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The distribution of generalized zeros of oscillatory solutions for second-order nonlinear neutral delay difference equations

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Abstract

In this paper, the distributions of generalized zeros of oscillatory solutions for second-order nonlinear neutral delay difference equations are studied. By means of inequality techniques, specific function sequences and non-increasing solutions for corresponding first-order difference inequality, some new estimates for the distribution of the zeros of oscillatory solutions are presented, which extend and improve some well-known results.

MSC: 35B05; 65Q10; 60E05

Keywords: Oscillation; Neutral difference equation; Distribution of zeros

1 Introduction

In recent years, the study of oscillation of differential equations has become more and more perfect, including various sufficient conditions, necessary conditions, the existence of non-oscillatory solutions, and even the zeros distribution of oscillatory solutions.

In 2017, Li et al. [1] studied the distribution of zeros of oscillatory solutions for second-order nonlinear neutral delay differential equation

$$\bigl(a(t)z'(t)\bigr)' + q(t)f\bigl(x(t-\sigma)\bigr) = 0, \quad t \ge t_0,$$

and obtained a sufficient condition for oscillation of differential equation.

However, most of references about oscillation of difference equations are concerned with sufficient or necessary conditions for oscillation; see [2–8]. We will also naturally ask some questions of difference equations: Are there any bounds for the distance between adjacent generalized zeros of oscillatory solutions when equations show oscillation? And how do we estimate these bounds? Therefore, we obtain the oscillation criteria of difference equations by studying the distribution of zeros.

The distribution of generalized zeros of oscillation solutions for first-order dynamic equations and second-order non-neutral dynamic equations on time scale can be found in [9–11]. However, most oscillatory results for second-order neutral dynamic equations are sufficient conditions for oscillation; see [12–19]. To the best of our knowledge, there is no paper on the generalized zero distribution of oscillation solutions for second-order neutral dynamic equations on time scale.

Motivated by the above papers, we consider the second-order neutral difference equation of the following form:

$$\Delta\big(a(t)\Delta z(t)\big) + q(t)f\big(x(t-\sigma)\big) = 0, \quad t \in [t_0,\infty)_{\mathbb{Z}}, \tag{1.1}$$

where Δ denotes the forward difference operator $\Delta x(t) = x(t+1) - x(t)$, $z(t) = x(t) + p(t)x(t-\tau)$, $\mathbb{Z}$ represents the set of all integers and

$$\sum_{s=t_0}^{\infty} \frac{1}{a(s)} = \infty.$$

Throughout this paper, we assume that the following hypotheses are satisfied:

(H_1) $a(t), q(t), p(t) \in (0,\infty)$, where $t \in [t_0,\infty)_{\mathbb{Z}}$.

(H_2) $\tau, \sigma \in \mathbb{R}^+$, where $\mathbb{R}^+$ represents the set of all positive real numbers, and $\sigma > \tau$.

(H_3) There exists a positive constant k such that $\frac{f(u)}{u} \geq k$ for all $u \neq 0$.

(H_4) There exists a function $H(t)$ which satisfies $H(t) \geq \frac{p(t-\sigma)q(t)}{q(t-\tau)}$ and $\Delta H(t) \leq 0$, $t \geq t_1$ for some $t_1 \geq t_0 + \sigma$, where $t \in \mathbb{Z}$.

In this paper, we relate the distance between adjacent generalized zeros of an oscillation solution of (1.1) to a positivity problem of certain solution for a first-order delay difference inequality

$$\Delta x(t) + P(t)x(t-r_1) \leq 0, \quad t \in [t_0,\infty)_{\mathbb{Z}}, \tag{1.2}$$

where $P(t) \in [0,1)$ which define by (2.1), r_1 is a constant satisfying $r_1 \geq 2$.

2 Preliminaries

In order to prove our main results, we establish some fundamental results in this section.

For convenience, we define a sequence $\{F_n(t)\} \in [0,1)$ by

$$\begin{aligned} F_0(t) = P(t) &:= \frac{kq(t)}{1+H(t+1)} \sum_{s=T_0}^{t-1} \frac{1}{a(s-\sigma)}, \quad t \in [t_0,\infty)_{\mathbb{Z}}, \\ F_n(t) &= F_{n-1}(t) \sum_{s=t-r}^{t-1} F_{n-1}(s) \prod_{\zeta=s-r}^{t} \frac{1}{1-F_{n-1}(\zeta)}, \quad t \in [t_0+2nr,\infty)_{\mathbb{Z}}, n = 1,2,\ldots, \end{aligned} \tag{2.1}$$

where T_0 satisfies $x(t) > 0$, $t \geq T_0$ when $x(t)$ is eventually positive solution.

If t_n is a generalized zero of solution of (1.1), then it satisfies $x([t_n]) \cdot x([t_n]+1) \leq 0$. Let $d_s(x)$ be the least upper bound of the distance between adjacent generalized zeros of a solution $x(t)$ of Eq. (1.1) on $[s,\infty)$.

Lemma 2.1 *Assume that $x(t)$ is an eventually positive solution of* (1.1), *and* $(H_1) \sim (H_3)$ *hold. Then $z(t)$ satisfies $z(t) > 0$, $\Delta z(t) > 0$, $\Delta(a(t)\Delta z(t)) < 0$.*

Proof If $x(t)$ is an eventually positive solution of Eq. (1.1), then there exists a $t_1 > t_0$ such that $x(t) > 0$, $x(t-\tau) > 0$ and $x(t-\sigma) > 0$ for all $t \geq t_1$. Thus $z(t) = x(t) + p(t)x(t-\tau) > 0$. From (1.1) and condition (H_3), we obtain

$$\Delta\big(a(t)\Delta z(t)\big) = -q(t)\frac{f(x(t-\sigma))}{x(t-\sigma)}x(t-\sigma) \leq -kq(t)x(t-\sigma) < 0, \quad t \geq t_1, \tag{2.2}$$

so we can conclude $a(t)\Delta z(t)$, $t \geq t_1$ is decreasing. It can be seen that there exists a $t_2 > t_1$ such that $\Delta z(t) > 0$ or $\Delta z(t) < 0$ for $t \geq t_2$. Now, we prove $\Delta z(t) > 0$, $t \geq t_2$. If not, assume that $\Delta z(t) < 0$, $t \geq t_2$, then also $a(t)\Delta z(t) < -c < 0$ and summing up it from t_2 to $t-1$, we have

$$z(t) - z(t_2) < -c \sum_{s=t_2}^{t-1} \frac{1}{a(s)}. \tag{2.3}$$

Taking limits of both sides for the above inequality, we have $\lim_{t\to\infty} z(t) = -\infty$, which is a contradiction. The proof is completed. □

In the following lemmas, let $r = [r_1] := \max\{a | a \leq r_1, a \in \mathbb{Z}\}$, where r_1 is the delay argument of (1.2). And δ is a constant satisfying $|\delta| \leq r$.

