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On more general boundary value problems involving sequential fractional derivatives

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Abstract
We investigate the existence of solutions for new boundary value problems of Caputo-type sequential fractional differential equations and inclusions supplemented with nonlocal integro-multipoint boundary conditions. We apply the modern techniques of functional analysis to obtain the main results. We emphasize that the results presented in this paper are new and specialize to some known theorems with an appropriate choice of the parameters involved in the problems at hand.

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1 Introduction

In this paper, we investigate the existence criteria for the solutions of Caputo-type sequential fractional differential equations and inclusions:

$$\left({}^cD^q + \mu\, {}^cD^{q-1}\right)x(t) = f\left(t, x(t), {}^cD^\kappa x(t)\right), \quad t \in [0,1], \tag{1.1}$$

$$\left({}^cD^q + \mu\, {}^cD^{q-1}\right)x(t) \in F\left(t, x(t), {}^cD^\kappa x(t)\right), \quad t \in [0,1], \tag{1.2}$$

supplemented with nonlocal integro-multipoint boundary conditions:

$$\begin{cases} \rho_1 x(0) + \rho_2 x(1) = \sum_{i=1}^{m-2} \alpha_i x(\sigma_i) + \sum_{j=1}^{p-2} r_j \int_{\xi_j}^{\eta_j} x(s)\,ds, \\ \rho_3 x'(0) + \rho_4 x'(1) = \sum_{i=1}^{m-2} \delta_i x'(\sigma_i) + \sum_{j=1}^{p-2} \gamma_j \int_{\xi_j}^{\eta_j} x'(s)\,ds, \\ 0 < \sigma_1 < \sigma_2 < \cdots < \sigma_{m-2} < \cdots < \xi_1 < \eta_1 < \xi_2 < \eta_2 < \cdots < \xi_{p-2} < \eta_{p-2} < 1, \end{cases} \tag{1.3}$$

where ${}^cD^q$, ${}^cD^\kappa$ denote the Caputo fractional derivatives of order $q \in (1,2]$ and $\kappa \in (0,1)$, respectively, $\mu > 0$, f is a given continuous function, $F : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map ($\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$), $\rho_1, \rho_2, \rho_3, \rho_4$ are real constants and α_i, δ_i ($i = 1, 2, \ldots, m-2$), r_j, γ_j ($j = 1, 2, \ldots, p-2$) are positive real constants.

The tools of fractional calculus such as fractional order differential and integral operators are found to be of great utility in developing the mathematical models related to dynamical systems involving fractals and chaos. The modelers' interest in such tools is due to the fact that fractional order operators are nonlocal in nature and can trace the hereditary characteristics of many materials and processes involved in the problem. For a detailed account of the subject, for example, see the books [1–6] and papers [7–10].

Fractional order single- and multivalued boundary value problems involving different kinds of boundary conditions attracted significant attention during the last two decades. The literature on this topic is now much enriched and contains a variety of results ranging from the existence theory to the methods of solution for such problems [11–28].

The present work is motivated by a recent article [29] dealing with a fractional differential equation: ${}^cD^q x(t) = f(t, x(t), {}^cD^\beta x(t))$, $1 < q \leq 2$, $0 < \beta < 1$, $t \in (0,1)$, equipped with boundary conditions (1.3).

The rest of the paper is arranged as follows. In Sect. 2, we outline the basic concepts of fractional calculus and prove an auxiliary lemma. Section 3 contains the main results for the problem (1.1) and (1.3) and illustrative examples for the obtained results. In Sect. 4, we prove the existence of solutions for the inclusion problem (1.2) and (1.3) for convex- as well as nonconvex-valued maps involved in the given problem. The paper concludes with some interesting observations.

2 Preliminaries

Let us begin this *section* with some preliminary concepts of fractional calculus [1, 4].

Definition 2.1 Let v be a locally integrable real-valued function on $-\infty \leq a < t < b \leq +\infty$. The Riemann–Liouville fractional integral I_a^α of v of order $\alpha \in \mathbb{R}$ ($\alpha > 0$) is defined as

$$I_a^\alpha v(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} v(s)\, ds,$$

where Γ denotes the Euler gamma function.

Definition 2.2 Let $v, v^{(m)} \in L^1[a,b]$ for $-\infty \leq a < t < b \leq +\infty$. The Riemann–Liouville fractional derivative D_a^α of v of order $\alpha > 0$ ($m-1 < \alpha < m$, $m \in \mathbb{N}$) is defined as

$$D_a^\alpha v(t) = \frac{d^m}{dt^m} I_a^{m-\alpha} v(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_a^t (t-s)^{m-1-\alpha} v(s)\, ds,$$

while the Caputo fractional derivative ${}^cD_a^\alpha$ of v of order $\alpha \in \mathbb{R}$ ($m-1 < \alpha < m$, $m \in \mathbb{N}$) is defined as

$${}^cD_a^\alpha v(t) = D_a^\alpha \left[v(t) - v(a) - v'(a)\frac{(t-a)}{1!} - \cdots - v^{(m-1)}(a)\frac{(t-a)^{m-1}}{(m-1)!} \right].$$

Remark 2.3 If $v \in C^m[a,b]$, then the Caputo fractional derivative ${}^cD_a^\alpha$ of order $\alpha \in \mathbb{R}$ ($m-1 < \alpha < m$, $m \in \mathbb{N}$) is defined as

$${}^cD_a^\alpha v(t) = I_a^{m-\alpha} v^{(m)}(t) = \frac{1}{\Gamma(m-\alpha)} \int_a^t (t-s)^{m-1-\alpha} v^{(m)}(s)\, ds.$$

In the present work, we denote the Riemann–Liouville fractional integral I_a^α and the Caputo fractional derivative ${}^cD_a^\alpha$ with $a = 0$ by I^α and ${}^cD^\alpha$, respectively.

To define the solution for problem (1.1) and (1.2), we consider the following lemma dealing with its linear variant.

Lemma 2.4 *Let $h \in C([0,1],\mathbb{R})$. Then the integral solution for the sequential fractional differential equation*

$$\left({}^cD^q + \mu {}^cD^{q-1}\right)x(t) = h(t), \tag{2.1}$$

supplemented with the boundary conditions (1.3) *is given by*

$$\begin{aligned} x(t) &= \sum_{i=1}^{m-2}\phi_i(t)\int_0^{\sigma_i} e^{\mu(s-\sigma_i)}\left(I^{q-1}h(s)\right)ds \\ &\quad + \sum_{j=1}^{p-2}\psi_j(t)\int_{\xi_j}^{\eta_j}\left(\int_0^s e^{\mu(m-s)}\left(I^{q-1}h(m)\right)dm\right)ds \\ &\quad + \Omega_1(t)\left[\sum_{i=1}^{m-2}\delta_i I^{q-1}h(\sigma_i) + \sum_{j=1}^{p-2}\gamma_j\int_{\xi_j}^{\eta_j}\left(I^{q-1}h(s)\right)ds - \rho_4 I^{q-1}h(1)\right] \\ &\quad + \Omega_2(t)\int_0^1 e^{\mu(s-1)}\left(I^{q-1}h(s)\right)ds + \int_0^t e^{\mu(s-t)}\left(I^{q-1}h(s)\right)ds, \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} \phi_i(t) &= \frac{1}{\mu\Delta}\left[\mu e^{-\mu t}(B_2\alpha_i + \mu B_1\delta_i) - \left(1-e^{-\mu t}\right)(A_2\alpha_i + \mu A_1\delta_i)\right], \\ \psi_j(t) &= \frac{1}{\mu\Delta}\left[\mu e^{-\mu t}(B_2 r_j + \mu B_1\gamma_j) - \left(1-e^{-\mu t}\right)(A_2 r_j + \mu A_1\gamma_j)\right], \end{aligned} \tag{2.3}$$

$$\begin{aligned} \Omega_1(t) &= \frac{1}{-\mu\Delta}\left[\mu B_1 e^{-\mu t} - A_1\left(1-e^{-\mu t}\right)\right], \\ \Omega_2(t) &= \frac{1}{-\mu\Delta}\left[\mu e^{-\mu t}(\rho_4\mu B_1 + \rho_2 B_2) - \left(1-e^{-\mu t}\right)(\rho_4\mu A_1 + \rho_2 A_2)\right], \end{aligned} \tag{2.4}$$

$$\begin{aligned} A_1 &= \rho_1 + \rho_2 e^{-\mu} - \sum_{i=1}^{m-2}\alpha_i e^{-\mu\sigma_i} + \sum_{j=1}^{p-2}\frac{r_j}{\mu}\left(e^{-\mu\eta_j} - e^{-\mu\xi_j}\right), \\ A_2 &= -\mu\left[\rho_3 + \rho_4 e^{-\mu} - \sum_{i=1}^{m-2}\delta_i e^{-\mu\sigma_i} + \frac{1}{\mu}\sum_{j=1}^{p-2}\gamma_j\left(e^{-\mu\eta_j} - e^{-\mu\xi_j}\right)\right], \end{aligned} \tag{2.5}$$

$$\begin{aligned} B_1 &= \frac{1}{\mu}\left\{\rho_2\left(1-e^{-\mu}\right) - \sum_{i=1}^{m-2}\alpha_i\left(1-e^{-\mu\sigma_i}\right) - \sum_{j=1}^{p-2}\left[r_j(\eta_j - \xi_j) + \frac{1}{\mu}\left(e^{-\mu\eta_j} - e^{-\mu\xi_j}\right)\right]\right\}, \\ B_2 &= \rho_3 + \rho_4 e^{-\mu} - \sum_{i=1}^{m-2}\delta_i e^{-\mu\sigma_i} + \frac{1}{\mu}\sum_{j=1}^{p-2}\left(e^{-\mu\eta_j} - e^{-\mu\xi_j}\right), \end{aligned} \tag{2.6}$$

with the assumption that

$$\Delta = A_1B_2 - B_1A_2 \neq 0. \tag{2.7}$$

Proof Applying the integral operator I^{q-1} on both sides of (2.1) and then solving the resulting equation, we get

$$x(t) = \omega_0 e^{-\mu t} + \frac{\omega_1}{\mu}\left(1-e^{-\mu t}\right) + \int_0^t e^{\mu(s-t)}\left(\int_0^s \frac{(s-\tau)^{q-2}}{\Gamma(q-1)}h(\tau)\,d\tau\right)ds, \tag{2.8}$$

where ω_i $(i = 0, 1)$ are unknown arbitrary constants. From (2.8), we have

$$x'(t) = -\mu\omega_0 e^{-\mu t} + \omega_1 e^{-\mu t} - \mu \int_0^t e^{\mu(s-t)} \left(\int_0^s \frac{(s-\tau)^{q-2}}{\Gamma(q-1)} h(\tau)\, d\tau \right) ds + \int_0^t \frac{(t-\tau)^{q-2}}{\Gamma(q-1)} h(\tau)\, d\tau. \tag{2.9}$$

Using (2.8) and (2.9) in the boundary conditions (1.2), we obtain

$$A_1\omega_0 + B_1\omega_1 = J_1, \qquad A_2\omega_0 + B_2\omega_1 = J_2, \tag{2.10}$$

where A_i and B_i $(i = 1, 2)$ are respectively given by (2.5) and (2.6), and

$$\begin{aligned} J_1 &= \sum_{i=1}^{m-2} \alpha_i \int_0^{\sigma_i} e^{\mu(s-\sigma_i)} \big(I^{q-1}h(s)\big)\, ds + \sum_{j=1}^{p-2} r_j \int_{\xi_j}^{\eta_j} \left(\int_0^s e^{\mu(m-s)} \big(I^{q-1}h(m)\big)\, dm \right) ds \\ &\quad - \rho_2 \int_0^1 e^{\mu(s-1)} \big(I^{q-1}h(s)\big)\, ds, \\ J_2 &= \sum_{i=1}^{m-2} \delta_i \left[I^{q-1}h(\sigma_i) - \mu \int_0^{\sigma_i} e^{\mu(s-\sigma_i)} \big(I^{q-1}h(s)\big)\, ds \right] \\ &\quad + \sum_{j=1}^{p-2} \gamma_j \left[\int_{\xi_j}^{\eta_j} \big(I^{q-1}h(s)\big)\, ds - \mu \int_{\xi_j}^{\eta_j} \left(\int_0^s e^{\mu(m-s)} \big(I^{q-1}h(m)\big)\, dm \right) ds \right] \\ &\quad + \rho_4 \left[\mu \int_0^1 e^{\mu(s-1)} \big(I^{q-1}h(s)\big)\, ds - I^{q-1}h(1) \right]. \end{aligned}$$

