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# An efficient method for solving fractional Riccati equations

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## Abstract

In this article, we present a method to approximate the solution of a fractional Riccati equation based on the ABFD in the Caputo sense. The proposed method depends on the fractional operational matrix of the fractional derivative. We present two examples. The results show the agreement between the exact and the approximate solutions with different choices of  $\gamma$ . Form the numerical results; we see that the proposed method give accurate results.

**Keywords:** ABFD; Fractional Riccati equation; Fractional Legendre functions

## 1 Introduction

Fractional calculus is used to describe several problems in biology, physics, engineering, and applied mathematics. Fractional derivatives are investigated in two major ways: as local and nonlocal derivatives. There are several definitions for the local derivatives. These definitions are considered as a direct generalization of the ordinary derivatives; see [1, 2]. However, these definitions do not consider the history of the functions during computing the derivative. Other definitions are the nonlocal fractional derivatives such as the Caputo and ABFD definitions; see [3, 4]. The fractional Riccati equation (FRE) is investigated based on different definitions of fractional derivatives. For example, see [5–7]. Akgul, et al., discussed several fractional problems using different methods such as the reproducing kernel Hilbert space method [8–16], while Alquran, et al., used a fractional power series method [17–24]. In this article, we consider the following class of equations:

$$\begin{aligned} \mathcal{I}_\gamma (u(x)) + \mu(x)u(x) + v(x)u^2(x) &= \xi(x), x \in (0, 1), \quad 0 < \gamma \leq 1, \\ u(0) &= \alpha, \end{aligned}$$

where  $u \in H^1(0, 1)$ ,  $\mu$ ,  $v$ ,  $\xi$  are continuous functions on  $[0, 1]$  We organize our paper as follows. In Sect. 2, we present the preliminaries which we used in this paper. Method of the solution is given in Sect. 3. Some numerical results are presented in Sect. 4 to show the efficiency of the proposed method. Finally, we draw some conclusions and closing remarks.

## 2 Preliminaries

We present the basic definitions and theorems which we use in this article.

**Definition 1** ([4]) Let  $u \in H^1(0, 1) = \{u \in L^2(0, 1) : u' \in L^2(0, 1)\}$  and  $0 < \gamma \leq 1$ . The ABFD in the Caputo sense of  $u$  of order  $\gamma$  is defined by

$$\mathfrak{I}_\gamma(u(x)) = \frac{B(\gamma)}{1-\gamma} \int_0^x u'(s) E_\gamma\left(-\gamma \frac{(x-s)^\gamma}{1-\gamma}\right) ds. \tag{1}$$

For simplicity, we choose  $B(\gamma) = 1$ . The fractional integral is defined as follows.

**Definition 2** ([4]) The fractional integral is given by

$$I^\gamma u(x) = \frac{1-\gamma}{B(\gamma)} u(x) + \frac{\gamma}{B(\gamma)\Gamma(\gamma)} \int_0^x u(s)(x-s)^{\gamma-1} ds. \tag{2}$$

**Theorem 3** Let  $u \in H^1(0, 1)$  such that  $\mathfrak{I}(u)$  exists. Then

$$I^\gamma (\mathfrak{I}_\gamma(u(x))) = u(x) - u(0). \tag{3}$$

*Proof* See [2]. □

The fractional Legendre polynomials are  $\{FL_r(x) : r = 0, 1, 2, \dots\}$  which are defined by

$$FL_r(x) = L_r(2x^\gamma - 1), \quad r = 0, 1, 2, \dots, \tag{4}$$

where  $\{L_r(x) : r = 0, 1, 2, \dots\}$  are the Legendre polynomials. Then

$$\int_0^1 FL_r(x) FL_s(x) w(x) dx = \begin{cases} \frac{1}{(2r+1)^\gamma}, & r = s, \\ 0, & r \neq s, \end{cases} \tag{5}$$

where  $w(x) = x^{\gamma-1}$ . Also,  $FL_r(x)$  is given as

$$FL_r(x) = \sum_{j=0}^r (-1)^{j+r} \frac{(r+j)!}{(r-j)! (j!)^2} x^{j\gamma}. \tag{6}$$

**Theorem 4** Let  $g \in C^1[0, 1]$ . Then

$$g(x) = \sum_{r=0}^\infty g_r FL_r(x), \tag{7}$$

where

$$g_r = (2r+1)\gamma \int_0^1 g(x) FL_r(x) w(x) dx. \tag{8}$$

For the proof of these properties, see [6].

