|\bar{v}_{x}\right\|_{L^{2}}^{2}+\left\|v_{x}\right\|_{L^{2}}^{2}\right)\right) \\
& \quad+M\left(\|\hat{v}\|_{L^{\infty}}^{2}\left\|\bar{u}_{x x}\right\|_{L^{2}}^{2}+\left\|\bar{u}_{x}\right\|_{L^{\infty}}^{2}\left\|\hat{v}_{x}\right\|_{L^{2}}^{2}+\left\|\hat{u}_{x}\right\|_{L^{2}}^{2}\right)  \tag{2.32}\\
& \quad+M\left(\|\hat{v}\|_{L^{\infty}}^{2}\left(\left\|\bar{v}_{x}\right\|_{L^{2}}^{2}+\left\|v_{x}\right\|_{L^{2}}^{2}\right)+\left\|\bar{\theta}_{x}\right\|_{L^{\infty}}^{2}\|\hat{v}\|_{L^{2}}^{2}+\left\|\hat{v}_{x}\right\|_{L^{2}}^{2}\right) .
\end{align*}
$$

Inserting (2.32) into (2.31) and taking $\epsilon>0$ sufficiently small, one obtains

$$
\frac{d}{d t}\left\|\hat{u}_{x}\right\|_{L^{2}}^{2}+\left\|\hat{u}_{x x}\right\|_{L^{2}}^{2}+\left\|\hat{u}_{t}\right\|_{L^{2}}^{2}
$$

$$
\begin{align*}
\leq & M\left(\left\|\bar{\theta}_{x}\right\|_{L^{\infty}}^{2}+\left\|v_{x}\right\|_{L^{2}}^{2}+\left\|\bar{v}_{x}\right\|_{L^{2}}^{2}+\left\|\bar{u}_{x x}\right\|_{L^{2}}^{2}+\left\|\bar{u}_{x}\right\|_{L^{\infty}}^{2}\right)\|\hat{v}\|_{L^{2}}^{2} \\
& +M\left(1+\left\|\bar{v}_{x}\right\|_{L^{2}}^{2}+\left\|v_{x}\right\|_{L^{2}}^{2}+\left\|\bar{u}_{x x}\right\|_{L^{2}}^{2}+\left\|u_{x}\right\|_{L^{\infty}}^{2}+\left\|\bar{u}_{x}\right\|_{L^{\infty}}^{2}\right)\left\|\hat{v}_{x}\right\|_{L^{2}}^{2} \\
& +M\left(\left\|\hat{\theta}_{x}\right\|_{L^{2}}^{2}+\|\hat{\theta}\|_{L^{2}}^{2}\left\|v_{x}\right\|_{L^{2}}^{2}+\left(1+\left\|u_{x}\right\|_{L^{\infty}}^{2}\right)\left\|\hat{u}_{x}\right\|_{L^{2}}^{2}\right) . \tag{2.33}
\end{align*}
$$

Combining (2.33) and (2.30) and noticing that $\hat{v}_{0 x}=\hat{u}_{0 x}=0$, by Lemma 2.13 and Gronwall's inequality, for any fixed $0<T<\infty$, one has

$$
\begin{align*}
& \sup _{t \in[0, T)}\left(\left\|\hat{v}_{x}\right\|_{L^{2}}^{2}+\left\|\hat{u}_{x}\right\|_{L^{2}}^{2}\right)+\int_{0}^{T}\left(\left\|\hat{u}_{x x}\right\|_{L^{2}}^{2}+\left\|\hat{u}_{t}\right\|_{L^{2}}^{2}\right) d t \\
& \quad \leq N\left(M \lambda^{1 / 2}+\lambda^{1 / 2} \int_{0}^{T}\left\|v_{x}\right\|_{L^{2}}^{2} d t+\int_{0}^{T}\left\|\hat{\theta}_{x}\right\|_{L^{2}}^{2} d t\right) \leq N \lambda^{1 / 2} \tag{2.34}
\end{align*}
$$

where $N$ is a constant independent of $\lambda$.
Multiplying $(2.28)_{3}$ by $\hat{\theta}_{t}$, integrating over [ 0,1$]$ on $x$, by the Cauchy-Schwarz inequality, we get

$$
\begin{aligned}
& \frac{1}{2} \frac{d}{d t}\left\|\hat{\theta}_{x}\right\|_{L^{2}}^{2}+\left\|\hat{\theta}_{t}\right\|_{L^{2}}^{2} d x \\
& \quad \leq \epsilon\left\|\hat{\theta}_{t}\right\|_{L^{2}}^{2}+C_{\epsilon}\left(1+\left\|u_{x}\right\|_{L^{\infty}}^{2}\right)\left\|\hat{\theta}_{x}\right\|_{L^{2}}^{2} \\
& \quad+C_{\epsilon}\left(\left\|u_{x}\right\|_{L^{\infty}}^{2}\|\hat{\theta}\|_{L^{2}}^{2}+\left\|\hat{u}_{x}\right\|_{L^{2}}^{2}+\left(\|\hat{v}\|_{L^{2}}^{2}+\left\|\hat{v}_{x}\right\|_{L^{2}}^{2}\right)\left\|\bar{\theta}_{x x}\right\|_{L^{2}}^{2}\right) \\
& \quad+C_{\epsilon}\left(\left\|\bar{\theta}_{x}\right\|_{L^{\infty}}^{2}\left\|\hat{v}_{x}\right\|_{L^{2}}^{2}+\left\|\bar{\theta}_{x}\right\|_{L^{\infty}}^{2}\left(\|\hat{v}\|_{L^{2}}^{2}+\left\|\hat{v}_{x}\right\|_{L^{2}}^{2}\right)\left(\left\|v_{x}\right\|_{L^{2}}^{2}+\left\|\bar{v}_{x}\right\|_{L^{2}}^{2}\right)\right) \\
& \quad+C_{\epsilon}\left(\left\|u_{x}\right\|_{L^{2}}^{2}\left\|u_{x}\right\|_{L^{\infty}}^{2}\left(\|\hat{v}\|_{L^{2}}^{2}+\left\|\hat{v}_{x}\right\|_{L^{2}}^{2}\right)+\left(\|\hat{v}\|_{L^{2}}^{2}+\left\|\hat{v}_{x}\right\|_{L^{2}}^{2}\right)\left\|\bar{u}_{x}\right\|_{L^{2}}^{2}\right) \\
& \quad+C_{\epsilon}\left(\left\|\hat{u}_{x}\right\|_{L^{\infty}}^{2}\left\|\hat{u}_{x}\right\|_{L^{2}}^{2}+\|\hat{z}\|_{L^{2}}^{2}\right) .
\end{aligned}
$$

