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Solution of singular one-dimensional Boussinesq equation by using double conformable Laplace decomposition method

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Abstract

In this study the method which was obtained from a combination of the conformable fractional double Laplace transform method and the Adomian decomposition method has been successfully applied to solve linear and nonlinear singular conformable fractional Boussinesq equations. Two examples are given to illustrate our method. Furthermore, the results show that the proposed method is effective and is easy to use for certain problems in physics and engineering.

Keywords: Conformable single and double Laplace transform; Inverse double Laplace transform; Singular conformable fractional Boussinesq equation

1 Introduction

There are numerous scientific, engineering, and technological processes that can be mathematically modeled by linear and nonlinear Boussinesq equations such as model flows of water in unconfined aquifers. The fractional Boussinesq equations are appropriate for discussing the water propagation through heterogeneous porous media. Many powerful methods have been modified and developed to obtain numerical and analytical solutions of fractional linear differential equations [1]. In [2] conformable fractional derivative was used to obtain the exact analytical solutions for the time fractional variant Boussinesq equations. In [3] the authors discussed the one- and two-dimensional heat diffusion models involving fractional order derivative in time and also considered the fractional orders that include Caputo's and the new fractional conformable derivatives. The conformable Laplace transform was initiated in [4] and studied and modified in [5]. The conformable Laplace transform is not only useful to solve local conformable fractional dynamical systems but also it can be employed to solve systems within nonlocal conformable fractional derivatives that were defined in [6] and used in [7]. Very recently, the authors in [8] defined and studied a more general version of generalized Laplace transforms with its corresponding convolution theory, which can be applied to solve systems of generalized fractional derivatives with a kernel depending on a certain function $g(t)$. In case $g(t) = \frac{(t-a)^\rho}{\rho}$ we can treat the fractional derivatives in [6, 7], and if $g(t) = \frac{t^\rho}{\rho}$ we can treat the so-called Katugampola fractional derivative [9]. Fractional double Laplace transform and its properties [10] and the fractional variational principles beside the semi-inverse technique are applied to derive the space-time fractional Boussinesq equation [11]. The space-time fractional Boussinesq equations in Caputo sense derivatives are studied by using the homotopy perturbation method [12]. The authors in [13] discussed the nonlinear conformable problem by using the $\exp(-\phi(\varepsilon))$-expansion and modified Kudryashov methods. In the present study, the new conformable fractional double Laplace transform decomposition method is recommended for developing the solutions of singular Boussinesq equation. In [14] the authors introduced a new definition of fractional calculus, which is called conformable fractional derivative of order α, $0 < \alpha < 1$, as follows:

$$\frac{d^\alpha}{dt^\alpha} f(t) = \lim_{\epsilon\to 0} \frac{f(t+\epsilon t^{1-\alpha}) - f(t)}{\epsilon}, \quad t > 0, 0 < \alpha \leq 1.$$

Here we briefly recall some definitions from the conformable Laplace transform which are used further in this work.

Definition 1 ([4, 15, 16]) Let $f : [a,\infty) \to R$ and $0 < \beta \leq 1$. Then the fractional Laplace transform of order β is defined by

$$L_t^\beta \left(f\left(\frac{t^\beta}{\beta}\right)\right) = \int_0^\infty e^{-s\frac{t^\beta}{\beta}} f\left(\frac{t^\beta}{\beta}\right) t^{\beta-1}\, dt. \tag{1.1}$$

Definition 2 ([17]) Let $u(x,t)$ be a piecewise continuous function on the interval $[0,\infty) \times [0,\infty)$ of exponential order, consider for some $a, b \in \mathbb{R}$ $\sup x, t > 0$, $\frac{|u(x,t)|}{e^{\frac{ax^\alpha}{\alpha}+\frac{bt^\beta}{\beta}}}$. Under these conditions a conformable double Laplace transform is defined by

$$L_x^\alpha L_t^\beta \left(f\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right)\right) = F_{\alpha,\beta}(p,s) = \int_0^\infty \int_0^\infty e^{-p\frac{x^\alpha}{\alpha} - s\frac{t^\beta}{\beta}} f\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right) t^{\beta-1} x^{\alpha-1}\, dt\, dx, \tag{1.2}$$

where the symbol $L_x^\alpha L_t^\beta$ indicates the conformable double Laplace transform and $p, s \in \mathbb{C}$, $0 < \alpha, \beta \leq 1$.

2 Properties of conformable fractional Laplace transform

The main objective of this section is to study the conformable double Laplace transform using three examples. In addition, we discuss the existence condition of the conformable double Laplace transform.

