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On boundedness and convergence of solutions for neutral stochastic functional differential equations driven by G-Brownian motion

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Abstract

The current article presents the study of neutral stochastic functional differential equations driven by G-Brownian motion in the phase space $C_q((-\infty, 0]; \mathbb{R}^n)$. The mean-square boundedness of solutions has been derived. The convergence of solutions with different initial data has been investigated. The boundedness and convergence of solution maps have been obtained. In addition, the L_G^2 and exponential estimates of solutions have been determined.

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Keywords: Boundedness; Convergence; The L_G^2 and exponential estimates; G-Brownian motion; Neutral stochastic functional differential equations

1 Introduction

Several stochastic dynamical systems not only rely on current and past values but also include derivatives with delays. Neutral stochastic functional differential equations (NSFDEs) are employed to express such type of systems. These equations and their applications in aeroelasticity and chemical engineering were introduced by Kolmanovskii, Nosov and Myshkis [10, 11]. Thenceforth, the theory of NSFDEs has attracted the attention of many authors [15, 16, 29, 32]. The existence-uniqueness and stability of solutions for neutral stochastic functional differential equations driven by G-Brownian motion (G-NSFDEs) with Lipschitz and non-Lipschitz conditions was, respectively, studied by Faiz [4] and Faiz et al. [7]. The pth moment exponential stability for solutions to G-NSFDEs with Markovian switching [13] and the asymptotic stability of Euler–Maruyama numerical solutions for G-NSFDEs [14] was given by Li and Yang. The quasi sure exponential stability for solutions to the stated equations was established by Zhu et al. [33]. The mean-square stability of delayed stochastic neural networks driven by G-Brownian motion and stabilization of SDEs driven by G-Brownian motion can be found in [20, 28]. For the text on stochastic functional differential equations driven by G-Brownian motion we refer the reader to see [3, 5, 18]. The existence theory and estimates for the difference between exact and approximate solutions of stochastic differential equations driven by G-Brownian motion can be found in [2, 8, 9]. Also see [21–26]. Unlike to the above briefly discussed literature, this article presents the study of G-NSFDEs with some suitable monotone type conditions in the phase space C_q defined below. We investigate the boundedness and convergence of solutions. We derive the convergence of any two solution maps with distinct initial conditions. Furthermore, the L_G^2 and exponential estimates for solutions to G-NSFGEs are determined. Let $\mathbb{R}^n$ be an n-dimensional Euclidean space and $C((-\infty,0];\mathbb{R}^n)$ be the collection of continuous functions from $(-\infty,0]$ to $\mathbb{R}^n$. For a given number $q>0$ the phase space with the fading memory $C_q((-\infty,0];\mathbb{R}^n)$ is defined by

$$C_q\big((-\infty,0];\mathbb{R}^n\big)=\Big\{\psi\in C\big((-\infty,0];\mathbb{R}^n\big):\lim_{\theta\to-\infty}e^{q\theta}\psi(\theta)\text{ exists in }\mathbb{R}^n\Big\}.$$

The space $C_q((-\infty,0];\mathbb{R}^n)$ endowed with the norm $\|\psi\|_q=\sup_{-\infty<\alpha\leq 0}e^{q\alpha}|\psi(\alpha)|<\infty$ is a Banach space of continuous and bounded functions and for any $0\leq q_1\leq q_2<\infty$, $C_{q_1}\subseteq C_{q_2}$ [12, 31]. Let $\mathcal{B}(C_q)$ be the σ-algebra generated by C_q and $C_q^0=\{\psi\in C_q:\lim_{\theta\to-\infty}e^{q\theta}\psi(\theta)=0\}$. Denote by $C^2(C_q)$ (resp. $C^2(C_q^0)$) the space of all $\mathcal{F}$-measurable C_q-valued (resp. C_q^0-valued) stochastic processes ψ such that $E\|\psi\|_q^2<\infty$. Let $(\Omega,\mathcal{F},\mathbb{P})$ be a complete probability space, $B(t)$ be a n-dimensional G-Brownian motion and $\mathcal{F}_t=\sigma\{B(s):0\leq s\leq t\}$ be the natural filtration. Let the filtration $\{\mathcal{F};t\geq 0\}$ satisfies the usual conditions. Let $\mathcal{P}$ be the collection of all probability measures on $(C_q,\mathcal{B}(C_q))$ and $C_b(C_q)$ be the set of all bounded continuous functionals. Let N_0 be the set of probability measures on $(-\infty,0]$ such that, for any $\mu\in N_0$, $\int_{-\infty}^0\mu(d\theta)=1$. For any $m>0$ we define N_m by

$$N_m=\Big\{\mu\in N_0:\mu^{(m)}=\int_{-\infty}^0 e^{-m\theta}\mu(d\theta)<\infty\Big\},$$

where for any $k\in(0,k_0)$, $N_{k_0}\subset N_k\subset N_0$ [31]. We study the G-NSFDE with infinite delay

$$d\big[z(t)-u(z_t)\big]=g(z_t)\,dt+h(z_t)\,d\langle B,B\rangle(t)+\gamma(z_t)\,dB(t),\tag{1.1}$$

on $t\geq 0$ with the given initial data $z_0=\zeta\in C_q((-\infty,0];\mathbb{R}^n)$ and $z_t=\{z(t+\theta):-\infty<\theta\leq 0\}$. The remaining article is divided in three sections. The basic notions and definitions can be found in Sect. 2. Section 3 includes some useful lemmas, boundedness and convergence of solutions and solution maps. The L_G^2 and exponential estimates are placed in the last Sect. 4.

2 Preliminaries

This section contains some basic notions and results required for our further study in the subsequent sections of this article [2, 6, 17–19, 27, 30]. Assume a sublinear expectation space $(S,\mathcal{W},\hat{\mathbb{E}})$ where $\mathcal{W}$ is a space of real mappings defined on a given non-empty set S. Assume that S denotes the collection of all $\mathbb{R}^n$-valued continuous trajectories $(z(t))_{t\geq 0}$ with $z(0)=0$ endowed with the distance

$$\rho\big(z^1,z^2\big)=\sum_{i=1}^{\infty}\frac{1}{2^i}\Big(\max_{t\in[0,i]}\big|z^1(t)-z^2(t)\big|\wedge 1\Big).$$

Let $B(t)=B(t,z)=z(t)$ [18] for any $z\in S$ and $t\geq 0$, be the canonical process. For a selected $T\in[0,\infty)$, we define

$$\mathcal{L}_{ip}(S_T)=\big\{\chi\big(B(t_1),B(t_2),\ldots,B(t_n)\big):n\geq 1,t_1,t_2,\ldots,t_n\in[0,T],\chi\in C_{b.\mathrm{Lip}}\big(\mathbb{R}^{n\times m}\big)\big\},$$

where $C_{b.\mathrm{Lip}}(\mathbb{R}^{n\times m})$ is a space of bounded Lipschitz mappings. For $t \le T$, $\mathcal{L}_{ip}(S_t) \subseteq \mathcal{L}_{ip}(S_T)$ and $\mathcal{L}_{ip}(S) = \bigcup_{n=1}^{\infty} \mathcal{L}_{ip}(S_n)$. Let $\mathcal{L}_G^p(S)$ denote the completion of $\mathcal{L}_{ip}(S)$ equipped with the Banach norm $\hat{\mathbb{E}}[|\cdot|^p]^{\frac{1}{p}}$, $p \ge 1$ and $\mathcal{L}_G^p(S_t) \subseteq \mathcal{L}_G^p(S_T) \subseteq \mathcal{L}_G^p(S)$ for $0 \le t \le T < \infty$. Let $\mathcal{F}_t = \sigma\{B(v), 0 \le v \le t\}$ indicate the filtration produced by the stated canonical process and $\mathcal{F} = \{\mathcal{F}_t\}_{t\ge 0}$. Let $\pi_T = \{t_0, t_1, \ldots, t_{\mathbb{Z}^+}\}$, $0 \le t_0 \le t_1 \le \cdots \le t_{\mathbb{Z}^+} \le \infty$ be a partition of $[0, T]$. For every $\mathbb{Z}^+ \ge 1$, $0 = t_0 < t_1 < \cdots < t_{\mathbb{Z}^+} = T$ and $i = 0, 1, \ldots, \mathbb{Z}^+ - 1$, define the space $\mathcal{M}_G^{p,0}([0, T])$, $p \ge 1$ of simple processes as

$$\mathcal{M}_G^{p,0}\bigl([0,T]\bigr) = \left\{ \eta_t(z) = \sum_{i=0}^{\mathbb{Z}^+-1} \xi_{t_i}(z) I_{[t_i,t_{i+1}]}(t); \xi_{t_i}(z) \in \mathcal{L}_G^p(\Omega_{t_i}) \right\}. \tag{2.1}$$

The space (2.1) is complete with the norm $\|\eta\| = \{\int_0^T \hat{\mathbb{E}}[|\eta(s)|^p]\, ds\}^{1/p}$ and is indicated by $\mathcal{M}_G^p(0, T)$, $p \ge 1$.