Lemma 2.2 *Let n be a positive integer such that*

$$\sum_{s=t-r}^{t-1} F_n(s)\,ds \geq 1, \quad t \in \big[t_0 + (2n+1)r, \infty\big)_{\mathbb{Z}}, n = 1, 2, \ldots. \tag{2.4}$$

If $x(t)$ is a non-increasing function on $[T_1 - \delta, T]_{\mathbb{Z}}$ which satisfies (1.2) on $[T_1, T]_{\mathbb{Z}}$, then $x(t)$ cannot be positive on $[T_1, T]_{\mathbb{Z}}$, where $T > T_1 + (3n+1)r + (n+1) - \delta$, $T_1 \geq t_0 + (2n+1)r$.

Proof Without loss of generality, we assume that $x(t)$ is positive on $[T_1, T]_{\mathbb{Z}}$. Summing up (1.2) from $t-r$ to $t-1$, we have

$$x(t) - x(t-r) + \sum_{s=t-r}^{t-1} P(s)x(s-r_1) \leq 0, \quad t \in [T_1 + r, T]_{\mathbb{Z}}.$$

Multiplying this inequality by $P(t)$ and using (1.2), we get

$$\Delta x(t) + P(t)x(t) + P(t)\sum_{s=t-r}^{t-1} P(s)x(s-r_1) \leq 0, \quad t \in [T_1 + r + 1 - \delta, T]_{\mathbb{Z}}, \tag{2.5}$$

so

$$\prod_{\zeta=t_0}^{t} \frac{1}{1-P(\zeta)} \left(\Delta x(t) + P(t)x(t) + P(t)\sum_{s=t-r}^{t-1} P(s)x(s-r_1)\right) \leq 0, \quad t \in [T_1 + r + 1 - \delta, T]_{\mathbb{Z}}.$$

Using $\Delta x(t) = x(t+1) - x(t)$, we get

$$\prod_{\zeta=t_0}^{t} \frac{1}{1-P(\zeta)} x(t+1) - \prod_{\zeta=t_0}^{t-1} \frac{1}{1-P(\zeta)} x(t) + \prod_{\zeta=t_0}^{t} \frac{1}{1-P(\zeta)} P(t)\sum_{s=t-r}^{t-1} P(s)x(s-r_1) \leq 0,$$
$$t \in [T_1 + r + 1 - \delta, T]_{\mathbb{Z}}. \tag{2.6}$$

Let $y_1(t) := x(t)\prod_{\zeta=t_0}^{t-1} \frac{1}{1-P(\zeta)}$. Then $y_1(t) > 0$ on $[T_1, T]_{\mathbb{Z}}$ and

$$\Delta y_1(t) = \prod_{\zeta=t_0}^{t} \frac{1}{1-P(\zeta)} x(t+1) - \prod_{\zeta=t_0}^{t-1} \frac{1}{1-P(\zeta)} x(t). \tag{2.7}$$

Using (2.7) in (2.6), we have

$$\Delta y_1(t) + P(t)\sum_{s=t-r}^{t-1} P(s)\left(x(s-r_1)\prod_{\zeta=t_0}^{t}\frac{1}{1-P(\zeta)}\right) \le 0, \quad t \in [T_1 + r + 1 - \delta, T]_{\mathbb{Z}},$$

i.e.

$$\Delta y_1(t) + P(t)\sum_{s=t-r}^{t-1}\left(P(s)x(s-r_1)\prod_{\zeta=t_0}^{s-r-1}\frac{1}{1-P(\zeta)}\prod_{\zeta=s-r}^{t}\frac{1}{1-P(\zeta)}\right)$$
$$\le 0, \quad t \in [T_1 + r + 1 - \delta, T]_{\mathbb{Z}}. \tag{2.8}$$

From the definition of $y_1(t)$ and $\Delta x(t) \le 0$, $t \in [T_1 + r + 1 - \delta, T]_{\mathbb{Z}}$ we obtain

$$\begin{aligned}
\Delta y_1(t) &= \Delta x(t)\prod_{\zeta=t_0}^{t}\frac{1}{1-P(\zeta)} + x(t)\Delta\left(\prod_{\zeta=t_0}^{t-1}\frac{1}{1-P(\zeta)}\right) \\
&= \Delta x(t)\prod_{\zeta=t_0}^{t}\frac{1}{1-P(\zeta)} + x(t)\prod_{\zeta=t_0}^{t}\frac{1}{1-P(\zeta)} - x(t)\prod_{\zeta=t_0}^{t-1}\frac{1}{1-P(\zeta)} \\
&= \prod_{\zeta=t_0}^{t}\frac{1}{1-P(\zeta)}\bigl(\Delta x(t) + x(t) - x(t)\bigl(1-P(t)\bigr)\bigr) \\
&= \prod_{\zeta=t_0}^{t}\frac{1}{1-P(\zeta)}\bigl(\Delta x(t) + P(t)x(t)\bigr) \\
&\le \prod_{\zeta=t_0}^{t}\frac{1}{1-P(\zeta)}\bigl(\Delta x(t) + P(t)x(t-r_1)\bigr).
\end{aligned}$$