Example 1 Conformable double Laplace transform of the function $f(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}) = e^{i(a\frac{x^\alpha}{\alpha}+b\frac{t^\beta}{\beta})}$ is denoted by

$$\begin{aligned} L_x^\alpha L_t^\beta \left[e^{i(a\frac{x^\alpha}{\alpha}+b\frac{t^\beta}{\beta})}\right] &= \frac{1}{(p-ai)(s-bi)} \\ &= \frac{(p+ai)(s+bi)}{(p-aui)(s-bi)(p+ai)(s+bi)} \\ &= \frac{ps - ab + (as+pb)i}{(p^2+a^2u^2)(s^2+b^2v^2)}. \end{aligned}$$

Hence,

$$L_x^\alpha L_t^\beta \left[\cos\left(a\frac{x^\alpha}{\alpha} + b\frac{t^\beta}{\beta}\right)\right] = \frac{ps-ab}{(p^2+a^2)(s^2+b^2)},$$

and

$$L_x^\alpha L_t^\beta\left[\sin\left(a\frac{x^\alpha}{\alpha}+b\frac{t^\beta}{\beta}\right)\right]=\frac{as+pb}{(p^2+a^2)(s^2+b^2)}.$$

Example 2 ([17]) By applying the conformable double Laplace transform for the function $f(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta})=(\frac{x^\alpha}{\alpha}\frac{t^\beta}{\beta})^n$, we have

$$L_x^\alpha L_t^\beta\left[\left(\frac{x^\alpha}{\alpha}\frac{t^\beta}{\beta}\right)^n\right]=\frac{(n!)^2}{p^{n+1}s^{n+1}},$$

where n is a positive integer. If the conditions of $a(>-1)$ and $b(>-1)$ are real numbers, then

$$L_x^\alpha L_t^\beta\left[\left(\frac{x^\alpha}{\alpha}\right)^a\left(\frac{t^\beta}{\beta}\right)^b\right]=\frac{\Gamma(a+1)\Gamma(b+1)}{p^{a+1}s^{b+1}},$$

then it follows from Eq. (1.2) that

$$\begin{aligned}L_x^\alpha L_t^\beta\left[\left(\frac{x^\alpha}{\alpha}\right)^a\left(\frac{t^\beta}{\beta}\right)^b\right]&=\int_0^\infty\int_0^\infty e^{-p\frac{x^\alpha}{\alpha}-s\frac{t^\beta}{\beta}}\left(\frac{x^\alpha}{\alpha}\right)^a\left(\frac{t^\beta}{\beta}\right)^b t^{\beta-1}x^{\alpha-1}\,dt\,dx\\&=\int_0^\infty e^{-p\frac{x^\alpha}{\alpha}}\left(\frac{x^\alpha}{\alpha}\right)^a x^{\alpha-1}\left(\int_0^\infty e^{-s\frac{t^\beta}{\beta}}\left(\frac{t^\beta}{\beta}\right)^b t^{\beta-1}\,dt\right)dx.\end{aligned}$$

Let $p\frac{x^\alpha}{\alpha}=\frac{r^\alpha}{\alpha}$ and $s\frac{t^\beta}{\beta}=\frac{q^\beta}{\beta}$, we get

$$\begin{aligned}L_x^\alpha L_t^\beta\left[\left(\frac{x^\alpha}{\alpha}\right)^a\left(\frac{t^\beta}{\beta}\right)^b\right]&=\int_0^\infty\left(\frac{1}{p}\frac{r^\alpha}{\alpha}\right)^a e^{-\frac{r^\alpha}{\alpha}}\frac{dr}{p^\alpha}\int_0^\infty e^{-\frac{q^\beta}{\beta}}\left(\frac{1}{s}\frac{q^\beta}{\beta}\right)^b\frac{1}{s^\beta}\,dq\\&=\frac{1}{p^{a+\alpha}}\frac{1}{s^{b+\beta}}\int_0^\infty\int_0^\infty\left(\frac{r^\alpha}{\alpha}\right)^a\left(\frac{q^\beta}{\beta}\right)^b e^{-\frac{r^\alpha}{\alpha}}e^{-\frac{q^\beta}{\beta}}\,dr\,dq\\&=\frac{\Gamma(a+1)\Gamma(b+1)}{p^{a+\alpha}s^{b+\beta}},\end{aligned}$$

where $p,s\in\mathbb{C}$, $0<\alpha,\beta\leq 1$, gamma functions of a and b are given by

$$\Gamma(a)\Gamma(b)=\int_0^\infty e^{-\frac{r^\alpha}{\alpha}}\left(\frac{r^\alpha}{\alpha}\right)^{a-1}dx\int_0^\infty e^{-\frac{q^\beta}{\beta}}\left(\frac{q^\beta}{\beta}\right)^{b-1}dt,\quad a>0,b>0.$$

