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The Minkowski inequalities via generalized proportional fractional integral operators

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Abstract

Recent research has gained more attention on conformable integrals and derivatives to derive the various type of inequalities. One of the recent advancements in the field of fractional calculus is the generalized nonlocal proportional fractional integrals and derivatives lately introduced by Jarad et al. (Eur. Phys. J. Special Topics 226:3457–3471, 2017) comprising the exponential functions in the kernels. The principal aim of this paper is to establish reverse Minkowski inequalities and some other fractional integral inequalities by utilizing generalized proportional fractional integrals. Also, two new theorems connected with this inequality as well as other inequalities associated with the generalized proportional fractional integrals are established.

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1 Introduction

Fractional calculus is a study of integrals and derivatives of arbitrary order which was a natural outgrowth of conventional definitions of calculus integral and derivative. Fractional integral has been comprehensively studied in the literature. The idea has been defined by numerous mathematicians with a slightly different formula, for example, Riemann–Liouville, Weyl, Erdélyi–Kober, Hadamard integral, Liouville and Katugampola fractional integral (see [18, 22, 23, 26, 34]). In the last few years, Khalil et al. [24] and Abdeljawad [1] established a new class of fractional derivatives and integrals called fractional conformable derivatives and integrals. Jarad et al. [21] introduced the fractional conformable integral operators. On the basis of that idea, one can obtain the generalizations of the inequalities: Hadamard, Hermite–Hadamard, Opial, Grüss, Ostrowski, Chebyshev, among others [19, 35, 37–39]).

Later on in [6], Anderson and Ulness improved the idea of the fractional conformable derivative by introducing the idea of local derivatives. In [2, 3, 7, 9, 27] researchers introduced new fractional derivative operators by using exponential and Mittag-Leffler functions in their kernels. In [20], Jarad et al. proposed the left and right generalized nonlocal proportional fractional integral and derivative operators. Such generalizations motivate future research to present more innovative ideas to unify the fractional operators and obtain the inequalities involving such fractional operators. The integral inequalities and their applications play an essential role in the theory of differential equations and applied mathematics. A variety of various types of some classical integral inequalities and their generalizations have been established by utilizing the classical fractional integral, fractional derivative operators (see, e.g., [4, 12, 14–17, 25, 28–30, 32, 33, 36, 41, 42, 46, 47]).

The reverse Minkowski fractional integral inequalities are perceived in [13]. Anber et al. [5] have gained some fractional integral inequalities by using Riemann–Liouville fractional integral. In [11], the authors established Minkowski inequalities and some other inequalities by employing Katugampola fractional integral operators. In [10, 45], the authors established the reverse Minkowski inequality for Hadamard fractional integral operators. In [31], Mubeen et al. recently established the reverse Minkowski inequalities and some related inequalities for generalized k-fractional conformable integrals.

This paper is organized as follows: In the second section, we present some known results and basic definitions. In the third section, the reverse Minkowski inequalities are presented. In the fourth section, some other related inequalities involving generalized nonlocal proportional fractional integrals are presented.

2 Preliminaries

This section is devoted to some known definitions and results associated with the classical Riemann–Liouville fractional integrals and their generalization involving the Riemann–Liouville fractional integrals. Set et al. [40] presented Hermite–Hadamard and reverse Minkowski inequalities for Riemann–Liouville fractional integrals. In [8], Bougoffa also presented Hardy's and reverse Minkowski inequalities. The following theorems involving the reverse Minkowski inequalities are the motivation of work performed so far, involving the classical Riemann integrals.

Theorem 2.1 ([40]) *Let $r \geq 1$ and let g, h be two positive functions on $[0,\infty)$. If $0 < m \leq \frac{g(\rho)}{h(\rho)} \leq M$, $\vartheta \in [a,b]$, then the following inequality holds*:

$$\left(\int_a^b g^r(\vartheta)\,d\vartheta\right)^{1/r} + \left(\int_a^b h^r(\vartheta)\,d\vartheta\right)^{1/r} \leq \frac{1+M(m+2)}{(m+1)(M+1)}\left(\int_a^b (g+h)^r(\vartheta)\,d\vartheta\right)^{1/r}. \tag{1}$$

Theorem 2.2 ([40]) *Let $r \geq 1$ and let g, h be two positive functions on $[0,\infty)$. If $0 < m \leq \frac{g(\rho)}{h(\rho)} \leq M$, $\vartheta \in [a,b]$, then the following inequality holds*:

$$\left(\int_a^b g^r(\vartheta)\,d\vartheta\right)^{2/r} + \left(\int_a^b h^r(\vartheta)\,d\vartheta\right)^{2/r} \geq \left(\frac{(M+1)(m+1)}{M} - 2\right)\left(\int_a^b g^r(\vartheta)\,d\vartheta\right)^{1/r}\left(\int_a^b h^r(\vartheta)\,d\vartheta\right)^{1/r}. \tag{2}$$

Definition 2.1 ([26, 34]) The left and right R-L fractional integrals of order λ are respectively defined by

$$\left({}_a\mathfrak{I}^{\lambda} g\right)(\vartheta) = \frac{1}{\Gamma(\lambda)}\int_a^{\vartheta} (\vartheta-\rho)^{\lambda-1} g(\rho)\,d\rho, \quad a < \vartheta \tag{3}$$

and

$$\left(\mathfrak{I}_b^{\lambda} g\right)(\vartheta) = \frac{1}{\Gamma(\lambda)} \int_{\vartheta}^{b} (\rho - \vartheta)^{\lambda-1} g(\rho)\, d\rho, \quad \vartheta < b, \tag{4}$$

where $\lambda \in \mathbb{C}$ and $\Re(\lambda) > 0$.

In [13], Dahmani introduced the following reverse Minkowski inequalities involving the R-L fractional integral operators.