Definition 2.1 Let $\eta_t \in \mathcal{M}_G^{2,0}(0, T)$. Then the G-Itô integral, say $J(\eta)$, is given by

$$J(\eta) = \int_0^T \eta(s)\, dB^\theta(s) = \sum_{i=0}^{\mathbb{Z}^+-1} \xi_i\bigl(B^\theta(t_{i+1}) - B^\theta(t_i)\bigr).$$

We can continuously extend the mapping $J : \mathcal{M}_G^{2,0}(0, T) \mapsto \mathcal{L}_G^2(\mathcal{F}_T)$ to $J : \mathcal{M}_G^2(0, T) \mapsto \mathcal{L}_G^2(\mathcal{F}_T)$. For $\eta \in \mathcal{M}_G^2(0, T)$ we can still give the G-Itô integral as

$$\int_0^T \eta(s)\, dB^\theta(s) = J(\eta).$$

Definition 2.2 The quadratic variation process $\{\langle B^\theta\rangle(t)\}_{t\ge 0}$ of G-Brownian motion is given by

$$\langle B^\theta\rangle(t) = \lim_{\mathbb{Z}^+\to\infty} \sum_{i=0}^{\mathbb{Z}^+-1} \bigl(B^\theta\bigl(t_{i+1}^{\mathbb{Z}^+}\bigr) - B^\theta\bigl(t_i^{\mathbb{Z}^+}\bigr)\bigr)^2 = B^\theta(t)^2 - 2\int_0^t B^\theta(s)\, dB^\theta(s).$$

The stated process is increasing, $\langle B^\theta\rangle(0) = 0$ and for any $0 \le s \le t$, $\langle B^\theta\rangle(t) - \langle B^\theta\rangle(s) \le \sigma_{\theta\theta^\tau}(t - s)$.

Assume that $\theta, \hat{\theta} \in \mathbb{R}^n$ be given vectors. The mutual variation process of $B^{\hat{\theta}}$ and B^θ is given by $\langle B^\theta, B^{\hat{\theta}}\rangle = \frac{1}{4}[\langle B^\theta + B^{\hat{\theta}}\rangle(t) - \langle B^\theta - B^{\hat{\theta}}\rangle(t)]$. A mapping $\mathcal{W}_{0,T} : \mathcal{M}_G^{0,1}(0, T) \mapsto \mathcal{L}_G^2(\mathcal{F}_T)$ is defined by

$$\mathcal{W}_{0,T}(\eta) = \int_0^T \eta(s)\, d\langle B^\theta\rangle(s) = \sum_{i=0}^{\mathbb{Z}^+-1} \xi_i\bigl(\langle B^\theta\rangle_{(t_{i+1})} - \langle B^\theta\rangle(t_i)\bigr).$$

We can continuously extend it to $\mathcal{M}_G^1(0, T)$ and for $\eta \in \mathcal{M}_G^1(0, T)$ this is still given by

$$\int_0^T \eta(s)\, d\langle B^\theta\rangle(s) = \mathcal{W}_{0,T}(\eta).$$

The concept of G-capacity and Lemma 2.3 can be found in [1]. Let $\mathcal{B}(S)$ be a Borel σ-algebra of S. Let $\mathcal{Q}$ represent the group of all probability measures on $(S,\mathcal{B}(S)$. The G-capacity $\hat{\nu}$ is given as follows:

$$\hat{\nu}(C) = \sup_{\mathbb{P}\in\mathcal{Q}} \mathbb{P}(C),$$

where set $C \in \mathcal{B}(S)$. If $\hat{\nu}(C) = 0$ then the set $C \in \mathcal{B}(S)$ is called polar. A characteristic holds quasi-surely when it holds external of the set C.

Lemma 2.3 *Let $z \in \mathcal{L}_G^p$ and $\hat{\mathbb{E}}|z|^p < \infty$. Then, for each $c > 0$, the G-Markov inequality is given by*

$$\hat{\nu}\bigl(|z| > c\bigr) \leq \frac{\hat{\mathbb{E}}[|z|^p]}{c}.$$

For the proof of Lemmas 2.4 and 2.5 see [9].

Lemma 2.4 *Let $\theta \in \mathbb{R}^n$, $\eta \in \mathcal{M}_G^2(0,T)$, $p \geq 2$ and $z(t) = \int_0^t \eta(s)\,dB^\theta(s)$. Then on some $\bar{S} \subset S$ with $\nu(\bar{S}^c) = 0$ and $\forall t \in [0,T]$, $\hat{\nu}(|z(t) - \bar{z}| \neq 0) = 0$ so that*

$$\hat{\mathbb{E}}\Bigl[\sup_{s\leq v\leq t} \bigl|\bar{z}(v) - \bar{z}(s)\bigr|^p\Bigr] \leq \hat{K}\sigma_{\theta\theta^\tau}^{\frac{p}{2}} \hat{\mathbb{E}}\biggl(\int_s^t \bigl|\eta(v)\bigr|^2\,dv\biggr)^{\frac{p}{2}},$$

where $0 < \hat{K} < \infty$ is a positive constant and $\bar{z}(t)$ is a modification of $z(t)$.

Lemma 2.5 *Let $\theta, \hat{\theta} \in \mathbb{R}^n$, $p \geq 1$ and $\eta \in \mathcal{M}_G^p(0,T)$. A continuous modification $\bar{z}^{\theta,\hat{\theta}}(t)$ of $z^{\theta,\hat{\theta}}(t) = \int_0^t \eta(s)\,d\langle B^\theta, B^{\hat{\theta}}\rangle(s)$ exists and for $0 \leq s \leq v \leq t \leq T$,*

$$\hat{\mathbb{E}}\Bigl[\sup_{0\leq s\leq v\leq t} \bigl|\bar{z}^{\theta,\hat{\theta}}(v) - \bar{z}^{\theta,\hat{\theta}}(s)\bigr|^p\Bigr] \leq \biggl(\frac{1}{4}\sigma_{(\theta+\hat{\theta})(\theta-\hat{\theta})^\tau}\biggr)^p (t-s)^{p-1} \hat{\mathbb{E}}\int_s^t \bigl|\eta(v)\bigr|^p\,dv.$$

Lemma 2.6 *Let $\lambda < 2q$ and $\mu_i \in N_m$, for any $i \in \mathbb{Z}^+$. Then, for any $\zeta \in C_q((-\infty,0];\mathbb{R}^n)$,*

$$\int_0^t \int_{-\infty}^0 \bigl|z(s+\theta)\bigr|^2 \mu_i(d\theta)\,ds \leq \frac{\mu_i^{(2q)}}{2q}\|\zeta\|_q^2 + \int_0^t \bigl|z(s)\bigr|^2\,ds, \tag{2.2}$$

$$\int_0^t \int_{-\infty}^0 e^{\lambda s}\bigl|z(s+\theta)\bigr|^2 \mu_i(d\theta)\,ds \leq \frac{\mu_i^{(2q)}}{2q-\lambda}\|\zeta\|_q^2 + \mu_i^{(2q)}\int_0^t e^{\lambda s}\bigl|z(s)\bigr|^2\,ds. \tag{2.3}$$