Solving system (2.10) for the unknown constants ω_0, ω_1 yields

$$\begin{aligned} \omega_0 &= \frac{1}{\Delta} \Bigg\{ \sum_{i=1}^{m-2} (B_2\alpha_i + \mu B_1\delta_i) \int_0^{\sigma_i} e^{\mu(s-\sigma_i)} \big(I^{q-1}h(s)\big)\, ds \\ &\quad + \sum_{j=1}^{p-2} (B_2 r_j + \mu B_1\gamma_j) \int_{\xi_j}^{\eta_j} \left(\int_0^s e^{\mu(m-s)} \big(I^{q-1}h(m)\big)\, dm \right) ds \\ &\quad - (\rho_4\mu B_1 + \rho_2 B_2) \int_0^1 \big(e^{\mu(s-1)} I^{q-1}h(s)\big)\, ds \\ &\quad - B_1 \sum_{i=1}^{m-2} \delta_i I^{q-1}h(\sigma_i) - B_1 \sum_{j=1}^{p-2} \gamma_j \int_{\xi_j}^{\eta_j} \big(I^{q-1}h(s)\big)\, ds - \rho_4 B_1 I^{q-1}h(1) \Bigg\}, \\ \omega_1 &= \frac{1}{\Delta} \Bigg\{ -\sum_{i=1}^{m-2} (A_2\alpha_i + \mu A_1\delta_i) \int_0^{\sigma_i} e^{\mu(s-\sigma_i)} \big(I^{q-1}h(s)\big)\, ds \\ &\quad - \sum_{j=1}^{p-2} (A_2 r_j + \mu A_1\gamma_j) \int_{\xi_j}^{\eta_j} \left(\int_0^s e^{\mu(m-s)} \big(I^{q-1}h(m)\big)\, dm \right) ds \\ &\quad + A_1 \sum_{i=1}^{m-2} \delta_i I^{q-1}h(\sigma_i) + A_1 \sum_{j=1}^{p-2} \gamma_j \int_{\xi_j}^{\eta_j} \big(I^{q-1}h(s)\big)\, ds - \rho_4 A_1 I^{q-1}h(1) \\ &\quad + (\rho_4\mu A_1 + \rho_2 A_2) \int_0^1 \big(e^{\mu(s-1)} I^{q-1}h(s)\big)\, ds \Bigg\}, \end{aligned}$$

where Δ is given by (2.7). Substituting the values of ω_0 and ω_1 into (2.8), together with the notations (2.3) and (2.4), we get the solution (2.2). The converse follows by direct computation. This completes the proof. □

3 Main results for the problem (1.1) and (1.3)

Let $X = \{x | x \in C([0,1],\mathbb{R}) \text{ and } {}^cD^\kappa x \in C([0,1],\mathbb{R})\}$ be a space equipped with the norm $\|x\|_X = \sup_{t\in[0,1]} |x(t)| + \sup_{t\in[0,1]} |{}^cD^\kappa x(t)| = \|x\| + \|{}^cD^\kappa x\|$, where ${}^cD^\kappa$ denotes the standard Caputo fractional derivative of order $0 < \kappa \leq 1$. As argued in [30], $(X, \|\cdot\|_X)$ is a Banach space.

By means of Lemma 2.4, we transform problem (1.1) and (1.3) into a fixed point problem as $x = \mathfrak{H}x$, where the operator $\mathfrak{H} : X \to X$ is defined by

$$\begin{aligned}(\mathfrak{H}x)(t) ={}& \sum_{i=1}^{m-2} \phi_i(t) \int_0^{\sigma_i} e^{\mu(s-\sigma_i)} \bigl(I^{q-1} f\bigl(s, x(s), {}^cD^\kappa x(s)\bigr)\bigr)\,ds \\ &+ \sum_{j=1}^{p-2} \psi_j(t) \int_{\xi_j}^{\eta_j} \biggl(\int_0^s e^{\mu(m-s)} \bigl(I^{q-1} f\bigl(m, x(m), {}^cD^\kappa x(m)\bigr)\bigr)\,dm\biggr)\,ds \\ &+ \Omega_1(t) \Biggl[\sum_{i=1}^{m-2} \delta_i \bigl(I^{q-1} f\bigl(\sigma_i, x(\sigma_i), {}^cD^\kappa x(\sigma_i)\bigr)\bigr) \\ &+ \sum_{j=1}^{p-2} \gamma_j \int_{\xi_j}^{\eta_j} \bigl(I^{q-1} f\bigl(s, x(s), {}^cD^\kappa x(s)\bigr)\bigr)\,ds - \rho_4 I^{q-1} f\bigl(1, x(1), {}^cD^\kappa x(1)\bigr)\Biggr] \\ &+ \Omega_2(t) \int_0^1 e^{\mu(s-1)} \bigl(I^{q-1} f\bigl(s, x(s), {}^cD^\kappa x(s)\bigr)\bigr)\,ds \\ &+ \int_0^t e^{\mu(s-t)} \bigl(I^{q-1} f\bigl(s, x(s), {}^cD^\kappa x(s)\bigr)\bigr)\,ds, \quad t \in [0,1], \end{aligned} \tag{3.1}$$

where ϕ_i, ψ_j and Ω_i $(i = 1, 2)$ are respectively given by (2.3) and (2.4). Furthermore, we have

$$\begin{aligned}(\mathfrak{H}x)'(t) ={}& \sum_{i=1}^{m-2} \phi_i'(t) \int_0^{\sigma_i} e^{\mu(s-\sigma_i)} \bigl(I^{q-1} f\bigl(s, x(s), {}^cD^\kappa x(s)\bigr)\bigr)\,ds \\ &+ \sum_{j=1}^{p-2} \psi_j'(t) \int_{\xi_j}^{\eta_j} \biggl(\int_0^s e^{\mu(m-s)} \bigl(I^{q-1} f\bigl(m, x(m), {}^cD^\kappa x(m)\bigr)\bigr)\,dm\biggr)\,ds \\ &+ \Omega_1'(t) \Biggl[\sum_{i=1}^{m-2} \delta_i \bigl(I^{q-1} f\bigl(\sigma_i, x(\sigma_i), {}^cD^\kappa x(\sigma_i)\bigr)\bigr) \\ &+ \sum_{j=1}^{p-2} \gamma_j \int_{\xi_j}^{\eta_j} \bigl(I^{q-1} f\bigl(s, x(s), {}^cD^\kappa x(s)\bigr)\bigr)\,ds - \rho_4 I^{q-1} f\bigl(1, x(1), {}^cD^\kappa x(1)\bigr)\Biggr] \\ &+ \Omega_2'(t) \int_0^1 e^{\mu(s-1)} \bigl(I^{q-1} f\bigl(s, x(s), {}^cD^\kappa x(s)\bigr)\bigr)\,ds \\ &- \mu \int_0^t e^{\mu(s-t)} \bigl(I^{q-1} f\bigl(s, x(s), {}^cD^\kappa x(s)\bigr)\bigr)\,ds \\ &+ I^{q-1} f\bigl(t, x(t), {}^cD^\kappa x(t)\bigr), \quad t \in [0,1], \end{aligned} \tag{3.2}$$

where

$$\phi_i'(t) = \frac{-e^{-\mu t}}{\Delta}\big[\mu(B_2\alpha_i + \mu B_1\delta_i) + A_2\alpha_i + \mu A_1\delta_i\big],$$
$$\psi_i'(t) = \frac{-e^{-\mu t}}{\Delta}\big[\mu(B_2 r_j + \mu B_1\gamma_j) + A_2 r_j + \mu A_1\gamma_j\big],$$
$$\Omega_1'(t) = \frac{e^{-\mu t}}{\Delta}[\mu B_1 + A_1],$$
$$\Omega_2'(t) = \frac{e^{-\mu t}}{\Delta}\big[\mu(\rho_4\mu B_1 + \rho_2 B_1) + \rho_4\mu A_1 + \rho_2 A_2\big].$$

Next we fix some quantities as follows:

$$\begin{aligned}\lambda &= \frac{1}{\mu\Gamma(q)}\sum_{i=1}^{m-2}\widehat{\phi}_i\sigma_i^{q-1}\big(1-e^{-\mu\sigma_i}\big) + \frac{1}{\mu\Gamma(q+1)}\sum_{j=1}^{p-2}\widehat{\psi}_j\big(\eta_j^q - \xi_j^q\big)\big(1-e^{-\mu\eta_j}\big)\\ &\quad + \frac{\widehat{\Omega}_1}{\Gamma(q+1)}\left[q\left(\sum_{i=1}^{m-2}\delta_i\sigma_i^{q-1} + |\rho_4|\right) + \sum_{j=1}^{p-2}\gamma_j\big(\eta_j^q - \xi_j^q\big)\right]\\ &\quad + \frac{1}{\mu\Gamma(q)}(\widehat{\Omega}_2 + 1)\big(1-e^{-\mu}\big), \end{aligned} \tag{3.3}$$

$$\widehat{\lambda} = \lambda - \frac{1}{\mu\Gamma(q)}\big(1-e^{-\mu}\big), \tag{3.4}$$

$$\widehat{\phi}_i = \max_{t\in[0,1]}\big|\phi_i(t)\big| = \frac{1}{\mu|\Delta|}\big[\mu|B_2\alpha_i + \mu B_1\delta_i| + \big(1-e^{-\mu}\big)|A_2\alpha_i + \mu A_1\delta_i|\big],$$
$$\widehat{\psi}_j = \max_{t\in[0,1]}\big|\psi_j(t)\big| = \frac{1}{\mu|\Delta|}\big[\mu|B_2 r_j + \mu B_1\gamma_j| + \big|(A_2 r_j + \mu A_1\gamma_j)\big|\big(1-e^{-\mu}\big)\big],$$
$$\widehat{\Omega}_1 = \max_{t\in[0,1]}\big|\Omega_1(t)\big| = \frac{1}{\mu|\Delta|}\big[\mu|B_1| + |A_1|\big(1-e^{-\mu}\big)\big],$$
$$\widehat{\Omega}_2 = \max_{t\in[0,1]}\big|\Omega_2(t)\big| = \frac{1}{\mu|\Delta|}\big[\mu|\rho_4\mu B_1 + \rho_2 B_2| + \big(1-e^{-\mu}\big)|\rho_4\mu A_1 + \rho_2 A_2|\big];$$

$$\begin{aligned}\lambda_1 &= \frac{1}{\mu\Gamma(q)}\sum_{i=1}^{m-2}\widehat{\phi}_i'\sigma_i^{q-1}\big(1-e^{-\mu\sigma_i}\big) + \frac{1}{\mu\Gamma(q+1)}\sum_{j=1}^{p-2}\widehat{\psi}_j'\big(\eta_j^q - \xi_j^q\big)\big(1-e^{-\mu\eta_j}\big)\\ &\quad + \frac{\widehat{\Omega}_1'}{\Gamma(q+1)}\left[q\left(\sum_{i=1}^{m-2}\delta_i\sigma_i^{q-1} + |\rho_4|\right) + \sum_{j=1}^{p-2}\gamma_j\big(\eta_j^q - \xi_j^q\big)\right]\\ &\quad + \frac{1}{\mu\Gamma(q)}\widehat{\Omega}_2'\big(1-e^{-\mu}\big) + \frac{2-e^{-\mu}}{\Gamma(q)}, \end{aligned} \tag{3.5}$$

$$\widehat{\lambda}_1 = \lambda_1 - \frac{2-e^{-\mu}}{\Gamma(q)}, \tag{3.6}$$

$$\widehat{\phi}_i' = \max_{t\in[0,1]}\big|\phi_i'(t)\big| = \frac{1}{|\Delta|}\big[\mu|B_2\alpha_i + \mu B_1\delta_i| + |A_2\alpha_i + \mu A_1\delta_i|\big],$$
$$\widehat{\psi}_j' = \max_{t\in[0,1]}\big|\psi_i'(t)\big| = \frac{1}{|\Delta|}\big[\mu|B_2 r_j + \mu B_1\gamma_j| + |A_2 r_j + \mu A_1\gamma_j|\big],$$
$$\widehat{\Omega}_1' = \max_{t\in[0,1]}\big|\Omega_1'(t)\big| = \frac{1}{|\Delta|}\big[\mu|B_1| + |A_1|\big],$$
$$\widehat{\Omega}_2' = \max_{t\in[0,1]}\big|\Omega_2'(t)\big| = \frac{1}{|\Delta|}\big[\mu|\rho_4\mu B_1 + \rho_2 B_2| + |\rho_4\mu A_1 + \rho_2 A_2|\big].$$

In the following, for brevity, we use the notation

$$\widehat{f}\big(x(\theta)\big) = f\big(\theta, x(\theta), {}^{c}D^{\kappa}x(\theta)\big). \tag{3.7}$$

3.1 Existence result via Krasnoselskii's fixed point theorem

Here we present an existence result for the problem (1.1) and (1.3), which relies on Krasnoselskii's fixed point theorem [31].