**Theorem 5**  $I^\gamma FL_r(x)$  is a linear combination of  $\{FL_0(x), FL_1(x), \dots, FL_{r+1}(x)\}$ .

*Proof* Simple calculations imply that

$$\begin{aligned}
 I^\gamma FL_r(x) &= \frac{1-\gamma}{B(\gamma)} FL_r(x) + \frac{\gamma}{B(\gamma)\Gamma(\gamma)} \int_0^x FL_r(s)(x-s)^{\gamma-1} ds \\
 &= \frac{1-\gamma}{B(\gamma)} FL_r(x) + \frac{\gamma}{B(\gamma)\Gamma(\gamma)} \int_0^x \sum_{j=0}^r (-1)^{j+r} \frac{(r+j)!}{(r-j)!(j!)^2} \frac{s^{j\gamma}}{(x-s)^{\gamma-1}} ds \\
 &= \frac{1-\gamma}{B(\gamma)} FL_r(x) + \frac{\gamma}{B(\gamma)\Gamma(\gamma)} \sum_{j=0}^r (-1)^{j+r} \frac{(r+j)!}{(r-j)!(j!)^2} \int_0^x s^{j\gamma} (x-s)^{\gamma-1} ds \\
 &= \frac{1-\gamma}{B(\gamma)} FL_r(x) + \frac{\gamma}{B(\gamma)} \sum_{j=0}^r (-1)^{j+r} \frac{(r+j)!}{(r-j)!(j!)^2} \frac{1}{\Gamma(1+j\gamma)} x^{(j+1)r}. \tag{9}
 \end{aligned}$$

Thus,  $I^\gamma FL_r(x)$  is a linear combination of  $\{FL_0(x), FL_1(x), \dots, FL_{r+1}(x)\}$ . □

### 3 Method of solution

We want to investigate the operational matrix of  $I^\gamma$ . Define the set of  $k$  block pulse functions on  $[0, 1)$  by

$$\{p_0(x), p_1(x), \dots, p_{k-1}(x)\},$$

where

$$p_r(x) = \begin{cases} 1, & \frac{r}{k} \leq x < \frac{r+1}{k}, \\ 0, & x \in [0, 1) - [\frac{r}{k}, \frac{r+1}{k}), \end{cases} \quad r = 0, 1, \dots, k-1. \tag{10}$$

Then

$$p_r(x)p_s(x) = \begin{cases} p_r(x), & r = s, \\ 0, & r \neq s, \end{cases} \tag{11}$$

and

$$\int_0^1 p_r(x)p_s(x) dx = \begin{cases} \frac{1}{k}, & r = s, \\ 0, & r \neq s, \end{cases} \tag{12}$$

where  $0 \leq r, s \leq k-1$ . If  $g \in L_2[0, 1]$ , then

$$g(x) = \sum_{r=0}^{k-1} g_r p_r(x). \tag{13}$$

Multiply both sides by  $p_s(x)$  then integrate from 0 to 1 to get

$$\begin{aligned}
 \int_0^1 g(x)p_s(x) dx &= \sum_{r=0}^{k-1} g_r \int_0^1 p_r(x)p_s(x) dx \\
 &= \frac{g_s}{k}, \tag{14}
 \end{aligned}$$

which implies that

$$g_s = k \int_0^1 g(x)p_s(x) dx. \tag{15}$$

Thus,

$$g(x) = G^T P(x), \tag{16}$$

where

$$G = k \begin{pmatrix} \int_0^1 g(x)p_0(x) dx \\ \int_0^1 g(x)p_1(x) dx \\ \vdots \\ \int_0^1 g(x)p_{k-1}(x) dx \end{pmatrix}, \quad p = \begin{pmatrix} p_0(x) \\ p_1(x) \\ \vdots \\ p_{k-1}(x) \end{pmatrix}.$$