Proof Let $\zeta \in C_q((-\infty,0];\mathbb{R}^n)$ and $\mu_i \in N_{2q}$ for any $i \in \mathbb{Z}^+$. By using the definition of norm and the Fubini theorem, we derive

$$\begin{aligned}
&\int_0^t \int_{-\infty}^0 \bigl|z(s+\theta)\bigr|^2 \mu_i(d\theta)\,ds \\
&\quad= \int_0^t \biggl[\int_{-\infty}^{-s} e^{2q(s+\theta)}\bigl|z(s+\theta)\bigr|^2 e^{-2q(s+\theta)}\mu_i(d\theta) + \int_{-s}^0 \bigl|z(s+\theta)\bigr|^2 \mu_i(d\theta)\biggr]\,ds \\
&\quad\leq \|\zeta\|_q^2 \int_0^t e^{-2qs}\,ds \int_{-\infty}^0 e^{-2q\theta}\mu_i(d\theta) + \int_{-\infty}^0 \mu_i(d\theta)\int_0^t \bigl|z(s)\bigr|^2\,ds,
\end{aligned}$$

by noticing that $\int_{-\infty}^{0}\mu_i(d\theta) = 1$ and $\int_{-\infty}^{0}e^{-2q\theta}\mu_i(d\theta) = \mu_i^{(2q)}$, $i \in \mathbb{Z}^+$, we derive

$$\int_0^t\int_{-\infty}^0 \bigl|z(s+\theta)\bigr|^2\mu_i(d\theta)\,ds \le \frac{\mu_i^{(2q)}}{2q}\|\zeta\|_q^2 + \int_0^t \bigl|z(s)\bigr|^2\,ds.$$

The proof of (2.2) is complete. The assertion (2.3) can be proved in a similar fashion as above. □

The book [16] is a good reference for the following three lemmas.

Lemma 2.7 *Let $a, b \ge 0$ and $\epsilon \in (0,1)$. Then*

$$(a+b)^2 \le \frac{a^2}{\epsilon} + \frac{b^2}{1-\epsilon}.$$

Lemma 2.8 *Assume $p \ge 2$ and $\hat{\epsilon}, a, b > 0$. Then the following two inequalities hold*:

(i) $a^{p-1}b \le \frac{(p-1)\hat{\epsilon}a^p}{p} + \frac{b^p}{p\hat{\epsilon}^{p-1}}$.

(ii) $a^{p-2}b^2 \le \frac{(p-2)\hat{\epsilon}a^p}{p} + \frac{2b^p}{p\hat{\epsilon}^{\frac{p-2}{2}}}$.

Lemma 2.9 *Let $a_1, a_2 \in \mathbb{R}$ and $\delta \in (0,1)$. Then for any $p > 1$*

$$|a_1 + a_2|^p \le \bigl[1 + \delta^{\frac{1}{p-1}}\bigr]^{p-1}\biggl(|a_1|^p + \frac{|a_2|^p}{\delta}\biggr).$$

3 Boundedness and convergence of solutions

Consider that problem (1.1) has a solution $z(t)$. All through this article we take $\lambda < pq$ for any $p \ge 1$. We assume the following two hypotheses:

(A_1) Let $y, z \in C_q((-\infty,0];\mathbb{R}^n)$ and $\mu_1, \mu_2, \mu_3 \in N_{2q}$. Then there are positive constants λ_i, $i = 1, 2, \ldots, 5$ so that

$$\begin{aligned}&\bigl[z(0)-y(0)-\bigl(u(z)-u(y)\bigr)\bigr]^T\bigl[g(z)-g(y)\bigr]\\&\quad\le -\lambda_1\bigl|z(0)-y(0)\bigr|^2 + \lambda_2\int_{-\infty}^0\bigl|z(\theta)-y(\theta)\bigr|^2\mu_1(d\theta),\\&\bigl[z(0)-y(0)-\bigl(u(z)-u(y)\bigr)\bigr]^T\bigl[h(z)-h(y)\bigr]\\&\quad\le -\lambda_3\bigl|z(0)-y(0)\bigr|^2 + \lambda_4\int_{-\infty}^0\bigl|z(\theta)-y(\theta)\bigr|^2\mu_2(d\theta),\end{aligned}$$

and

$$\bigl|\gamma(z)-\gamma(y)\bigr|^2 \le \lambda_5\int_{-\infty}^0\bigl|z(\theta)-y(\theta)\bigr|^2\mu_3(d\theta).$$

(A_2) Let $z \in C_q((-\infty,0];\mathbb{R}^n)$ and $\mu_4 \in N_{2q}$ with $\mu_4^{(2q)} < 1$. Then there is a constant $0 < \hat{b} < 1$ so that

$$\bigl|u(z)\bigr|^2 \le \hat{b}\int_{-\infty}^0\bigl|z(\theta)\bigr|^2\mu_4(d\theta). \tag{3.1}$$

Let $p > 1$, $z \in C_q((-\infty, 0]; \mathbb{R}^n)$ and $\mu_4 \in N_{pq}$ with $\mu_4^{(pq)} < 1$. Then there is a constant $0 < b < 1$ so that

$$\left|u(z)\right|^p \leq b^p \int_{-\infty}^{0} \left|z(\theta)\right|^p \mu_4(d\theta). \tag{3.2}$$

Obviously, if $p = 2$ and letting $b^2 = \hat{b}$ then (3.2) is the same as (3.1). Firstly, we give the prove of some important lemmas.

Lemma 3.1 *Let $p > 1$, $q > 0$ and $\zeta \in C_q((-\infty, 0]; \mathbb{R}^n)$. Let condition 3.2 hold. Then*

$$\sup_{0<s\leq t} \left|z(s)\right|^p \leq \frac{b\mu_4^{(pq)}}{1-b} e^{-pqs} \|\zeta\|_q^p + \frac{1}{(1-b)^p} \sup_{0<s\leq t} \left|z(s) - u(z_s)\right|^p,$$

where $0 < b < 1$.

Proof By using Lemma 2.9 and condition (3.2) for any $\delta > 0$, it follows that

$$\begin{aligned}
\left|z(t)\right|^p &= \left|u(z_t) + z(t) - u(z_t)\right|^p \\
&\leq \left[1 + \delta^{\frac{1}{p-1}}\right]^{p-1} \left(\frac{b^p}{\delta} \int_{-\infty}^{0} \left|z(t+\theta)\right|^p \mu_4 \, d(\theta) + \left|z(t) - u(z_t)\right|^p \right).
\end{aligned}$$

Taking $\delta = (\frac{b}{1-b})^{p-1}$ and using the definition of norm, we obtain

$$\begin{aligned}
\sup_{0<s\leq t} \left|z(s)\right|^p &\leq b \int_{-\infty}^{0} \sup_{0<s\leq t} \left|z(s+\theta)\right|^p \mu_4(d\theta) + \frac{1}{(1-b)^{p-1}} \sup_{0<s\leq t} \left|z(s) - u(z_s)\right|^p \\
&= b \int_{-\infty}^{-s} \sup_{0<s\leq t} e^{pq(s+\theta)} \left|z(s+\theta)\right|^p e^{-pq(s+\theta)} \mu_4(d\theta) + b \int_{-s}^{0} \sup_{0<s\leq t} \left|z(s+\theta)\right|^p \mu_4(d\theta) \\
&\quad + \frac{1}{(1-b)^{p-1}} \sup_{0<s\leq t} \left|z(s) - u(z_s)\right|^p \\
&\leq b e^{-pqs} \|\zeta\|_q^p \int_{-\infty}^{-s} e^{-pq\theta} \mu_4(d\theta) + b \sup_{0<s\leq t} \left|z(s)\right|^p \int_{-s}^{0} \mu_4(d\theta) \\
&\quad + \frac{1}{(1-b)^{p-1}} \sup_{0<s\leq t} \left|z(s) - u(z_s)\right|^p \\
&\leq b e^{-pqs} \|\zeta\|_q^p \int_{-\infty}^{0} e^{-pq\theta} \mu_4(d\theta) + b \sup_{0<s\leq t} \left|z(s)\right|^p \int_{-\infty}^{0} \mu_4(d\theta) \\
&\quad + \frac{1}{(1-b)^{p-1}} \sup_{0<s\leq t} \left|z(s) - u(z_s)\right|^p \\
&\leq b\mu_4^{(pq)} e^{-pqs} \|\zeta\|_q^p + b \sup_{0<s\leq t} \left|z(s)\right|^p + \frac{1}{(1-b)^{p-1}} \sup_{0<s\leq t} \left|z(s) - u(z_s)\right|^p,
\end{aligned}$$

simplification yields the desired assertion. The proof is complete. □

Lemma 3.2 *Let $p > 1$, $q > 0$ and $\zeta \in C_q((-\infty, 0]; \mathbb{R}^n)$. Let condition 3.2 hold. Then there exists a constant $0 < b < 1$ such that*

$$\left|\zeta(0) - u(\zeta)\right|^p \leq (1+b)^{p-1} \left(1 + b\mu_4^{(pq)}\right) \|\zeta\|_q^p.$$