Example 3 The conformable double Laplace transform for the function

$$f\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)=H\left(\frac{x^\alpha}{\alpha}\right)\otimes H\left(\frac{t^\beta}{\beta}\right)\ln\frac{x^\alpha}{\alpha}\ln\frac{t^\beta}{\beta}$$

is as follows:

$$\begin{aligned}&L_x^\alpha L_t^\beta\left[H\left(\frac{x^\alpha}{\alpha}\right)\otimes H\left(\frac{t^\beta}{\beta}\right)\ln\frac{x^\alpha}{\alpha}\ln\frac{t^\beta}{\beta}\right]\\&\quad=\int_0^\infty\int_0^\infty e^{-p\frac{x^\alpha}{\alpha}-s\frac{t^\beta}{\beta}}\ln\frac{x^\alpha}{\alpha}\ln\frac{t^\beta}{\beta}t^{\beta-1}x^{\alpha-1}\,dt\,dx,\end{aligned}\tag{2.1}$$

where $H(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}) = H(\frac{x^\alpha}{\alpha}) \otimes H(\frac{t^\beta}{\beta})$ is a Heaviside function and the symbol $\otimes$ indicates the tensor product, see [18]. By substituting $\frac{\zeta^\alpha}{\alpha} = p\frac{x^\alpha}{\alpha}$ and $\frac{\eta^\beta}{\beta} = s\frac{t^\beta}{\beta}$ in Eq. (2.1), we have

$$\begin{aligned} &L_x^\alpha L_t^\beta\left[H\left(\frac{x^\alpha}{\alpha}\right) \otimes H\left(\frac{t^\beta}{\beta}\right) \ln\frac{x^\alpha}{\alpha} \ln\left(\frac{t^\beta}{\beta}\right)\right] \\ &= \frac{1}{ps}\int_0^\infty e^{-\frac{\eta^\beta}{\beta}} \ln\left(\frac{1}{s}\frac{\eta^\beta}{\beta}\right)\left(\int_0^\infty e^{-\frac{\zeta^\alpha}{\alpha}} \ln\left(\frac{1}{p}\frac{\zeta^\alpha}{\alpha}\right)\zeta^{\alpha-1}\,d\zeta\right)\eta^{\beta-1}\,d\eta \\ &= \frac{1}{ps}(\gamma + \ln p)(\gamma + \ln s), \end{aligned} \tag{2.2}$$

where the symbol $\gamma = 0.5772\cdot\ldots$ is Euler's constant.

Existence condition for the conformable double Laplace transform

If $f(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta})$ is an exponential order a and b as $\frac{x^\alpha}{\alpha} \to \infty$, $\frac{t^\beta}{\beta} \to \infty$, if there exists a constant $K > 0$, then for all $\frac{x^\alpha}{\alpha} > X$ and $\frac{t^\beta}{\beta} > T$

$$\left|f\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right)\right| \le Ke^{a\frac{x^\alpha}{\alpha} + b\frac{t^\beta}{\beta}}, \tag{2.3}$$

one can get

$$f\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right) = O\left(e^{a\frac{x^\alpha}{\alpha} + b\frac{t^\beta}{\beta}}\right) \quad \text{as } \frac{x^\alpha}{\alpha} \to \infty, \frac{t^\beta}{\beta} \to \infty,$$

which is equivalent to

$$\lim_{\substack{\frac{x^\alpha}{\alpha}\to\infty \\ \frac{t^\beta}{\beta}\to\infty}} e^{-\mu\frac{x^\alpha}{\alpha} - \eta\frac{t^\beta}{\beta}}\left|f\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right)\right| = K\lim_{\substack{\frac{x^\alpha}{\alpha}\to\infty \\ \frac{t^\beta}{\beta}\to\infty}} e^{-(\mu-a)\frac{x^\alpha}{\alpha} - (\eta-b)\frac{t^\beta}{\beta}} = 0,$$

where $\mu > a$ and $\eta > a$. The function $f(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta})$ does not grow faster than $Ke^{a\frac{x^\alpha}{\alpha} + b\frac{t^\beta}{\beta}}$ as $\frac{x^\alpha}{\alpha} \to \infty$, $\frac{t^\beta}{\beta} \to \infty$.