Since $\Delta x(t) + P(t)x(t-r_1) \le 0$, we can conclude $\Delta y_1(t) \le 0$, $t \in [T_1 + r + 1 - \delta, T]_{\mathbb{Z}}$, and from (2.8), we have

$$\Delta y_1(t) + F_1(t)y_1(t-r) \le 0, \quad t \in [T_1 + 3r + 1 - \delta, T]_{\mathbb{Z}}.$$

Repeating the above procedure to this inequality, we get

$$\Delta y_1(t) + F_1(t)y_1(t) + F_1(t)\sum_{s=t-r}^{t-1} F_1(s)y_1(s-r) \le 0, \quad t \in [T_1 + 4r + 1 - \delta, T]_{\mathbb{Z}}. \tag{2.9}$$

Let $y_2(t) := y_1(t)\prod_{\zeta=t_0+2r}^{t-1}\frac{1}{1-F_1(\zeta)}$. It follows from (2.9) that

$$\Delta y_2(t) + F_1(t)\sum_{s=t-r}^{t-1} F_1(s)y_2(s-r)\prod_{\zeta=s-r}^{t}\frac{1}{1-F_1(s)} \le 0, \quad t \in [T_1 + 4r + 1 - \delta, T]_{\mathbb{Z}},$$

where $\Delta y_2(t) \le 0$ for $t \in [T_1 + 4r + 2 - \delta, T]_{\mathbb{Z}}$ and hence

$$\Delta y_2(t) + F_2(t)y_2(t-r) \le 0, \quad t \in [T_1 + 6r + 2 - \delta, T]_{\mathbb{Z}}.$$

Repeating this argument n times, we obtain

$$\Delta y_n(t) + F_n(t)y_n(t-r) \le 0, \quad t \in [T_1 + 3nr + n - \delta, T]_{\mathbb{Z}}, \tag{2.10}$$

where $\Delta y_n(t) \le 0$ for $t \in [T_1 + (3n-2)r + n - \delta, T]_{\mathbb{Z}}$. Now, summing up (2.10) from $t-r$ to $t-1 \in [T_1 + (3n+1)r + n - \delta, T]_{\mathbb{Z}}$, we have

$$y_n(t) - y_n(t-r) + \sum_{s=t-r}^{t-1} F_n(s)y_n(s-r) \le 0.$$

Since $y(t)$ is decreasing, we obtain

$$y_n(t) + y_n(t-r)\left[\sum_{s=t-r}^{t-1} F_n(s) - 1\right] \le 0, \quad t \in \left[T_1 + (3n+1)r + (n+1) - \delta, T\right]_{\mathbb{Z}},$$

which is a contradiction with hypothesis (2.4). The proof of Lemma 2.2 is complete. □

Lemma 2.3 *Assume that* $\sum_{s=t-r}^{t-2} P(s) \ge \beta$ *for* $0 < \beta < 1$ *and there exist* $T_2 \ge t_0 + r$, $T \ge T_2 + (1+n)r - \delta$, $n = 1, 2, \ldots$ *and a function* $x(t)$ *satisfying inequality* (1.2) *on* $[T_2, T]_{\mathbb{Z}}$ *with* $\Delta x(t) \le 0$ *for* $t \in [T_2 - \delta, T]_{\mathbb{Z}}$. *If* $x(t)$ *is positive on* $[T_2, T]_{\mathbb{Z}}$, *then*

$$\frac{x(t-r)}{x(t)} \ge f_n(\beta) > 0, \quad t \in \left[T_2 + (1+n)r - \delta, T\right]_{\mathbb{Z}}, \tag{2.11}$$

for some integer $n \ge 0$, *where* $f_n(\beta)$ *is defined by*

$$f_0(\beta) = 1, \qquad f_1(\beta) = \frac{1}{1-\beta}, \qquad f_{n+2}(\beta) = \frac{r - \beta f_n(\beta)}{r - \beta f_n(\beta) - \beta + \frac{\beta f_n(\beta)}{f_{n+1}(\beta)}}.$$

Proof Since $x(t)$ is non-increasing on $[T_2 - \delta, T]_{\mathbb{Z}}$, we find

$$\frac{x(t-r)}{x(t)} \ge f_0(\beta) = 1, \quad t \in [T_2 + r - \delta, T]_{\mathbb{Z}}. \tag{2.12}$$

Summing inequality (1.2) from $t-r+1$ to $t-1$, where $t \in [T_2 + 2r - \delta, T]_{\mathbb{Z}}$, we obtain

$$x(t-r) \ge x(t) + \sum_{s=t-r+1}^{t-1} P(s)x(s-r_1) \ge x(t) + \beta x(t-r). \tag{2.13}$$

Therefore

$$\frac{x(t-r)}{x(t)} \ge \frac{1}{1-\beta} = f_1(\beta) > 0, \quad t \in [T_2 + 2r - \delta, T]_{\mathbb{Z}}.$$

On the other hand, dividing inequality (1.2) by $x(t)$,

$$\frac{\Delta x(t)}{x(t)} = -P(t)\frac{x(t-r_1)}{x(t)},$$

because of $\Delta x(t) < 0$,

$$\frac{x(t+1)}{x(t)} \le 1 - P(t)\frac{x(t-r)}{x(t)}.$$

Multiplying from $s-r$ to $t-r-1$ where $s \in [t-r+1, t-1]_{\mathbb{Z}}$, we find

$$\frac{x(s-r)}{x(t-r)} \ge \prod_{u=s-r}^{t-r-1} \frac{1}{1 - P(u)\frac{x(u-r)}{x(u)}}, \quad t \in [T_2 + 3r - \delta, T]_{\mathbb{Z}}.$$

According to (2.12), this yields

$$\frac{x(s-r)}{x(t-r)} \ge \prod_{u=s-r}^{t-r-1} \frac{1}{1 - f_0(\beta)P(u)}, \quad t \in [T_2 + 3r - \delta, T]_{\mathbb{Z}}. \tag{2.14}$$