Proof In view of Lemma 2.9 and condition (3.2), we obtain

$$\begin{aligned}\left|\zeta(0)-u(\zeta)\right|^p &\le \left[1+\delta^{\frac{1}{p-1}}\right]^{p-1}\left(\left|\zeta(0)\right|^p+\frac{|u(\zeta)|^p}{\delta}\right)\\ &\le \left[1+\delta^{\frac{1}{p-1}}\right]^{p-1}\left(\left|\zeta(0)\right|^p+\frac{b^p}{\delta}\int_{-\infty}^{0}\left|\zeta(\theta)\right|^p\mu_4(d\theta)\right).\end{aligned}$$

Observing $|\zeta(0)|^p \le \sup_{-\infty<\alpha\le 0} e^{pq\theta}|\zeta(\theta)|^p = \|\zeta\|_q^p$ and substituting $\delta = b^{p-1}$ we have

$$\begin{aligned}\left|\zeta(0)-u(\zeta)\right|^p &\le (1+b)^{p-1}\|\zeta\|_q^p + b(1+b)^{p-1}\int_{-\infty}^{0}\left|\zeta(\theta)\right|^p\mu_4(d\theta)\\ &\le (1+b)^{p-1}\|\zeta\|_q^p + b(1+b)^{p-1}\int_{-\infty}^{0}\sup_{-\infty<\theta\le 0} e^{pq\theta}\left|\zeta(\theta)\right|^p e^{-pq\theta}\mu_4(d\theta)\\ &= (1+b)^{p-1}\|\zeta\|_q^p + b(1+b)^{p-1}\|\zeta\|_q^p\int_{-\infty}^{0} e^{-pq\theta}\mu_4(d\theta)\\ &= (1+b)^{p-1}\|\zeta\|_q^p + b(1+b)^{p-1}\|\zeta\|_q^p\mu_4^{(pq)}\\ &= (1+b)^{p-1}\left(1+b\mu_4^{(pq)}\right)\|\zeta\|_q^p.\end{aligned}$$

The proof is completed. □

Lemma 3.3 *Let* $p>1$, $q>0$ *and* $\zeta\in C_q((-\infty,0];\mathbb{R}^n)$. *Let condition* 3.2 *hold. Then*

$$\left|z(t)-u(z_t)\right|^p \le (1+b)^{p-1}\left|z(t)\right|^p + \frac{(1+b)^{p-1}}{b}\int_{-\infty}^{0}\left|z(t+\theta)\right|^p\mu_4(d\theta),$$

where $0<b<1$.

We omit the proof. It can be proved in a similar procedure to the above last lemma. Now let us see one of the main results.

Theorem 3.4 *Under assumptions* A_1 *and* A_2, *if for any* $\zeta\in C_q$, λ_i, $i=1,2,\dots,5$ *satisfy* $\lambda_1 > \mu_1^{(2q)}\lambda_2 - k_2\lambda_3 + k_2\mu_2^{(2q)}\lambda_4 + (2k_1^2+k_2)\mu_3^{(2q)}\lambda_5$ *then there exists* $\lambda\in(0, \frac{1}{(1+b_1\mu_4^{(2q)})}(\lambda_1 + k_2\lambda_3 - \mu_1^{(2q)}\lambda_2 - k_2\mu_2^{(2q)}\lambda_4 - (2k_1^2+k_2)\mu_3^{(2q)}\lambda_5)\wedge 2q)$ *such that*

$$\hat{\mathbb{E}}\Big[\sup_{0<s\le t}\left|z(s)\right|^2\Big] \le C + Ke^{-\lambda t},$$

where $C=b_3c_1$, $K=(b_2+b_3c_2)\hat{\mathbb{E}}\|\zeta\|_q^2$, $b_1 = 1+b^{-1}$, $b_2 = b\mu_4^{(2q)}(1-b)^{-1}$, $b_3=(1-b)^{-2}$, $c_1 = \frac{1}{\lambda}[\frac{1}{\epsilon_1}|g(0)|^2 + k_2\frac{1}{\epsilon_1}|h(0)|^2 + (k_2+k_3)\frac{1}{\epsilon_3}|\gamma(0)|^2]$, $c_2 = \frac{2}{2q-\lambda}(4(2q-\lambda) + 2\lambda_2\mu_1^{(2q)} + 2k_2\lambda_4\mu_2^{(2q)} + (2k_1^2+k_2)\mu_3^{(2q)} + b_1(\lambda+\epsilon_1+\epsilon_1k_2)\mu_4^{(2q)})$, k_1, k_2 *are positive constants and* ϵ_1,ϵ_2 *are sufficiently small constants such that*

$$\begin{aligned}&\lambda_1 + k_2\lambda_3 - \mu_1^{(2q)}\lambda_2 - k_2\mu_2^{(2q)}\lambda_4 - \left(2k_1^2+k_2\right)\mu_3^{(2q)}\frac{1}{1-\epsilon_2}\lambda_5\\ &\quad - \left(1+b_1\mu_4^{(2q)}\right)(\lambda+1+k_2)\epsilon_1 > 0.\end{aligned}$$

Proof By virtue of the G-Itô formula, for any $t \in [0, T]$, it follows that

$$\begin{aligned} & e^{\lambda t}\left|z(t)-u(z_t)\right|^2 \\ & \quad \leq \left|z(0)-u(z_0)\right|^2+\int_0^t e^{\lambda s}\left[\lambda\left|z(s)-u(z_s)\right|^2+2\left|z(s)-u(z_s)\right|^T g(z_s)\right] ds \\ & \qquad +\int_0^t e^{\lambda s}\left[2\left|z(s)-u(z_s)\right|^T h(z_s)+\left|\gamma(z_s)\right|^2\right] d\langle B,B\rangle(s) \\ & \qquad +\int_0^t 2e^{\lambda s}\left|z(s)-u(z_s)\right|^T \gamma(z_s)\, dB(s). \end{aligned}$$

Applying the G-expectation on both sides, utilizing Lemma 2.5, Lemma 2.4 and Lemma 3.2, there exist $k_1 > 0$ and $k_2 > 0$ so that

$$\begin{aligned} & \hat{\mathbb{E}}\Big[\sup_{0\leq s\leq t} e^{\lambda s}\left|z(s)-u(z_s)\right|^2\Big] \\ & \quad \leq 4\hat{\mathbb{E}}\|\zeta\|_q^2+\hat{\mathbb{E}}\int_0^t e^{\lambda s}\left[\lambda\left|z(s)-u(z_s)\right|^2+2\left|z(s)-u(z_s)\right|^T g(z_s)\right] ds \\ & \qquad +k_2\hat{\mathbb{E}}\int_0^t e^{\lambda s}\left[2\left|z(s)-u(z_s)\right|^T h(z_s)+\left|\gamma(z_s)\right|^2\right] ds \\ & \qquad +2k_1\hat{\mathbb{E}}\Big[\int_0^t \left(e^{\lambda s}\left|z(s)-u(z_s)\right|^T\left|\gamma(z_s)\right|\right)^2\Big]^{\frac{1}{2}} ds \\ & \quad \leq 4\hat{\mathbb{E}}\|\zeta\|_q^2+\hat{\mathbb{E}}\int_0^t e^{\lambda s}\left[\lambda\left|z(s)-u(z_s)\right|^2+2\left|z(s)-u(z_s)\right|^T g(z_s)\right] ds \\ & \qquad +2k_1^2\hat{\mathbb{E}}\int_0^t e^{\lambda s}\left|\gamma(z_s)\right|^2 ds+\frac{1}{2}\hat{\mathbb{E}}\Big(\sup_{0<s\leq t} e^{\lambda s}\left|z(s)-u(z_s)\right|^2\Big) \\ & \qquad +k_2\hat{\mathbb{E}}\int_0^t e^{\lambda s}\left[2\left|z(s)-u(z_s)\right|^T h(z_s)+\left|\gamma(z_s)\right|^2\right] ds. \end{aligned} \tag{3.3}$$