Theorem 3.1 *Let $f : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the conditions*:

- (A_1) *There exists a constant $L > 0$ such that $|f(t,x_1,y_1) - f(t,x_2,y_2)| \le L(|x_1 - x_2| + |y_1 - y_2|)$ for all $t \in [0,1]$, $x_i, y_i \in \mathbb{R}$, $i = 1,2$;*
- (B_1) *There exists a function $m \in C([0,1],\mathbb{R}^+)$ such that $|f(t,x,y)| \le m(t)$ for all $(t,x,y) \in [0,1] \times \mathbb{R} \times \mathbb{R}$.*

Then the problem (1.1) and (1.3) has at least one solution on $[0,1]$ provided that

$$L\left(\widehat{\lambda} + \frac{1}{\Gamma(2-\kappa)}\widehat{\lambda}_1\right) < 1, \tag{3.8}$$

where $\widehat{\lambda}$, $\widehat{\lambda}_1$ are given by (3.4) and (3.6), respectively.

Proof For $\rho > \|m\|(\lambda + \frac{\lambda_1}{\Gamma(2-\kappa)})$, we consider the closed ball $B_\rho = \{x \in X : \|x\|_X \le \rho\}$ and introduce the operators A and B on B_ρ as follows:

$$\begin{aligned}
(Ax)(t) ={}& \sum_{i=1}^{m-2} \phi_i(t) \int_0^{\sigma_i} e^{\mu(s-\sigma_i)} \big(I^{q-1}\widehat{f}\big(x(s)\big)\big)\, ds \\
&+ \sum_{j=1}^{p-2} \psi_j(t) \int_{\xi_j}^{\eta_j} \left(\int_0^s e^{\mu(m-s)} \big(I^{q-1}\widehat{f}\big(x(m)\big)\big)\, dm \right) ds \\
&+ \Omega_1(t) \Bigg[\sum_{i=1}^{m-2} \delta_i \big(I^{q-1}\widehat{f}\big(x(\sigma_i)\big)\big) \\
&+ \sum_{j=1}^{p-2} \gamma_j \int_{\xi_j}^{\eta_j} \big(I^{q-1}\widehat{f}\big(x(s)\big)\big)\, ds - \rho_4 I^{q-1}\widehat{f}\big(x(1)\big) \Bigg] \\
&+ \Omega_2(t) \int_0^1 e^{\mu(s-1)} \big(I^{q-1}\widehat{f}\big(x(s)\big)\big)\, ds, \quad t \in [0,1],
\end{aligned} \tag{3.9}$$

$$(Bx)(t) = \int_0^t e^{\mu(s-t)} \big(I^{q-1}\widehat{f}\big(x(s)\big)\big)\, ds, \quad t \in [0,1]. \tag{3.10}$$

For any $x, y \in B_\rho$, it is straightforward to show that

$$\big\|(\mathfrak{H}x)\big\|_X = \|Ax + Bx\|_X \le \|m\| \left(\lambda + \frac{\lambda_1}{\Gamma(2-\kappa)}\right) < \rho,$$

which implies that $Ax + By \in B_\rho$. Also, the operator B is completely continuous. Indeed, B is uniformly bounded on B_r as

$$\|Bx\|_X \le \frac{1 - e^{-\mu}}{\mu \Gamma(q)} + \frac{2 - e^{-\mu}}{\Gamma(2-\kappa)\Gamma(q)}.$$

Observe that

$$\begin{aligned}
&\left|Bx(t_2) - Bx(t_1)\right| \\
&\quad \le \left|\int_0^{t_1} \left[e^{\mu(s-t_2)} - e^{\mu(s-t_1)}\right]\left(I^{q-1}\left|\widehat{f}\left(x(s)\right)\right|\right) ds + \int_{t_1}^{t_2} e^{\mu(s-t_2)}\left(I^{q-1}\left|\widehat{f}\left(x(s)\right)\right|\right) ds\right| \\
&\quad \le \|m\|\left|\int_0^{t_1} \left[e^{\mu(s-t_2)} - e^{\mu(s-t_1)}\right]\left(I^{q-1}1\right) ds + \int_{t_1}^{t_2} e^{\mu(s-t_2)}\left(I^{q-1}1\right) ds\right| \to 0, \quad \text{as } t_1 \to t_2.
\end{aligned}$$

Also

$$\begin{aligned}
&\left|Bx'(t_2) - Bx'(t_1)\right| \\
&\quad \le \|m\|\mu\left|\int_0^{t_1} \left[e^{\mu(s-t_2)} - e^{\mu(s-t_1)}\right]\left(I^{q-1}1\right) ds + \int_{t_1}^{t_2} e^{\mu(s-t_2)}\left(I^{q-1}1\right) ds\right| \\
&\qquad + \frac{\|m\|}{\Gamma(q-1)}\left|\int_0^{t_1} \left[(t_2-s)^{q-2} - (t_1-s)^{q-2}\right] ds + \int_{t_1}^{t_2} (t_2-s)^{q-2} ds\right| \to 0,
\end{aligned}$$

and

$$\begin{aligned}
\left|{}^cD^\kappa Bx(t_2) - {}^cD^\kappa Bx(t_1)\right| &\le \int_0^t \frac{(t-s)^{-\kappa}}{\Gamma(1-\kappa)}\left|Bx'(t_2) - Bx'(t_1)\right| ds \\
&\le \frac{1}{\Gamma(2-\kappa)}\left|Bx'(t_2) - Bx'(t_1)\right| \to 0,
\end{aligned}$$

as $t_1 \to t_2$ independent of x. So B is equicontinuous. Using assumption (A_1) and condition (3.8), it is easy to establish that operator A is a contraction. Thus the hypotheses of Krasnoselskii's fixed point theorem [31] hold true, and consequently we deduce by its conclusion that the problem (1.1) and (1.3) has at least one solution on $[0,1]$. This completes the proof. □

3.2 Uniqueness of solutions

Here we establish the uniqueness of solutions for the problem (1.1) and (1.3) by applying a fixed point theorem due to Banach.

Theorem 3.2 *Let assumption* (A_1) *be satisfied and*

$$L\left(\lambda + \frac{\lambda_1}{\Gamma(2-q)}\right) < 1, \tag{3.11}$$

where λ *and* λ_1 *are given by* (3.3) *and* (3.5), *respectively. Then there exists a unique solution for the problem* (1.1) *and* (1.3) *on* $[0,1]$.

Proof Let us define $\sup_{t\in[0,1]} |f(t,0,0)| = M$ and select

$$\bar{r} \ge \frac{M(\lambda + \frac{\lambda_1}{\Gamma(2-q)})}{1 - L(\lambda + \frac{\lambda_1}{\Gamma(2-q)})}$$

to show that $\mathfrak{H}B_{\bar{r}} \subset B_{\bar{r}}$, where $B_{\bar{r}} = \{x \in X : \|x\|_X \le \bar{r}\}$ and $\mathfrak{H}$ is defined by (3.1).

Using condition (A_1), we have

$$\begin{aligned}\big|\widehat{f}\big(x(\theta)\big)\big| &= \big|f\big(\theta, x(\theta), {}^cD^\kappa x(\theta)\big)\big| = \big|f\big(\theta, x(\theta), {}^cD^\kappa x(\theta)\big) - f(\theta,0,0)\big| + \big|f(\theta,0,0)\big| \\ &\le L\big(\|x\| + \big\|{}^cD^\kappa x\big\|\big) + M \le L\|x\|_X + M \le L\bar{r} + M.\end{aligned}$$

Then, for $x \in B_{\bar{r}}$, we obtain

$$\begin{aligned}\big\|(\mathfrak{H}x)\big\| &= \sup_{t\in[0,1]}\big|(\mathfrak{H}x)(t)\big| \\ &\le \sup_{t\in[0,1]}\Bigg\{\sum_{i=1}^{m-2}\big|\phi_i(t)\big|\int_0^{\sigma_i} e^{\mu(s-\sigma_i)}\big(I^{q-1}\big|\widehat{f}\big(x(s)\big)\big|\big)\,ds \\ &\quad + \sum_{j=1}^{p-2}\big|\psi_j(t)\big|\int_{\xi_j}^{\eta_j}\bigg(\int_0^s e^{\mu(m-s)}\big(I^{q-1}\big|\widehat{f}\big(x(m)\big)\big|\big)\,dm\bigg)\,ds \\ &\quad + \big|\Omega_1(t)\big|\Bigg[\sum_{i=1}^{m-2}|\delta_i|\big(I^{q-1}\big|\widehat{f}\big(x(\sigma_i)\big|\big) \\ &\quad + \sum_{j=1}^{p-2}|\gamma_j|\int_{\xi_j}^{\eta_j}\big(I^{q-1}\big|\widehat{f}\big(x(s)\big)\big|\big)\,ds + |\rho_4|I^{q-1}\big|\widehat{f}\big(x(1)\big)\big|\Bigg] \\ &\quad + \big|\Omega_2(t)\big|\int_0^1 e^{\mu(s-1)}\big(I^{q-1}\big|\widehat{f}\big(x(s)\big)\big|\big)\,ds + \int_0^t e^{\mu(s-t)}\big(I^{q-1}\big|\widehat{f}\big(x(s)\big)\big|\big)\,ds\Bigg\} \\ &\le (L\bar{r}+M)\Bigg\{\sum_{i=1}^{m-2}\widehat{\phi}_i\int_0^{\sigma_i} e^{\mu(s-\sigma_i)}\big(I^{q-1}1\big)\,ds \\ &\quad + \sum_{j=1}^{p-2}\widehat{\psi}_j\int_{\xi_j}^{\eta_j}\bigg(\int_0^s e^{\mu(m-s)}\big(I^{q-1}1\big)\,dm\bigg)\,ds \\ &\quad + \widehat{\Omega}_1\Bigg[\sum_{i=1}^{m-2}\delta_i\big(I^{q-1}1\big) + \sum_{j=1}^{p-2}\gamma_j\int_{\xi_j}^{\eta_j}\big(I^{q-1}1\big)\,ds - \rho_4 I^{q-1}1\Bigg] \\ &\quad + \widehat{\Omega}_2\int_0^1 e^{\mu(s-1)}\big(I^{q-1}1\big)\,ds + \int_0^t e^{\mu(s-t)}\big(I^{q-1}1\big)\,ds\Bigg\} \\ &\le (L\bar{r}+M)\lambda.\end{aligned}$$