We can easily obtain

$$\begin{aligned}
&\Delta\left(-\frac{1}{f_0(\beta)} \prod_{u=t-r}^{s-1} \bigl(1 - f_0(\beta)P(u)\bigr)\right) \\
&\quad = -\frac{1}{f_0(\beta)} \prod_{u=t-r}^{s} \bigl(1 - f_0(\beta)P(u)\bigr) + \frac{1}{f_0(\beta)} \prod_{u=t-r}^{s-1} \bigl(1 - f_0(\beta)P(u)\bigr) \\
&\quad = -\frac{1}{f_0(\beta)} \bigl(1 - f_0(\beta)P(s) - 1\bigr) \prod_{u=t-r}^{s-1} \bigl(1 - f_0(\beta)P(u)\bigr) \\
&\quad = P(s) \prod_{u=t-r}^{s-1} \bigl(1 - f_0(\beta)P(u)\bigr).
\end{aligned} \tag{2.15}$$

Combining (2.14), (2.15) with (2.13), and because of the fact

$$\frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}} \le \sqrt[n]{a_1 a_2 \cdots a_n} \le \frac{a_1 + a_2 + \cdots + a_n}{n}, \quad a_i > 0, i = 1, 2, \ldots, n,$$

we have

$$\begin{aligned}
x(t-r) - x(t) &\ge x(t-r) \sum_{s=t-r+1}^{t-1} P(s) \frac{x(s-r)}{x(t-r)} \\
&\ge x(t-r) \sum_{s=t-r+1}^{t-1} P(s) \left(\prod_{u=s-r}^{t-r-1} \frac{1}{1 - f_0(\beta)P(u)} \right) \\
&\ge x(t-r) \sum_{s=t-r+1}^{t-1} P(s) \left(\prod_{u=s-r}^{s-1} \frac{1}{1 - f_0(\beta)P(u)} \prod_{u=t-r}^{s-1} 1 - f_0(\beta)P(u) \right) \\
&\ge x(t-r) \frac{r}{r - \beta f_0(\beta)} \sum_{s=t-r+1}^{t-1} P(s) \left(\prod_{u=t-r}^{s-1} \bigl(1 - f_0(\beta)P(u)\bigr) \right) \\
&= x(t-r) \frac{r}{r - \beta f_0(\beta)} \sum_{s=t-r+1}^{t-1} \Delta\left(-\frac{1}{f_0(\beta)} \prod_{u=t-r}^{s-1} \bigl(1 - f_0(\beta)P(u)\bigr) \right)
\end{aligned}$$

$$
\begin{aligned}
&= x(t-r)\frac{r}{r-\beta f_0(\beta)}\left(-\frac{1}{f_0(\beta)}\prod_{u=t-r}^{t-1}\bigl(1-f_0(\beta)P(u)\bigr)\right. \\
&\quad \left.+\frac{1}{f_0(\beta)}\prod_{u=t-r}^{t-r}\bigl(1-f_0(\beta)P(u)\bigr)\right) \\
&\geq x(t-r)\frac{r}{r-\beta f_0(\beta)}\frac{1}{f_0(\beta)}\bigl(1-f_0(\beta)P(t-r)\bigr)\left(1-\frac{r-1-\beta f_0(\beta)}{r-1}\right) \\
&\geq x(t-r)\frac{r}{r-\beta f_0(\beta)}\frac{1}{f_0(\beta)}\left(1-\frac{f_0(\beta)}{f_1(\beta)}\right)\left(1-\frac{r-1-\beta f_0(\beta)}{r-1}\right) \\
&\geq x(t-r)\frac{\beta}{r-\beta f_0(\beta)}\left(1-\frac{f_0(\beta)}{f_1(\beta)}\right).
\end{aligned}
$$

Thus

$$
\frac{x(t-r)}{x(t)} \geq \frac{r-\beta f_0(\beta)}{r-\beta f_0(\beta)-\beta+\frac{\beta f_0(\beta)}{f_1(\beta)}} = f_2(\beta) > 1, \quad t \in [T_2+3r-\delta, T]_{\mathbb{Z}}.
$$

Repeating this argument, it follows by induction that

$$
\frac{x(t-r)}{x(t)} \geq f_n(\beta) > 0, \quad t \in \bigl[T_2+(n+1)r-\delta, T\bigr]_{\mathbb{Z}}.
$$

The proof is complete. □

Remark It can easily be seen that either $f_n(\beta)$ satisfies $\lim_{t\to\infty} f_n(\beta) = 1$ or $f_n(\beta)$ is nondecreasing and $\lim_{t\to\infty} f_n(\beta) = \infty$ or $f_n(\beta) \to \infty$ after finite number of terms or $f_n(\beta)$ is negative.

Lemma 2.4 *Assume that* $\sum_{s=t-r}^{t-2} P(s) \geq \beta$, $t \geq t_0$ *holds for some* $0 < \beta < 1$ *and there exists a function* $x(t)$ *satisfying inequality* (1.2) *on* $[T_2, T+Nr+1]_{\mathbb{Z}}$ *for some positive integer* N *such that* $\Delta x(t) \leq 0$ *on* $[T_2-\delta, T_2+Nr+1]_{\mathbb{Z}}$ *where* $T_2 \geq t_0 + r$. *If* $x(t)$ *is positive on* $[T_2, T_2 + Nr+1]_{\mathbb{Z}}$, *then*

$$
0 < \frac{x(t-r)}{x(t)} \leq g_m(\beta), \quad t \in \bigl[T_2+2r-\delta, T_2+(N-m)r+1\bigr]_{\mathbb{Z}}, \tag{2.16}
$$

where m *is a positive integer,* $N \geq m+2-\frac{\delta}{r}$, *and* $g_m(\beta)$ *is defined by*

$$
g_1(\beta) = \frac{2(1-\beta)}{\beta^2}, \qquad g_{m+1}(\beta) = \frac{2(1-\beta-\frac{1}{g_m(\beta)})}{\beta^2}, \quad m = 1,2,\ldots.
$$

Proof From $\sum_{s=t-r}^{t-1} P(s) \geq \beta$, $t \geq t_0$, we see that $\sum_{s=t}^{t+r-1} P(s) \geq \beta$ for $t \geq T_2$. Summing both sides of (1.2) from t to $t+r-1$, we obtain

$$
x(t) - x(t+r) \geq \sum_{s=t}^{t+r-1} P(s)x(s-r_1), \quad t \in \bigl[T_2+r, T_2+(N-1)r+1\bigr]_{\mathbb{Z}}. \tag{2.17}
$$