By using assumption A_1 and Lemma 2.8 we derive

$$\begin{aligned} & 2\left|z(s)-u(z_s)\right|^\tau g(z_s)\, ds \\ & \quad \leq 2\left|z(s)-u(z_s)\right|^\tau g(0)-2\lambda_1\left|z(s)\right|^2+2\lambda_2\int_{-\infty}^0\left|z(s+\theta)\right|^2\mu_1(d\theta) \\ & \quad \leq \epsilon_1\left|z(s)-u(z_s)\right|^2+\frac{1}{\epsilon_1}\left|g(0)\right|^2-2\lambda_1\left|z(s)\right|^2+2\lambda_2\int_{-\infty}^0\left|z(s+\theta)\right|^2\mu_1(d\theta), \end{aligned} \tag{3.4}$$

similar arguments give

$$\begin{aligned} & 2\left|z(s)-u(z_s)\right|^\tau h(z_s)\, ds \\ & \quad \leq \epsilon_1\left|z(s)-u(z_s)\right|^2+\frac{1}{\epsilon_1}\left|h(0)\right|^2-2\lambda_3\left|z(s)\right|^2 \\ & \qquad +2\lambda_4\int_{-\infty}^0\left|z(s+\theta)\right|^2\mu_2(d\theta). \end{aligned} \tag{3.5}$$

In view of assumption A_1 and Lemma 2.7 we derive

$$\left|\gamma(z_s)\right|^2 \leq \frac{1}{\epsilon_2}\left|\gamma(0)\right|^2 + \frac{\lambda_5}{1-\epsilon_2}\int_{-\infty}^{0}\left|z(s+\theta)\right|^2\mu_3(d\theta). \tag{3.6}$$

By substituting (3.4), (3.5) and (3.6) in (3.3) and using Lemma 3.3 we get

$$\begin{aligned}
&\hat{\mathbb{E}}\Big[\sup_{0\leq s\leq t} e^{\lambda s}\left|z(s)-u(z_s)\right|^2\Big] \\
&\quad\leq 8\hat{\mathbb{E}}\|\zeta\|_q^2 + 4[-\lambda_1 - k_2\lambda_3 + \lambda + \epsilon_1 + \epsilon_1 k_2]\hat{\mathbb{E}}\int_0^t e^{\lambda s}\left|z(s)\right|^2 ds \\
&\qquad + c_1\big(e^{\lambda t}-1\big) + 4\lambda_2\hat{\mathbb{E}}\int_0^t\int_{-\infty}^0 e^{\lambda s}\left|z(s+\theta)\right|^2\mu_1(d\theta)\,ds \\
&\qquad + 4k_2\lambda_4\hat{\mathbb{E}}\int_0^t\int_{-\infty}^0 e^{\lambda s}\left|z(s+\theta)\right|^2\mu_2(d\theta)\,ds \\
&\qquad + 2\big(2k_1^2+k_2\big)\frac{\lambda_5}{1-\epsilon_2}\hat{\mathbb{E}}\int_0^t\int_{-\infty}^0 e^{\lambda s}\left|z(s+\theta)\right|^2\mu_3(d\theta)\,ds \\
&\qquad + 2b_1(\lambda+\epsilon_1+\epsilon_1 k_2)\hat{\mathbb{E}}\int_0^t\int_{-\infty}^0 e^{\lambda s}\left|z(s+\theta)\right|^2\mu_4(d\theta)\,ds,
\end{aligned}$$

where $c_1 = \frac{2}{\lambda}[\frac{1}{\epsilon_1}|g(0)|^2 + k_2\frac{1}{\epsilon_1}|h(0)|^2 + (k_2+k_3)\frac{1}{\epsilon_2}|\gamma(0)|^2]$ and $b_1 = 1 + b^{-1}$. By virtue of Lemma 2.6, it follows that

$$\begin{aligned}
&\hat{\mathbb{E}}\Big[\sup_{0\leq s\leq t} e^{\lambda s}\left|z(s)-u(z_s)\right|^2\Big] \\
&\quad\leq c_1\big(e^{\lambda t}-1\big) + c_2\hat{\mathbb{E}}\|\zeta\|_q^2 + 2\bigg[12\lambda_1 - 2k_2\epsilon_3 + 2\lambda + 2\epsilon_1 + 2\epsilon_1 k_2 \\
&\qquad + 2\lambda_2\mu_1^{(2q)} + 2k_2\lambda_4\mu_2^{(2q)} + \big(2k_1^2 2 + k_2\big)\frac{\lambda_5}{1-\epsilon_2}\mu_3^{(2q)} \\
&\qquad + b_1(\lambda+\epsilon_1+\epsilon_1 k_2)\mu_4^{(2q)}\bigg]\hat{\mathbb{E}}\int_0^t e^{\lambda s}\left|z(s)\right|^2 ds,
\end{aligned}$$

where $c_2 = \frac{2}{2q-\lambda}(4(2q-\lambda) + 2\lambda_2\mu_1^{(2q)} + 2k_2\lambda_4\mu_2^{(2q)} + (2k_1^2+k_2)\mu_3^{(2q)} + b_1(\lambda+\epsilon_1+\epsilon_1 k_2)\mu_4^{(2q)})$. Next, we use Lemma 3.1 to derive

$$\begin{aligned}
&\hat{\mathbb{E}}\Big[\sup_{0< s\leq t} e^{\lambda s}\left|z(s)\right|^2\Big] \\
&\quad\leq (b_2+b_3c_2)\hat{\mathbb{E}}\|\zeta\|_q^2 + b_3c_1\big(e^{\lambda t}-1\big) - 4b_3\bigg[\lambda_1 + k_2\lambda_3 - \mu_1^{(2q)}\lambda_2 - k_2\mu_2^{(2q)}\lambda_4 \\
&\qquad - \big(2k_1^2+k_2\big)\mu_3^{(2q)}\frac{1}{1-\epsilon_2}\lambda_5 - \big(1+b_1\mu_4^{(2q)}\big)\lambda \\
&\qquad - (1+k_2)\big(1+b_1\mu_4^{(2q)}\big)\epsilon_1\bigg]\hat{\mathbb{E}}\int_0^t e^{\lambda s}\left|z(s)\right|^2 ds,
\end{aligned}\tag{3.7}$$

where $b_2 = b\mu_4^{(2q)}(1-b)^{-1}$, $b_3 = (1-b)^{-2}$ and $e^{-(2q-\lambda)s} < 1$. From the assumptions, we notice that $\lambda_1 > \mu_1^{(2q)}\lambda_2 - k_2\lambda_3 + k_2\lambda_4\mu_2^{(2q)} + (2k_1^2+k_2)\mu_3^{(2q)}\lambda_5$ and $\lambda \in (0, \frac{1}{(1+b_1\mu_4^{(2q)})}(\lambda_1 + k_2\lambda_3 -$

$\mu_1^{(2q)}\lambda_2 - k_2\mu_2^{(2q)}\lambda_4 - (2k_1^2 + k_2)\mu_3^{(2q)}\lambda_5) \wedge 2q)$. Choosing ϵ_1 and ϵ_2 sufficiently small such that

$$\lambda_1 + k_2\lambda_3 - \mu_1^{(2q)}\lambda_2 - k_2\mu_2^{(2q)}\lambda_4 - \bigl(2k_1^2 + k_2\bigr)\mu_3^{(2q)}\frac{1}{1-\epsilon_2}\lambda_5 - \bigl(1 + b_1\mu_4^{(2q)}\bigr)(\lambda + 1 + k_2)\epsilon_1 > 0,$$

we get the desired result. The proof is completed. □

Next, let us see the convergence of any two solutions of G-NSFDEs with different initial data.