Also we have

$$\begin{aligned}\big|(\mathfrak{H}x)'(t)\big| &\le \sum_{i=1}^{m-2}\big|\phi_i'(t)\big|\int_0^{\sigma_i} e^{\mu(s-\sigma_i)}\big(I^{q-1}\big|\widehat{f}\big(x(s)\big)\big|\big)\,ds \\ &\quad + \sum_{j=1}^{p-2}\big|\psi_j'(t)\big|\int_{\xi_j}^{\eta_j}\bigg(\int_0^s e^{\mu(m-s)}\big(I^{q-1}\big|\widehat{f}\big(x(m)\big)\big|\big)\,dm\bigg)\,ds \\ &\quad + \big|\Omega_1'(t)\big|\Bigg[\sum_{i=1}^{m-2}|\delta_i|\big(I^{q-1}\big|\widehat{f}\big(x(\sigma_i)\big)\big|\big)\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{j=1}^{p-2} |\gamma_j| \int_{\xi_j}^{\eta_j} \left(I^{q-1}\left|\widehat{f}\big(x(s)\big)\right|\right) ds + |\rho_4| I^{q-1}\left|\widehat{f}\big(x(1)\big)\right| \Bigg] \\
&+ \left|\Omega_2'(t)\right| \int_0^1 e^{\mu(s-1)}\left(I^{q-1}\left|\widehat{f}\big(x(s)\big)\right|\right) ds \\
&+ \mu \int_0^t e^{\mu(s-t)}\left(I^{q-1}\left|\widehat{f}\big(x(s)\big)\right|\right) ds + I^{q-1}\left|\widehat{f}\big(x(s)\big)\right| \\
&\le (L\bar{r} + M)\lambda_1.
\end{aligned}
$$

Using the above inequality in the definition of Caputo fractional derivative with $0 < \kappa \le 1$, we get

$$
\begin{aligned}
\left|{}^cD^\kappa(\mathfrak{H}x)(t)\right| &\le \int_0^t \frac{(t-s)^{-\kappa}}{\Gamma(1-\kappa)}\left|(\mathfrak{H}x)'(s)\right| \le (L\bar{r}+M)\lambda_1 \int_0^t \frac{(t-s)^{-\kappa}}{\Gamma(1-\kappa)}\, ds \\
&\le \frac{1}{\Gamma(2-\kappa)}(L\bar{r}+M)\lambda_1.
\end{aligned}
$$

Hence

$$
\left\|(\mathfrak{H}x)\right\|_X = \left\|(\mathfrak{H}x)\right\| + \left\|{}^cD^\kappa(\mathfrak{H}x)\right\| \le (L\bar{r}+M)\lambda + \frac{1}{\Gamma(2-\kappa)}(L\bar{r}+M)\lambda_1 < \bar{r},
$$

which clearly shows that $\mathfrak{H}x \in B_{\bar{r}}$ for any $x \in B_{\bar{r}}$. Thus $\mathfrak{H}B_{\bar{r}} \subset B_{\bar{r}}$.

Now, for $x, y \in X$ and for each $t \in [0,1]$, we have

$$
\begin{aligned}
&\left|(\mathfrak{H}x)(t) - (\mathfrak{H}y)(t)\right| \\
&\le \sum_{i=1}^{m-2}\left|\phi_i(t)\right| \int_0^{\sigma_i} e^{\mu(s-\sigma_i)}\left(I^{q-1}\left|\widehat{f}\big(x(s)\big) - \widehat{f}\big(y(s)\big)\right|\right) ds \\
&+ \sum_{j=1}^{p-2}\left|\psi_j(t)\right| \int_{\xi_j}^{\eta_j}\left(\int_0^s e^{\mu(m-s)}\left(I^{q-1}\left|\widehat{f}\big(x(m)\big) - \widehat{f}\big(y(m)\big)\right|\right) dm\right) ds \\
&+ \left|\Omega_1(t)\right| \Bigg[\sum_{i=1}^{m-2}|\delta_i|\left(I^{q-1}\left|\widehat{f}\big(x(\sigma_i)\big) - \widehat{f}\big(y(\sigma_i)\big)\right|\right) \\
&+ \sum_{j=1}^{p-2}|\gamma_j| \int_{\xi_j}^{\eta_j}\left(I^{q-1}\left|\widehat{f}\big(x(s)\big) - \widehat{f}\big(y(s)\big)\right|\right) ds \\
&+ |\rho_4| I^{q-1}\left|\widehat{f}\big(x(1)\big) - \widehat{f}\big(y(1)\big)\right|\Bigg] + \left|\Omega_2(t)\right| \int_0^1 e^{\mu(s-1)}\left(I^{q-1}\left|\widehat{f}\big(x(s)\big) - \widehat{f}\big(y(s)\big)\right|\right) ds \\
&+ \int_0^t e^{\mu(s-t)}\left(I^{q-1}\left|\widehat{f}\big(x(s)\big) - \widehat{f}\big(y(s)\big)\right|\right) ds \\
&\le L\lambda\|x-y\|_X.
\end{aligned}
$$

Also

$$
\begin{aligned}
&\left|(\mathfrak{H}x)'(t) - (\mathfrak{H}y)'(t)\right| \\
&\le \sum_{i=1}^{m-2}\left|\phi_i'(t)\right| \int_0^{\sigma_i} e^{\mu(s-\sigma_i)}\left(I^{q-1}\left|\widehat{f}\big(x(s)\big) - \widehat{f}\big(y(s)\big)\right|\right) ds
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{j=1}^{p-2}\left|\psi_j'(t)\right|\int_{\xi_j}^{\eta_j}\left(\int_0^s e^{\mu(m-s)}\left(I^{q-1}\left|\widehat{f}\big(x(m)\big)-\widehat{f}\big(y(m)\big)\right|\right)dm\right)ds\\
&+\left|\Omega_1'(t)\right|\left[\sum_{i=1}^{m-2}|\delta_i|\left(I^{q-1}\left|\widehat{f}\big(x(\sigma_i)\big)-\widehat{f}\big(y(\sigma_i)\big)\right|\right)\right.\\
&\left.+\sum_{j=1}^{p-2}|\gamma_j|\int_{\xi_j}^{\eta_j}\left(I^{q-1}\left|\widehat{f}\big(x(s)\big)-\widehat{f}\big(y(s)\big)\right|\right)ds+|\rho_4|I^{q-1}\left|\widehat{f}\big(x(1)\big)-\widehat{f}\big(y(1)\big)\right|\right]\\
&+\left|\Omega_2'(t)\right|\int_0^1 e^{\mu(s-1)}\left(I^{q-1}\left|\widehat{f}\big(x(s)\big)-\widehat{f}\big(y(s)\big)\right|\right)ds\\
&+\mu\int_0^t e^{\mu(s-t)}\left(I^{q-1}\left|\widehat{f}\big(x(s)\big)-\widehat{f}\big(y(s)\big)\right|\right)ds+I^{q-1}\left|\widehat{f}\big(x(s)\big)-\widehat{f}\big(y(s)\big)\right|\\
\le{}& L\lambda_1\|x-y\|_X,
\end{aligned}
$$

and moreover,

$$
\left|{}^cD^\kappa(\mathfrak{H}x)(t)-{}^cD^\kappa(\mathfrak{H}y)(t)\right|\le\frac{1}{\Gamma(2-\kappa)}L\lambda_1\|x-y\|_X.
$$

Consequently, we have

$$
\left\|(\mathfrak{H}x)-(\mathfrak{H}y)\right\|_X\le L\left(\lambda+\frac{\lambda_1}{\Gamma(2-q)}\right)\|x-y\|_X,
$$

which shows that $\mathfrak{H}$ is a contraction by condition (3.11). Thus the operator $\mathfrak{H}$ has a unique fixed point by Banach fixed point theorem, which corresponds to a unique solution of the problem (1.1) and (1.3) on $[0,1]$. This completes the proof. □

3.3 An example

Consider the following nonlinear sequential fractional differential equation:

$$
\begin{aligned}
\left(D^{7/5}+D^{2/5}\right)x(t)={}&\frac{1}{t^2+25}\\
&+\frac{e^{-t}}{t^4+16}\left(\tan^{-1}\big(x(t)\big)+\frac{|{}^cD^{1/2}x(t)|}{(1+|{}^cD^{1/2}x(t)|)}\right),\quad t\in[0,1],
\end{aligned}
\tag{3.12}
$$

supplemented with the integro-multipoint boundary conditions:

$$
\begin{aligned}
&\rho_1x(0)+\rho_2x(1)=\sum_{i=1}^{3}\alpha_ix(\sigma_i)+\sum_{j=1}^{5}r_j\int_{\xi_j}^{\eta_i}x(s)\,ds,\\
&\rho_3x'(0)+\rho_4x'(1)=\sum_{i=1}^{3}\delta_ix'(\sigma_i)+\sum_{j=1}^{5}\gamma_j\int_{\xi_j}^{\eta_j}x'(s)\,ds.
\end{aligned}
\tag{3.13}
$$

Here $q=7/5$, $\kappa=1/2$, $\mu=1$, $m=5$, $p=7$, $\sigma_1=1/18$, $\sigma_2=1/9$, $\sigma_3=1/6$, $\xi_1=1/5$, $\eta_1=1/4$, $\xi_2=3/10$, $\eta_2=7/20$, $\xi_3=2/5$, $\eta_3=9/20$, $\xi_4=1/2$, $\eta_4=11/20$, $\xi_5=3/5$, $\eta_5=13/20$, $\alpha_1=1$, $\alpha_2=1/2$, $\alpha_3=1$, $\delta_1=3/4$, $\delta_2=1$, $\delta_3=3/2$, $r_1=1/3$, $r_2=2/3$, $r_3=1$, $r_4=-1$, $r_5=2$, $\gamma_1=-1/2$, $\gamma_2=1/2$, $\gamma_3=2$, $\gamma_4=1$, $\gamma_5=3/2$, $\rho_1=1/2$, $\rho_2=3/4$, $\rho_3=-1$, $\rho_4=2$. Using the given data, we find that $\Delta = 4.3972$, $\lambda = 2.4121$, $\widehat{\lambda} = 1.6997$, $\lambda_1 = 4.1293$, $\widehat{\lambda}_1 = 2.2898$ (Δ, λ, $\widehat{\lambda}$, λ_1 and $\widehat{\lambda}_1$ are respectively given by (2.7), (3.3), (3.4), (3.5), and (3.6)). Further we have

$$m(t) = \frac{1}{t^2+25} + \frac{(\pi+2)e^{-t}}{2(t^4+16)}, \qquad L = \frac{1}{16}, \qquad L\left(\widehat{\lambda} + \frac{1}{\Gamma(2-\kappa)}\widehat{\lambda}_1\right) \approx 0.2677 < 1.$$

Clearly, the hypothesis of Theorem 3.1 holds true. In consequence, there *exists* a solution to the problem (3.12)–(3.13) on $[0,1]$ by Theorem 3.1. Also

$$L\left(\lambda + \frac{\lambda_1}{\Gamma(2-\kappa)}\right) \approx 0.4420 < 1,$$

which implies that the problem (3.12)–(3.13) has a unique solution on $[0,1]$ by Theorem 3.2.

4 The case of inclusions

In this section, we investigate the existence of solutions for the multivalued (inclusion) boundary value problem (1.2) and (1.3). Our first result deals with the case when the multivalued map F has convex values, while the second result is concerned with the nonconvex multivalued maps.

For each $x \in C([0,1],\mathbb{R})$, the set

$$S_{F,x} = \left\{f \in L^1\big([0,1],\mathbb{R}\big) : f(t) \in F\big(t, x(t), {}^cD^\kappa x(t)\big) \text{ for a.e. } t \in [0,1]\right\}$$

is known as the set of selections of the multivalued map F.