**Theorem 6**  $I^\gamma P = \Omega P$  where

$$\Omega = \begin{pmatrix} a & b & b & \dots & b \\ b & a & b & \dots & b \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ b & b & b & \dots & a \end{pmatrix},$$

where

$$a = \frac{1-\gamma}{B(\gamma)} + \frac{1}{B(\gamma)\Gamma(\gamma)k^\gamma}, \quad b = \frac{1}{B(\gamma)\Gamma(\gamma)k^\gamma}.$$

*Proof* For any  $0 \leq r \leq k-1$

$$\begin{aligned} I^\gamma p_r(x) &= \frac{1-\gamma}{B(\gamma)} p_r(x) + \frac{\gamma}{B(\gamma)\Gamma(\gamma)} \int_0^x p_r(s)(x-s)^{\gamma-1} ds \\ &= \frac{1-\gamma}{B(\gamma)} p_r(x) + \frac{\gamma}{B(\gamma)\Gamma(\gamma)} \int_{\frac{x}{k}}^{\frac{r+1}{k}} \left(\frac{r+1}{k} - s\right)^{\gamma-1} ds \\ &= \frac{1-\gamma}{B(\gamma)} p_r(x) + \frac{1}{B(\gamma)\Gamma(\gamma)k^\gamma} \\ &= \frac{1-\gamma}{B(\gamma)} p_r(x) + \frac{1}{B(\gamma)\Gamma(\gamma)k^\gamma} \sum_{r=0}^{k-1} p_r(x). \end{aligned} \tag{17}$$

Thus,

$$I^\gamma P = \Omega P, \tag{18}$$

where

$$\Omega = \begin{pmatrix} a & b & b & \cdots & b \\ b & a & b & \cdots & b \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ a & a & a & \cdots & a \end{pmatrix} \tag{19}$$

and

$$a = \frac{1 - \gamma}{B(\gamma)} + \frac{1}{B(\gamma)\Gamma(\gamma)k^\gamma}, \quad b = \frac{1}{B(\gamma)\Gamma(\gamma)k^\gamma}. \tag{20}$$

□

**Theorem 7** *Let*

$$FL(x) = \begin{pmatrix} FL_0(x) \\ FL_1(x) \\ \vdots \\ FL_{k-1}(x) \end{pmatrix}.$$

*Then*

$$FL(x) = FPP(x), \tag{21}$$

where  $FP$  is a  $k \times k$  matrix with

$$(FP)_{r,s} = \sum_{j=0}^r (-1)^{j+r} \frac{(r+j)!}{(r-j)! (j!)^2} \left( \frac{(s+1)^{j\gamma+1} - s^{j\gamma+1}}{(j\gamma+1)k^{j\gamma}} \right). \tag{22}$$

*Proof* For any  $r \in \{0, 1, \dots, k-1\}$ ,  $FL_r(x) \in L_2[0, 1]$ . Thus from Eq. (14), we get

$$FL_r(x) = \sum_{s=0}^{k-1} (FP)_{r,s} p_s(x). \tag{23}$$

Hence, from Eq. (15), we get

$$\begin{aligned} (FP)_{r,s} &= k \int_0^1 FL_r(x) p_s(x) dx \\ &= k \sum_{j=0}^r (-1)^{j+r} \frac{(r+j)!}{(r-j)! (j!)^2} \int_{\frac{s}{k}}^{\frac{s+1}{k}} x^{j\gamma} dx \\ &= \sum_{j=0}^r (-1)^{j+r} \frac{(r+j)!}{(r-j)! (j!)^2} \left( \frac{(s+1)^{j\gamma+1} - s^{j\gamma+1}}{(j\gamma+1)k^{j\gamma}} \right). \end{aligned} \tag{24}$$