Theorem 1 *The function $f(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta})$ is defined on $(0, X)$ and $(0, T)$ and of exponential order $e^{a\frac{x^\alpha}{\alpha} + b\frac{t^\beta}{\beta}}$, then the conformable Laplace transform of $f(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta})$ exists for all $\Re(p) > \mu$, $\Re(s) > \eta$.*

Proof From Eq. (1.2), we obtain

$$\begin{aligned} \left|F_{\alpha,\beta}(p,s)\right| &= \left|\int_0^\infty\int_0^\infty e^{-p\frac{x^\alpha}{\alpha} - s\frac{t^\beta}{\beta}} f\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right) t^{\beta-1}x^{\alpha-1}\,dt\,dx\right| \\ &\le K\left|\int_0^\infty\int_0^\infty e^{-(p-a)\frac{x^\alpha}{\alpha} - (s-b)\frac{t^\beta}{\beta}} t^{\beta-1}x^{\alpha-1}\,dt\,dx\right| \\ &= \frac{1}{(p-a)(s-b)}. \end{aligned} \tag{2.4}$$

For $\Re(p) > \mu$, $\Re(s) > \eta$, from Eq. (2.4), we get

$$\lim_{\substack{p\to\infty\\ s\to\infty}} \left|F_{\alpha,\beta}(p,s)\right| = 0 \quad \text{or} \quad \lim_{\substack{p\to\infty\\ s\to\infty}} F_{\alpha,\beta}(p,s) = 0. \qquad \square$$

Theorem 2 *If $f(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta})$ is a periodic function of the periods $\lambda > 0$ and $\mu > 0$ such that $f(\frac{x^\alpha}{\alpha} + \frac{\lambda^\alpha}{\alpha}, \frac{t^\beta}{\beta} + \frac{\mu^\beta}{\beta}) = f(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta})$ for all $\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta} \in [0,\infty)$ if conformable fractional double Laplace of $L_x^\alpha L_t^\beta(f(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta})) = F_{\alpha,\beta}(p,s)$ exists, then*

$$F_{\alpha,\beta}(p,s) = \frac{\int_0^{\frac{\lambda^\alpha}{\alpha}} \int_0^{\frac{\mu^\beta}{\beta}} e^{-p\frac{x^\alpha}{\alpha}-s\frac{t^\beta}{\beta}} f(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}) t^{\beta-1} x^{\alpha-1}\,dt\,dx}{(1 - e^{-p\frac{\lambda^\alpha}{\alpha}-s\frac{\mu^\beta}{\beta}})}.$$

Proof By using the definition of conformable fractional double Laplace transform, we have

$$\begin{aligned} F_{\alpha,\beta}(p,s) &= \int_0^\infty \int_0^\infty e^{-p\frac{x^\alpha}{\alpha}-s\frac{t^\beta}{\beta}} f\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right) t^{\beta-1} x^{\alpha-1}\,dt\,dx \\ &= \int_0^{\frac{\lambda^\alpha}{\alpha}} \int_0^{\frac{\mu^\beta}{\beta}} e^{-p\frac{x^\alpha}{\alpha}-s\frac{t^\beta}{\beta}} f\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right) t^{\beta-1} x^{\alpha-1}\,dt\,dx \\ &\quad + \int_{\frac{\lambda^\alpha}{\alpha}}^\infty \int_{\frac{\mu^\beta}{\beta}}^\infty e^{-p\frac{x^\alpha}{\alpha}-s\frac{t^\beta}{\beta}} f\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right) t^{\beta-1} x^{\alpha-1}\,dt\,dx. \end{aligned}$$

Let $\frac{x^\alpha}{\alpha} = \frac{u^\alpha}{\alpha} + \frac{\lambda^\alpha}{\alpha}$ and $\frac{t^\beta}{\beta} = \frac{v^\beta}{\beta} + \frac{\mu^\beta}{\beta}$ in the second double integral, we get