4.1 The case of upper semicontinuous (convex-valued) maps

Here we present an existence result for the problem (1.2) and (1.3) when the multivalued map F is convex-valued. We make use of nonlinear alternative for Kakutani maps [32] to derive the desired result.

Theorem 4.1 *Assume that*:

- (C_1) $F : [0,1] \times \mathbb{R}^2 \to \mathcal{P}(\mathbb{R})$ *is* L^1*-Carathéodory and has nonempty, compact, and convex values*;
- (C_2) *There exist a function* $g \in C([0,1],\mathbb{R}^+)$ *with* $\|g\| = \sup_{t\in[0,1]}|g(t)|$ *and nondecreasing function* $Q : \mathbb{R}^+ \to \mathbb{R}^+$ *such that*

$$\begin{aligned}\big\|F(t,x,y)\big\|_{\mathcal{P}} &:= \sup\big\{|w| : w \in F(t,x,y)\big\}\\ &\le g(t)Q\big(\|x\| + \|y\|\big), \quad \forall (t,x,y) \in [0,1]\times\mathbb{R}\times\mathbb{R};\end{aligned}$$

- (C_3) *There exists a constant* $K > 0$ *such that*

$$\frac{K}{\|g\|Q(K)(\lambda + \frac{\lambda_1}{\Gamma(2-\kappa)})} > 1,$$

where λ, λ_1 *are are given by* (3.3) *and* (3.5), *respectively.*

Then the problem (1.2)–(1.3) *has at least one solution on* $[0,1]$.

Proof We transform the problem (1.2)–(1.3) into a fixed point problem. Consider the multivalued map: $N : X \to \mathcal{P}(X)$ defined by

$$N(x) = \left\{ \begin{array}{c} h \in X : \\ h(t) = \left\{ \begin{array}{l} \sum_{i=1}^{m-2} \phi_i(t) \int_0^{\sigma_i} e^{\mu(s-\sigma_i)} (I^{q-1} v(s))\, ds \\ + \sum_{j=1}^{p-2} \psi_j(t) \int_{\xi_j}^{\eta_j} (\int_0^s e^{\mu(m-s)} (I^{q-1} v(m))\, dm)\, ds \\ + \Omega_1(t)[\sum_{i=1}^{m-2} \delta_i (I^{q-1} v(\sigma_i)) \\ + \sum_{j=1}^{p-2} \gamma_j \int_{\xi_j}^{\eta_j} (I^{q-1} v(s))\, ds - \rho_4 I^{q-1} v(1)] \\ + \Omega_2(t) \int_0^1 e^{\mu(s-1)} (I^{q-1} v(s))\, ds \\ + \int_0^t e^{\mu(s-t)} (I^{q-1} v(s))\, ds, \quad t \in [0,1], v \in S_{F,x}. \end{array} \right. \end{array} \right\}$$

It is clear that the fixed points of N are solutions of problem (1.2)–(1.3). Now we proceed to show that operator N satisfies all the conditions of the nonlinear alternative for Kakutani maps [32]. The proof is given in several steps.

Step 1. $N(x)$ is convex for each $x \in X$.

Indeed, if h_1, h_2 belong to $N(x)$, then there exist $v_1, v_2 \in S_{F,x}$ such that for each $t \in [0,1]$, we have

$$\begin{aligned} h_k(t) &= \sum_{i=1}^{m-2} \phi_i(t) \int_0^{\sigma_i} e^{\mu(s-\sigma_i)} \bigl(I^{q-1} v_k(s)\bigr)\, ds \\ &\quad + \sum_{j=1}^{p-2} \psi_j(t) \int_{\xi_j}^{\eta_j} \biggl(\int_0^s e^{\mu(m-s)} \bigl(I^{q-1} v_k(m)\bigr)\, dm \biggr)\, ds + \Omega_1(t) \Biggl[\sum_{i=1}^{m-2} \delta_i \bigl(I^{q-1} v_k(\sigma_i)\bigr) \\ &\quad + \sum_{j=1}^{p-2} \gamma_j \int_{\xi_j}^{\eta_j} \bigl(I^{q-1} v_k(s)\bigr)\, ds - \rho_4 I^{q-1} v_k(1) \Biggr] \\ &\quad + \Omega_2(t) \int_0^1 e^{\mu(s-1)} \bigl(I^{q-1} v_k(s)\bigr)\, ds + \int_0^t e^{\mu(s-t)} \bigl(I^{q-1} v_k(s)\bigr)\, ds, \quad k = 1,2. \end{aligned}$$

Let $0 \leq \mu \leq 1$. Then for each $t \in [0,1]$, we have

$$\begin{aligned} &\bigl[\mu h_1 + (1-\mu) h_2\bigr](t) \\ &\quad = \sum_{i=1}^{m-2} \phi_i(t) \int_0^{\sigma_i} e^{\mu(s-\sigma_i)} \bigl(I^{q-1}\bigl[\mu v_1(s) + (1-\mu) v_2(s)\bigr]\bigr)\, ds \\ &\qquad + \sum_{j=1}^{p-2} \psi_j(t) \int_{\xi_j}^{\eta_j} \biggl(\int_0^s e^{\mu(m-s)} \bigl(I^{q-1}\bigl[\mu v_1(m) + (1-\mu) v_2(m)\bigr]\bigr)\, dm \biggr)\, ds \\ &\qquad + \Omega_1(t) \Biggl[\sum_{i=1}^{m-2} \delta_i \bigl(I^{q-1}\bigl[\mu v_1(\sigma_i) + (1-\mu) v_2(\sigma_i)\bigr]\bigr) \\ &\qquad + \sum_{j=1}^{p-2} \gamma_j \int_{\xi_j}^{\eta_j} \bigl(I^{q-1}\bigl[\mu v_1(s) + (1-\mu) v_2(s)\bigr]\bigr)\, ds \end{aligned}$$

$$
\begin{aligned}
&- \rho_4 I^{q-1}\big[\mu \nu_1(1) + (1-\mu)\nu_2(1)\big]\Bigg] \\
&+ \Omega_2(t)\int_0^1 e^{\mu(s-1)}\big(I^{q-1}\big[\mu\nu_1(s) + (1-\mu)\nu_2(s)\big]\big)\,ds \\
&+ \int_0^t e^{\mu(s-t)}\big(I^{q-1}\big[\mu\nu_1(s) + (1-\mu)\nu_2(s)\big]\big)\,ds.
\end{aligned}
$$

Since F has convex values ($S_{F,x}$ is convex),

$$\mu h_1 + (1-\mu)h_2 \in N(x).$$

Step 2. $N(x)$ maps bounded sets (balls) into bounded sets in X.
For a positive number r, let $\mathcal{B}_r = \{x \in X : \|x\|_X \leq r\}$ be a bounded set in X. Then, for $h \in N(x)$, $x \in B_r$, there exists $\nu \in S_{F,x}$ such that

$$
\begin{aligned}
h(t) = {} & \sum_{i=1}^{m-2}\phi_i(t)\int_0^{\sigma_i} e^{\mu(s-\sigma_i)}\big(I^{q-1}\nu(s)\big)\,ds \\
&+ \sum_{j=1}^{p-2}\psi_j(t)\int_{\xi_j}^{\eta_j}\bigg(\int_0^s e^{\mu(m-s)}\big(I^{q-1}\nu(m)\big)\,dm\bigg)\,ds + \Omega_1(t)\Bigg[\sum_{i=1}^{m-2}\delta_i\big(I^{q-1}\nu(\sigma_i)\big) \\
&+ \sum_{j=1}^{p-2}\gamma_j\int_{\xi_j}^{\eta_j}\big(I^{q-1}\nu(s)\big)\,ds - \rho_4 I^{q-1}\nu(1)\Bigg] \\
&+ \Omega_2(t)\int_0^1 e^{\mu(s-1)}\big(I^{q-1}\nu(s)\big)\,ds + \int_0^t e^{\mu(s-t)}\big(I^{q-1}\nu(s)\big)\,ds,
\end{aligned}
$$

for some $\nu \in S_{F,x_r}$. Then, for $t \in [0,1]$, we have

$$
\begin{aligned}
\big|h(t)\big| \leq {} & \sum_{i=1}^{m-2}\widehat{\phi}_i\int_0^{\sigma_i} e^{\mu(s-\sigma_i)}\big(I^{q-1}\big|\nu(s)\big|\big)\,ds \\
&+ \sum_{j=1}^{p-2}\widehat{\psi}_j\int_{\xi_j}^{\eta_j}\bigg(\int_0^s e^{\mu(m-s)}\big(I^{q-1}\big|\nu(m)\big|\big)\,dm\bigg)\,ds \\
&+ \widehat{\Omega}_1\Bigg[\sum_{i=1}^{m-2}\delta_i\big(I^{q-1}\big|\nu(s)\big|\big) + \sum_{j=1}^{p-2}\gamma_j\int_{\xi_j}^{\eta_j}\big(I^{q-1}\big|\nu(1)\big|\big)\,ds - \rho_4 I^{q-1}\big|\nu(1)\big|\Bigg] \\
&+ \widehat{\Omega}_2\int_0^1 e^{\mu(s-1)}\big(I^{q-1}\big|\nu(s)\big|\big)\,ds + \int_0^t e^{\mu(s-t)}\big(I^{q-1}\big|\nu(s)\big|\big)\,ds \\
\leq {} & \|g\|Q\big(\|x\| + \big\|{}^cD^\kappa x\big\|\big)\Bigg\{\sum_{i=1}^{m-2}\widehat{\phi}_i\int_0^{\sigma_i} e^{\mu(s-\sigma_i)}\big(I^{q-1}1\big)\,ds \\
&+ \sum_{j=1}^{p-2}\widehat{\psi}_j\int_{\xi_j}^{\eta_j}\bigg(\int_0^s e^{\mu(m-s)}\big(I^{q-1}1\big)\,dm\bigg)\,ds \\
&+ \widehat{\Omega}_1\Bigg[\sum_{i=1}^{m-2}\delta_i\big(I^{q-1}1\big) + \sum_{j=1}^{p-2}\gamma_j\int_{\xi_j}^{\eta_j}\big(I^{q-1}1\big)\,ds - \rho_4 I^{q-1}1\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&+ \widehat{\Omega}_2 \int_0^1 e^{\mu(s-1)}\bigl(I^{q-1}1\bigr)\,ds + \int_0^t e^{\mu(s-t)}\bigl(I^{q-1}1\bigr)\,ds\Bigr\} \\
&\le \|g\|Q\bigl(\|x\|_X\bigr)\lambda,
\end{aligned}
$$

which, when taking the norm for $t \in [0,1]$, yields $\|h\| \le \|g\|Q(r)\lambda$. Also we have

$$
\begin{aligned}
\bigl|h'(t)\bigr| \le{}& \sum_{i=1}^{m-2}\bigl|\phi_i'(t)\bigr|\int_0^{\sigma_i} e^{\mu(s-\sigma_i)}\bigl(I^{q-1}\bigl|\nu(s)\bigr|\bigr)\,ds \\
&+ \sum_{j=1}^{p-2}\bigl|\psi_j'(t)\bigr|\int_{\xi_j}^{\eta_j}\left(\int_0^s e^{\mu(m-s)}\bigl(I^{q-1}\bigl|\nu(m)\bigr|\bigr)\,dm\right)ds \\
&+ \bigl|\Omega_1'(t)\bigr|\left[\sum_{i=1}^{m-2}|\delta_i|\bigl(I^{q-1}\bigl|\nu(\sigma_i)\bigr|\bigr)\right. \\
&+ \left.\sum_{j=1}^{p-2}|\gamma_j|\int_{\xi_j}^{\eta_j}\bigl(I^{q-1}\bigl|\nu(s)\bigr|\bigr)\,ds + |\rho_4|I^{q-1}\bigl|\nu(1)\bigr|\right] \\
&+ \bigl|\Omega_2'(t)\bigr|\int_0^1 e^{\mu(s-1)}\bigl(I^{q-1}\bigl|\nu(s)\bigr|\bigr)\,ds \\
&+ \mu\int_0^t e^{\mu(s-t)}\bigl(I^{q-1}\bigl|\nu(s)\bigr|\bigr)\,ds + I^{q-1}\bigl|\nu(s)\bigr| \\
\le{}& \|g\|Q\bigl(\|x\|_X\bigr)\lambda_1 \le \|g\|Q(r)\lambda_1.
\end{aligned}
$$