Theorem 2.3 ([13]) *Let $\lambda \in \mathbb{C}$, $\Re(\lambda) > 0$, $r \geq 1$, and let g, h be two positive functions on $[0,\infty)$ such that, for all $\vartheta > 0$, $\mathfrak{I}^{\lambda} g^r(\vartheta) < \infty$, $\mathfrak{I}^{\lambda} h^r(\vartheta) < \infty$. If $0 < m \leq \frac{g(\rho)}{h(\rho)} \leq M$, $\rho \in [a,\vartheta]$, then the following inequality holds*:

$$\left(\mathfrak{I}^{\lambda} g^r(\vartheta)\right)^{1/r} + \left(\mathfrak{I}^{\lambda} h^r(\vartheta)\right)^{1/r} \leq \frac{1 + M(m+2)}{(m+1)(M+1)} \left(\mathfrak{I}^{\lambda}(g+h)^r(\vartheta)\right)^{1/r}. \tag{5}$$

Theorem 2.4 ([13]) *Let $\lambda \in \mathbb{C}$, $\Re(\lambda) > 0$, $r \geq 1$, and let g, h be two positive functions on $[0,\infty)$ such that, for all $\vartheta > 0$, $\mathfrak{I}^{\lambda} g^r(\vartheta) < \infty$, $\mathfrak{I}^{\lambda} h^r(\vartheta) < \infty$. If $0 < m \leq \frac{g(\rho)}{h(\rho)} \leq M$, $\rho \in [a,\vartheta]$, then the following inequality holds*:

$$\begin{aligned} &\left(\mathfrak{I}^{\lambda} g^r(\vartheta)\right)^{2/r} + \left(\mathfrak{I}^{\lambda} h^r(\vartheta)\right)^{2/r} \\ &\quad \geq \left(\frac{(M+1)(m+1)}{M} - 2\right) \left(\mathfrak{I}^{\lambda} g^r(\vartheta)\right)^{1/r} \left(\mathfrak{I}^{\lambda} h^r(\vartheta)\right)^{1/r}. \end{aligned} \tag{6}$$

Definition 2.2 ([20]) The left and right generalized nonlocal proportional integral operators are respectively defined by

$$\left({}_a\mathfrak{I}^{\lambda,\eta} g\right)(\vartheta) = \frac{1}{\eta^{\lambda}\Gamma(\lambda)} \int_{a}^{\vartheta} \exp\left[\frac{\eta-1}{\eta}(\vartheta - \rho)\right] (\vartheta - \rho)^{\lambda-1} g(\rho)\, d\rho \tag{7}$$

and

$$\left(\mathfrak{I}_b^{\lambda,\eta} g\right)(\vartheta) = \frac{1}{\eta^{\lambda}\Gamma(\lambda)} \int_{\vartheta}^{b} \exp\left[\frac{\eta-1}{\eta}(\rho - \vartheta)\right] (\rho - \vartheta)^{\lambda-1} g(\rho)\, d\rho, \tag{8}$$

where $\eta \in (0,1]$ and $\lambda \in \mathbb{C}$ and $\Re(\lambda) > 0$.

Remark 2.1 If we consider $\eta = 1$ in (7) and (8), then we get the left and right Riemann–Liouville (3) and (4) respectively.

3 Reverse Minkowski inequalities via generalized proportional fractional integral operator

In this section, we use generalized nonlocal proportional fractional integral operator to develop reverse Minkowski integral inequalities. The reverse Minkowski fractional integral inequality is presented in the following theorem.

Theorem 3.1 *Let $\eta \in (0,1]$, $\lambda \in \mathbb{C}$, $\Re(\lambda) > 0$, $r \geq 1$, and let g, h be two positive functions on $[0,\infty)$ such that, for all $\vartheta > 0$, ${}_a\mathfrak{I}^{\lambda,\eta} g^r(\vartheta) < \infty$, ${}_a\mathfrak{I}^{\lambda,\eta} h^r(\vartheta) < \infty$. If $0 < m \leq \frac{g(\rho)}{h(\rho)} \leq M$,*

$\rho \in [a, \vartheta]$, *then the following inequality holds*:

$$\left({}_a\mathfrak{I}^{\lambda,\eta} g^r(\vartheta)\right)^{1/r} + \left({}_a\mathfrak{I}^{\lambda,\eta} h^r(\vartheta)\right)^{1/r} \le \frac{1 + M(m+2)}{(m+1)(M+1)} \left({}_a\mathfrak{I}^{\lambda,\eta} (g+h)^r(\vartheta)\right)^{1/r}. \tag{9}$$

Proof Under the condition stated in Theorem 3.1, $\frac{g(\rho)}{h(\rho)} \le M$, $\rho \in [0, \vartheta]$, $\vartheta > 0$, we have

$$(M+1)^r g^r(\rho) \le M^r (g+h)^r(\rho). \tag{10}$$

Consider a function

$$\begin{aligned} \mathfrak{F}(\vartheta, \rho) &= \frac{1}{\eta^\lambda \Gamma(\lambda)} \exp\left[\frac{\eta-1}{\eta}(\vartheta-\rho)\right](\vartheta-\rho)^{\lambda-1} \\ &= \frac{1}{\eta^\lambda \Gamma(\lambda)} (\vartheta-\rho)^{\lambda-1}\left[1 + \frac{\eta-1}{\eta}(\vartheta-\rho) + \frac{(\frac{\eta-1}{\eta}(\vartheta-\rho))^2}{2} + \cdots\right]. \end{aligned} \tag{11}$$

We observe that the function $\mathfrak{F}(\vartheta, \rho)$ remains positive for all $\rho \in (a, \vartheta)$, $a < \vartheta \le b$, since each term of the above function is positive in view of conditions stated in Theorem 3.1.