Taking $\epsilon>0$ sufficiently small, by the $L^{2}$-estimates, (2.34), and Gronwall's inequality, for any fixed $0<T<\infty$, we have

$$
\begin{aligned}
& \sup _{t \in[0, T)}\left\|\hat{\theta}_{x}\right\|_{L^{2}}^{2}+\int_{0}^{T}\left\|\hat{\theta}_{t}\right\|_{L^{2}}^{2} d t \\
& \quad \leq N \lambda^{1 / 2}+M\left\|\hat{u}_{x}\right\|_{L^{2}}^{2} \int_{0}^{T}\left(\left\|\bar{u}_{x}\right\|_{L^{\infty}}^{2}+\left\|u_{x}\right\|_{L^{\infty}}^{2}\right) d t \leq N \lambda^{1 / 2}
\end{aligned}
$$

where $N$ is a positive constant independent of $\lambda$.

Lemma 2.15 Under the conditions of Theorem 1.3, for any fixed $0<T<\infty$, let $(\hat{v}, \hat{u}, \hat{\theta}, \hat{z})$, which is defined on $(0,1) \times[0, T)$, be the solution of problem (2.28). Then it holds that

$$
\sup _{t \in[0, T)}\left\|\hat{z}_{x}(t)\right\|_{L^{2}} \leq N \lambda^{1 / 2}
$$

Proof Differentiating $(2.28)_{4}$ with respect to $x$, multiplying it by $\hat{z}_{x}$, and integrating the resulting equation over $[0,1]$, one has

$$
\frac{1}{2} \frac{d}{d t}\left\|\hat{z}_{x}\right\|_{L^{2}}^{2} d x=-\int_{0}^{1}\left(k \phi^{\prime}(\theta) \theta_{x} z-k \phi^{\prime}(\bar{\theta}) \bar{\theta}_{x} \bar{z}+k \phi(\theta) z_{x}-k \phi(\bar{\theta}) \bar{z}_{x}\right) \hat{z}_{x} d x
$$

$$
\begin{equation*}
+\int_{0}^{1}\left(\frac{\lambda}{v^{2}} z_{x}\right)_{x x} \hat{z}_{x} d x=: \sum_{i=1}^{2} I_{i} \tag{2.35}
\end{equation*}
$$

By the Cauchy-Schwarz inequality, we have

$$
\begin{equation*}
I_{1} \leq M \int_{0}^{1} \hat{z}_{x}^{2} d x+M\left(\left\|\theta_{x}\right\|_{L^{\infty}}^{2}+\left\|\bar{\theta}_{x}\right\|_{L^{\infty}}^{2}\right) \int_{0}^{1} \hat{z}^{2} d x \tag{2.36}
\end{equation*}
$$

With the help of Lemmas 2.1-2.11 and the Cauchy-Schwarz inequality, we obtain

$$
\begin{align*}
I_{2} & \leq M \int_{0}^{1}\left|\frac{\lambda}{v^{2}} z_{x x x}+\frac{\lambda}{v^{4}} v_{x}^{2} z_{x}-\frac{\lambda}{\nu^{3}} v_{x x} z_{x}-\frac{\lambda}{\nu^{3}} v_{x} z_{x x}\right|^{2} d x+M \int_{0}^{1} \hat{z}_{x}^{2} d x \\
& \leq M \lambda\left(\lambda \int_{0}^{1} z_{x x x}^{2} d x\right)+M \lambda\left(\lambda \int_{0}^{1} z_{x x}^{2} d x\right)+M \lambda^{2} \int_{0}^{1} v_{x x}^{2} d x+M \int_{0}^{1} \hat{z}_{x}^{2} d x \tag{2.37}
\end{align*}
$$

Combining with (2.35), (2.36), and (2.37), one has

$$
\begin{aligned}
\frac{1}{2} \frac{d}{d t} \int_{0}^{1} \hat{z}_{x}^{2} d x \leq & M \int_{0}^{1} \hat{z}_{x}^{2} d x+M\left(\left\|\theta_{x}\right\|_{L^{\infty}}^{2}+\left\|\bar{\theta}_{x}\right\|_{L^{\infty}}^{2}\right) \int_{0}^{1} \hat{z}^{2} d x \\
& +M \lambda\left(\lambda \int_{0}^{1} z_{x x}^{2} d x+\lambda \int_{0}^{1} z_{x x x}^{2} d x\right)+M \lambda^{2} \int_{0}^{1} v_{x x}^{2} d x .
\end{aligned}
$$

By Gronwall's inequality and $\hat{z}-L^{2}$ norm estimates, we deduce

$$
\begin{aligned}
& \sup _{t \in[0, T)}\left\|\hat{z}_{x}\right\|_{L^{2}}^{2} d x \\
& \quad \leq M \lambda\left(\lambda \int_{0}^{T} \int_{0}^{1}\left(z_{x x}^{2}+z_{x x x}^{2}+v_{x x}^{2}\right) d x d t\right)+M \lambda^{1 / 2} \int_{0}^{T}\left(\left\|\theta_{x}\right\|_{L^{\infty}}^{2}+\left\|\bar{\theta}_{x}\right\|_{L^{\infty}}^{2}\right) d t \\
& \quad \leq M \lambda^{1 / 2}+M \lambda+M \lambda^{2} \leq M \lambda^{1 / 2} .
\end{aligned}
$$