By definition of Caputo fractional derivative with $0 < \kappa \le 1$, we get

$$
\bigl|{}^cD^\kappa h(t)\bigr| \le \int_0^t \frac{(t-s)^{-\kappa}}{\Gamma(1-\kappa)}\bigl|h'(t)\bigr| \le \|g\|Q(r)\lambda_1\int_0^t \frac{(t-s)^{-\kappa}}{\Gamma(1-\kappa)}\,ds \le \frac{1}{\Gamma(2-\kappa)}\|g\|Q(r)\lambda_1.
$$

Hence

$$
\|h\|_X = \|h\| + \bigl\|{}^cD^\kappa h\bigr\| \le \|g\|Q(r)\left(\lambda + \frac{\lambda_1}{\Gamma(2-\kappa)}\right). \tag{4.1}
$$

Step 3. N maps bounded sets into equicontinuous sets of X.
Next we show that N maps bounded sets into equicontinuous sets of X. Let $t_1, t_2 \in [0,1]$ with $t_1 < t_2$ and $y \in \mathcal{B}_r$, where $\mathcal{B}_r$ is a bounded set of X. Then we obtain

$$
\begin{aligned}
\bigl|h(t_2) - h(t_1)\bigr| \le{}& \sum_{i=1}^{m-2}\bigl|\phi_i(t_2) - \phi_i(t_1)\bigr|\int_0^{\sigma_i} e^{\mu(s-\sigma_i)}\bigl(I^{q-1}\bigl|\nu(s)\bigr|\bigr)\,ds \\
&+ \sum_{j=1}^{p-2}\bigl|\psi_j(t_2) - \psi_j(t_1)\bigr|\int_{\xi_j}^{\eta_j}\left(\int_0^s e^{\mu(m-s)}\bigl(I^{q-1}\bigl|\nu(m)\bigr|\bigr)\,dm\right)ds \\
&+ \bigl|\Omega_1(t_2) - \Omega_1(t_1)\bigr|\left[\sum_{i=1}^{m-2}|\delta_i|\bigl(I^{q-1}\bigl|\nu(\sigma_i)\bigr|\bigr) + \sum_{j=1}^{p-2}|\gamma_j|\int_{\xi_j}^{\eta_j}\bigl(I^{q-1}\bigl|\nu(s)\bigr|\bigr)\,ds\right. \\
&+ \left.|\rho_4|I^{q-1}\bigl|\nu(1)\bigr|\right] + \bigl|\Omega_2(t_2) - \Omega_2(t_1)\bigr|\int_0^1 e^{\mu(s-1)}\bigl(I^{q-1}\bigl|\nu(s)\bigr|\bigr)\,ds
\end{aligned}
$$

$$
\begin{aligned}
&+\left|\int_0^{t_1}\left[e^{\mu(s-t_2)}-e^{\mu(s-t_1)}\right]\left(I^{q-1}\left|\nu(s)\right|\right)ds+\int_{t_1}^{t_2}e^{\mu(s-t_2)}\left(I^{q-1}\left|\nu(s)\right|\right)ds\right| \\
&\le \|g\|Q(r)\left\{\sum_{i=1}^{m-2}\left|\phi_i(t_2)-\phi_i(t_1)\right|\int_0^{\sigma_i}e^{\mu(s-\sigma_i)}\left(I^{q-1}1\right)ds\right. \\
&+\sum_{j=1}^{p-2}\left|\psi_j(t_2)-\psi_j(t_1)\right|\int_{\xi_j}^{\eta_j}\left(\int_0^s e^{\mu(m-s)}\left(I^{q-1}1\right)dm\right)ds \\
&+\left|\Omega_1(t_2)-\Omega_1(t_1)\right|\left[\sum_{i=1}^{m-2}|\delta_i|\left(I^{q-1}1\right)+\sum_{j=1}^{p-2}|\gamma_j|\int_{\xi_j}^{\eta_j}\left(I^{q-1}1\right)ds\right. \\
&\left.+|\rho_4|I^{q-1}1\right]+\left|\Omega_2(t_2)-\Omega_2(t_1)\right|\int_0^1 e^{\mu(s-1)}\left(I^{q-1}\right)ds \\
&\left.+\left|\int_0^{t_1}\left[e^{\mu(s-t_2)}-e^{\mu(s-t_1)}\right]\left(I^{q-1}1\right)ds+\int_{t_1}^{t_2}e^{\mu(s-t_2)}\left(I^{q-1}1\right)ds\right|\right\},
\end{aligned}
$$

$$
\begin{aligned}
\left|h'(t_2)-h'(t_1)\right| &\le \|g\|Q(r)\left\{\sum_{i=1}^{m-2}\left|\phi_i'(t_2)-\phi_i'(t_1)\right|\int_0^{\sigma_i}e^{\mu(s-\sigma_i)}\left(I^{q-1}1\right)ds\right. \\
&+\sum_{j=1}^{p-2}\left|\psi_j'(t_2)-\psi_j'(t_1)\right|\int_{\xi_j}^{\eta_j}\left(\int_0^s e^{\mu(m-s)}\left(I^{q-1}1\right)dm\right)ds \\
&+\left|\Omega_1'(t_2)-\Omega_1'(t_1)\right|\left[\sum_{i=1}^{m-2}|\delta_i|\left(I^{q-1}1\right)+\sum_{j=1}^{p-2}|\gamma_j|\int_{\xi_j}^{\eta_j}\left(I^{q-1}1\right)ds\right. \\
&\left.+|\rho_4|I^{q-1}1\right]+\left|\Omega_2'(t_2)-\Omega_2'(t_1)\right|\int_0^1 e^{\mu(s-1)}\left(I^{q-1}1\right)ds \\
&+\mu\left|\int_0^{t_1}\left[e^{\mu(s-t_2)}-e^{\mu(s-t_1)}\right]\left(I^{q-1}1\right)ds+\int_{t_1}^{t_2}e^{\mu(s-t_2)}\left(I^{q-1}1\right)ds\right| \\
&\left.+\frac{1}{\Gamma(q-1)}\left|\int_0^{t_1}\left[(t_2-s)^{q-2}-(t_1-s)^{q-2}\right]ds+\int_{t_1}^{t_2}(t_2-s)^{q-2}ds\right|\right\},
\end{aligned}
$$

and

$$
\begin{aligned}
\left|{}^cD^\kappa h(t_2)-{}^cD^\kappa h(t_1)\right| &\le \int_0^t\frac{(t-s)^{-\kappa}}{\Gamma(1-\kappa)}\left|h'(t_2)-h'(t_1)\right|ds \\
&\le \frac{1}{\Gamma(2-\kappa)}\left|h'(t_2)-h'(t_1)\right|.
\end{aligned}
$$

Obviously, the right-hand sides of the above inequalities tend to zero independently of $x \in \mathcal{B}_r$ as $t_2 - t_1 \to 0$. As N satisfies the above assumptions, it follows by the Arzelá–Ascoli theorem that $N : X \to X$ is completely continuous.

By virtue of Proposition 1.2 in [33], it is enough to establish that operator $N : X \to X$ has a closed graph, which will imply that N is u.s.c. as it is already shown to be completely continuous. This is done in our next step.

Step 4. N has a closed graph.

Let $x_n \to x_*$, $h_n \in N(x_n)$, and $h_n \to h_*$. We need to show that $h_* \in N(x_*)$. Now $h_n \in N(x_n)$ implies that there exists $v_n \in S_{F,x_n}$ such that for each $t \in [0,1]$,

$$\begin{aligned} h_n(t) &= \sum_{i=1}^{m-2} \phi_i(t) \int_0^{\sigma_i} e^{\mu(s-\sigma_i)} \bigl(I^{q-1} v_n(s)\bigr)\, ds \\ &\quad + \sum_{j=1}^{p-2} \psi_j(t) \int_{\xi_j}^{\eta_j} \biggl(\int_0^s e^{\mu(m-s)} \bigl(I^{q-1} v_n(m)\bigr)\, dm \biggr)\, ds \\ &\quad + \Omega_1(t) \Biggl[\sum_{i=1}^{m-2} \delta_i \bigl(I^{q-1} v_n(\sigma_i)\bigr) + \sum_{j=1}^{p-2} \gamma_j \int_{\xi_j}^{\eta_j} \bigl(I^{q-1} v_n(s)\bigr)\, ds - \rho_4 I^{q-1} v_n(1) \Biggr] \\ &\quad + \Omega_2(t) \int_0^1 e^{\mu(s-1)} \bigl(I^{q-1} v_n(s)\bigr)\, ds + \int_0^t e^{\mu(s-t)} \bigl(I^{q-1} v_n(s)\bigr)\, ds. \end{aligned}$$

We must show that there exists $v_* \in S_{F,x_*}$ such that for each $t \in [0,1]$,

$$\begin{aligned} h_*(t) &= \sum_{i=1}^{m-2} \phi_i(t) \int_0^{\sigma_i} e^{\mu(s-\sigma_i)} \bigl(I^{q-1} v_*(s)\bigr)\, ds \\ &\quad + \sum_{j=1}^{p-2} \psi_j(t) \int_{\xi_j}^{\eta_j} \biggl(\int_0^s e^{\mu(m-s)} \bigl(I^{q-1} v_*(m)\bigr)\, dm \biggr)\, ds + \Omega_1(t) \Biggl[\sum_{i=1}^{m-2} \delta_i \bigl(I^{q-1} v_*(\sigma_i)\bigr) \\ &\quad + \sum_{j=1}^{p-2} \gamma_j \int_{\xi_j}^{\eta_j} \bigl(I^{q-1} v_*(s)\bigr)\, ds - \rho_4 I^{q-1} v_*(1) \Biggr] \\ &\quad + \Omega_2(t) \int_0^1 e^{\mu(s-1)} \bigl(I^{q-1} v_*(s)\bigr)\, ds + \int_0^t e^{\mu(s-t)} \bigl(I^{q-1} v_*(s)\bigr)\, ds. \end{aligned}$$

Consider the continuous linear operator $\Theta : L^1([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ given by

$$\begin{aligned} v \mapsto \Theta(v)(t) &= \sum_{i=1}^{m-2} \phi_i(t) \int_0^{\sigma_i} e^{\mu(s-\sigma_i)} \bigl(I^{q-1} v(s)\bigr)\, ds \\ &\quad + \sum_{j=1}^{p-2} \psi_j(t) \int_{\xi_j}^{\eta_j} \biggl(\int_0^s e^{\mu(m-s)} \bigl(I^{q-1} v(m)\bigr)\, dm \biggr)\, ds \\ &\quad + \Omega_1(t) \Biggl[\sum_{i=1}^{m-2} \delta_i \bigl(I^{q-1} v(\sigma_i)\bigr) + \sum_{j=1}^{p-2} \gamma_j \int_{\xi_j}^{\eta_j} \bigl(I^{q-1} v(s)\bigr)\, ds - \rho_4 I^{q-1} v(1) \Biggr] \\ &\quad + \Omega_2(t) \int_0^1 e^{\mu(s-1)} \bigl(I^{q-1} v(s)\bigr)\, ds + \int_0^t e^{\mu(s-t)} \bigl(I^{q-1} v(s)\bigr)\, ds. \end{aligned}$$

Since $\|h_n - h_*\|_X \to 0$ as $n \to \infty$, it follows from a closed graph result obtained in [34] that $\Theta \circ S_{F,x}$ is a closed graph operator. Moreover, we have $h_n \in \Theta(S_{F,x_n})$. Since $x_n \to x_*$, we obtain

$$h_*(t) = \sum_{i=1}^{m-2} \phi_i(t) \int_0^{\sigma_i} e^{\mu(s-\sigma_i)} \bigl(I^{q-1} v_*(s)\bigr)\, ds$$

$$+\sum_{j=1}^{p-2}\psi_j(t)\int_{\xi_j}^{\eta_j}\left(\int_0^s e^{\mu(m-s)}\big(I^{q-1}v_*(m)\big)\,dm\right)ds$$
$$+\Omega_1(t)\left[\sum_{i=1}^{m-2}\delta_i\big(I^{q-1}v_*(\sigma_i)\big)+\sum_{j=1}^{p-2}\gamma_j\int_{\xi_j}^{\eta_j}\big(I^{q-1}v_*(s)\big)\,ds-\rho_4 I^{q-1}v_*(1)\right]$$
$$+\Omega_2(t)\int_0^1 e^{\mu(s-1)}\big(I^{q-1}v_*(s)\big)\,ds+\int_0^t e^{\mu(s-t)}\big(I^{q-1}v_*(s)\big)\,ds,$$

for some $v_* \in S_{F,x_*}$

Step 5. We show there exists an open set $U \subseteq C(J,\mathbb{R})$ with $x \notin \theta N(x)$ for any $\theta \in (0,1)$ and all $x \in \partial U$.