Multiplying both sides of (10) by $\mathfrak{F}(\vartheta, \rho)$ and integrating the resultant inequality with respect to ρ from a to ϑ, we have

$$\begin{aligned} &\frac{(M+1)^r}{\eta^\lambda \Gamma(\lambda)} \int_a^\vartheta \exp\left[\frac{\eta-1}{\eta}(\vartheta-\rho)\right](\vartheta-\rho)^{\lambda-1} g^r(\rho)\, d\rho \\ &\quad \le \frac{M^r}{\eta^\lambda \Gamma(\lambda)} \int_a^\vartheta \exp\left[\frac{\eta-1}{\eta}(\vartheta-\rho)\right](\vartheta-\rho)^{\lambda-1} (g+h)^r(\rho)\, d\rho, \end{aligned}$$

which can be written as

$${}_a\mathfrak{I}^{\lambda,\eta} g^r(\vartheta) \le \frac{M^r}{(M+1)^r}\, {}_a\mathfrak{I}^{\lambda,\eta} (g+h)^r(\vartheta).$$

Hence, it follows that

$$\left({}_a\mathfrak{I}^{\lambda,\eta} g^r(\vartheta)\right)^{1/r} \le \frac{M}{(M+1)} \left({}_a\mathfrak{I}^{\lambda,\eta} (g+h)^r(\vartheta)\right)^{1/r}. \tag{12}$$

Now, using the condition $mg(\rho) \le h(\rho)$, we have

$$\left(1 + \frac{1}{m}\right) h(\rho) \le \frac{1}{m}\left(g(\rho) + h(\rho)\right),$$

it follows that

$$\left(1 + \frac{1}{m}\right)^r h^r(\rho) \le \left(\frac{1}{m}\right)^r \left(g(\rho) + h(\rho)\right)^r. \tag{13}$$

Multiplying both sides of (13) by $\mathfrak{F}(\vartheta, \rho)$ and integrating the resultant inequality with respect to ρ from a to ϑ, we have

$$\left({}_a\mathfrak{I}^{\lambda,\eta} h^r(\vartheta)\right)^{1/r} \le \frac{1}{(m+1)} \left({}_a\mathfrak{I}^{\lambda,\eta} (g+h)^r(\vartheta)\right)^{1/r}. \tag{14}$$

Thus adding inequalities (12) and (14) yields the desired inequality. □

Theorem 3.2 *Let $\eta \in (0,1]$, $\lambda \in \mathbb{C}$, $\Re(\lambda) > 0$, $r \geq 1$, and let g, h be two positive functions on $[0,\infty)$ such that, for all $\vartheta > 0$, ${}_a\Im^{\lambda,\eta} g^r(\vartheta) < \infty$, ${}_a\Im^{\lambda,\eta} h^r(\vartheta) < \infty$. If $0 < m \leq \frac{g(\rho)}{h(\rho)} \leq M$, $\rho \in [a,\vartheta]$, then the following inequality holds:*

$$\begin{aligned}&\bigl({}_a\Im^{\lambda,\eta} g^r(\vartheta)\bigr)^{2/r} + \bigl({}_a\Im^{\lambda,\eta} h^r(\vartheta)\bigr)^{2/r}\\&\quad\geq \biggl(\frac{(M+1)(m+1)}{M} - 2\biggr)\bigl({}_a\Im^{\lambda,\eta} g^r(\vartheta)\bigr)^{1/r}\bigl({}_a\Im^{\lambda,\eta} h^r(\vartheta)\bigr)^{1/r}.\end{aligned} \tag{15}$$

Proof The multiplication of inequalities (12) and (14) yields

$$\biggl(\frac{(M+1)(m+1)}{M}\biggr)\bigl({}_a\Im^{\lambda,\eta} g^r(\vartheta)\bigr)^{1/r}\bigl({}_a\Im^{\lambda,\eta} h^r(\vartheta)\bigr)^{1/r} \leq \bigl[\bigl({}_a\Im^{\lambda,\eta}\bigl(g(\vartheta)+h(\vartheta)\bigr)^r\bigr)^{1/r}\bigr]^2. \tag{16}$$

Now, applying the Minkowski inequality to the right-hand side of (16), we obtain

$$\begin{aligned}&\bigl[\bigl({}_a\Im^{\lambda,\eta}\bigl(g(\vartheta)+h(\vartheta)\bigr)^r\bigr)^{1/r}\bigr]^2\\&\quad\leq \bigl[\bigl({}_a\Im^{\lambda,\eta} g^r(\vartheta)\bigr)^{1/r} + \bigl({}_a\Im^{\lambda,\eta} h^r(\vartheta)\bigr)^{1/r}\bigr]^2\\&\quad\leq \bigl({}_a\Im^{\lambda,\eta} g^r(\vartheta)\bigr)^{2/r} + \bigl({}_a\Im^{\lambda,\eta} h^r(\vartheta)\bigr)^{2/r} + 2\bigl({}_a\Im^{\lambda,\eta} g^r(\vartheta)\bigr)^{1/r}\bigl({}_a\Im^{\lambda,\eta} h^r(\vartheta)\bigr)^{1/r}.\end{aligned} \tag{17}$$

Thus, from inequalities (16) and (17), we get the desired inequality (15). □

4 Certain related inequalities via generalized proportional fractional integral operator

This section is devoted to deriving certain related inequalities involving a generalized proportional fractional integral operator.

Theorem 4.1 *Let $\eta \in (0,1]$, $\lambda \in \mathbb{C}$, $\Re(\lambda) > 0$, $r > 1$, $1/r + 1/s = 1$, and let g, h be two positive functions on $[0,\infty)$ such that ${}_a\Im^{\lambda,\eta}[g(\vartheta)] < \infty$, ${}_a\Im^{\lambda,\eta}[h(\vartheta)] < \infty$. If $0 < m \leq \frac{g(\rho)}{h(\rho)} \leq M < \infty$, $\rho \in [a,\vartheta]$, $\vartheta > a$, we have*

$$\bigl({}_a\Im^{\lambda,\eta} g(\vartheta)\bigr)^{1/r}\bigl({}_a\Im^{\lambda,\eta} h(\vartheta)\bigr)^{1/s} \leq \biggl(\frac{M}{m}\biggr)^{1/rs}\bigl({}_a\Im^{\lambda,\eta}\bigl[g(\vartheta)\bigr]^{1/r}\bigl[h(\vartheta)\bigr]^{1/s}\bigr). \tag{18}$$