Theorem 3.5 *Let the assumptions of Theorem 3.4 hold. Let $y(t)$ and $z(t)$ be any two solutions of problem (1.1) with the respective initial conditions ζ and ξ. Then*

$$\hat{\mathbb{E}}\Bigl[\sup_{0\le s\le t}\bigl|z(t) - y(t)\bigr|^2\Bigr] \le L\hat{\mathbb{E}}\|\zeta - \xi\|_q^2 e^{-\lambda t},$$

where $L = b_2 + b_3c_3$, $b_2 = b\mu_4^{(2q)}(1-b)^{-1}$, $b_3 = (1-b)^{-2}$, $c_3 = \frac{2}{2q-\lambda}[4(2q-\lambda) + 2\lambda_2\mu_1^{(2q)} + 2k_2\lambda_4\mu_2^{(2q)} + (k_2 + 2k_1^2)\lambda_5\mu_3^{(2q)} + \lambda b_1\mu_4^{(2q)}]$, $b_1 = 1 + b^{-1}$, k_1 and k_2 are positive constants.

Proof Define $\Lambda(t) = z(t) - y(t)$, $\hat{u}(t) = u(z_t) - u(y_t)$, $\hat{g}(t) = g(z_t) - g(y_t)$, $\hat{h}(t) = h(z_t) - h(y_t)$, $\hat{\gamma}(t) = \gamma(z_t) - \gamma(y_t)$. By the G-Itó formula and similar arguments to Theorem 3.4 it follows that

$$\begin{aligned}
&\hat{\mathbb{E}}\Bigl[\sup_{0\le s\le t} e^{\lambda s}\bigl|\Lambda(s) - \hat{u}(s)\bigr|^2\Bigr] \\
&\quad\le 8\hat{\mathbb{E}}\|\zeta - \xi\|_q^2 + 2\hat{\mathbb{E}}\Bigl[\int_0^t e^{\lambda s}\bigl[\lambda\bigl|\Lambda(s) - \hat{u}(s)\bigr|^2 + 2\bigl|\Lambda(s) - \hat{u}(s)\bigr|^T\hat{g}(s)\bigr]\,ds\Bigr] \\
&\qquad + 2k_2\hat{\mathbb{E}}\Bigl[\int_0^t e^{\lambda s}\bigl[2\bigl|\Lambda(s) - \hat{u}(s)\bigr|^T\hat{h}(s) + \bigl|\hat{\gamma}(s)\bigr|^2\bigr]\,ds\Bigr] + 4k_1^2\hat{\mathbb{E}}\int_0^t e^{\lambda s}\bigl|\hat{\gamma}(s)\bigr|^2\,ds.
\end{aligned} \tag{3.8}$$

By virtue of assumptions A_1, we have

$$\begin{aligned}
&\bigl|\Lambda(s) - \hat{u}(s)\bigr|^T\hat{g}(s) \le -\lambda_1\bigl|\Lambda(s)\bigr|^2 + \lambda_2\int_{-\infty}^0\bigl|\Lambda(s+\theta)\bigr|^2\mu_1(d\theta), \\
&\bigl|\Lambda(s) - \hat{u}(s)\bigr|^T\hat{h}(s) \le -\lambda_3\bigl|\Lambda(s)\bigr|^2 + \lambda_4\int_{-\infty}^0\bigl|\Lambda(s+\theta)\bigr|^2\mu_2(d\theta), \\
&\bigl|\hat{\gamma}(s)\bigr|^2 \le \lambda_5\int_{-\infty}^0\bigl|\Lambda(s+\theta)\bigr|^2\mu_3(d\theta).
\end{aligned}$$

Substituting this in (3.8) and using Lemma 3.3 we obtain

$$\begin{aligned}
&\hat{\mathbb{E}}\Bigl[\sup_{0\le s\le t} e^{\lambda s}\bigl|\Lambda(s) - \hat{u}(s)\bigr|^2\Bigr] \\
&\quad\le 8\hat{\mathbb{E}}\|\zeta - \xi\|_q^2 + 4(\lambda - \lambda_1 - k_2\lambda_3)\hat{\mathbb{E}}\int_0^t e^{\lambda s}\bigl|\Lambda(s)\bigr|^2\,ds \\
&\qquad + 4\lambda_2\hat{\mathbb{E}}\int_0^t\int_{-\infty}^0 e^{\lambda s}\bigl|\Lambda(s+\theta)\bigr|^2\mu_1(d\theta)\,ds
\end{aligned}$$

$$+ 4k_2\lambda_4\hat{\mathbb{E}}\int_0^t\int_{-\infty}^0 e^{\lambda s}\big|\Lambda(s+\theta)\big|^2\mu_2(d\theta)\,ds$$
$$+ 2\big(k_2+2k_1^2\big)\lambda_5\hat{\mathbb{E}}\int_0^t\int_{-\infty}^0 e^{\lambda s}\big|\Lambda(s+\theta)\big|^2\mu_3(d\theta)\,ds$$
$$+ 2\lambda b_1\hat{\mathbb{E}}\int_0^t\int_{-\infty}^0 e^{\lambda s}\big|\Lambda(s+\theta)\big|^2\mu_4(d\theta)\,ds,$$

where $b_1 = 1 + b^{-1}$. By using Lemma 2.6, it follows that

$$\begin{aligned}
&\hat{\mathbb{E}}\Big[\sup_{0\le s\le t} e^{\lambda s}\big|\Lambda(s)-\hat{u}(s)\big|^2\Big]\\
&\quad\le c_3\hat{\mathbb{E}}\|\zeta-\xi\|_q^2 - 4\big[\lambda_1 + k_2\lambda_3 - \mu_1^{(2q)}\lambda_2 - k_2\mu_2^{(2q)}\lambda_4\\
&\qquad - \big(k_2+2k_1^2\big)\mu_3^{(2q)}\lambda_5 - \big(1+b_1\mu_4^{(2q)}\big)\lambda\big]\hat{\mathbb{E}}\int_0^t e^{\lambda s}\big|\Lambda(s)\big|^2\,ds,
\end{aligned}$$

where $c_3 = \frac{2}{2q-\lambda}(4(2q-\lambda) + 2\lambda_2\mu_1^{(2q)} + 2k_2\lambda_4\mu_2^{(2q)} + (k_2+2k_1^2)\lambda_5\mu_3^{(2q)} + \lambda b_1\mu_4^{(2q)})$. By using Lemma 3.1, we have

$$\begin{aligned}
&\hat{\mathbb{E}}\Big[e^{\lambda s}\sup_{0\le s\le t}\big|\Lambda(s)\big|^2\Big]\\
&\quad\le (b_2+b_3c_3)E\|\zeta-\xi\|_q^2 - 4b_3\big[\lambda_1 + k_2\lambda_3 - \mu_1^{(2q)}\lambda_2 - k_2\mu_2^{(2q)}\lambda_4\\
&\qquad - \big(k_2+2k_1^2\big)\mu_3^{(2q)}\lambda_5 - \big(1+b_1\mu_4^{(2q)}\big)\lambda\big]E\int_0^t e^{\lambda s}\big|\Lambda(s)\big|^2\,ds,
\end{aligned}\tag{3.9}$$

where $b_2 = b\mu_4^{(2q)}(1-b)^{-1}$, $b_3 = (1-b)^{-2}$ and $e^{-(2q-\lambda)s} < 1$. By using the assumptions $\lambda_1 > \mu_1^{(2q)}\lambda_2 - k_2\lambda_3 + k_2\mu_2^{(2q)}\lambda_4 + (2k_1^2+k_2)\mu_3^{(2q)}\lambda_5$ and $\lambda \in (0, \frac{1}{(1+b_1\mu_4^{(2q)})}(\lambda_1 + k_2\lambda_3 - \mu_1^{(2q)}\lambda_2 - k_2\mu_2^{(2q)}\lambda_4 - (2k_1^2+k_2)\mu_3^{(2q)}\lambda_5) \wedge 2q)$, we derive the desired assertion. The proof is completed. □

The following two theorems show that the solutions maps of G-NSFDEs are bounded and any two solutions maps with different initial data are convergent, respectively.