Let $\theta \in (0,1)$ and $x \in \theta N(x)$. Then there exists $v \in L^1([0,1],\mathbb{R})$ with $v \in S_{F,x}$ such that, for $t \in J$, we have

$$x(t)=\theta\sum_{i=1}^{m-2}\phi_i(t)\int_0^{\sigma_i}e^{\mu(s-\sigma_i)}\big(I^{q-1}v(s)\big)\,ds$$
$$+\theta\sum_{j=1}^{p-2}\psi_j(t)\int_{\xi_j}^{\eta_j}\left(\int_0^s e^{\mu(m-s)}\big(I^{q-1}v(m)\big)\,dm\right)ds+\theta\Omega_1(t)\left[\sum_{i=1}^{m-2}\delta_i\big(I^{q-1}v(\sigma_i)\big)\right.$$
$$\left.+\sum_{j=1}^{p-2}\gamma_j\int_{\xi_j}^{\eta_j}\big(I^{q-1}v(s)\big)\,ds-\rho_4 I^{q-1}v(1)\right]$$
$$+\theta\Omega_2(t)\int_0^1 e^{\mu(s-1)}\big(I^{q-1}v(s)\big)\,ds+\int_0^t e^{\mu(s-t)}\big(I^{q-1}v(s)\big)\,ds.$$

As in the first step, we can find that

$$\frac{\|x\|_X}{\|g\|Q(\|x\|_X)(\lambda+\frac{\lambda_1}{\Gamma(2-\kappa)})}\le 1. \tag{4.2}$$

By condition (C_3), there does not exist any solution x such that $\|x\|_X \neq K$. Let us introduce a set $W=\{x\in X:\|x\|_X<K\}$. The operator $N:\overline{W}\to X$ is continuous and completely continuous. From the choice of W, there is no $w\in\partial W$ such that $w=\theta N(w)$ for some $\theta\in(0,1)$. In consequence, we deduce by the nonlinear alternative for Kakutani maps [32] that operator N has a fixed point $w\in\overline{W}$ which is a solution of the problem (1.2) and (1.3). The proof is completed. □

4.2 The case of Lipschitz maps

Now we prove the existence of solutions for the boundary value problem (1.2) and (1.3) with a nonconvex valued right-hand side by applying a fixed point theorem for multivalued maps due to Covitz and Nadler [35].

Let (X,d) be a metric space induced from the normed space $(X;\|\cdot\|)$. Consider $H_d:\mathcal{P}(X)\times\mathcal{P}(X)\to\mathbb{R}\cup\{\infty\}$ given by

$$H_d(U,V)=\max\Big\{\sup_{u\in U}d(u,V),\sup_{v\in V}d(U,v)\Big\},$$

where $d(U,v)=\inf_{u\in U}d(u;v)$ and $d(u,V)=\inf_{v\in V}d(u;v)$. Then $(\mathcal{P}_{cl,b}(X),H_d)$ is a metric space (see [36]), where $\mathcal{P}_{cl,b}(X)=\{\mathcal{Y}\in\mathcal{P}(X):\mathcal{Y}\text{ is closed and bounded}\}$.

Definition 4.2 A multivalued operator $N : X \to \mathcal{P}_{cl}(X)$ is called (a) θ-Lipschitz if and only if there exists $\theta > 0$ such that $H_d(N(x), N(y)) \leq \theta d(x, y)$ for each $x, y \in X$; and (b) a contraction if and only if it is θ-Lipschitz with $\theta < 1$.

Lemma 4.3 ([35]) *Let (X, d) be a complete metric space and $\mathcal{P}_{cl}(X) = \{\mathcal{Y} \in \mathcal{P}(X) : \mathcal{Y}$ is closed$\}$. If $N : X \to \mathcal{P}_{cl}(X)$ is a contraction, then* $\operatorname{Fix} N \neq \emptyset$.

Theorem 4.4 *Assume that the following conditions hold*:

(K_1) $F : [0,1] \times \mathbb{R}^2 \to \mathcal{P}_{cp}(\mathbb{R})$ *is such that* $F(\cdot, x, y) : [0,1] \to \mathcal{P}_{cp}(\mathbb{R})$ *is measurable for each* $x, y \in \mathbb{R}$;

(K_2) $H_d(F(t,x,y), F(t,\bar{x},\bar{y})) \leq m(t)(|x-\bar{x}| + |y-\bar{y}|)$ *for almost all* $t \in [0,1]$ *and* $x, \bar{x}, y, \bar{y} \in \mathbb{R}$ *with* $m \in C([0,1], \mathbb{R}^+)$ *and* $d(0, F(t,0,0)) \leq m(t)$ *for almost all* $t \in [0,1]$.

Then the inclusion problem (1.2) *and* (1.3) *has at least one solution on* $[0,1]$ *if*

$$\|m\|\left(\lambda + \frac{\lambda_1}{\Gamma(2-\kappa)}\right) < 1, \tag{4.3}$$

where λ, λ_1 are *given by* (3.3) *and* (3.5), *respectively.*

Proof In order to show that the operator $N : C([0,1], \mathbb{R}) \to \mathcal{P}(C([0,1], \mathbb{R}))$ (introduced in the beginning of the proof of Theorem 4.1) satisfies the assumptions of Lemma 4.3, we proceed as follows.

Step I. $N(x)$ is nonempty and closed for every $v \in S_{F,x}$.

Notice that $S_{F,x} \neq \emptyset$ for each x by assumption (K_2), and thus we can find a measurable selection for F (see [37, Theorem III.6]). In order to show that $N(x) \in \mathcal{P}_{cl}(X)$ for each $x \in X$, let $\{u_n\}_{n \geq 0} \in N(x)$ be such that $u_n \to u$ ($n \to \infty$) in X. Then $u \in C(J, \mathbb{R})$ and there exists $v_n \in S_{F,x_n}$ such that, for each $t \in [0,1]$, we have

$$\begin{aligned} u_n(t) = {} & \sum_{i=1}^{m-2} \phi_i(t) \int_0^{\sigma_i} e^{\mu(s-\sigma_i)} \big(I^{q-1} v_n(s)\big)\, ds \\ & + \sum_{j=1}^{p-2} \psi_j(t) \int_{\xi_j}^{\eta_j} \left(\int_0^s e^{\mu(m-s)} \big(I^{q-1} v_n(m)\big)\, dm \right) ds + \Omega_1(t) \left[\sum_{i=1}^{m-2} \delta_i \big(I^{q-1} v_n(\sigma_i)\big) \right. \\ & \left. + \sum_{j=1}^{p-2} \gamma_j \int_{\xi_j}^{\eta_j} \big(I^{q-1} v_n(s)\big)\, ds - \rho_4 I^{q-1} v_n(1) \right] \\ & + \Omega_2(t) \int_0^1 e^{\mu(s-1)} \big(I^{q-1} v_n(s)\big)\, ds + \int_0^t e^{\mu(s-t)} \big(I^{q-1} v_n(s)\big)\, ds. \end{aligned}$$

Since F has compact values, one can pass onto a subsequence (if necessary) to find that v_n converges to v in $L^1([0,1], \mathbb{R})$. Thus $v \in S_{F,x}$ and for each $t \in [0,1]$, we have

$$\begin{aligned} u_n(t) \to u(t) = {} & \sum_{i=1}^{m-2} \phi_i(t) \int_0^{\sigma_i} e^{\mu(s-\sigma_i)} \big(I^{q-1} v(s)\big)\, ds \\ & + \sum_{j=1}^{p-2} \psi_j(t) \int_{\xi_j}^{\eta_j} \left(\int_0^s e^{\mu(m-s)} \big(I^{q-1} v(m)\big)\, dm \right) ds \end{aligned}$$

$$+ \Omega_1(t)\left[\sum_{i=1}^{m-2}\delta_i\big(I^{q-1}v(\sigma_i)\big) + \sum_{j=1}^{p-2}\gamma_j\int_{\xi_j}^{\eta_j}\big(I^{q-1}v(s)\big)\,ds - \rho_4 I^{q-1}v(1)\right]$$
$$+ \Omega_2(t)\int_0^1 e^{\mu(s-1)}\big(I^{q-1}v(s)\big)\,ds + \int_0^t e^{\mu(s-t)}\big(I^{q-1}v(s)\big)\,ds,$$

which implies that $u \in N(x)$.

Step II. There exists $\bar{\theta} < 1$ ($\bar{\theta}$ is given by (4.3)) such that

$$H_d\big(N(x),N(\bar{x})\big) \leq \bar{\theta}\|x-\bar{x}\|_X \quad \text{for each } x,\bar{x} \in C\big([0,1],\mathbb{R}\big).$$

Let $x,\bar{x} \in X$ and $h_1 \in N(x)$. Then we can find $v_1(t) \in F(t,x(t),y(t))$ such that, for each $t \in [0,1]$,

$$h_1(t) = \sum_{i=1}^{m-2}\phi_i(t)\int_0^{\sigma_i} e^{\mu(s-\sigma_i)}\big(I^{q-1}v_1(s)\big)\,ds$$
$$+ \sum_{j=1}^{p-2}\psi_j(t)\int_{\xi_j}^{\eta_j}\left(\int_0^s e^{\mu(m-s)}\big(I^{q-1}v_1(m)\big)\,dm\right)ds + \Omega_1(t)\left[\sum_{i=1}^{m-2}\delta_i\big(I^{q-1}v_1(\sigma_i)\big)\right.$$
$$\left.+ \sum_{j=1}^{p-2}\gamma_j\int_{\xi_j}^{\eta_j}\big(I^{q-1}v_1(s)\big)\,ds - \rho_4 I^{q-1}v_1(1)\right]$$
$$+ \Omega_2(t)\int_0^1 e^{\mu(s-1)}\big(I^{q-1}v_1(s)\big)\,ds + \int_0^t e^{\mu(s-t)}\big(I^{q-1}v_1(s)\big)\,ds.$$

By (K_2), we have

$$H_d(F(t,x,y),F(t,\bar{x},\bar{y}) \leq m(t)\big(\big|x(t)-\bar{x}(t)\big| + \big|y(t)-\bar{y}(t)\big|\big).$$

So there exists $w(t) \in F(t,\bar{x}(t),\bar{y}(t))$ such that

$$\big|v_1(t)-w\big| \leq m(t)\big(\big|x(t)-\bar{x}(t)\big| + \big|y(t)-\bar{y}(t)\big|\big), \quad t \in [0,1].$$