$$\begin{aligned} F_{\alpha,\beta}(p,s) &= \int_0^{\frac{\lambda^\alpha}{\alpha}} \int_0^{\frac{\mu^\beta}{\beta}} e^{-p\frac{x^\alpha}{\alpha}-s\frac{t^\beta}{\beta}} f\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right) t^{\beta-1} x^{\alpha-1}\,dt\,dx \\ &\quad + \int_0^\infty \int_0^\infty e^{-p(\frac{u^\alpha}{\alpha}+\frac{\lambda^\alpha}{\alpha})-s(\frac{v^\beta}{\beta}+\frac{\mu^\beta}{\beta})} f\left(\frac{u^\alpha}{\alpha} + \frac{\lambda^\alpha}{\alpha}, \frac{v^\beta}{\beta} + \frac{\mu^\beta}{\beta}\right) v^{\beta-1} u^{\alpha-1}\,dv\,du \\ &= \int_0^{\frac{\lambda^\alpha}{\alpha}} \int_0^{\frac{\mu^\beta}{\beta}} e^{-p\frac{x^\alpha}{\alpha}-s\frac{t^\beta}{\beta}} f\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right) t^{\beta-1} x^{\alpha-1}\,dt\,dx \\ &\quad + e^{-p\frac{\lambda^\alpha}{\alpha}-s\frac{\mu^\beta}{\beta}} \int_0^\infty \int_0^\infty e^{-p\frac{u^\alpha}{\alpha}-s(\frac{v^\beta}{\beta}+\frac{\mu^\beta}{\beta})} f\left(\frac{u^\alpha}{\alpha}, \frac{v^\beta}{\beta}\right) v^{\beta-1} u^{\alpha-1}\,dv\,du. \end{aligned}$$

Consequently,

$$F_{\alpha,\beta}(p,s) = \int_0^{\frac{\lambda^\alpha}{\alpha}} \int_0^{\frac{\mu^\beta}{\beta}} e^{-p\frac{x^\alpha}{\alpha}-s\frac{t^\beta}{\beta}} f\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right) t^{\beta-1} x^{\alpha-1}\,dt\,dx + e^{-p\frac{\lambda^\alpha}{\alpha}-s\frac{\mu^\beta}{\beta}} F_{\alpha,\beta}(p,s),$$

we have

$$F_{\alpha,\beta}(p,s) = \frac{\int_0^{\frac{\lambda^\alpha}{\alpha}} \int_0^{\frac{\mu^\beta}{\beta}} e^{-p\frac{x^\alpha}{\alpha}-s\frac{t^\beta}{\beta}} f(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}) t^{\beta-1} x^{\alpha-1}\,dt\,dx}{1 - e^{-p\frac{\lambda^\alpha}{\alpha}-s\frac{\mu^\beta}{\beta}}}. \qquad \square$$

3 The conformable Laplace transform of the convolution product

Theorem 3 *Let $f(\frac{t^\beta}{\beta})$ and $g(\frac{t^\beta}{\beta})$ be integrable functions. If the convolution of $f(\frac{t^\beta}{\beta})$ and $g(\frac{t^\beta}{\beta})$ is denoted by*

$$(f*g)\left(\frac{t^\beta}{\beta}\right)=\int_0^{\frac{t^\beta}{\beta}} f\left(\frac{x^\beta}{\beta}\right)g\left(\frac{t^\beta}{\beta}-\frac{x^\beta}{\beta}\right)x^{\beta-1}\,dx, \tag{3.1}$$

then the conformable fractional Laplace transform of the convolution product is defined as follows:

$$L_x^\alpha L_t^\beta\left[f(t)*g\left(\frac{t^\beta}{\beta}\right)\right]=F_\beta(s)G_\beta(s), \tag{3.2}$$

where symbols $F_\beta(s)$ and $G_\beta(s)$ indicate the conformable fractional Laplace transforms of $f(\frac{t^\beta}{\beta})$ and $g(\frac{t^\beta}{\beta})$ respectively.

Theorem 4 *Let $f(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta})$ and $g(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta})$ have a conformable fractional double Laplace transform. Then the conformable fractional double Laplace transform of the double convolution of $f(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta})$ and $g(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta})$ is*

$$(f**g)\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)=\int_0^{\frac{x^\alpha}{\alpha}}\int_0^{\frac{t^\beta}{\beta}} f\left(\frac{\zeta^\alpha}{\alpha},\frac{\eta^\beta}{\beta}\right)g\left(\frac{x^\alpha}{\alpha}-\frac{\zeta^\alpha}{\alpha},\frac{t^\beta}{\beta}-\frac{\eta^\beta}{\beta}\right)\zeta^{\alpha-1}\eta^{\beta-1}\,d\zeta\,d\eta. \tag{3.3}$$

Therefore one has the equality

$$L_x^\alpha L_t^\beta\left[(f**g)\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)\right]=F_{\alpha,\beta}(p,s)G_{\alpha,\beta}(p,s), \tag{3.4}$$

where $F_{\alpha,\beta}(p,s)$ and $G_{\alpha,\beta}(p,s)$ indicate the conformable fractional double Laplace transforms of $f(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta})$ and $g(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta})$ respectively.