Since $T_2 + r \le t \le s \le t + r - 1$, it follows $T_2 \le t - r \le s - r \le t - 1$. Again, summing (1.2) from $s - r$ to t yields

$$x(s-r) - x(t) \ge \sum_{u=s-r}^{t-1} P(u)x(u-r_1).$$

It is clear that $x(u - r_1)$ is non-increasing on $[s - r, t + 1]_{\mathbb{Z}} \subseteq [T_2 + r - \delta, T_2 + (N - 1)r + 1]_{\mathbb{Z}}$. Thus,

$$\begin{aligned} x(s-r) &\ge x(t) + \sum_{u=s-r}^{t-1} P(u)x(u-r_1) \\ &\ge x(t) + x(t-r)\sum_{u=s-r}^{t-1} P(u) \\ &= x(t) + x(t-r)\left[\sum_{u=s-r}^{s} P(u) - \sum_{u=t}^{s} P(u)\right] \\ &\ge x(t) + x(t-r)\left[\beta - \sum_{u=t}^{s} P(u)\right]. \end{aligned}$$

In view of the last inequality and (2.17), we obtain

$$\begin{aligned} x(t) &\ge x(t+r) + \sum_{s=t}^{t+r-1} P(s)x(s-r_1) \\ &\ge x(t+r) + \sum_{s=t}^{t+r-1} P(s)\left[x(t) + x(t-r)\left(\beta - \sum_{u=t}^{s} P(u)\right)\right] \\ &\ge x(t+r) + \beta x(t) + \beta^2 x(t-r) - x(t-r)\sum_{s=t}^{t+r-1} P(s)\left(\sum_{u=t}^{s} P(u)\right), \end{aligned} \tag{2.18}$$

for all $t \in [T_2 + 2r - \delta, T_2 + (N - 1)r + 1]_{\mathbb{Z}}$. As is well known, we have the identity

$$\sum_{s=t}^{t+r-1}\sum_{u=t}^{s}\bigl(P(s)P(u)\bigr) = \sum_{u=t}^{t+r-1}\sum_{s=u}^{t+r-1}\bigl(P(u)P(s)\bigr) = \sum_{s=t}^{t+r-1}\sum_{u=s}^{t+r-1}\bigl(P(s)P(u)\bigr).$$

Consequently,

$$\sum_{s=t}^{t+r-1}\sum_{u=t}^{s}\bigl(P(s)P(u)\bigr) > \frac{1}{2}\sum_{s=t}^{t+r-1}\sum_{u=t}^{t+r-1}\bigl(P(s)P(u)\bigr) = \frac{1}{2}\left(\sum_{s=t}^{t+r-1} P(s)\right)^2 \ge \frac{\beta^2}{2}.$$

Substituting into (2.18),

$$x(t) \ge x(t+r) + \beta x(t) + \frac{\beta^2}{2}x(t-r), \quad t \in \bigl[T_2 + 2r - \delta, T_2 + (N-1)r + 1\bigr]_{\mathbb{Z}}. \tag{2.19}$$

Since $x(t + r) > 0$ on $[T_2 + 2r - \delta, T_2 + (N - 1)r]_{\mathbb{Z}}$, we get

$$\frac{x(t-r)}{x(t)} < \frac{2(1-\beta)}{\beta^2} = g_1(\beta), \quad t \in \bigl[T_2 + 2r - \delta, T_2 + (N-1)r + 1\bigr]_{\mathbb{Z}}. \tag{2.20}$$

On the other hand, when $t \in [T_2 + 2r - \delta, T_2 + (N-2)r + 1]_{\mathbb{Z}}$, we have $T_2 + 2r - \delta \le t \le t + r \le T_2 + (N-1)r$. So (2.20) leads to

$$x(t+r) > \frac{1}{g_1(\beta)} x(t), \quad t \in \left[T_1 + 2r - \delta, T_1 + (N-2)r + 1\right]_{\mathbb{Z}}.$$

Since $x(t)$ is non-increasing on $[T_2 - \delta, T + Nr + 1]$, it follows that

$$x(t+r) > \frac{1}{g_1(\beta)} x(t) \ge \frac{1}{g_1(\beta)} x(t), \quad t \in \left[T_2 + 2r - \delta, T_2 + (N-2)r + 1\right]_{\mathbb{Z}}.$$

From this inequality and (2.19), we obtain

$$x(t) \ge \frac{1}{g_1(\beta)} x(t) + \beta x(t) + \frac{\beta^2}{2} x(t-r), \quad t \in \left[T_2 + 2r - \delta, T_2 + (N-2)r + 1\right]_{\mathbb{Z}}.$$

Rearranging,

$$\frac{x(t-r)}{x(t)} < \frac{2(1 - \beta - \frac{1}{g_1(\beta)})}{\beta^2} = g_2(\beta), \quad t \in \left[T_2 + 2r - \delta, T_2 + (N-2)r + 1\right]_{\mathbb{Z}}.$$

Repeating the above procedure, we get

$$\frac{x(t-r)}{x(t)} < \frac{2(1 - \beta - \frac{1}{g_{m-1}(\beta)})}{\beta^2} = g_m(\beta), \quad t \in \left[T_2 + 2r - \delta, T_2 + (N-m)r + 1\right]_{\mathbb{Z}}.$$

The proof of Lemma 2.4 is complete. □

Remark Wu and Xu [18] proved that $g_m(\beta)$ is decreasing. They found also that $g_{m+1}(\beta) > \frac{1-\beta}{\beta_2}$ for $m = 1, 2, \ldots$. So when $0 < \beta \le \sqrt{2} - 1$, there exists a function $g(\beta) = \frac{2(1-\beta-\frac{1}{g(\beta)})}{\beta_2}$ such that $\lim_{m\to\infty} g_m(\beta) = g(\beta)$.