## 3 The species diffusion and the rate of reactant limits

### 3.1 Global $\lambda, k$-independent estimates of (1.1)-(1.4)

Based on Sect. 2, the global well-posedness of solutions to problem (1.1)-(1.4) can be shown in the same way as in $[1,18,20]$. Our main purpose, in this section, is to obtain the global $\lambda, k$-independent estimates of solutions, which will be used to justify the vanishing rate of reactant limit. In order to get our results, we assume that the conditions of Theorem 1.2 hold, and ( $v^{\lambda, k}, u^{\lambda, k}, \theta^{\lambda, k}, z^{\lambda, k}$ ), defined on $(0,1) \times[0, \infty)$, is a solution of problems (1.1)-(1.4). For simplicity, we still use $(v, u, \theta, z)$ to denote the solution of problems (1.1)-(1.4), use $M^{*}$ to denote the generic positive constant which may depend on $a, \mu, v, q,\|\phi\|_{L^{\infty}}$, but not on $\lambda, k$, and $t$.
From equation (1.1) ${ }_{4}$, we know that

$$
\begin{equation*}
\int_{0}^{1} z(x, t) d x+k \int_{0}^{\infty} \int_{0}^{1} \phi(\theta) z(x, t) d x d \tau=\int_{0}^{1} z_{0}(x) d x \tag{3.1}
\end{equation*}
$$

On the other hand, the rate function $\phi(\theta)$ is smooth, we have

$$
\begin{equation*}
k^{2} \int_{0}^{1} \phi(\theta)^{2} z^{2} d x \leq k \int_{0}^{1} \phi(\theta) z^{2} d x \tag{3.2}
\end{equation*}
$$

From (3.1)-(3.2), and based on Sect. 2, we know that the global $\lambda, k$-independent estimates of solutions $(v, u, \theta)$ are similar to the estimates of problems (1.1)-(1.4) in Sect. 2. Our main purpose in this section is to obtain the global $\lambda, k$-independent estimates of $z$.

Multiplying $(1.1)_{4}$ by $z$ and integrating over $\Omega$, we have

$$
\begin{equation*}
\sup _{t \in[0, \infty)}\|z\|_{L^{2}}^{2}+\lambda \int_{0}^{\infty}\left\|z_{x}\right\|_{L^{2}}^{2} d t+k \int_{0}^{\infty} \int_{0}^{1} \phi(\theta) z^{2} d x d t \leq M^{*} . \tag{3.3}
\end{equation*}
$$

Next, multiplying both sides of $(1.1)_{4}$ by $z_{x x}$, it follows

$$
\begin{aligned}
& \frac{1}{2} \frac{d}{d t} \int_{0}^{1} z_{x}^{2} d x+k \int_{0}^{1} \phi(\theta) z_{x}^{2} d x+\lambda \int_{0}^{1} z_{x x}^{2} d x \\
& \quad \leq \epsilon k \int_{0}^{1} z_{x}^{2} d x+C_{\epsilon} \int_{0}^{1} \theta_{x}^{2} d x+\frac{1}{2} \lambda \int_{0}^{1} z_{x x}^{2} d x+M^{*} \lambda \int_{0}^{1} z_{x}^{2} d x
\end{aligned}
$$

Taking $\epsilon>0$ suitably small, by the Cauchy-Schwarz inequality and (3.3), we have

$$
\begin{equation*}
\int_{0}^{1} z_{x}^{2} d x+\lambda \int_{0}^{\infty} \int_{0}^{1} z_{x x}^{2} d x d \tau+k \int_{0}^{\infty} \int_{0}^{1} \phi(\theta) z_{x}^{2} d x d \tau \leq M^{*} \tag{3.4}
\end{equation*}
$$

Finally, differentiating (1.1) ${ }_{4}$ with respect to $x$ and setting $\omega=z_{x}$, we get

$$
\begin{equation*}
\omega_{t}+k \phi(\theta) \omega+k \phi^{\prime}(\theta) \theta_{x} z \omega=\left(\frac{\lambda}{v^{2}} \omega\right)_{x x} . \tag{3.5}
\end{equation*}
$$

Differentiating (3.5) with respect to $x$, multiplying it by $\omega_{x}$, and integrating the resulting equation over $[0,1]$, then

$$
\begin{align*}
& \frac{1}{2} \frac{d}{d t} \int_{0}^{1} \omega_{x}^{2} d x+k \int_{0}^{1} \phi(\theta) \omega_{x}^{2} d x+\int_{0}^{1} \frac{\lambda}{v^{2}} \omega_{x x}^{2} d x \\
& \leq \int_{0}^{1}\left|\left(\left(\frac{\lambda}{v^{2}}\right)_{x x} \omega d x+2\left(\frac{\lambda}{v^{2}}\right)_{x} \omega_{x}\right) \omega_{x x}\right| d x \\
& \quad+k \int_{0}^{1}\left|\phi^{\prime}(\theta) \theta_{x} \omega \omega_{x}\right| d x+k \int_{0}^{1}\left|\left(\phi^{\prime}(\theta) \theta_{x} z \omega\right)_{x} \omega_{x}\right| d x \triangleq \sum_{i=1}^{3} J_{i} . \tag{3.6}
\end{align*}
$$

With the help of the estimates in Sect. 2, the Cauchy-Schwarz inequality, and Sobolev's imbedding theorem, we have

$$
\begin{align*}
J_{1} & \leq \epsilon \lambda \int_{0}^{1} \omega_{x x}^{2} d x+C_{\epsilon} \int_{0}^{1}\left(v_{x}^{4}+v_{x x}^{2}\right) \omega^{2}+v_{x}^{2} \omega_{x}^{2} d x \\
& \leq \epsilon \lambda \int_{0}^{1} \omega_{x x}^{2} d x+C_{\epsilon}\|\omega\|_{L^{\infty}}^{2}\left(\left\|v_{x}\right\|_{L^{\infty}}^{2} \int_{0}^{1} v_{x}^{2} d x+\int_{0}^{1} v_{x x}^{2} d x\right)+C_{\epsilon}\left\|v_{x}\right\|_{L^{\infty}}^{2} \int_{0}^{1} \omega_{x}^{2} d x \\
& \leq \epsilon \lambda \int_{0}^{1} \omega_{x x}^{2} d x+C_{\epsilon}\left(\left\|v_{x x}\right\|_{L^{2}}^{2}+\left\|v_{x}\right\|_{L^{2}}^{2}\right) \int_{0}^{1} \omega_{x}^{2} d x \tag{3.7}
\end{align*}
$$