Proof If we apply the definition of conformable fractional double Laplace transform and fractional double convolution above, then we obtain

$$\begin{aligned}
&L_x^\alpha L_t^\beta\left[(f**g)\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)\right]\\
&=\int_0^\infty\int_0^\infty e^{-p\frac{x^\alpha}{\alpha}-s\frac{t^\beta}{\beta}}(f**g)\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)t^{\beta-1}x^{\alpha-1}\,dt\,dx\\
&=\int_0^\infty\int_0^\infty e^{-p\frac{x^\alpha}{\alpha}-s\frac{t^\beta}{\beta}}\left(\int_0^{\frac{x^\alpha}{\alpha}}\int_0^{\frac{t^\beta}{\beta}} f\left(\frac{\zeta^\alpha}{\alpha},\frac{\eta^\beta}{\beta}\right)g\left(\frac{x^\alpha}{\alpha}-\frac{\zeta^\alpha}{\alpha},\frac{t^\beta}{\beta}-\frac{\eta^\beta}{\beta}\right)\zeta^{\alpha-1}\eta^{\beta-1}\,d\zeta\,d\eta\right)\\
&\quad\times t^{\beta-1}x^{\alpha-1}\,dt\,dx.
\end{aligned}$$

Let $\frac{\mu^\alpha}{\alpha} = \frac{x^\alpha}{\alpha} - \frac{\zeta^\alpha}{\alpha}$ and $\frac{\nu^\beta}{\beta} = \frac{t^\beta}{\beta} - \frac{\eta^\beta}{\beta}$, then $x^{\alpha-1}\,dx = \mu^{\alpha-1}\,d\mu$, $t^{\beta-1}\,dt = \nu^{\beta-1}\,d\nu$, and applying the valid extension of upper bound of integrals to $\frac{x^\alpha}{\alpha} \to \infty$ and $\frac{t^\beta}{\beta} \to \infty$, we get

$$L_x^\alpha L_t^\beta\left[(f**g)\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)\right] = \left(\int_0^\infty\int_0^\infty e^{-p\frac{\zeta^\alpha}{\alpha}-s\frac{\eta^\beta}{\beta}} f\left(\frac{\zeta^\alpha}{\alpha},\frac{\eta^\beta}{\beta}\right)\zeta^{\alpha-1}\eta^{\beta-1}\,d\zeta\,d\eta\right) \times\left(\int_{-\frac{\zeta^\alpha}{\alpha}}^\infty\int_{-\frac{\eta^\beta}{\beta}}^\infty g\left(\frac{\mu^\alpha}{\alpha},\frac{\nu^\beta}{\beta}\right)\nu^{\beta-1}\mu^{\alpha-1}\,d\nu\,d\mu\right).$$

Both functions $f(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta})$ and $g(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta})$ have zero value for $\frac{t^\beta}{\beta} < 0$ and $\frac{x^\alpha}{\alpha} < 0$, it follows with respect to the lower limit of integrations that

$$L_x^\alpha L_t^\beta\left[(f**g)\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)\right] = \left(\int_0^\infty\int_0^\infty e^{-p\frac{\zeta^\alpha}{\alpha}-s\frac{\eta^\beta}{\beta}} f\left(\frac{\zeta^\alpha}{\alpha},\frac{\eta^\beta}{\beta}\right)\zeta^{\alpha-1}\eta^{\beta-1}\,d\zeta\,d\eta\right) \times\left(\int_{-\frac{\zeta^\alpha}{\alpha}}^\infty\int_{-\frac{\eta^\beta}{\beta}}^\infty g\left(\frac{\mu^\alpha}{\alpha},\frac{\nu^\beta}{\beta}\right)\nu^{\beta-1}\mu^{\alpha-1}\,d\nu\,d\mu\right).$$

Therefore,

$$L_x^\alpha L_t^\beta\left[(f**g)\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)\right] = F_{\alpha,\beta}(p,s)G_{\alpha,\beta}(p,s). \qquad \square$$