Theorem 3.6 *Let all the assumptions of Theorem 3.5 hold. Then for any initial data* $\zeta \in C_q$

$$\hat{\mathbb{E}}\|z_t\| \le C_1 + K_1e^{-\lambda t},$$

where $C_1 = b_3c_1$, $K_1 = (1+b_2+b_3c_2)E\|\zeta\|_q^2$ *and* b_2, b_3, c_1, c_2 *are defined in Theorem 3.4.*

Proof By virtue of the definition of norm $\|\cdot\|$ and observing that $2q > \lambda$ we have

$$\begin{aligned}
\hat{\mathbb{E}}\|z_t\| &= \hat{\mathbb{E}}\Big[\sup_{-\infty<\theta\le 0} e^{q\theta}\big|z(t+\theta)\big|\Big]^2 \le \hat{\mathbb{E}}\Big[\sup_{-\infty<\theta\le 0} e^{\lambda\theta}\big|z(t+\theta)\big|^2\Big]\\
&\le \hat{\mathbb{E}}\Big[\sup_{-\infty<s\le 0} e^{-\lambda(t-s)}\big|z(s)\big|^2\Big] + \hat{\mathbb{E}}\Big[\sup_{0<s\le t} e^{-\lambda(t-s)}\big|z(s)\big|^2\Big],
\end{aligned}$$

consequently,

$$\hat{\mathbb{E}}\|z_t\| \le e^{-\lambda t}\hat{\mathbb{E}}\|\zeta\|_q^2 + e^{-\lambda t}\hat{\mathbb{E}}\Big[\sup_{0<s\le t} e^{\lambda s}\big|z(s)\big|^2\Big].\tag{3.10}$$

But from (3.7) we have

$$\begin{aligned}
&\hat{\mathbb{E}}\Big[\sup_{0<s\leq t} e^{\lambda s}\big|z(s)\big|^2\Big] \\
&\quad\leq (b_2+b_3c_2)\hat{\mathbb{E}}\|\zeta\|_q^2 + b_3c_1\big(e^{\lambda t}-1\big) - 4b_3\Big[\lambda_1 + k_2\lambda_3 - \mu_1^{(2q)}\lambda_2 - k_2\mu_2^{(2q)}\lambda_4 \\
&\qquad - \big(2k_1^2+k_2\big)\mu_3^{(2q)}\frac{1}{1-\epsilon_2}\lambda_5 - \big(1+b_1\mu_4^{(2q)}\big)\lambda \\
&\qquad - (1+k_2)\big(1+b_1\mu_4^{(2q)}\big)\epsilon_1\Big]\hat{\mathbb{E}}\int_0^t e^{\lambda s}\big|z(s)\big|^2\,ds,
\end{aligned}$$

using the assumptions of Theorem 3.5, it gives

$$\hat{\mathbb{E}}\Big[\sup_{0<s\leq t} e^{\lambda s}\big|z(s)\big|^2\Big] \leq (b_2+b_3c_2)\hat{\mathbb{E}}\|\zeta\|_q^2 + c_1b_3\big(e^{\lambda t}-1\big). \tag{3.11}$$

By substituting (3.11) in (3.10), we derive

$$\hat{\mathbb{E}}\|z_t\| \leq b_3c_1 + (1+b_2+b_3c_2)\hat{\mathbb{E}}\|\zeta\|_q^2 e^{-\lambda t},$$

which yields the desired assertion. The proof is completed. □

Theorem 3.7 *Under the assumptions of Theorem 3.5, different solution maps z_t and y_t of problem (1.1) with respective different initial data ζ and ξ converge, i.e.,*

$$\hat{\mathbb{E}}\big[|z_t-y_t|^2\big] \leq L_1\hat{\mathbb{E}}\|\zeta-\xi\|_q^2 e^{-\lambda t},$$

where $L_1 = (1+b_2+b_3c_3)$ and b_2, b_3, c_3 are defined in Theorem 3.5.

Proof Using similar arguments of Theorem 3.6 we derive

$$\hat{\mathbb{E}}\big\|z_t(\zeta)-y_t(\xi)\big\| \leq e^{-\lambda t}\hat{\mathbb{E}}\|\zeta-\xi\|_q^2 + e^{-\lambda t}\hat{\mathbb{E}}\Big(\sup_{0<s\leq t} e^{\lambda s}\big|z(s)-y(s)\big|^2\Big). \tag{3.12}$$

But from (3.9), using the assertion of Theorem 3.5, we derive

$$\hat{\mathbb{E}}\Big[\sup_{0\leq s\leq t} e^{\lambda s}\big|z(s)-y(s)\big|^2\Big] \leq (b_2+b_3c_3)\hat{\mathbb{E}}\|\zeta-\xi\|_q^2,$$

which on substituting in (3.12) yields the desired assertion. The proof is completed. □

4 Exponential estimate

For the purpose of exponential estimate one needs to assume that problem (1.1) admits a unique solution $z(t)$ on $t\geq 0$. In the following theorem, we give the L_G^2 and exponential estimates for the solutions of neutral stochastic functional differential equation driven by G-Brownian motion.

Theorem 4.1 *Let problem (1.1) has a unique solution $z(t)$ on $t\geq 0$ and $\hat{\mathbb{E}}\|\zeta\|_q^2<\infty$. Assume that A_1 and A_2 are satisfied. Then the following results hold:*

$$\hat{\mathbb{E}}\Big[\sup_{-\infty<s\leq t}\big|z(s)\big|^2\Big] \leq C_2e^{K_2t}, \quad t\geq 0, \tag{4.1}$$

where $C_2 = 2b_3[4q + \mu_1^{2q} + k_2\lambda_4\mu_2^{2q} + \lambda_5(k_2 + 2k_1^2)\mu_3^{2q} + (1 + k_2)(1 + b^{-1})\mu_4^{2q}]\frac{1}{q}\hat{\mathbb{E}}\|\zeta\|_q^2 + (1 + b_2)\hat{\mathbb{E}}\|\zeta\|_q^2 + b_3c^*$, $K_2 = 4b_3(-\lambda_1 - k_2\lambda_3 + \lambda_2 + \lambda_4k_2 + \lambda_5(k_2 + 2k_1^2) + (2 + k_2)(1 + b^{-1}))$, $b_2 = b\mu_4^{(2q)}(1 - b)^{-1}$ *and* $b_3 = (1 - b)^{-2}$ *are positive constants. Furthermore,*

$$\lim_{t\to\infty}\sup \frac{1}{t}\log\left|z(t)\right| \le M, \tag{4.2}$$

where $M = 2b_3(-\lambda_1 - k_2\lambda_3 + \lambda_2 + \lambda_4k_2 + \lambda_5(k_2 + 2k_1^2) + (2 + k_2)(1 + b^{-1}))$.

Proof To prove (4.1) in a similar fashion to Theorem 3.5 we derive

$$\begin{aligned}
&\hat{\mathbb{E}}\Big[\sup_{0\le s\le t}\left|z(s) - u(z_s)\right|^2\Big] \\
&\quad\le 4\hat{\mathbb{E}}\|\zeta\|_q^2 + 2\hat{\mathbb{E}}\int_0^t \left|z(s) - u(z_s)\right|^T g(z_s)\,ds + \frac{1}{2}\hat{\mathbb{E}}\Big[\sup_{0\le s\le t}\left|z(s) - u(s)\right|^2\Big] \\
&\qquad + k_2\hat{\mathbb{E}}\int_0^t \Big[2\left|z(s) - u(z_s)\right|^T h(z_s) + \left|\gamma(z_s)\right|^2\Big]\,ds + 2k_1^2 \sup_{0\le s\le t}\hat{\mathbb{E}}\int_0^t \gamma(z_s)\,ds,
\end{aligned}$$

which yields

$$\begin{aligned}
&\hat{\mathbb{E}}\Big[\sup_{0\le s\le t}\left|z(s) - u(z_s)\right|^2\Big] \\
&\quad\le 8\hat{\mathbb{E}}\|\zeta\|_q^2 + 4\hat{\mathbb{E}}\int_0^t \left|z(s) - u(z_s)\right|^T g(z_s)\,ds \\
&\qquad + 2k_2\hat{\mathbb{E}}\int_0^t \Big[2\left|z(s) - u(z_s)\right|^T h(z_s)\Big]\,ds + 2\big(k_2 + 2k_1^2\big)\hat{\mathbb{E}}\int_0^t \left|\gamma(z_s)\right|^2\,ds.
\end{aligned} \tag{4.3}$$