□

**Theorem 8**  $FP$  is an invertible matrix.

*Proof* From Theorem 7,

$$FL(x)(FL(x))^T = FP P(x)(P(x))^T FP^T. \tag{25}$$

Hence,

$$\int_0^1 FL(x)(FL(x))^T x^{\gamma-1} dx = FP \int_0^1 P(x)(P(x))^T x^{\gamma-1} dx FP^T. \tag{26}$$

From Eqs. (6) and (12), we have

$$\begin{aligned} \int_0^1 FL(x)(FL(x))^T x^{\gamma-1} dx &= \frac{1}{\gamma} \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & \frac{1}{3} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \frac{1}{2k-1} \end{pmatrix} \\ &= \Lambda_1 \end{aligned} \tag{27}$$

and

$$\begin{aligned} \int_0^1 P(x)(P(x))^T x^{\gamma-1} dx &= \frac{1}{\gamma k^\gamma} \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 2^\gamma - 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & k^\gamma - (k-1)^\gamma \end{pmatrix} \\ &= \Lambda_2. \end{aligned} \tag{28}$$

Therefore,

$$\Lambda_1 = FP \Lambda_2 FP^T. \tag{29}$$

Thus,

$$\det(FP) = \sqrt{\frac{\prod_{r=0}^{k-1} \frac{1}{\gamma(2r+1)}}{\prod_{r=0}^{k-1} \frac{(r+1)^\gamma - r^\gamma}{\gamma k^\gamma}}} \neq 0. \tag{30}$$

Hence,  $FP$  is invertible. □

**Theorem 9**  $I^\gamma FL(x) = \Psi FL(x)$  where

$$\Psi = FP \Omega FP^{-1}. \tag{31}$$

*Proof* Let

$$I^\gamma FL(x) = \Psi FL(x). \tag{32}$$

From Eqs. (19) and (22)

$$I^\gamma FL(x) = \Psi FPP(x) \tag{33}$$

and

$$\begin{aligned} I^\gamma FL(x) &= FPI^\gamma P(x) \\ &= FP\Omega P(x). \end{aligned} \tag{34}$$

Hence,

$$\Psi FPP(x) = FP\Omega P(x), \tag{35}$$

which implies that

$$\Psi FP = FP\Omega \tag{36}$$

or

$$\Psi = FP\Omega FP^{-1}. \tag{37}$$

Now, let  $u \in C^1[0, 1]$ , then

$$u(x) = \sum_{r=0}^{\infty} u_r FL_r(x), \tag{38}$$

where

$$u_r = (2r + 1)\gamma \int_0^1 u(x) FL_r(x) x^{\gamma-1} dx. \tag{39}$$

Let

$$U_k(x) = \sum_{r=0}^{k-1} u_r FL_r(x) = U^T FL(x), \tag{40}$$

where

$$U = \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_{k-1} \end{pmatrix}.$$

Thus,

$$\mathfrak{L}_r(u(x)) = U^T FL(x) = U^T FPP(x). \tag{41}$$

Using Theorem 3, we get

$$\begin{aligned}
 u(x) - \alpha &= I^\nu U^T FL(x) \\
 &= U^T I^\nu FL(x) \\
 &= U^T \Psi FL(x) \\
 &= U^T \Psi FPP(x),
 \end{aligned} \tag{42}$$

which implies that

$$u(x) = U^T \Psi FPP(x) + \alpha. \tag{43}$$

Thus, Eq. (1) implies that

$$U^T FPP(x) + \mu(x)(U^T \Psi FPP(x) + \alpha) + \nu(x)(U^T \Psi FPP(x) + \alpha)^2 = \xi FPP(x), \tag{44}$$

where  $\xi(x) = \xi FPP(x)$ . If

$$U^T \Psi FP = (\varkappa_1 \quad \varkappa_2 \quad \cdots \quad \varkappa_k), \tag{45}$$

then

$$(U^T \Psi FPP(x) + \alpha)^2 = (\varkappa_1^2 + 2\varkappa_1\alpha \quad \varkappa_2^2 + 2\varkappa_2\alpha \quad \cdots \quad \varkappa_k^2 + 2\varkappa_k\alpha) P(x) + \alpha^2. \tag{46}$$