By the Cauchy-Schwarz inequality, one has

$$
\begin{equation*}
J_{2} \leq k \int_{0}^{1}\left|\phi^{\prime}(\theta) \theta_{x} \omega \omega_{x}\right| d x \leq \epsilon k \int_{0}^{1} \omega_{x}^{2} d x+C_{\epsilon}\left\|\theta_{x}\right\|_{L^{\infty}}^{2} \int_{0}^{1} \omega^{2} d x \tag{3.8}
\end{equation*}
$$

Noticing that $\left.\omega\right|_{x=0,1}=0$ and by the interpolation inequality, we deduce

$$
\begin{align*}
J_{3} & \leq k \int_{0}^{1}\left|\left(\left(\phi^{\prime}(\theta) \theta_{x}\right)_{x} z \omega+\phi^{\prime}(\theta) \theta_{x} \omega^{2}+\phi^{\prime}(\theta) \theta_{x} z \omega_{x}\right) \omega_{x}\right| d x \\
& \leq \epsilon k \int_{0}^{1} \omega_{x}^{2} d x+C_{\epsilon}\left(\left\|\theta_{x}\right\|_{L^{\infty}}^{2}+\int_{0}^{1} \theta_{x x}^{2} d x\right) \int_{0}^{1} \omega_{x}^{2} d x . \tag{3.9}
\end{align*}
$$

With the aid of (3.3)-(3.4), taking $\epsilon>0$ suitably small and applying Gronwall's inequality for (3.6), we obtain

$$
\begin{equation*}
\sup _{t \in[0, \infty)}\left\|\omega_{x}\right\|_{L^{2}}^{2}+k \int_{0}^{\infty} \int_{0}^{1} \phi(\theta) \omega_{x}^{2}(x, t) d x d t+\lambda \int_{0}^{\infty} \int_{0}^{1} \omega_{x x}^{2}(x, t) d x d t \leq M^{*} \tag{3.10}
\end{equation*}
$$

which implies

$$
\begin{equation*}
\sup _{t \in[0, \infty)}\left\|z_{x x}\right\|_{L^{2}}^{2}+k \int_{0}^{\infty} \int_{0}^{1} \phi(\theta) z_{x x}^{2}(x, t) d x d t+\lambda \int_{0}^{\infty} \int_{0}^{1} z_{x x x}^{2}(x, t) d x d t \leq M^{*} \tag{3.11}
\end{equation*}
$$

With the help of (3.1)-(3.11) and Sect. 2, we have the following results, which imply Theorem 1.2(i).

Lemma 3.1 Suppose that

$$
0<v_{0}, 0<\theta_{0}, \quad\left(v_{0}, u_{0}, \theta_{0}, z_{0}\right)(x) \in H^{1}
$$

Then, for each fixed $\lambda, k>0$, there exists a unique global solution $(v, u, \theta, z)$ to the initialboundary value problem (1.1)-(1.4) on $(0,1) \times[0, \infty)$ such that

$$
\begin{aligned}
& M^{*-1} \leq v(x, t), \quad \theta(x, t) \leq M^{*} \quad \text { for all }(x, t) \in[0,1] \times[0, \infty) \\
& \sup _{t \in[0, \infty)}\left(\|(v-\tilde{v}, u)\|_{H^{2}}^{2}+\|\theta-\tilde{\theta}\|_{H^{1}}^{2}+\left\|\left(v_{t}, u_{t}\right)\right\|_{L^{2}}^{2}\right) \\
& \quad+\int_{0}^{\infty}\left(\left\|\left(v_{x}, u_{x}, \theta_{x}\right)\right\|_{H^{1}}^{2}+\left\|\left(v_{x t}, u_{x t}\right)\right\|_{L^{2}}^{2}\right) d t+\int_{0}^{\infty}\left(\left\|\left(v_{t}, u_{t}, \theta_{t}, z_{t}\right)\right\|_{L^{2}}^{2}\right) d t \leq M^{*}, \\
& \sup _{t \in[0, \infty)}\|z(t)\|_{H^{2}}^{2}+\lambda \int_{0}^{\infty}\left\|z_{x}\right\|_{H^{2}}^{2} d t+k \int_{0}^{\infty} \int_{0}^{1} \phi(\theta)\left(z^{2}+z_{x}^{2}+z_{x x}^{2}\right) d x d t \leq M^{*}
\end{aligned}
$$

where the positive constants $\bar{v}$ and $\bar{\theta}$ are defined in Theorem 1.1, and $M^{*}$ denotes the generic positive constant which may depend on $a, \mu, v, q,\|\phi\|_{L^{\infty}}$, but not depend on $\lambda, k$, and $t$.

### 3.2 Global estimates of (1.9)-(1.12)

In this section, our purpose is to derive the global estimates of the solutions to the initialboundary value problem of (1.9)-(1.12) under the conditions of Theorem 1.2. For simplicity, in this section, we still use $(v, u, \theta, z)$ to denote the solution of problem (1.9)-(1.12), $M^{*}$ denotes the generic positive constant which may depend on $a, \mu, \nu, q,\|\phi\|_{L^{\infty}}$, but not on $\lambda, k$, and $t$.
The following elementary estimates are easily derived from (1.9)-(1.12) by the same way as Lemmas 2.1-2.11.