Lemma 2.5 *Assume that* $\sum_{s=t-r}^{t-2} P(s) \ge \beta$ *holds for some* $\beta > \sqrt{2} - 1$ *and* $x(t)$ *is a function satisfying inequality* (1.2) *on* $[T_2, T]_{\mathbb{Z}}$ *with* $\Delta x(t) \le 0$ *for* $[T_2 - \delta, T]_{\mathbb{Z}}$, $T \ge T_2 + (k_\beta + 1)r - \delta$, $T_2 \ge t_0 + r$ *and* k_β *is defined by*

$$\begin{aligned} k_\beta &= \begin{cases} 1, & \beta \ge 1, \\ \min\{\alpha, \gamma\}, & \sqrt{2} - 1 < \beta < 1, \end{cases} \\ \alpha &= \min_{n\ge1, m\ge1} \left\{ n + m \mid f_n(\beta) \ge g_m(\beta) \right\}, \\ \gamma &= 1 + \min_{n\ge1} \left\{ n \mid f_n(\beta) < 0 \text{ or } f_{n+1}(\beta) = \infty \right\}. \end{aligned} \tag{2.21}$$

Then $x(t)$ *is positive on* $[T_2, T]_{\mathbb{Z}}$.

Proof Suppose, for the sake of contradiction, that $x(t)$ is positive on $[T_2, T]$. We consider two cases:

Case 1: $\beta \geq 1$. In this case $k_\beta = 1$ and $T \geq T_2 + 2r - \delta$. Since $\Delta x(t) \leq 0$ on $[T_2 - \delta, T]_{\mathbb{Z}}$, we obtain

$$x(t) \geq x(T_2 + r - \delta), \quad t \in [T_2 - \delta, T_2 + r - \delta]_{\mathbb{Z}}.$$

Summing both sides of (1.2) from $T_2 + r - \delta$ to $T_2 + 2r - \delta - 1$ and using the above inequality, we obtain

$$\begin{aligned} x(T_2 + 2r - \delta) &\leq x(T_2 + r - \delta) - \sum_{s=T_2+r-\delta}^{T_2+2r-\delta-1} P(s)x(s-r) \\ &\leq x(T_2 + r - \delta) - x(T_2 + r - \delta) \sum_{s=T_2+r-\delta}^{T_2+2r-\delta-1} P(s) \\ &= x(T_2 + r - \delta)\left[1 - \sum_{s=T_2+r-\delta}^{T_2+2r-\delta-1} P(s)\right] < 0, \end{aligned}$$

which is a contradiction.

Case 2: $\sqrt{2} - 1 < \beta < 1$. If $k_\beta = n^* + m^*$, then

$$f_{n^*}(\beta) \geq g_{m^*}(\beta). \tag{2.22}$$

From Lemma 2.3, it follows that

$$\frac{x(t-r)}{x(t)} \geq f_{n^*}(\beta), \quad t \in \left[T_2 + \left(n^* + 1\right)r - \delta, T\right]_{\mathbb{Z}}. \tag{2.23}$$

On the other hand, by Lemma 2.4 we find

$$\frac{x(t-r)}{x(t)} < g_{m^*}(\beta), \quad t \in \left[T_2 + 2r - \delta, T_2 + \left(N - m^* r\right) + 1\right]_{\mathbb{Z}}. \tag{2.24}$$

So, when $t = T_2 + (n^* + 1)r - \delta$ in (2.23) and (2.24), it follows that

$$f_{n^*}(\beta) \leq \frac{x(T_2 + n^* r - \delta)}{x(T_2 + (n^* + 1)r - \delta)} < g_{m^*}(\beta),$$

which contradicts (2.22). If

$$k_\beta = 1 + \min_{n \geq 1}\left\{n | f_{n+1}(\beta) < 0 \text{ or } f_{n+1}(\beta) = \infty\right\},$$

then Lemma 2.3 implies a contradiction and the proof is complete. □

3 Main results

In this section, we obtain sufficient oscillation conditions for Eq. (1.1) about the distribution of generalized zeros.

Theorem 3.1 *Let* (H_1)*–*(H_4) *and* (2.4) *establish for some positive integer* n *with* $r_1 = \sigma - \tau$. *Then the equation* (1.1) *oscillates and* $d_{\tilde{t}}(x) \leq 2\sigma + 3n(\sigma - \tau)$, *where* $\tilde{t} = t_1 + (2n + 1)(\sigma - \tau)$.

Proof If Eq. (1.1) has a non-oscillatory solution $x(t)$, and $-x(t)$ is also the solution of Eq. (1.1), so we only consider the situation of the solution of (1.1) is eventually positive. We assume $x(t) > 0$ on $[T_0, T]_{\mathbb{Z}}$ for some integer $T_0 \geq \tilde{t}$ where $T > T_0 + 2\sigma + 3n(\sigma - \tau)$. Since $z(t) = x(t) + p(t)x(t - \tau)$ for $t \in [T_0 + \tau, T]$, $z(t) > 0$ on $[T_0 + \tau, T]_{\mathbb{Z}}$. From inequality (2.2), we have

$$\begin{aligned} &\Delta\bigl(a(t)\Delta z(t)\bigr) \\ &\quad \leq -kq(t)z(t-\sigma) + kq(t)p(t-\sigma)x(t-\tau-\sigma), \quad t \in [T_0+\sigma+\tau, T]_{\mathbb{Z}}. \end{aligned} \tag{3.1}$$

Also from (2.2), we obtain $x(t - \tau - \sigma) \leq -\frac{\Delta(a(t-\tau)\Delta z(t-\tau))}{kq(t-\tau)}$, $t \in [T_0 + \sigma + \tau, T]_{\mathbb{Z}}$. Then inequality (3.1) can be rewritten as

$$\begin{aligned} &\Delta\bigl(a(t)\Delta z(t)\bigr) \\ &\quad \leq -kq(t)z(t-\sigma) - kq(t)p(t-\sigma)\frac{\Delta(a(t-\tau)\Delta z(t-\tau))}{kq(t-\tau)} \\ &\quad = -kq(t)z(t-\sigma) - p(t-\sigma)\frac{q(t)}{q(t-\tau)}\Delta\bigl(a(t-\tau)\Delta z(t-\tau)\bigr), \quad t \in [T_0+\sigma, T]_{\mathbb{Z}}, \end{aligned}$$

i.e.