Proof Since $\frac{g(\rho)}{h(\rho)} \leq M < \infty$, $\rho \in [a,\vartheta]$, $\vartheta > a$, therefore we have

$$\bigl[h(\rho)\bigr]^{1/s} \geq M^{-1/s}\bigl[g(\rho)\bigr]^{1/s}. \tag{19}$$

It follows that

$$\begin{aligned}\bigl[g(\rho)\bigr]^{1/r}\bigl[h(\rho)\bigr]^{1/s} &\geq M^{-1/r}\bigl[g(\rho)\bigr]^{1/r}\bigl[g(\rho)\bigr]^{1/s}\\&\geq M^{-1/s}\bigl[g(\rho)\bigr]^{\frac{1}{r}+1/s}\\&\geq M^{-1/r}\bigl[g(\rho)\bigr].\end{aligned} \tag{20}$$

Multiplying both sides of (20) by $\mathfrak{F}(\vartheta,\rho)$ where $\mathfrak{F}(\vartheta,\rho)$ is defined by (11) and integrating the resultant inequality with respect to ρ from a to ϑ, we have

$$\begin{aligned}&\frac{1}{\eta^{\lambda}\Gamma(\lambda)}\int_a^{\vartheta}\exp\biggl[\frac{\eta-1}{\eta}(\vartheta-\rho)\biggr](\vartheta-\rho)^{\lambda-1}\bigl[g(\rho)\bigr]^{1/r}\bigl[h(\rho)\bigr]^{1/s}\,d\rho\\&\quad\geq\frac{M^{-1/r}}{\eta^{\lambda}\Gamma(\lambda)}\int_a^{\vartheta}\exp\biggl[\frac{\eta-1}{\eta}(\vartheta-\rho)\biggr](\vartheta-\rho)^{\lambda-1}g(\rho)\,d\rho.\end{aligned}\tag{21}$$

It follows that

$${}_a\mathfrak{I}^{\lambda,\eta}\bigl[\bigl[g(\vartheta)\bigr]^{1/r}\bigl[h(\vartheta)\bigr]^{1/s}\bigr]\geq M^{\frac{-1}{r}}\bigl[{}_a\mathfrak{I}^{\lambda,\eta}g(\vartheta)\bigr].\tag{22}$$

Consequently, we have

$$\bigl({}_a\mathfrak{I}^{\lambda,\eta}\bigl[\bigl[g(\vartheta)\bigr]^{1/r}\bigl[h(\vartheta)\bigr]^{1/s}\bigr]\bigr)^{1/r}\geq M^{\frac{-1}{rs}}\bigl[{}_a\mathfrak{I}^{\lambda,\eta}h(\vartheta)\bigr]^{1/r}.\tag{23}$$

On the other hand, $mg(\rho)\leq h(\rho)$, $\rho\in[a,\vartheta]$, $\vartheta>a$, therefore we have

$$\bigl[g(\rho)\bigr]^{1/r}\geq m^{1/r}\bigl[h(\rho)\bigr]^{1/r}.\tag{24}$$

It follows that

$$\begin{aligned}\bigl[g(\rho)\bigr]^{1/r}\bigl[h(\rho)\bigr]^{1/s}&\geq m^{1/r}\bigl[g(\rho)\bigr]^{1/r}\bigl[h(\rho)\bigr]^{1/s}\\&\geq m^{1/r}\bigl[h(\rho)\bigr]^{\frac{1}{r}+1/s}\\&\geq m^{1/r}\bigl[h(\rho)\bigr].\end{aligned}\tag{25}$$

Again, multiplying both sides of (25) by $\mathfrak{F}(\vartheta,\rho)$ where $\mathfrak{F}(\vartheta,\rho)$ is defined by (11) and integrating the resultant inequality with respect to ρ from a to ϑ, we have

$$\begin{aligned}&\frac{1}{\eta^{\lambda}\Gamma(\lambda)}\int_a^{\vartheta}\exp\biggl[\frac{\eta-1}{\eta}(\vartheta-\rho)\biggr](\vartheta-\rho)^{\lambda-1}\bigl[g(\rho)\bigr]^{1/r}\bigl[h(\rho)\bigr]^{1/s}\,d\rho\\&\quad\geq\frac{m^{1/r}}{\eta^{\lambda}\Gamma(\lambda)}\int_a^{\vartheta}\exp\biggl[\frac{\eta-1}{\eta}(\vartheta-\rho)\biggr](\vartheta-\rho)^{\lambda-1}h(\rho)\,d\rho.\end{aligned}\tag{26}$$

Hence, we can write

$$\bigl({}_a\mathfrak{I}^{\lambda,\eta}\bigl[\bigl[g(\vartheta)\bigr]^{1/r}\bigl[h(\vartheta)\bigr]^{1/s}\bigr]\bigr)^{1/r}\geq m^{\frac{1}{rs}}\bigl[{}_a\mathfrak{I}^{\lambda,\eta}g(\vartheta)\bigr]^{1/s}.\tag{27}$$

Multiplying (23) and (27), we get the desired inequality. □

Theorem 4.2 *Let $\eta\in(0,1]$, $\lambda\in\mathbb{C}$, $\Re(\lambda)>0$, $r>1$, $\frac{1}{r}+1/s=1$, and let g, h be two positive functions on $[0,\infty)$ such that ${}_a\mathfrak{I}^{\lambda,\eta}[g^r(\vartheta)]<\infty$, ${}_a\mathfrak{I}^{\lambda,\eta}[h^s(\vartheta)]<\infty$. If $0<m\leq\frac{g(\rho)^r}{h(\rho)^s}\leq M<\infty$, $\rho\in[a,\vartheta]$, $\vartheta>a$, we have*

$$\bigl({}_a\mathfrak{I}^{\lambda,\eta}g^r(\vartheta)\bigr)^{1/r}\bigl({}_a\mathfrak{I}^{\lambda,\eta}h^s(\vartheta)\bigr)^{1/s}\leq\biggl(\frac{M}{m}\biggr)^{\frac{1}{rs}}\bigl({}_a\mathfrak{I}^{\lambda,\eta}\bigl[g(\vartheta)\bigr]^{1/r}\bigl[h(\vartheta)\bigr]^{1/s}\bigr).\tag{28}$$

Proof Replacing $g(\vartheta)$ and $h(\vartheta)$ by $g^r(\vartheta)$ and $h^r(\vartheta)$, $a < \vartheta \leq b$ in Theorem 4.1, we get the desired inequality (28). □