By using assumption A_1 and the basic inequality $2a_1a_2 \le \sum_{i=1}^2 a_i^2$ we derive

$$\begin{aligned}
&2\left|z(s) - u(z_s)\right|^T g(z_s) \\
&\quad\le 2\left|z(s) - u(z_s)\right|g(0) - 2\lambda_1\left|z(s)\right|^2 + 2\lambda_2\int_{-\infty}^0 \left|z(s+\theta)\right|^2\mu_1(d\theta) \\
&\quad\le \left|z(s) - u(z_s)\right|^2 + \left|g(0)\right|^2 - 2\lambda_1\left|z(s)\right|^2 + 2\lambda_2\int_{-\infty}^0 \left|z(s+\theta)\right|^2\mu_1(d\theta).
\end{aligned} \tag{4.4}$$

Similar arguments to above give the following:

$$\begin{aligned}
2\left|z(s) - u(z_s)\right|^T h(z_s) \le{}& \left|z(s) - u(z_s)\right|^2 + \left|h(0)\right|^2 - 2\lambda_3\left|z(s)\right|^2 \\
&+ 2\lambda_4\int_{-\infty}^0 \left|z(s+\theta)\right|^2\mu_2(d\theta).
\end{aligned} \tag{4.5}$$

In view of assumption A_1 and the basic inequality $|\sum_{i=1}^2 a_i|^2 \le \sum_{i=1}^2 |a_i|^2$, we obtain

$$\left|\gamma(z_s)\right|^2 \le 2\left|\gamma(0)\right|^2 + 2\lambda_5\int_{-\infty}^0 \left|z(s+\theta)\right|^2\mu_3(d\theta). \tag{4.6}$$

Substituting (4.4), (4.5) and (4.6) in (4.3) and using Lemma 3.3 we have

$$\begin{aligned}
&\hat{\mathbb{E}}\Big[\sup_{0\le s\le t}\big|z(s)-u(z_s)\big|^2\Big] \\
&\le 8\hat{\mathbb{E}}\|\zeta\|_q^2 + c^* + 4(1+k_2-\lambda_1-k_2\lambda_3)\hat{\mathbb{E}}\int_0^t \big|z(s)\big|^2\,ds \\
&\quad + 4\lambda_2\hat{\mathbb{E}}\int_0^t\int_{-\infty}^0 \big|z(s+\theta)\big|^2\mu_1(d\theta)\,ds + 4\lambda_4k_2\hat{\mathbb{E}}\int_0^t\int_{-\infty}^0 \big|z(s+\theta)\big|^2\mu_2(d\theta)\,ds \\
&\quad + 4\lambda_5\big(k_2+2k_1^2\big)\hat{\mathbb{E}}\int_0^t\int_{-\infty}^0 \big|z(s+\theta)\big|^2\mu_3(d\theta)\,ds \\
&\quad + 2(1+k_2)b_1\hat{\mathbb{E}}\int_0^t\int_{-\infty}^0 \big|z(s+\theta)\big|^2\mu_4(d\theta)\,ds,
\end{aligned}$$

where $c^* = 2(|g(0)|^2 + k_2|h(0)|^2 + 2(k_2+2k_1^2)|\gamma(0)|^2)T$. By using Lemma 2.6, it follows that

$$\begin{aligned}
&\hat{\mathbb{E}}\Big[\sup_{0\le s\le t}\big|z(s)-u(z_s)\big|^2\Big] \\
&\le 2\big[4q+\mu_1^{2q}+k_2\lambda_4\mu_2^{2q}+\lambda_5\big(k_2+2k_1^2\big)\mu_3^{2q}+(1+k_2)b_1\mu_4^{2q}\big]\frac{1}{q}\hat{\mathbb{E}}\|\zeta\|_q^2 + c^* \\
&\quad + 4\big(-\lambda_1-k_2\lambda_3+\lambda_2+\lambda_4k_2+\lambda_5\big(k_2+2k_1^2\big)+(2+k_2)b_1\big)\int_0^t \hat{\mathbb{E}}\Big[\sup_{0\le s\le t}\big|z(s)\big|^2\Big]\,ds.
\end{aligned}$$

By using Lemma 3.1, we derive

$$\begin{aligned}
&\hat{\mathbb{E}}\Big[\sup_{0<s\le t}\big|z(s)\big|^2\Big] \\
&\le 2b_3\big[4q+\mu_1^{2q}+k_2\lambda_4\mu_2^{2q}+\lambda_5\big(k_2+2k_1^2\big)\mu_3^{2q}+(1+k_2)b_1\mu_4^{2q}\big]\frac{1}{q}\hat{\mathbb{E}}\|\zeta\|_q^2 \\
&\quad + b_2\hat{\mathbb{E}}\|\zeta\|_q^2 + b_3c^* + 4b_3\big(-\lambda_1-k_2\lambda_3+\lambda_2+\lambda_4k_2+\lambda_5\big(k_2+2k_1^2\big) \\
&\quad + (2+k_2)\big(1+b^{-1}\big)\big)\int_0^t \hat{\mathbb{E}}\Big[\sup_{0\le s\le t}\big|z(s)\big|^2\Big]\,ds,
\end{aligned} \tag{4.7}$$

where $b_2 = b\mu_4^{(2q)}(1-b)^{-1}$, $b_3 = (1-b)^{-2}$ and $e^{-2qs} < 1$. Noticing that $\hat{\mathbb{E}}[\sup_{-\infty<s\le t}|z(s)|^2] \le \hat{\mathbb{E}}\|\zeta\|_q^2 + \hat{\mathbb{E}}[\sup_{0<s\le t}|z(s)|^2]$ and using the Gronwall inequality, we get the desired assertion. The proof is complete.

To prove (4.2) by the Gronwall inequality from (4.7) we have

$$\hat{\mathbb{E}}\Big[\sup_{0\le s\le t}\big|z(s)\big|^2\Big] \le \alpha e^{\beta t}, \tag{4.8}$$

where $\alpha = 2b_3[4q+\mu_1^{2q}+k_2\lambda_4\mu_2^{2q}+\lambda_5(k_2+2k_1^2)\mu_3^{2q}+(1+k_2)(1+b^{-1})\mu_4^{2q}]\frac{1}{q}E\|\zeta\|_q^2 + b_2\hat{\mathbb{E}}\|\zeta\|_q^2 + b_3c^*$ and $\beta = 4b_3(-\lambda_1-k_2\lambda_3+\lambda_2+\lambda_4k_2+\lambda_5(k_2+2k_1^2)+(2+k_2)(1+b^{-1}))$. By virtue of the above result (4.8), for each $k = 1,2,3,\ldots$, we have

$$\hat{\mathbb{E}}\Big[\sup_{k-1\le t\le k}\big|z(t)\big|^2\Big] \le \alpha e^{\beta m}.$$

For any $\epsilon > 0$, by using Lemma 2.3 we get

$$\hat{\nu}\Big(\delta : \sup_{k-1\le t\le k} \big|z(t)\big|^2 > e^{(\beta+\epsilon)k}\Big) \le \frac{\hat{\mathbb{E}}[\sup_{k-1\le t\le k} |z(t)|^2]}{e^{(\beta+\epsilon)k}} \le \alpha e^{-\epsilon k}.$$

But, for almost all $\delta \in \Omega$, the Borel–Cantelli lemma shows that there is $k_0 = k_0(\delta)$ so that

$$\sup_{k-1\le t\le k} \big|z(t)\big|^2 \le e^{(\beta+\epsilon)k}, \quad \text{whenever } k \ge k_0,$$

and consequently we derive the desired assertion. The proof is completed. □

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Under some useful conditions, we have proved the boundedness of solutions to neutral stochastic functional differential equations driven by G-Brownian motion in the phase space $C_q((-\infty,0];\mathbb{R}^n)$. The convergence of any two solutions with different initial data has been explored. We have also obtained the boundedness and convergent of solution maps in the mentioned space. Moreover, we have determined the L_G^2 and exponential estimates for solutions to G-NSFDEs. Furthermore, our work can be further generalized to neutral stochastic functional differential equations with Markovian switching and NSFDES with Levy process, which could be a new direction of our further work. All authors read and approved the final manuscript.

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