$$\begin{aligned} &\Delta\bigl(a(t)\Delta z(t)\bigr) + p(t-\sigma)\frac{q(t)}{q(t-\tau)}\Delta\bigl(a(t-\tau)\Delta z(t-\tau)\bigr) + kq(t)z(t-\sigma) \\ &\quad \leq 0, \quad t \in [T_0+\sigma, T]_{\mathbb{Z}}. \end{aligned}$$

We can conclude from condition (H_4) and $\Delta(a(t)\Delta z(t)) < 0$,

$$\begin{aligned} &\Delta\bigl(a(t)\Delta z(t)\bigr) + H(t)\Delta\bigl(a(t-\tau)\Delta z(t-\tau)\bigr) + kq(t)z(t-\sigma) \\ &\quad \leq 0, \quad t \in [T_0+\sigma, T]_{\mathbb{Z}}. \end{aligned} \tag{3.2}$$

Let

$$w(t) = a(t)\Delta z(t) + H(t)\bigl(a(t-\tau)\Delta z(t-\tau)\bigr), \quad t \in [T_0+\sigma+\tau, T]_{\mathbb{Z}}. \tag{3.3}$$

So

$$w(t) \leq \bigl(1 + H(t)\bigr)\bigl(a(t-\tau)\Delta z(t-\tau)\bigr). \tag{3.4}$$

Differentiating both sides of (3.3), and because of (3.2), and $\Delta(a(t)\Delta z(t)) < 0$, we obtain

$$\begin{aligned} \Delta w(t) &\leq \Delta H(t)\bigl(a(t-\tau+1)\Delta z(t-\tau+1)\bigr) - kq(t)z(t-\sigma) \\ &< \Delta H(t)\bigl(a(t-\tau)\Delta z(t-\tau)\bigr) - kq(t)z(t-\sigma), \quad t \in [T_0+\sigma, T]_{\mathbb{Z}}. \end{aligned} \tag{3.5}$$

From (3.4), we get

$$\Delta z(t-\sigma) = \Delta z(t+\tau-\sigma-\tau) \geq \frac{w(t+\tau-\sigma)}{a(t-\sigma)(1+H(t+\tau-\sigma))}, \quad t \in [T_0+2\sigma, T]_{\mathbb{Z}}.$$

Summing up the above form from T_0 to $t-1$, we have

$$z(t-\sigma)-z(T_0-\sigma)\geq\sum_{s=T_0}^{t-1}\frac{w(s+\tau-\sigma)}{1+H(s+\tau-\sigma)}\frac{1}{a(s-\sigma)},$$

therefore

$$z(t-\sigma)\geq\sum_{s=T_0}^{t-1}\frac{w(s+\tau-\sigma)}{1+H(s+\tau-\sigma)}\frac{1}{a(s-\sigma)}. \tag{3.6}$$

Let

$$y(t)=\frac{w(t)}{1+H(t)}>0,\quad t\in[T_0+\sigma+\tau,T]_{\mathbb{Z}}. \tag{3.7}$$

Then

$$\Delta y(t)=\frac{\Delta w(t)(1+H(t))-w(t)\Delta(1+H(t))}{(1+H(t+1))(1+H(t))},\quad t\in[T_0+\sigma+\tau,T]_{\mathbb{Z}}. \tag{3.8}$$

Adding (3.3) and (3.5) to (3.8), we have

$$\Delta y(t)+\frac{kq(t)z(t-\sigma)}{1+H(t+1)}<0,\quad t\in[T_0+\sigma+\tau,T]_{\mathbb{Z}}. \tag{3.9}$$

From (3.6), (3.7) and the decreasing of $y(t)$, we get

$$z(t-\sigma)\geq\sum_{s=T_0}^{t-1}\frac{1}{a(s-\sigma)}y(s+\tau-\sigma)\geq y(t+\tau-\sigma)\sum_{s=T_0}^{t-1}\frac{1}{a(s-\sigma)},\quad t\in[T_0+\sigma+\tau,T]_{\mathbb{Z}}.$$

Substituting the above inequality into (3.9), we obtain

$$\Delta y(t)+\frac{kq(t)y(t+\tau-\sigma)}{1+H(t+1)}\sum_{s=T_0}^{t-1}\frac{1}{a(s-\sigma)}<0,\quad t\in[T_0+\sigma+\tau,T]_{\mathbb{Z}}. \tag{3.10}$$

Set $r_1=\sigma-\tau$, $T_1=T_0+\sigma+\tau$ and $P(t)=\frac{kq(t)}{1+H(t+1)}\sum_{s=T_0}^{t-1}\frac{1}{a(s-\sigma)}$, we conclude

$$\Delta y(t)+P(t)y(t-r_1)<0,\quad t\in[T_1,T]_{\mathbb{Z}}.$$

What is more, (2.1) holds and $y(t)$ is decreasing. Then we can conclude from Lemma 2.2 that $y(t)$ cannot be positive on $[T_1,T]_{\mathbb{Z}}$ when $r_1=\sigma-\tau$, where $T>T_1+3n(\sigma-\tau)$. This is a contradiction with (3.7). The proof is completed. □

Assume the following condition holds:
(H_5) $\sum_{s=t-r}^{t-1}\frac{q(s)}{1+H(s+1)}\sum_{v=T_0}^{s-1}\frac{1}{a(v-\sigma)}\geq\beta$, $t\geq t_2$ for some $t_2\geq t_0+2\sigma-\tau$.
Then we can obtain some further conclusions by means of Theorem 3.1.