Theorem 4.3 *Let* $\eta \in (0,1]$, $\lambda \in \mathbb{C}$, $\Re(\lambda) > 0$, $r > 1$, $\frac{1}{r} + 1/s = 1$, *and let* g, h *be two positive functions on* $[0,\infty)$ *such that* ${}_a\mathfrak{I}^{\lambda,\eta}[g^r(\vartheta)] < \infty$, ${}_a\mathfrak{I}^{\lambda,\eta}[h^s(\vartheta)] < \infty$. *If* $0 < m \leq \frac{g^r(\rho)}{h^s(\rho)} \leq M < \infty$ *where* $m, M \in \mathbb{R}$, $\rho \in [a,\vartheta]$, $\vartheta > a$, *then the following inequality for left generalized proportional fractional integral holds*:

$$ {}_a\mathfrak{I}^{\lambda,\eta}\bigl[g(\vartheta)h(\vartheta)\bigr] \leq \frac{2^{r-1}M^r}{r(M+1)^r}\,{}_a\mathfrak{I}^{\lambda,\eta}\bigl[g^r + h^p\bigr](\vartheta) + \frac{2^{s-1}}{s(m+1)^s}\,{}_a\mathfrak{I}^{\lambda,\eta}\bigl[g^s + h^s\bigr](\vartheta). \tag{29} $$

Proof By the given hypothesis $\frac{g(\rho)}{h(\rho)} \leq M$, we have

$$ (M+1)^r g^r(\rho) \leq M^r[g+h]^r(\rho). \tag{30} $$

Multiplying both sides of inequality (30) by $\mathfrak{F}(\vartheta,\rho)$ where $\mathfrak{F}(\vartheta,\rho)$ is defined by (11) and integrating the resultant identity with respect to ρ over (a,ϑ), we get

$$ \begin{aligned} &\frac{(M+1)^r}{\eta^\lambda \Gamma(\lambda)} \int_a^\vartheta \exp\biggl[\frac{\eta-1}{\eta}(\vartheta-\rho)\biggr](\vartheta-\rho)^{\lambda-1} g^r(\rho)\,d\rho \\ &\quad \leq \frac{M^r}{\eta^\lambda \Gamma(\lambda)} \int_a^\vartheta \exp\biggl[\frac{\eta-1}{\eta}(\vartheta-\rho)\biggr](\vartheta-\rho)^{\lambda-1}[g+h]^r(\rho)\,d\rho. \end{aligned} \tag{31} $$

It follows that

$$ {}_a\mathfrak{I}^{\lambda,\eta} g^r(\vartheta) \leq \frac{M^r}{(M+1)^r}\,{}_a\mathfrak{I}^{\lambda,\eta}[g+h]^r(\vartheta). \tag{32} $$

On the other hand, using $m \leq \frac{g(\rho)}{h(\rho)}$, $a < t < \vartheta$, we have

$$ (m+1)^s h^s(\rho) \leq [g+h]^s(\rho). \tag{33} $$

Again, multiplying both sides of inequality (33) by $\mathfrak{F}(\vartheta,\rho)$ where $\mathfrak{F}(\vartheta,\rho)$ is defined by (11) and integrating the resultant identity with respect to ρ over (a,ϑ), we get

$$ {}_a\mathfrak{I}^{\lambda,\eta} h^s(\vartheta) \leq \frac{1}{(m+1)^s}\,{}_a\mathfrak{I}^{\lambda,\eta}[g+h]^s(\vartheta). \tag{34} $$

Now, using Young's inequality, we have

$$ g(\rho)h(\rho) \leq \frac{g^r(\rho)}{r} + \frac{g^s(\rho)}{s}. \tag{35} $$

Multiplying both sides of inequality (33) by $\mathfrak{F}(\vartheta,\rho)$ where $\mathfrak{F}(\vartheta,\rho)$ is defined by (11) and integrating the resultant identity with respect to ρ over (a,ϑ), we get

$$ {}_a\mathfrak{I}^{\lambda,\eta} g(\vartheta)h(\vartheta) \leq \frac{1}{r}\bigl({}_a\mathfrak{I}^{\lambda,\eta} g^r(\vartheta)\bigr) + 1/s\bigl({}_a\mathfrak{I}^{\lambda,\eta} g^s(\vartheta)\bigr). \tag{36} $$

With the aid of (32) and (34), (36) can be written as

$$\begin{aligned} {}_a\mathfrak{I}^{\lambda,\eta}g(\vartheta)h(\vartheta) &\leq \frac{1}{r}\big({}_a\mathfrak{I}^{\lambda,\eta}g^r(\vartheta)\big) + 1/s\big({}_a\mathfrak{I}^{\lambda,\eta}g^s(\vartheta)\big) \\ &\leq \frac{M^r}{r(M+1)^r}{}_a\mathfrak{I}^{\lambda,\eta}[g+h]^r(\vartheta) + \frac{1}{s(m+1)^s}{}_a\mathfrak{I}^{\lambda,\eta}[g+h]^s(\vartheta). \end{aligned} \tag{37}$$

Now, using the inequality $(\rho+\omega)^r \leq 2^{s-1}(\rho^r+\omega^r)$, $r>1$, $\rho,\omega>0$, one can obtain

$$ {}_a\mathfrak{I}^{\lambda,\eta}[g+h]^r(\vartheta) \leq {}_a\mathfrak{I}^{\lambda,\eta}\big[g^r+h^r\big](\vartheta) \tag{38}$$

and

$$ {}_a\mathfrak{I}^{\lambda,\eta}[g+h]^s(\vartheta) \leq {}_a\mathfrak{I}^{\lambda,\eta}\big[g^s+h^s\big](\vartheta). \tag{39}$$