Lemma 3.2 Under the conditions of Theorem 1.2, assume that $(v, u, \theta, z)$ is the solution of (1.9)-(1.12) defined on $[0,1] \times[0, \infty)$. Then

$$
M^{*-1} \leq v(x, t), \quad \theta(x, t) \leq M^{*} \quad \text { for all } x \in[0,1], t \in[0, \infty)
$$

and

$$
\begin{aligned}
& \sup _{t \in[0, \infty)}\left(\|(v-\tilde{v}, u, \theta-\tilde{\theta}, z)(t)\|_{H^{1}}^{2}+\left\|v_{t}\right\|_{L^{2}}^{2}+\left\|u_{t}\right\|_{L^{2}}^{2}\right) \\
& \quad+\int_{0}^{\infty}\left(\left\|v_{x}\right\|_{L^{2}}^{2}+\left\|u_{x}\right\|_{L^{2}}^{2}+\left\|\theta_{x}\right\|_{L^{2}}^{2}+\left\|z_{x}\right\|_{L^{2}}^{2}\right) d t \leq M^{*}
\end{aligned}
$$

where the positive constants $\tilde{v}$ and $\tilde{\theta}$ are defined in Theorem 1.1.

Lemma 3.3 Let the conditions of Theorem 1.2 be in force. Assume that $(v, u, \theta, z)$ is the solution of (1.9)-(1.12) defined on $[0,1] \times[0, \infty)$. Then

$$
\begin{aligned}
& \sup _{t \in[0, \infty)}\left\|\left(v_{x}, u_{x}, z_{x}\right)(t)\right\|_{H^{1}}^{2}+\int_{0}^{\infty}\left(\left\|v_{t}\right\|_{L^{2}}^{2}+\left\|u_{t}\right\|_{L^{2}}^{2}+\left\|\theta_{t}\right\|_{L^{2}}^{2}+\left\|z_{t}\right\|_{L^{2}}^{2}\right) d t \\
& \quad+\int_{0}^{\infty}\left(\left\|v_{x x}\right\|_{L^{2}}^{2}+\left\|u_{x x}\right\|_{L^{2}}^{2}+\left\|\theta_{x x}\right\|_{L^{2}}^{2}+\left\|v_{x t}\right\|_{L^{2}}^{2}+\left\|u_{x t}\right\|_{L^{2}}^{2}\right) d t \leq M^{*} .
\end{aligned}
$$

### 3.3 The species diffusion and rate of reactant limits and convergence rates

In this section, we use the previous estimates to prove the species diffusion and rate of reactant limit and the convergence rates. Assume that $(v, u, \theta, z)$ and $(\bar{v}, \bar{u}, \bar{\theta}, \bar{z})$ are the solutions of problems (1.1)-(1.4) and (1.9)-(1.12) defined on $[0,1] \times[0, \infty)$, respectively. Let

$$
\hat{v}=v-\bar{v}, \quad \hat{u}=u-\bar{u}, \quad \hat{\theta}=\theta-\bar{\theta}, \quad \hat{z}=z-\bar{z} .
$$

Then $(\hat{v}, \hat{u}, \hat{\theta}, \hat{z})$ satisfies

$$
\left\{\begin{array}{l}
\hat{v}_{t}-\hat{u}_{x}=0  \tag{3.12}\\
\hat{u}_{t}+\left(\frac{a \hat{\theta}}{v}\right)_{x}+\left(\frac{\mu \bar{u}_{x}}{\bar{v}} \frac{\hat{v}}{v}\right)_{x}=\left(\frac{\mu \hat{u}_{x}}{v}\right)_{x}+\left(\frac{a \bar{\theta}}{\bar{v}} \frac{\hat{v}}{v}\right)_{x}, \\
\hat{\theta}_{t}+\frac{a \hat{\theta} u_{x}}{v}+\frac{a \bar{\theta} \hat{u}_{x}}{v}+\left(\frac{v \bar{\theta}_{x}}{\bar{v}} \frac{\hat{v}}{v}\right)_{x}+\frac{\mu \bar{u}_{x}^{2} \hat{v}}{\bar{v}} \frac{a \bar{\theta} \bar{u}_{x}}{\bar{v}} \frac{\hat{v}}{v}+\left(\frac{v \hat{\theta}_{x}}{v}\right)_{x}+\frac{\mu \hat{u}_{x}^{2}}{v}+q k \phi(\theta) z, \\
\hat{z}_{t}+k \phi(\theta) z=\left(\frac{\lambda z_{x}}{v^{2}}\right)_{x} .
\end{array}\right.
$$

Next, we have the following four lemmas to show the species diffusion and rate of reactant limits, and the convergence rates with $L^{2}$-norm and $H^{1}$-norm, respectively, and this can be illustrated by Theorem 1.4.

Lemma 3.4 Under the conditions of Theorem 1.4, for any fixed $0<T<\infty$, let $(\hat{v}, \hat{u}, \hat{\theta}, \hat{z})$, defined on $(0,1) \times[0, T)$, be the solution of problem (3.12). Then

$$
\begin{equation*}
\sup _{t \in[0, T)}\|(\hat{v}, \hat{u}, \hat{\theta}, \hat{z})(\cdot, t)\|_{L^{2}}^{2}+\int_{0}^{T}\left(\|\left(\hat{u}_{x}\left\|_{L^{2}}^{2}+\right\| \hat{\theta}_{x} \|_{L^{2}}^{2}\right) d t \leq N^{*}\left(\lambda^{1 / 2}+k^{1 / 2}\right)\right. \tag{3.13}
\end{equation*}
$$

where $N^{*}$ is a positive constant independent of $\lambda, k$.