Hence,

$$(F_1(U) + \mu(x)F_2(U) + \nu(x)F_3(U))P(x) = F_4(x), \tag{47}$$

where

$$F_1(U) = U^T FP, \tag{48}$$

$$F_2(U) = U^T \Psi FP, \tag{49}$$

$$F_3(U) = (\varkappa_1^2 + 2\varkappa_1\alpha \quad \varkappa_2^2 + 2\varkappa_2\alpha \quad \cdots \quad \varkappa_k^2 + 2\varkappa_k\alpha), \tag{50}$$

$$F_4(x) = \xi FPP(x) - \alpha\mu(x) - \alpha^2\nu(x). \tag{51}$$

To solve Eq. (48), we use the collocation points

$$t_r = \frac{r+1}{k+1}, \quad r = 0, 1, \dots, k-1. \tag{52}$$

Then we solve the generated nonlinear system to find  $U$  using Mathematica. □

#### 4 Numerical results

We present two examples to show the efficiency of the proposed method.



*Example 1* Consider the following problem:

$$\mathbb{I}_{\frac{1}{2}}(u(x)) + u(x) + u^2(x) = f(x), \quad 0 < x < 1,$$

$$u(0) = 1,$$

where

$$f(x) = x^2 + 1 + (x^2 + 1)^2 - 4 - 4x + \frac{8\sqrt{x}(3 + 2x)}{3\sqrt{\pi}} + 4e^x \operatorname{Erfc}(\sqrt{x}).$$

The exact solution is

$$u(x) = x^2 + 1.$$

Let  $k = 7$ . Let  $Q = \{p_0(x), p_1(x), \dots, p_6(x)\}$  be the set of block pulse functions on  $[0, 1)$  where

$$p_r(x) = \begin{cases} 1, & \frac{r}{7} \leq x < \frac{r+1}{7}, \\ 0, & \text{otherwise,} \end{cases} \quad r = 0, 1, \dots, 6.$$

Let

$$f(x) = \sum_{r=0}^6 g_r p_r(x) = G^T P(x).$$

From Eq. (22), we have

$$G = \begin{pmatrix} 2.03186 \\ 2.21914 \\ 2.59375 \\ 3.16995 \\ 3.97389 \\ 5.04267 \\ 6.42401 \end{pmatrix}, \quad P(x) = \begin{pmatrix} p_0(x) \\ p_1(x) \\ p_2(x) \\ p_3(x) \\ p_4(x) \\ p_5(x) \\ p_6(x) \end{pmatrix}.$$