Corollary 3.1 *Suppose conditions* (H_1)–(H_5) *hold and a sequence* $\{\beta_n\}$ *is defined by*

$$\beta_0=\beta>0,\qquad\beta_n={\beta_0}^{n+1},\quad n=0,1,2,\ldots.$$

If there is some positive constant $n_0 \in \mathbb{N}$ such that $1 \leq \beta < r$, then Eq. (1.1) *is oscillatory and $d_{\tilde{t}} \leq 2\sigma + 3n_0(\sigma - \tau)$, where $\tilde{t} = t_1 + (2n_0 + 1)(\sigma - \tau)$.*

Proof According to condition (H_5), we have $\sum_{s=t-r}^{t-1} F_0(s) \geq \beta$. In addition, from the iterative sequence $\{F_n(t)\}$, we get

$$\begin{aligned}
\sum_{s=t-r}^{t-1} F_1(s) &= \sum_{s=t-r}^{t-1} F_0(s) \sum_{v=s-r}^{s-1} F_0(s) \prod_{\zeta=v-r}^{s-1} \frac{1}{1 - F_0(\zeta)} \\
&\geq \sum_{s=t-r}^{t-1} F_0(s) \sum_{v=s-r}^{s-1} F_0(s) \prod_{\zeta=v-r}^{v-1} \frac{1}{1 - F_0(\zeta)} \prod_{\zeta=v}^{s-1} \frac{1}{1 - F_0(\zeta)} \\
&\geq \frac{r}{r-\beta} \sum_{s=t-r}^{t-1} F_0(s) \sum_{v=s-r}^{s-1} F_0(s) \\
&= \frac{r}{r-\beta} \beta^2 \geq \beta^2.
\end{aligned}$$

In the same way, continuing the calculation n times, we obtain $\sum_{s=t-r}^{t-1} F_n(s) \geq \beta_n$ for $n = 2, 3, \ldots$. Therefore, by mathematical induction, we have

$$\sum_{s=t-r}^{t-1} F_n(s) \geq \beta_n, \quad \text{for all } n \in \mathbb{N}.$$

Let $n = n_0$. According to Theorem 3.1, the proof is completed. □

Theorem 3.2 *Let (H_1)–(H_5) hold. Then Eq.* (1.1) *oscillates and $d_{t_2}(x) \leq 2\sigma + k_\beta(\sigma - \tau)$, where k_β is defined by* (2.21).

Proof As usual, we assume (1.1) has a solution $x(t) > 0$ on $[T_0, T]_{\mathbb{Z}}$ where $T > T_0 + 2\sigma + k_\beta(\sigma - \tau)$ and $T_0 \geq t_2$. Proceed as in the proof of Theorem 3.1, when $T_1 = T_0 + 2\sigma$. It follows that

$$\Delta y(t) + p(t) y(t + \tau - \sigma) < 0, \quad t \in [T_1, T]_{\mathbb{Z}},$$

where

$$y(t) > 0, \quad t \in \bigl[T_1 - 2(\sigma - \tau), T\bigr]_{\mathbb{Z}}.$$

Also from (3.10) we obtain

$$\Delta y(t) < 0, \quad t \in \bigl[T_1 - (\sigma - \tau), T\bigr]_{\mathbb{Z}}.$$

Since (H_5) holds, we conclude from Lemma 2.5 with $\delta = \sigma - \tau$ that $y(t)$ cannot be positive on $[T_1, T]_{\mathbb{Z}}$, where $T > T_1 + k_\beta(\sigma - \tau)$. This contradiction completes the proof. □

4 Example

In this section, we will present an example to illustrate main results.

Example 4.1 We consider the delay difference equation

$$\Delta\bigl(\Delta\bigl(x(t)+x(t-1)\bigr)\bigr)+\frac{7}{48(t-2)}x(t-2)=0,\quad t\geq 1. \tag{4.1}$$

Compared with (1.1), we denote $a(t)=1$ where $c>t-1$, $p(t)=1$, $q(t)=\frac{7}{48(t-2)}$, $\tau=1$, $\sigma=3$, $r=\sigma-\tau=2$, $f(u)=u$ for $u\neq 0$. It is easy to verify that conditions (H_1) and (H_2). Since $f(u)=u$, we get $\frac{f(u)}{u}=1$. (H_3) holds. Now take $H(t)\equiv 1$ which satisfies (H_4). According to (2.1), we obtain $F_0(t)=\frac{kq(t)}{1+H(t+1)}\sum_{s=T_0}^{t-1}\frac{1}{a(s-\sigma)}=\frac{7}{2\times 48(t-2)}\sum_{s=2}^{t-1}1=\frac{7}{24}$, $t\geq 1$ and $\frac{1}{1-F_0(\zeta)}=\frac{24}{17}$, so

$$\begin{aligned} F_1(t)&=F_0(t)\sum_{s=t-r}^{t-1}F_0(s)\prod_{\zeta=s-r}^{t}\frac{1}{1-F_0(\zeta)}\\ &=\frac{7}{24}\frac{7}{24}\left(\frac{24}{17}\right)^5+\frac{7}{24}\frac{7}{24}\left(\frac{24}{17}\right)^4\approx 0.815,\quad t\geq 6. \end{aligned}$$

Therefore, $\sum_{s=t-2}^{t-1}F_1(s)\,ds\geq 1$ for all $t\geq 6$. Here, all conditions of Theorem 3.1 are satisfied with $n=1$, then we derive that (4.1) shows oscillatory and $d_{\tilde{t}}(x)\leq 2\sigma+3n(\sigma-\tau)=12$, where $\tilde{t}=t_1+(2n+1)(\sigma-\tau)=t_1+6$ and $t_1\geq t_0+\sigma=4$.

5 Conclusion

In this paper, two theorems on the distribution of oscillation zeros for second-order nonlinear neutral delay difference equations are obtained by means of inequality techniques, specific function sequences and non-increasing solutions for corresponding first-order difference inequality. Comparing with the corresponding differential equation, it is more complex to deal with the lower bound of summation. Function $\frac{1}{a(t)}\prod_{s=1}^{t-1}(1+a(s))$ is invariant after derivation in difference equation, which is equivalent to e^x in differential equation. That is the difficulty we address and the innovation of this paper. We study a second-order equation under the canonical form, and it is also of great significance for the study of non-canonical forms. Moreover, this paper can be extended to the dynamic equation on time scale.

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