Proof Multiplying (3.12) ${ }_{1}$ and $(3.12)_{2}$ by $\hat{v}$ and $\hat{u}$, respectively and integrating it over [ 0,1$]$, by the Cauchy-Schwarz inequality, we can deduce

$$
\begin{equation*}
\frac{1}{2} \frac{d}{d t} \int_{0}^{1} \hat{u}^{2}+\hat{v}^{2} d x+\mu \int_{0}^{1} \hat{u}_{x}^{2} d x \leq \epsilon \int_{0}^{1} \hat{u}_{x}^{2} d x+C_{\epsilon}\left(\left\|\bar{u}_{x}\right\|_{L^{\infty}}^{2}+1\right) \int_{0}^{1} \hat{v}^{2} d x \tag{3.14}
\end{equation*}
$$

Multiplying both sides of $(3.12)_{3}$ by $\hat{\theta}$ and integrating it over [ 0,1 ], by the CauchySchwarz inequality, we have

$$
\begin{aligned}
& \frac{1}{2} \frac{d}{d t} \int_{0}^{1} \hat{\theta}^{2} d x+v \int_{0}^{1} \hat{\theta}_{x}^{2} d x \\
& \quad \leq M^{*} k^{2} \int_{0}^{1} \hat{z}^{2} d x+M^{*}\left\|\bar{u}_{x}\right\|_{L^{\infty}}^{2} \int_{0}^{1} \hat{v}^{2} d x+M^{*} \int_{0}^{1} \hat{\theta}^{2} d x+\epsilon \int_{0}^{1} \hat{u}_{x}^{2} d x \\
& \quad+C_{\epsilon}\left(\left\|\bar{u}_{x}\right\|_{L^{\infty}}^{2}+\left\|u_{x}\right\|_{L^{\infty}}^{2}\right) \int_{0}^{1} \hat{\theta}^{2} d x+C_{\epsilon} \int_{0}^{1} \hat{\theta}^{2} d x+\epsilon \int_{0}^{1} \hat{\theta}_{x}^{2} d x \\
& \quad+C_{\epsilon}\left\|\bar{\theta}_{x}\right\|_{L^{\infty}}^{2} \int_{0}^{1} \hat{v}^{2} d x+M^{*}\left\|\bar{u}_{x}\right\|_{L^{\infty}}^{2} \int_{0}^{1}\left(\hat{v}^{2}+\hat{\theta}^{2}\right) d x
\end{aligned}
$$

Multiplying (3.12) ${ }_{4}$ by $\hat{z}$, integrating it over [ 0,1 ], by the Cauchy-Schwarz inequality, we obtain

$$
\begin{align*}
\frac{1}{2} \frac{d}{d t} \int_{0}^{1} \hat{z}^{2} d x= & -\int_{0}^{1} \frac{\lambda}{v^{2}} z_{x}(z-\bar{z})_{x} d x-k \int_{0}^{1} \phi(\theta) z(z-\bar{z}) d x \\
\leq & M^{*} \lambda^{1 / 2}\left(\lambda \int_{0}^{1} z_{x}^{2} d x+\int_{0}^{1} \bar{z}_{x}^{2} d x\right) \\
& +M^{*} k^{1 / 2}\left(k \int_{0}^{1} z^{2} d x+\int_{0}^{1} \bar{z}^{2} d x\right) \tag{3.15}
\end{align*}
$$

Combining (3.14)-(3.15) and taking $\epsilon>0$ sufficiently small, noticing that $\hat{v}_{0}(x)=\hat{u}_{0}(x)=$ $\hat{\theta}_{0}(x)=\hat{z}_{0}(x)=0$, by previous estimates and Gronwall's inequality, we can obtain (3.13).

Next, by using Lemma 3.4, one can establish the species diffusion and rate of reactant limit and convergence rate with $H^{1}$-norm as follows.

Lemma 3.5 Under the conditions of Theorem 1.4, for any fixed $0<T<\infty$, let $(\hat{v}, \hat{u}, \hat{\theta}, \hat{z})$, defined on $(0,1) \times[0, T)$, be the solution of problem (3.12). Then

$$
\sup _{t \in[0, T)}\left(\left\|\hat{v}_{x}\right\|_{L^{2}}^{2}+\left\|\hat{u}_{x}\right\|_{L^{2}}^{2}+\left\|\hat{\theta}_{x}\right\|_{L^{2}}^{2}\right)+\int_{0}^{T}\left(\left\|\hat{u}_{x x}\right\|_{L^{2}}^{2}+\left\|\hat{u}_{t}\right\|_{L^{2}}^{2}+\left\|\hat{\theta}_{t}\right\|_{L^{2}}^{2}\right) \leq N^{*}\left(\lambda^{1 / 2}+k^{1 / 2}\right),
$$

where $N^{*}$ is a positive constant independent of $\lambda, k$.

Proof By (2.30) and (2.33) and Lemma 3.4, we have

$$
\begin{aligned}
& \sup _{t \in[0, \infty)}\left(\left\|\hat{v}_{x}\right\|_{L^{2}}^{2}+\left\|\hat{u}_{x}\right\|_{L^{2}}^{2}\right)+\int_{0}^{\infty}\left(\left\|\hat{u}_{x x}\right\|_{L^{2}}^{2}+\left\|\hat{u}_{t}\right\|_{L^{2}}^{2}\right) d t \\
& \quad \leq N^{*}\left(M^{*} \lambda^{1 / 2}+\lambda^{1 / 2} \int_{0}^{\infty}\left\|v_{x}\right\|_{L^{2}}^{2} d t+\int_{0}^{\infty}\left\|\hat{\theta}_{x}\right\|_{L^{2}}^{2} d t\right)
\end{aligned}
$$

$$
\leq N^{*}\left(\lambda^{1 / 2}+k^{1 / 2}\right)
$$

Next, multiplying both sides of $(3.12)_{3}$ by $\hat{\theta}_{t}$ and integrating it over $[0,1]$, using the Cauchy-Schwarz inequality, we deduce