From Eqs. (23) and (24), we have

$$\Omega = \begin{pmatrix} \frac{1}{2} + \frac{1}{\sqrt{6\pi}} & \frac{1}{\sqrt{6\pi}} & \frac{1}{\sqrt{6\pi}} & \frac{1}{\sqrt{6\pi}} & \frac{1}{\sqrt{6\pi}} & \frac{1}{\sqrt{6\pi}} & \frac{1}{\sqrt{6\pi}} \\ \frac{1}{\sqrt{6\pi}} & \frac{1}{2} + \frac{1}{\sqrt{6\pi}} & \frac{1}{\sqrt{6\pi}} & \frac{1}{\sqrt{6\pi}} & \frac{1}{\sqrt{6\pi}} & \frac{1}{\sqrt{6\pi}} & \frac{1}{\sqrt{6\pi}} \\ \frac{1}{\sqrt{6\pi}} & \frac{1}{\sqrt{6\pi}} & \frac{1}{2} + \frac{1}{\sqrt{6\pi}} & \frac{1}{\sqrt{6\pi}} & \frac{1}{\sqrt{6\pi}} & \frac{1}{\sqrt{6\pi}} & \frac{1}{\sqrt{6\pi}} \\ \frac{1}{\sqrt{6\pi}} & \frac{1}{\sqrt{6\pi}} & \frac{1}{\sqrt{6\pi}} & \frac{1}{2} + \frac{1}{\sqrt{6\pi}} & \frac{1}{\sqrt{6\pi}} & \frac{1}{\sqrt{6\pi}} & \frac{1}{\sqrt{6\pi}} \\ \frac{1}{\sqrt{6\pi}} & \frac{1}{\sqrt{6\pi}} & \frac{1}{\sqrt{6\pi}} & \frac{1}{\sqrt{6\pi}} & \frac{1}{2} + \frac{1}{\sqrt{6\pi}} & \frac{1}{\sqrt{6\pi}} & \frac{1}{\sqrt{6\pi}} \\ \frac{1}{\sqrt{6\pi}} & \frac{1}{\sqrt{6\pi}} & \frac{1}{\sqrt{6\pi}} & \frac{1}{\sqrt{6\pi}} & \frac{1}{\sqrt{6\pi}} & \frac{1}{2} + \frac{1}{\sqrt{6\pi}} & \frac{1}{\sqrt{6\pi}} \\ \frac{1}{\sqrt{6\pi}} & \frac{1}{\sqrt{6\pi}} & \frac{1}{\sqrt{6\pi}} & \frac{1}{\sqrt{6\pi}} & \frac{1}{\sqrt{6\pi}} & \frac{1}{\sqrt{6\pi}} & \frac{1}{2} + \frac{1}{\sqrt{6\pi}} \end{pmatrix}.$$

From Eq. (32), we have

$$FP = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -0.496 & -0.079 & 0.193 & 0.413 & 0.603 & 0.772 & 0.927 \\ -0.083 & -0.479 & -0.437 & -0.239 & 0.049 & 0.3977 & 0.791 \\ 0.313 & 0.112 & -0.265 & -0.433 & -0.345 & 0.005 & 0.613 \\ -0.158 & 0.324 & 0.228 & -0.135 & -0.395 & -0.280 & 0.416 \\ -0.076 & -0.126 & 0.280 & 0.235 & -0.155 & -0.384 & 0.227 \\ 0.125 & -0.228 & -0.083 & 0.281 & 0.146 & -0.309 & 0.068 \end{pmatrix}.$$

It is easy to see that  $FP$  is invertible, since  $\det(FP) = 1.4895 \times 10^{-3} \neq 0$ . From Eq. (37), we have

$$\Psi = FP \Omega FP^{-1} = \begin{pmatrix} 2.1 & -6.3 \times 10^{-4} & 2.1 \times 10^{-4} & 1.1 \times 10^{-2} & 4.2 \times 10^{-4} & -7.6 \times 10^{-4} & -8.2 \times 10^{-4} \\ 0.5 & 0.50 & 6.6 \times 10^{-4} & 3.5 \times 10^{-3} & 5.4 \times 10^{-4} & -4.5 \times 10^{-4} & -2.3 \times 10^{-4} \\ 4.5 \times 10^{-5} & -4.4 \times 10^{-5} & 0.50 & -6.4 \times 10^{-4} & 5.3 \times 10^{-5} & 2.3 \times 10^{-5} & 1.8 \times 10^{-5} \\ 3.7 \times 10^{-5} & 2 \times 10^{-4} & -2.3 \times 10^{-5} & 0.50 & -1.4 \times 10^{-4} & 1.4 \times 10^{-4} & 4.4 \times 10^{-5} \\ -8 \times 10^{-5} & 1.1 \times 10^{-4} & -2.1 \times 10^{-4} & -3.5 \times 10^{-4} & 0.50 & 8.6 \times 10^{-5} & 9.9 \times 10^{-6} \\ 1.4 \times 10^{-4} & -1.1 \times 10^{-4} & -9.7 \times 10^{-5} & 5.3 \times 10^{-4} & 6.3 \times 10^{-5} & 0.50 & -3.4 \times 10^{-5} \\ 1.2 \times 10^{-5} & -1.4 \times 10^{-4} & 4.0 \times 10^{-5} & 6.2 \times 10^{-4} & 1.1 \times 10^{-4} & -1.1 \times 10^{-4} & 0.50 \end{pmatrix}.$$

By Eq. (47), we get

$$U = \begin{pmatrix} 1.2 \\ 0.4 \\ 0.28571428569 \\ 0.1 \\ 0.01428571428 \\ 0 \\ 0 \end{pmatrix}.$$

Thus,

$$u_7(x) = \sum_{r=0}^6 u_r FL_r(x) = 1 + x^2.$$

Hence, we get the exact solution.