Hence the proof of (29) can follow from (37), (38), and (39). □

Theorem 4.4 *Let $\eta \in (0,1]$, $\lambda \in \mathbb{C}$, $\Re(\lambda)>0$, $r \geq 1$, and let g, h be two positive functions on $[0,\infty)$ such that ${}_a\mathfrak{I}^{\lambda,\eta}[g^r(\vartheta)]<\infty$, ${}_a\mathfrak{I}^{\lambda,\eta}[h^r(\vartheta)]<\infty$. If $0<k<m\leq \frac{g(\rho)}{h(\rho)} \leq M<\infty$, where $m,M\in\mathbb{R}$, $\rho\in[a,\vartheta]$, $\vartheta>a$, then the following inequality for left generalized proportional fractional integral holds*:

$$\begin{aligned} \frac{M+1}{M-k}\big({}_a\mathfrak{I}^{\lambda,\eta}\big[g(\vartheta)-kh(\vartheta)\big]\big) &\leq \big({}_a\mathfrak{I}^{\lambda,\eta}g^r(\vartheta)\big)^{1/r} + \big({}_a\mathfrak{I}^{\lambda,\eta}h^r(\vartheta)\big)^{1/r} \\ &\leq \frac{m+1}{m-k}\big({}_a\mathfrak{I}^{\lambda,\eta}\big[g(\vartheta)-kh(\vartheta)\big]\big)^{1/r}. \end{aligned} \tag{40}$$

Proof Under the given hypothesis $0<k<m\leq \frac{g^r(\rho)}{h^s(\rho)} \leq M<\infty$, we have

$$\begin{aligned} mk \leq Mk \quad &\Rightarrow \quad mk+m \leq mk+M \leq Mk+M \\ &\Rightarrow \quad (M+1)(m-k) \leq (m+1)(M-k). \end{aligned}$$

It can be written as

$$\frac{(M+1)}{(M-k)} \leq \frac{(m+1)}{(m-k)}.$$

Also, we have

$$m-k \leq \frac{g(\rho)-kh(\rho)}{h(\rho)} \leq M-k.$$

It follows that

$$\frac{(g(\rho)-kh(\rho))^r}{(M-k)^r} \leq h^r(\rho) \leq \frac{(g(\rho)-kh(\rho))^r}{(m-k)^r}. \tag{41}$$

Also, we have

$$\frac{1}{M} \leq \frac{h(\rho)}{g(\rho)} \leq \frac{1}{m} \quad \Rightarrow \quad \frac{m-k}{km} \leq \frac{g(\rho)-kh(\rho)}{kg(\rho)} \leq \frac{M-k}{kM}.$$

It follows that

$$\left(\frac{M}{M-k}\right)^r \leq \big(g(\rho)-kh(\rho)\big)^r \leq g^r(\rho) \leq \left(\frac{m}{m-k}\right)^r \leq \big(g(\rho)-kh(\rho)\big)^r. \tag{42}$$

Multiplying both sides of inequality (41) by $\mathfrak{F}(\vartheta,\rho)$ where $\mathfrak{F}(\vartheta,\rho)$ is defined by (11) and integrating the resultant identity with respect to ρ over (a,ϑ), we get

$$\begin{aligned}
&\frac{1}{(M-k)^r\eta^\lambda\Gamma(\lambda)}\int_a^\vartheta \exp\left[\frac{\eta-1}{\eta}(\vartheta-\rho)\right](\vartheta-\rho)^{\lambda-1}\big(g(\rho)-kh(\rho)\big)^r\,d\rho \\
&\quad\leq \frac{1}{\eta^\lambda\Gamma(\lambda)}\int_a^\vartheta \exp\left[\frac{\eta-1}{\eta}(\vartheta-\rho)\right](\vartheta-\rho)^{\lambda-1}h^r(\rho)\,d\rho \\
&\quad\leq \frac{1}{(m-k)^r\eta^\lambda\Gamma(\lambda)}\int_a^\vartheta \exp\left[\frac{\eta-1}{\eta}(\vartheta-\rho)\right](\vartheta-\rho)^{\lambda-1}\big(g(\rho)-kh(\rho)\big)^r\,d\rho.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\frac{1}{(M-k)}\left({}_a\mathfrak{I}^{\lambda,\eta}\big(g(\vartheta)-kh(\vartheta)\big)^r\right)^{1/r} \leq \left({}_a\mathfrak{I}^{\lambda,\eta}h^r(\vartheta)\right)^{1/r} \\
&\qquad\qquad \leq \frac{1}{(m-k)}\left({}_a\mathfrak{I}^{\lambda,\eta}\big(g(\vartheta)-kh(\vartheta)\big)^r\right)^{1/r}.
\end{aligned} \tag{43}$$

Again, multiplying both sides of inequality (42) by $\mathfrak{F}(\vartheta,\rho)$ where $\mathfrak{F}(\vartheta,\rho)$ is defined by (11) and integrating the resultant identity with respect to ρ over (a,ϑ), we get

$$\begin{aligned}
&\left(\frac{M}{M-k}\right)\left({}_a\mathfrak{I}^{\lambda,\eta}\big(g(\vartheta)-kh(\vartheta)\big)^r\right)^{1/r} \leq \left({}_a\mathfrak{I}^{\lambda,\eta}g^r(\vartheta)\right)^{1/r} \\
&\qquad\qquad \leq \left(\frac{m}{m-k}\right)\left({}_a\mathfrak{I}^{\lambda,\eta}\big(g(\vartheta)-kh(\vartheta)\big)^r\right)^{1/r}.
\end{aligned} \tag{44}$$

Hence, by adding inequalities (43) and (44), we get the desired inequality (40). □

Theorem 4.5 *Let $\eta \in (0,1]$, $\lambda \in \mathbb{C}$, $\Re(\lambda) > 0$, $r \geq 1$, and let g, h be two positive functions on $[0,\infty)$ such that ${}_a\mathfrak{I}^{\lambda,\eta}[g^r(\vartheta)] < \infty$, ${}_a\mathfrak{I}^{\lambda,\eta}[h^r(\vartheta)] < \infty$. If $0 \leq \alpha \leq g(\rho) \leq \mathcal{A}$ and $0 \leq \sigma \leq h(\rho) \leq \mathcal{B}$ for all $\rho \in [a,\vartheta]$, $\vartheta > a$, then the following inequality for left generalized proportional fractional integral holds*:

$$\left({}_a\mathfrak{I}^{\lambda,\eta}g^r(\vartheta)\right)^{1/r} + \left({}_a\mathfrak{I}^{\lambda,\eta}h^r(\vartheta)\right)^{1/r} \leq \frac{\mathcal{A}(\alpha+\mathcal{B})+\mathcal{B}(\sigma+\mathcal{A})}{(\mathcal{A}+\sigma)(\mathcal{B}+\alpha)}\left({}_a\mathfrak{I}^{\lambda,\eta}[g+h]^r(\vartheta)\right)^{1/r}. \tag{45}$$