Define $\mathcal{V} : [0,1] \to \mathcal{P}(\mathbb{R})$ by

$$\mathcal{V}(t) = \big\{w \in \mathbb{R} : \big|v_1(t)-w\big| \leq m(t)\big(\big|x(t)-\bar{x}(t)\big| + \big|y(t)-\bar{y}(t)\big|\big)\big\}.$$

In view of the fact that the multivalued operator $\mathcal{V}(t) \cap F(t,\bar{x}(t),\bar{y}(t))$ is measurable (see [37, Proposition III.4]), we can find a function $v_2(t)$ which is a measurable selection for $\mathcal{V}$ and such that $v_2(t) \in F(t,\bar{x}(t),\bar{y}(t))$. Then, for each $t \in [0,1]$, we have $|v_1(t)-v_2(t)| \leq m(t)(|x(t)-\bar{x}(t)| + |y(t)-\bar{y}(t)|)$. For each $t \in [0,1]$, we define

$$h_2(t) = \sum_{i=1}^{m-2}\phi_i(t)\int_0^{\sigma_i} e^{\mu(s-\sigma_i)}\big(I^{q-1}v_2(s)\big)\,ds$$
$$+ \sum_{j=1}^{p-2}\psi_j(t)\int_{\xi_j}^{\eta_j}\left(\int_0^s e^{\mu(m-s)}\big(I^{q-1}v_2(m)\big)\,dm\right)ds + \Omega_1(t)\left[\sum_{i=1}^{m-2}\delta_i\big(I^{q-1}v_2(\sigma_i)\big)\right.$$

$$+\sum_{j=1}^{p-2}\gamma_j\int_{\xi_j}^{\eta_j}\big(I^{q-1}\nu_2(s)\big)\,ds-\rho_4 I^{q-1}\nu_2(1)\Bigg]$$
$$+\,\Omega_2(t)\int_0^1 e^{\mu(s-1)}\big(I^{q-1}\nu_2(s)\big)\,ds+\int_0^t e^{\mu(s-t)}\big(I^{q-1}\nu_2(s)\big)\,ds.$$

Thus,

$$\begin{aligned}\big|h_1(t)-h_2(t)\big| &\le \sum_{i=1}^{m-2}\big|\phi_i(t)\big|\int_0^{\sigma_i} e^{\mu(s-\sigma_i)}\big(I^{q-1}\big|\nu_2(s)-\nu_1(s)\big|\big)\,ds\\
&\quad+\sum_{j=1}^{p-2}\big|\psi_j(t)\big|\int_{\xi_j}^{\eta_j}\left(\int_0^s e^{\mu(m-s)}\big(I^{q-1}\big|\nu_2(m)-\nu_1(m)\big|\big)\,dm\right)ds\\
&\quad+\big|\Omega_1(t)\big|\Bigg[\sum_{i=1}^{m-2}|\delta_i|\big(I^{q-1}\big|\nu_2(\sigma)-\nu_1(\sigma)\big|\big)\\
&\quad+\sum_{j=1}^{p-2}|\gamma_j|\int_{\xi_j}^{\eta_j}\big(I^{q-1}\big|\nu_2(s)-\nu_1(s)\big|\big)\,ds-\rho_4 I^{q-1}\big|\nu_2(1)-\nu_1(1)\big|\Bigg]\\
&\quad+\big|\Omega_2(t)\big|\int_0^1 e^{\mu(s-1)}\big(I^{q-1}\big|\nu_2(s)-\nu_1(s)\big|\big)\,ds\\
&\quad+\int_0^t e^{\mu(s-t)}\big(I^{q-1}\big|\nu_2(s)-\nu_1(s)\big|\big)\,ds\\
&\le\Bigg\{\sum_{i=1}^{m-2}\widehat{\phi}_i\int_0^{\sigma_i} e^{\mu(s-\sigma_i)}\big(I^{q-1}1\big)\,ds\\
&\quad+\sum_{j=1}^{p-2}\widehat{\psi}_j\int_{\xi_j}^{\eta_j}\left(\int_0^s e^{\mu(m-s)}\big(I^{q-1}1\big)\,dm\right)ds\\
&\quad+\widehat{\Omega}_1\Bigg[\sum_{i=1}^{m-2}\delta_i\big(I^{q-1}1\big)+\sum_{j=1}^{p-2}\gamma_j\int_{\xi_j}^{\eta_j}\big(I^{q-1}1\big)\,ds-\rho_4 I^{q-1}1\Bigg]\\
&\quad+\widehat{\Omega}_2\int_0^1 e^{\mu(s-1)}\big(I^{q-1}1\big)\,ds+\int_0^t e^{\mu(s-t)}\big(I^{q-1}1\big)\,ds\Bigg\}\|m\|\|x-\overline{x}\|\\
&\le\lambda\|m\|\|x-\overline{x}\|_X.\end{aligned}$$

Hence $\|h_1-h_2\|\le\lambda\|m\|\|x-\overline{x}\|_X$. In a similar manner, we obtain

$$\begin{aligned}\big|h_1'(t)-h_2'(t)\big| &\le\Bigg\{\sum_{i=1}^{m-2}\big|\phi_i'(t)\big|\int_0^{\sigma_i} e^{\mu(s-\sigma_i)}\big(I^{q-1}1\big)\,ds\\
&\quad+\sum_{j=1}^{p-2}\big|\psi_j'(t)\big|\int_{\xi_j}^{\eta_j}\left(\int_0^s e^{\mu(m-s)}\big(I^{q-1}1\big)\,dm\right)ds+\big|\Omega_1'(t)\big|\Bigg[\sum_{i=1}^{m-2}|\delta_i|\big(I^{q-1}1\big)\\
&\quad+\sum_{j=1}^{p-2}|\gamma_j|\int_{\xi_j}^{\eta_j}\big(I^{q-1}1\big)\,ds+|\rho_4|I^{q-1}1\Bigg]+\big|\Omega_2'(t)\big|\int_0^1 e^{\mu(s-1)}\big(I^{q-1}1\big)\,ds\end{aligned}$$

$$
\begin{aligned}
&+\mu\int_0^t e^{\mu(s-t)}\left(I^{q-1}1\right)ds + I^{q-1}1\bigg\}\|m\|\|x-\overline{x}\|_X \\
&\le \lambda_1\|m\|\|x-\overline{x}\|_X,
\end{aligned}
$$

and

$$
\left|{}^cD^\kappa h_1(t) - {}^cD^\kappa h_2(t)\right| \le \int_0^t \frac{(t-s)^{-\kappa}}{\Gamma(1-\kappa)}\left|h_1'(s) - h_2'(s)\right| ds \le \frac{1}{\Gamma(2-\kappa)}\lambda_1\|m\|\|x-\overline{x}\|_X.
$$

Thus

$$
\|h_1 - h_2\|_X \le \|m\|\left(\lambda + \frac{\lambda_1}{\Gamma(2-\kappa)}\right)\|x-\bar{x}\|_X.
$$

Analogously, interchanging the roles of x and $\overline{x}$, we obtain

$$
H_d\big(N(x), N(\bar{x})\big) \le \|m\|\left(\lambda + \frac{\lambda_1}{\Gamma(2-\kappa)}\right)\|x-\bar{x}\|_X.
$$

From the foregoing arguments, we deduce that N is a contraction. Thus it follows by Lemma 4.3 that N has a fixed point x which is a solution of (1.2) and (1.3). This completes the proof. □

4.3 Examples

Consider the sequential fractional differential inclusion

$$
\left({}^cD^{7/5} + D^{2/5}\right)x(t) \in F\left(t, x(t), {}^cD^{1/2}x(t)\right), \quad t \in [0,1], \tag{4.4}
$$

equipped with conditions (3.13).

In order to illustrate Theorem 4.1, we take

$$
\begin{aligned}
F\left(t, x(t), {}^cD^{1/2}x(t)\right) = \Bigg[&\frac{1}{\sqrt{t^2+144}}\left(\frac{1}{3}\sin\big(x(t)\big) + \frac{1}{2}\frac{|{}^cD^{1/2}x(t)|}{(1+|{}^cD^{1/2}x(t)|)} + \frac{1}{2}\right), \\
&\frac{1}{(t+15)}\left(\frac{1}{16}e^{-x^4(t)} + \frac{1}{5}\sin\left({}^cD^{1/2}x(t)\right) + \frac{1}{2}\right)\Bigg].
\end{aligned} \tag{4.5}
$$

It is easy to find that $g(t) = \frac{1}{\sqrt{t^2+144}}$ with $\|g\| = 1/12$, $\Omega(K) = (K+3)/3$. By condition (C_3), we find that $K > 0.7333$. Thus all the assumptions of Theorem 4.1 hold, and consequently the inclusion (4.4) equipped with boundary conditions (3.13) and F given by (4.5) has a solution on $[0,1]$.

For illustrating Theorem 4.4, we consider the following multivalued map:

$$
\begin{aligned}
F\left(t, x(t), {}^cD^{1/2}x(t)\right) = \Bigg[&\frac{1}{t+10}x(t) + \frac{1}{t^2+15}\cos\left(D^{1/2}x(t)\right), \\
&\frac{1}{t^2+25}\tan^{-1}\big(x(t)\big) + \frac{1}{t^2+49}{}^cD^{1/2}x(t) + \frac{1}{98}\Bigg].
\end{aligned} \tag{4.6}
$$

By condition (D_2), we get $m(t) = 1/(t+10)$ with $\|m\| = 1/10$. Moreover,

$$
\|m\|\left(\lambda + \frac{\lambda_1}{\Gamma(2-\kappa)}\right) \approx 0.70715 < 1
$$

(the values of λ and λ_1 are taken from Sect. 3.1). Clearly, all the hypotheses of Theorem 4.4 are satisfied. Therefore there exists at least one solution on $[0,1]$ for the inclusion (4.4) with F given by (4.6), equipped with boundary conditions (3.13).

5 Conclusions

In this paper, we have addressed new problems of sequential fractional differential equations and inclusions supplemented with nonlocal integro-multipoint boundary conditions. The single- and multivalued maps involved in the given problems depend on the unknown function, together with its lower-order fractional derivative. Our results are not only new in the given configuration but also yield several new results as special cases by fixing the parameters appearing in the problems. Some of these results are mentioned below.

- Our results correspond to purely nonlocal multipoint and multistrip boundary conditions if we take $\rho_i = 0$ $(i = 1, 2, 3, 4)$ in the given problems.
- For $\rho_1 = \rho_3 = 0$, $\rho_2 = \rho_4 = 1$, we obtain the results for terminal nonlocal multipoint and multistrip conditions:

$$x(1) = \sum_{i=1}^{m-2} \alpha_i x(\sigma_i) + \sum_{j=1}^{p-2} r_j \int_{\xi_j}^{\eta_i} x(s)\,ds, \qquad x'(1) = \sum_{i=1}^{m-2} \delta_i x'(\sigma_i) + \sum_{j=1}^{p-2} \gamma_j \int_{\xi_j}^{\eta_j} x'(s)\,ds.$$

- The results for sequential fractional differential equations and inclusions equipped with periodic/antiperiodic type boundary conditions of the form: $x(0) = -(\rho_2/\rho_1)x(1)$, $x'(0) = -(\rho_4/\rho_3)x'(1)$ follow by fixing $r_j = \gamma_j = \alpha_j = \delta_j = 0$, $j = 1, \ldots, p$. Further, the results for antiperiodic boundary conditions can be recorded by taking $\rho_2/\rho_1 = 1 = \rho_4/\rho_3$.
- The results associated with nonseparated nonlocal multipoint and multistrip conditions can be obtained by letting $r_j = 0 = \gamma_j$, $j = 1, \ldots, p$ and $\alpha_j = 0 = \delta_j$, $j = 1, \ldots, p$, respectively.

In the context of sequential fractional differential equations and inclusions together with nonlocal integro-multipoint boundary conditions, the present work is quite versatile in nature and significantly contributes to the existing literature on the topic. Moreover, several new results follow as special cases of the present ones.

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