If the conformable fractional double Laplace transform of the function $f(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta})$ is given by $L_x^\alpha L_t^\beta[u(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta})] = U_{\alpha,\beta}(p,s)$, then the conformable fractional double Laplace transforms of $\frac{\partial^\alpha u}{\partial x^\alpha}$, $\frac{\partial^{2\alpha} u}{\partial x^{2\alpha}}$, $\frac{\partial^\beta u(x,t)}{\partial t^\beta}$, and $\frac{\partial^{2\beta} u(x,t)}{\partial t^{2\beta}}$ are given by

$$L_x^\alpha L_t^\beta\left[\frac{\partial^\alpha u}{\partial x^\alpha}\right] = pU_{\alpha,\beta}(p,s) - U_\beta(0,s), \tag{3.5}$$

$$L_x^\alpha L_t^\beta\left(\frac{\partial^{2\alpha} u}{\partial x^{2\alpha}}\right) = p^2U_{\alpha,\beta}(p,s) - pU_\beta(0,s) - L_t^\beta\left(\frac{\partial^\alpha}{\partial x^\alpha}u\left(0,\frac{t^\beta}{\beta}\right)\right) \tag{3.6}$$

and

$$L_x^\alpha L_t^\beta\left(\frac{\partial^\beta u}{\partial t^\beta}\right) = sU_{\alpha,\beta}(p,s) - U_\alpha(p,0), \tag{3.7}$$

$$L_x^\alpha L_t^\beta\left(\frac{\partial^{2\beta} u}{\partial t^{2\beta}}\right) = s^2U_{\alpha,\beta}(p,s) - sU_\alpha(p,0) - L_x^\alpha\left(\frac{\partial^\beta}{\partial t^\beta}u\left(\frac{x^\alpha}{\alpha},0\right)\right). \tag{3.8}$$

Next, we generalize the conformable fractional double Laplace transform for m, n times conformable fractional derivatives.

Theorem 5 *Let $0 < \alpha,\beta \le 1$ and $m,n \in \mathbb{N}$ such that $u(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}) \in C^l(\mathbb{R}^+\times\mathbb{R}^+)$, $l = \max(m,n)$. Also, let the conformable fractional Laplace transforms of the functions $u(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta})$, $\frac{\partial^{m\alpha} u}{\partial x^{m\alpha}}$, and*

$\frac{\partial^{n\beta}u}{\partial t^{n\beta}}$. *Then*

$$L_x^\alpha L_t^\beta\left(\frac{\partial^{m\alpha}}{\partial x^{m\alpha}}u\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)\right)$$
$$=p^m U_{\alpha,\beta}(p,s)-p^{m-1}U_\beta(0,s)-\sum_{i=1}^{m-1}p^{m-1-i}L_t^\beta\left(\frac{\partial^{i\alpha}}{\partial x^{i\alpha}}u\left(0,\frac{t^\beta}{\beta}\right)\right),$$
$$L_x^\alpha L_t^\beta\left(\frac{\partial^{n\beta}}{\partial t^{n\beta}}u\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)\right)$$
$$=s^n U_{\alpha,\beta}(p,s)-s^{n-1}U_\alpha(p,0)-\sum_{j=1}^{n-1}s^{n-1-j}L_x^\alpha\left(\frac{\partial^{j\beta}}{\partial t^{j\beta}}u\left(\frac{x^\alpha}{\alpha},0\right)\right),$$

where $\frac{\partial^{m\alpha}u}{\partial x^{m\alpha}}$ *and* $\frac{\partial^{n\beta}v}{\partial t^{n\beta}}$ *denote m, n times conformable fractional derivatives of function* $u(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta})$ *with order b and a respectively.*

The conformable fractional double Laplace transforms of functions $\frac{x^\alpha}{\alpha}f(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta})$ and $(\frac{x^\alpha}{\alpha})^n\frac{\partial^\beta f}{\partial t^\beta}$ are given by

$$(-1)^n\frac{d^n}{dp^n}\left(L_x^\alpha L_t^\beta\left[f\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)\right]\right)=L_x^\alpha L_t^\beta\left[\left(\frac{x^\alpha}{\alpha}\right)^n f\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)\right], \tag{3.9}$$
$$(-1)^n\frac{d^n}{dp^n}\left(L_x^\alpha L_t^\beta\left[\frac{\partial^\beta f}{\partial t^\beta}\right]\right)=L_x^\alpha L_t^\beta\left[\left(\frac{x^\alpha}{\alpha}\right)^n\frac{\partial^\beta f}{\partial t^\beta}\right]. \tag{3.10}$$

4 Conformable double Laplace transform decomposition method applied to singular Boussinesq equation

The aim of this section is to discuss the use of the conformable double Laplace transform decomposition method (CFDLDM) for the linear and nonlinear singular one-dimensional Boussinesq equations. In this work, we define the conformable double Laplace transform of the function $u(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta})$ by $U_{\alpha,\beta}(p,s)$. We suggest here two important problems.