Proof Under the given hypothesis, we have

$$\frac{1}{\mathcal{B}} \leq \frac{1}{h(\rho)} \leq \frac{1}{\sigma}. \tag{46}$$

The product of inequality (46) with $0 \leq \alpha \leq g(\rho) \leq \mathcal{A}$ yields

$$\frac{\alpha}{\mathcal{B}} \leq \frac{g(\rho)}{h(\rho)} \leq \frac{\mathcal{A}}{\sigma}. \tag{47}$$

From (47), we obtain

$$h^r(\rho) \leq \left(\frac{\mathcal{B}}{\alpha + \mathcal{B}}\right)^r \bigl(g(\rho) + h(\rho)\bigr)^r \tag{48}$$

and

$$g^r(\rho) \leq \left(\frac{\mathcal{A}}{\sigma + \mathcal{A}}\right)^r \bigl(g(\rho) + h(\rho)\bigr)^r. \tag{49}$$

Now, multiplying both sides of inequalities (48) and (49) respectively by $\mathfrak{F}(\vartheta, \rho)$ where $\mathfrak{F}(\vartheta, \rho)$ is defined by (11) and integrating the resultant identity with respect to ρ over (a, ϑ), we obtain

$$\bigl({}_a\mathfrak{J}^{\lambda,\eta} h^r(\vartheta)\bigr)^{1/r} \leq \left(\frac{\mathcal{B}}{\alpha + \mathcal{B}}\right) \bigl({}_a\mathfrak{J}^{\lambda,\eta} \bigl(g(\vartheta) + h(\vartheta)\bigr)^r\bigr)^{1/r} \tag{50}$$

and

$$\bigl({}_a\mathfrak{J}^{\lambda,\eta} g^r(\vartheta)\bigr)^{1/r} \leq \left(\frac{\mathcal{A}}{\sigma + \mathcal{A}}\right) \bigl({}_a\mathfrak{J}^{\lambda,\eta} \bigl(g(\vartheta) + h(\vartheta)\bigr)^r\bigr)^{1/r}. \tag{51}$$

Hence, by adding (50) and (51), we get the desired proof. □

Theorem 4.6 *Let $\eta \in (0,1]$, $\lambda \in \mathbb{C}$, $\Re(\lambda) > 0$, $r \geq 1$, and let g, h be two positive functions on $[0,\infty)$ such that ${}_a\mathfrak{J}^{\lambda,\eta}[g(\vartheta)] < \infty$, ${}_a\mathfrak{J}^{\lambda,\eta}[h(\vartheta)] < \infty$. If $0 < m \leq \frac{g(\rho)}{h(\rho)} \leq M$ where $m, M \in \mathbb{R}$ for all $\rho \in [a, \vartheta]$, $\vartheta > a$, then the following inequality for the left generalized proportional fractional integral holds*:

$$\begin{aligned} \frac{1}{M}\bigl({}_a\mathfrak{J}^{\lambda,\eta} g(\vartheta)h(\vartheta)\bigr) &\leq \frac{1}{(m+1)(M+1)}\bigl({}_a\mathfrak{J}^{\lambda,\eta}\bigl(g(\vartheta) + h(\vartheta)\bigr)^2\bigr) \\ &\leq \frac{1}{m}\bigl({}_a\mathfrak{J}^{\lambda,\eta} g(\vartheta)h(\vartheta)\bigr). \end{aligned} \tag{52}$$

Proof Under the given hypothesis, $0 < m \leq \frac{g(\rho)}{h(\rho)} \leq M$, we have

$$h(\rho)(m+1) \leq h(\rho) + g(\rho) \leq h(\rho)(M+1). \tag{53}$$

Also, we have $\frac{1}{M} \leq \frac{h(\rho)}{g(\rho)} \leq \frac{1}{m}$, which gives

$$g(\rho)\left(\frac{M+1}{M}\right) \leq g(\rho) + h(\rho) \leq g(\rho)\left(\frac{m+1}{m}\right). \tag{54}$$

The multiplication of (53) and (54) yields

$$\frac{g(\rho)h(\rho)}{M} \leq \frac{(g(\rho) + h(\rho))^2}{(m+1)(M+1)} \leq \frac{g(\rho)h(\rho)}{m}. \tag{55}$$

Now, multiplying both sides of inequality (55) by $\mathfrak{F}(\vartheta,\rho)$ where $\mathfrak{F}(\vartheta,\rho)$ is defined by (11) and integrating the resultant identity with respect to ρ over (a,ϑ), we have

$$\begin{aligned}
&\frac{1}{M\eta^{\lambda}\Gamma(\lambda)}\int_a^{\vartheta}\exp\biggl[\frac{\eta-1}{\eta}(\vartheta-\rho)\biggr](\vartheta-\rho)^{\lambda-1}g(\rho)h(\rho)\,d\rho\\
&\quad\le\frac{1}{(m+1)(M+1)\eta^{\lambda}\Gamma(\lambda)}\int_a^{\vartheta}\exp\biggl[\frac{\eta-1}{\eta}(\vartheta-\rho)\biggr](\vartheta-\rho)^{\lambda-1}\bigl(g(\rho)+h(\rho)\bigr)^2\,d\rho\\
&\quad\le\frac{1}{m\eta^{\lambda}\Gamma(\lambda)}\int_a^{\vartheta}\exp\biggl[\frac{\eta-1}{\eta}(\vartheta-\rho)\biggr](\vartheta-\rho)^{\lambda-1}g(\rho)h(\rho)\,d\rho.
\end{aligned}\tag{56}$$