$$
\begin{aligned}
& \frac{1}{2} \frac{d}{d t} \int_{0}^{1} \hat{\theta}_{x}^{2} d x+\int_{0}^{1} \hat{\theta}_{t}^{2} d x \\
& \leq \epsilon\left\|\hat{\theta}_{t}\right\|_{L^{2}}^{2}+C_{\epsilon} k \int_{0}^{1} \phi(\theta) z^{2} d x+M^{*}\left(1+\left\|u_{x}\right\|_{L^{\infty}}^{2}\right)\left\|\hat{\theta}_{x}\right\|_{L^{2}}^{2} \\
&+C_{\epsilon}\left(\left\|u_{x}\right\|_{L^{\infty}}^{2}\|\hat{\theta}\|_{L^{2}}^{2}+\left\|\hat{u}_{x}\right\|_{L^{2}}^{2}+\left(\|\hat{v}\|_{L^{2}}^{2}+\left\|\hat{v}_{x}\right\|_{L^{2}}^{2}\right)\left\|\bar{\theta}_{x x}\right\|_{L^{2}}^{2}\right) \\
& \quad+C_{\epsilon}\left(\left\|\bar{\theta}_{x}\right\|_{L^{\infty}}^{2}\left\|\hat{v}_{x}\right\|_{L^{2}}^{2}+\left\|\bar{\theta}_{x}\right\|_{L^{\infty}}^{2}\left(\|\hat{v}\|_{L^{2}}^{2}+\left\|\hat{v}_{x}\right\|_{L^{2}}^{2}\right)\left(\left\|v_{x}\right\|_{L^{2}}^{2}+\left\|\bar{v}_{x}\right\|_{L^{2}}^{2}\right)\right) \\
& \quad+C_{\epsilon}\left(\left\|u_{x}\right\|_{L^{2}}^{2}\left\|u_{x}\right\|_{L^{\infty}}^{2}\left(\|\hat{v}\|_{L^{2}}^{2}+\left\|\hat{v}_{x}\right\|_{L^{2}}^{2}\right)+\left(\|\hat{v}\|_{L^{2}}^{2}+\left\|\hat{v}_{x}\right\|_{L^{2}}^{2}\right)\left\|\bar{u}_{x}\right\|_{L^{2}}^{2}\right) \\
& \quad+C_{\epsilon}\left(\left\|\hat{u}_{x}\right\|_{L^{\infty}}^{2}\left\|\hat{u}_{x}\right\|_{L^{2}}^{2}\right) .
\end{aligned}
$$

Taking $\epsilon>0$ sufficiently small and using Gronwall's inequality, for any fixed $0<T<\infty$, it follows

$$
\begin{aligned}
\sup _{t \in[0, T)}\left\|\hat{\theta}_{x}\right\|_{L^{2}}^{2}+\int_{0}^{T}\left\|\hat{\theta}_{t}\right\|_{L^{2}}^{2} d t & \leq N^{*} \lambda^{1 / 2}+M^{*}\left\|\hat{u}_{x}\right\|_{L^{2}}^{2} \int_{0}^{T}\left(\left\|\bar{u}_{x}\right\|_{L^{\infty}}^{2}+\left\|u_{x}\right\|_{L^{\infty}}^{2}\right) d t \\
& \leq N^{*}\left(\lambda^{1 / 2}+k^{1 / 2}\right) .
\end{aligned}
$$

Lemma 3.6 Under the conditions of Theorem 1.4, for any fixed $0<T<\infty$, let $(\hat{v}, \hat{u}, \hat{\theta}, \hat{z})$, defined on $(0,1) \times[0, T)$, be the solution of problem (3.12). Then it holds that

$$
\sup _{t \in[0, T)}\left\|\hat{z}_{x}(t)\right\|_{L^{2}} \leq M^{*}\left(\lambda^{1 / 2}+k^{1 / 2}\right)
$$

Proof Differentiating $(3.12)_{4}$ with respect to $x$, multiplying it by $\hat{z}_{x}$, and integrating the resulting equation over $[0,1]$, we obtain

$$
\frac{1}{2} \frac{d}{d t} \int_{0}^{1} \hat{z}_{x}^{2} d x=-k \int_{0}^{1} \phi^{\prime}(\theta) \theta_{x} z \hat{z}_{x} d x-k \int_{0}^{1} \phi(\theta) z_{x} \hat{z}_{x} d x+\int_{0}^{1}\left(\frac{\lambda}{v^{2}} z_{x}\right)_{x x} \hat{z}_{x} d x=\sum_{i=1}^{3} J_{i}
$$

By the Cauchy-Schwarz inequality, we have

$$
\begin{equation*}
J_{1}=-k \int_{0}^{1} \phi^{\prime}(\theta) \theta_{x} z \hat{z}_{x} d x \leq M^{*} \int_{0}^{1} \hat{z}_{x}^{2} d x+M^{*} k^{2}\left\|\theta_{x}\right\|_{L^{\infty}}^{2} \int_{0}^{1} \hat{z}^{2} d x \tag{3.16}
\end{equation*}
$$

and

$$
\begin{aligned}
J_{2} & =-k \int_{0}^{1} \phi(\theta) z_{x}^{2} d x+k \int_{0}^{1} \phi(\theta) z_{x} \bar{z}_{x} d x \\
& \leq \int_{0}^{1} k^{3 / 4}\left|z_{x}\right| k^{1 / 4}\left|\bar{z}_{x}\right| d x \leq M^{*} k^{1 / 2}\left(k \int_{0}^{1} z_{x}^{2} d x+\int_{0}^{1} \bar{z}_{x}^{2} d x\right)
\end{aligned}
$$

With the help of Lemmas 2.1-2.11 and Young's inequality, we deduce

$$
\begin{align*}
J_{3} \leq & M^{*} \int_{0}^{1}\left|\frac{\lambda}{\nu^{2}} z_{x x x}+\frac{\lambda}{\nu^{2}} v_{x}^{2} z_{x}-\frac{\lambda}{\nu^{3}} v_{x x} z_{x}-\frac{\lambda}{\nu^{3}} v_{x} z_{x x}\right|^{2} d x+M^{*} \int_{0}^{1} \hat{z}_{x}^{2} d x \\
\leq & M^{*} \lambda^{2} \int_{0}^{1}\left(z_{x x}^{2}+z_{x x x}^{2}+v_{x x}^{2}\right) d x+M^{*} \lambda^{2}\left\|v_{x}\right\|_{L^{2}}^{2} \int_{0}^{1} z_{x x}^{2} d x+M^{*} \int_{0}^{1} \hat{z}_{x}^{2} d x \\
\leq & M^{*} \lambda\left(\lambda \int_{0}^{1} z_{x x x}^{2} d x\right)+M^{*} \lambda\left(\lambda \int_{0}^{1} z_{x x}^{2} d x\right) \\
& +M^{*} \lambda^{2} \int_{0}^{1} v_{x x}^{2} d x+M^{*} \int_{0}^{1} \hat{z}_{x}^{2} d x \tag{3.17}
\end{align*}
$$