*Example 2* Consider the following problem:

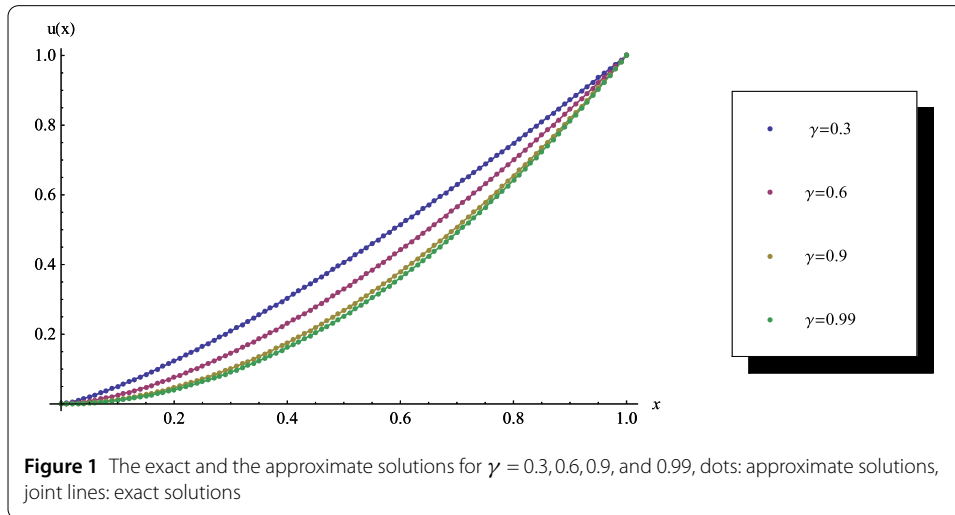
$$\begin{aligned} \mathfrak{L}_\gamma(u(x)) + u(x) + u^2(x) &= f(x), \quad 0 < x < 1, 0 < \gamma < 1, \\ u(0) &= 1, \end{aligned}$$

where

$$f(x) = x^{\gamma+1} + x^{2\gamma+2} + \frac{1+\gamma}{1-\gamma} \Gamma(1+\gamma)x^{1+\gamma} E_{\gamma,\gamma+2}\left(\frac{-\gamma x^\gamma}{1-\gamma}\right).$$

**Table 1** Absolute errors

$\gamma$	$R(\gamma)$
0.3	$1.21 * 10^{-13}$
0.6	$2.10 * 10^{-13}$
0.9	$2.32 * 10^{-13}$
0.99	$2.79 * 10^{-13}$



The exact solution is

$$u(x) = x^{\gamma+1}.$$

Let

$$R(\gamma) = \max_{r=0,1,\dots,100} \left| u_{30} \left( \frac{r}{100} \right) - u \left( \frac{r}{100} \right) \right|.$$

Then the absolute errors for different choices of  $\gamma$  are given in Table 1.

The graph of the exact and the approximate solutions for  $\gamma = 0.3, 0.6, 0.9,$  and  $0.99$  are given in Fig. 1.

### 5 Closing remarks

In this article, we present a method to approximate the solution of FRE based on the ABFD in Caputo sense. The numerical method is based on the fractional operational matrix of the fractional derivative. We present two examples. In the first example, we get the exact solution. In the second one, the absolute error is of order  $10^{-13}$ . Results are given in Table 1. Figure 1 presents the agreement between the exact and the approximate solutions in Example 2 for different choices of  $\gamma$ . From the numerical results, we see that the proposed method gives accurate results. It is advisable to use it for other nonlinear fractional differential equations.

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