It follows that

$$\begin{aligned}
\frac{1}{M}\bigl({}_a\mathfrak{I}^{\lambda,\eta}g(\vartheta)h(\vartheta)\bigr)&\le\frac{1}{(m+1)(M+1)}\bigl({}_a\mathfrak{I}^{\lambda,\eta}\bigl(g(\vartheta)+h(\vartheta)\bigr)^2\bigr)\\
&\le\frac{1}{m}\bigl({}_a\mathfrak{I}^{\lambda,\eta}g(\vartheta)h(\vartheta)\bigr),
\end{aligned}$$

which completes the desired proof. □

Theorem 4.7 *Let $\eta\in(0,1]$, $\lambda\in\mathbb{C}$, $\Re(\lambda)>0$, $r\ge1$, and let g, h be two positive functions on $[0,\infty)$ such that ${}_a\mathfrak{I}^{\lambda,\eta}[g(\vartheta)]<\infty$, ${}_a\mathfrak{I}^{\lambda,\eta}[h(\vartheta)]<\infty$. If $0<m\le\frac{g(\rho)}{h(\rho)}\le M$, where $m,M\in\mathbb{R}$ for all $\rho\in[a,\vartheta]$, $\vartheta>a$, then the following inequality for the left generalized proportional fractional integral holds*:

$$\bigl({}_a\mathfrak{I}^{\lambda,\eta}g^r(\vartheta)\bigr)^{1/r}+\bigl({}_a\mathfrak{I}^{\lambda,\eta}h^r(\vartheta)\bigr)^{1/r}\le2\bigl({}_a\mathfrak{I}^{\lambda,\eta}h^r\bigl(g(\vartheta),h(\vartheta)\bigr)\bigr),\tag{57}$$

where $h(g(\vartheta),h(\vartheta))=\max\{M[(\frac{M}{m}+1)g(\rho)-Mh(\rho)],\frac{(m+M)h(\rho)-g(\rho)}{m}\}$.

Proof Under the given hypothesis $0<m\le\frac{g(\rho)}{h(\rho)}\le M$, where $\rho\in[a,\vartheta]$, $\vartheta>a$, we have

$$0<m\le M+m-\frac{g(\rho)}{h(\rho)}\tag{58}$$

and

$$M+m-\frac{g(\rho)}{h(\rho)}\le M.\tag{59}$$

From (58) and (59), we have

$$h(\rho)<\frac{(M+m)h(\rho)-g(\rho)}{m}\le h\bigl(g(\rho),h(\rho)\bigr),\tag{60}$$

where $h(g(\vartheta),h(\vartheta))=\max\{M[(\frac{M}{m}+1)g(\rho)-Mh(\rho)],\frac{(m+M)h(\rho)-g(\rho)}{m}\}$. Also, from the given hypothesis $0<\frac{1}{M}\le\frac{h(\rho)}{g(\rho)}\le\frac{1}{m}$, we have

$$\frac{1}{M}\le\frac{1}{M}+\frac{1}{m}-\frac{h(\rho)}{g(\rho)}\tag{61}$$

and

$$\frac{1}{M}+\frac{1}{m}-\frac{h(\rho)}{g(\rho)}\le\frac{1}{m}. \tag{62}$$

From (61) and (62), we obtain

$$\frac{1}{M}\le\frac{(\frac{1}{M}+\frac{1}{m})g(\rho)-h(\rho)}{g(\rho)}\le\frac{1}{m}. \tag{63}$$

It follows that

$$\begin{aligned} g(\rho)&=M\left(\frac{1}{M}+\frac{1}{m}\right)g(\rho)-Mh(\rho)\\ &=\frac{M(M+m)g(\rho)-M^2mh(\rho)}{mM}\\ &=\left(\frac{M}{m}+1\right)g(\rho)-Mh(\rho)\\ &=M\left[\left(\frac{M}{m}+1\right)g(\rho)-Mh(\rho)\right]\\ &\le h\bigl(g(\rho),h(\rho)\bigr). \end{aligned} \tag{64}$$

From (60) and (64), we can write

$$g^r(\rho)\le h\bigl(g(\rho),h(\rho)\bigr) \tag{65}$$

and

$$h^r(\rho)\le h^r\bigl(g(\rho),h(\rho)\bigr). \tag{66}$$

Now, multiplying both sides of inequalities (65) and (62) respectively by $\mathfrak{F}(\vartheta,\rho)$ where $\mathfrak{F}(\vartheta,\rho)$ is defined by (11) and integrating the resultant identity with respect to ρ over (a,ϑ), we get

$$\begin{aligned} &\frac{1}{\eta^\lambda\Gamma(\lambda)}\int_a^\vartheta \exp\left[\frac{\eta-1}{\eta}(\vartheta-\rho)\right](\vartheta-\rho)^{\lambda-1}g^r(\rho)\,d\rho\\ &\quad\le\frac{1}{\eta^\lambda\Gamma(\lambda)}\int_a^\vartheta \exp\left[\frac{\eta-1}{\eta}(\vartheta-\rho)\right](\vartheta-\rho)^{\lambda-1}h\bigl(g(\rho),h(\rho)\bigr)\,d\rho. \end{aligned} \tag{67}$$

It follows that

$$\bigl({}_a\mathfrak{I}^{\lambda,\eta}g^r(\vartheta)\bigr)^{1/r}\le\bigl({}_a\mathfrak{I}^{\lambda,\eta}h\bigl(g(\vartheta),h(\vartheta)\bigr)\bigr)^{1/r}. \tag{68}$$

Similarly, from (62), we obtain

$$\bigl({}_a\mathfrak{I}^{\lambda,\eta}h^r(\vartheta)\bigr)^{1/r}\le\bigl({}_a\mathfrak{I}^{\lambda,\eta}h^r\bigl(g(\vartheta),h(\vartheta)\bigr)\bigr)^{1/r}. \tag{69}$$

Hence, by adding (68) and (69), we get the desired proof. □

5 Concluding remarks

In this paper, we presented the Minkowski inequalities and some other related inequalities via generalized nonlocal proportional fractional integral operators. The results exhibited in Sect. 3 generalized the work earlier done by Dahmani [13] for Riemann–Liouville fractional integral operator. Also, the special cases of the results presented in Sect. 3 are found in [40]. The inequalities established in Sect. 4 generalized the inequalities earlier obtained by Suliman [44]. Also, our result will reduce to some classical results which are found in the work of Sroysang [43].

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