Combining with (3.16)-(3.17), we have

$$
\begin{aligned}
\frac{1}{2} \frac{d}{d t} \int_{0}^{1} \hat{z}_{x}^{2} d x \leq & M^{*} k^{1 / 2}\left(k \int_{0}^{1} z_{x}^{2} d x+\int_{0}^{1} \bar{z}_{x}^{2} d x\right)+M^{*} k^{2}\left\|\theta_{x}\right\|_{L^{\infty}}^{2} \int_{0}^{1} \hat{z}^{2} d x \\
& +M^{*} \lambda\left(\lambda \int_{0}^{1} z_{x x}^{2} d x+\lambda \int_{0}^{1} z_{x x x}^{2} d x\right)+M^{*} \lambda^{2} \int_{0}^{1} v_{x x}^{2} d x
\end{aligned}
$$

By Gronwall's inequality and $\hat{z}-L^{2}$ norm estimates, we obtain

$$
\begin{aligned}
\sup _{t \in[0, \infty)}\left\|\hat{z}_{x}\right\|_{L^{2}}^{2} d x & \leq M^{*} \lambda^{1 / 2}+M^{*} \lambda+M^{*} \lambda^{2}+M^{*} k^{1 / 2}+M^{*} k^{2} \\
& \leq M^{*}\left(\lambda^{1 / 2}+k^{1 / 2}\right) .
\end{aligned}
$$

This completes the proof of Lemma 3.6.

With the help of Lemmas 3.4-3.6, we can obtain Theorem 1.4.

## Acknowledgements

The author would like to thank the anonymous referee for his/her helpful comments, which have improved the presentation of the paper.

## Funding

There is no fund to support this work.

## Availability of data and materials

Not applicable.

## Competing interests

The author declares that she has no competing interests.

## Authors' contributions

This entire work has been completed by the author, Dr. MZ. The author read and approved the finial manuscript.

## Authors' information

The author of this article is Dr. Zhang. She comes from the School of Mathematics Science, Xiamen University.

## Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

## References

1. Chen, G.-Q.: Global solutions to the compressible Navier-Stokes equations for a reacting mixture of nonlinear. SIAM J. Math. Anal. 23, 609-634 (1992)
2. Li, J., Liang, Z.-L.: Some uniform estimates and large-time behavior of solutions to one-dimensional compressible Navier-Stokes system in unbounded domains with large data. Arch. Ration. Mech. Anal. 220, 1195-1208 (2016)
3. Kazhikhov, A.V.: On the theory of initial-boundary-value problems for the equations of one-dimensional nonstationary motion of a viscous heat-conductive gas. Din. Sploš. Sredy 50, 37-62 (1981)
4. Bebernes, J., Eberly, D.: Mathematical Problems from Combustion Theory. Springer, New York (1989)
5. Williams, F.A.: Combustion Theory: The Fundamental Theory of Chemically Reacting Flow System, 2nd edn. Benjamin-Cummings, San Francisco (1985)
6. Gardner, R.A.: On the detonation of a combustible gas. Trans. Am. Math. Soc. 277, 431-468 (1983)
7. Wagner, D.H.: The existence and behavior of viscous structure for plane detonation waves. SIAM J. Math. Anal. 20, 1035-1054 (1989)
8. Williams, F.A.: Combustion Theory. Addison-Wesley, Reading (1965)
9. Collela, P., Majda, A., Roytburd, V.: Theoretical and numerical structure for reacting shock waves. SIAM J. Sci. Stat. Comput. 7, 1059-1080 (1986)
10. Majda, A.: High mach number combustion. Lect. Notes Appl. Math. 24, 109-184 (1986)
11. Ludford, G.S.S.: Low mach number combustion. Lect. Notes Appl. Math. 24, 3-74 (1986)
12. Okada, M., Kawashima, S.: On the equations of one-dimensional motion of compressible viscous fluids. J. Math. Kyoto Univ. 23, 55-71 (1983)
13. Matsumura, A., Nishida, T.: The initial-value problem for the equations of motion of viscous and heat-conductive gas. J. Math. Kyoto Univ. 20, 67-104 (1980)
14. Kanel', Y.A.: On a model system of equations of one-dimensional gas motion. Differ. Equ. 4, 374-380 (1968)
15. Itaya, N.: On the Cauchy problem for the system of fundamental equations describing the movement of compressible fluid. Kodai Math. Semin. Rep. 23, 60-120 (1971)
16. Ducomet, B.: Hydrodynamical models of gaseous stars. Rev. Math. Phys. 8, 957-1000 (1996)
17. Ducomet, B.: A model of thermal dissipation for a one-dimensional viscous reactive and radiative gas. Math. Methods Appl. Sci. 22, 1323-1349 (1999)
18. Chen, G.-Q., Hoff, D., Trivisa, K.: Global solution to a model for exothermically reacting compressible flows with large discontinuous data. Arch. Ration. Mech. Anal. 166, 321-358 (2003)
19. Zlotnik, A.: Weak solutions to the equations of motion of viscous compressible reacting binary mixture: uniqueness and Lipschitz-continuous dependence on data. Math. Notes. 75, 307-311 (2004)
20. Kazhikhov, A.V., Shelukhin, V.V.: Unique global solution with respect to time of the initial-boundary value problems for one-dimensional equations of a viscous gas. J. Appl. Math. Mech. 41, 282-291 (1977)
21. Nishida, T.: Equations of motion of compressible viscous fluids. In: Nishida, T., Mimura, M., Fujii, H. (eds.) Pattern and Waves, North-Holland, Amsterdam pp. 97-128 (1986)

## Submit your manuscript to a SpringerOpen ${ }^{\ominus}$ journal and benefit from:

- Convenient online submission
- Rigorous peer review
- Open access: articles freely available online
- High visibility within the field
- Retaining the copyright to your article

Submit your next manuscript at $>$ springeropen.com