First problem: Consider the linear Boussinesq equation with initial conditions in the following form:

$$\frac{\partial^{2\beta}u}{\partial t^{2\beta}}-\frac{\alpha}{x^\alpha}\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha u}{\partial x^\alpha}\right)+a\left(\frac{x^\alpha}{\alpha}\right)\frac{\partial^{4\alpha}u}{\partial x^{4\alpha}}+b\left(\frac{x^\alpha}{\alpha}\right)\frac{\partial^{2\alpha+2\beta}u}{\partial x^{2\alpha}\partial t^{2\beta}}=f\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right), \tag{4.1}$$

subject to

$$u\left(\frac{x^\alpha}{\alpha},0\right)=f_1\left(\frac{x^\alpha}{\alpha}\right),\qquad \frac{\partial^\beta u(\frac{x^\alpha}{\alpha},0)}{\partial t^\beta}=f_2\left(\frac{x^\alpha}{\alpha}\right), \tag{4.2}$$

where the functions $f(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta})$, $f_1(\frac{x^\alpha}{\alpha})$, $f_2(\frac{x^\alpha}{\alpha})$, $a(\frac{x^\alpha}{\alpha})$, and $b(\frac{x^\alpha}{\alpha})$ are given. Firstly, multiply Eq. (4.1) by $\frac{x^\alpha}{\alpha}$, and then using the properties of partial derivative of the conformable fractional double Laplace transform and single conformable fractional transform for Eq. (4.1) and Eq. (4.2) respectively and using Eq. (3.9) and Eq. (3.10), we obtain

$$-\frac{d}{dp}\left[s^2U_{\alpha,\beta}(p,s)-sU_\alpha(p,0)-\frac{\partial^\beta u(p,0)}{\partial t^\beta}\right]$$
$$=L_x^\alpha L_t^\beta[\Phi]-\frac{d}{dp}\left[L_x^\alpha L_t^\beta\left(f\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)\right)\right], \tag{4.3}$$

where the conformable Laplace transforms of $u(\frac{x^\alpha}{\alpha},0)$ and $\frac{\partial^\beta u(\frac{x^\alpha}{\alpha},0)}{\partial t^\beta}$ are denoted by $U_\alpha(p,0)=F_1(p,0)$, $\frac{\partial^\beta u(p,0)}{\partial t^\beta}=F_2(p,0)$ respectively and

$$\Phi=\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha u}{\partial x^\alpha}\right)-\frac{x^\alpha}{\alpha}a\left(\frac{x^\alpha}{\alpha}\right)\frac{\partial^\alpha u}{\partial x^\alpha}-\frac{x^\alpha}{\alpha}b\left(\frac{x^\alpha}{\alpha}\right)\frac{\partial^\alpha u}{\partial x^\alpha}.$$

Arranging Eq. (4.3), we get

$$\frac{d}{dp}\left[U_{\alpha,\beta}(p,s)\right]=\frac{1}{s}\frac{d}{dp}F_1(p,0)+\frac{1}{s^2}\frac{d}{dp}F_2(p,0)-\frac{1}{s^2}L_x^\alpha L_t^\beta[\Phi]+\frac{1}{s^2}\frac{d}{dp}F_{\alpha,\beta}(p,s). \tag{4.4}$$

By applying the integral for both sides of Eq. (4.4), from 0 to p with respect to p, we have

$$U_{\alpha,\beta}(p,s)=\frac{F_1(p,0)}{s}+\frac{F_2(p,0)}{s^2}-\frac{1}{s^2}\int_0^p L_x^\alpha L_t^\beta[\Phi]\,dp$$
$$+\frac{1}{s^2}F_{\alpha,\beta}(p,s), \tag{4.5}$$

where $F_{\alpha,\beta}(p,s)$, $F_1(p,0)$, and $F_2(p,0)$ are conformable fractional Laplace transforms of the functions $f(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta})$, $f_1(\frac{x^\alpha}{\alpha})$, and $f_2(\frac{x^\alpha}{\alpha})$ respectively. The solution is obtained by taking the inverse conformable fractional double Laplace transform for Eq. (4.5)

$$u\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)=f_1\left(\frac{x^\alpha}{\alpha}\right)+\frac{t^\beta}{\beta}f_2\left(\frac{x^\alpha}{\alpha}\right)+L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}F_{\alpha,\beta}(p,s)\right]$$
$$-L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x^\alpha L_t^\beta[\Phi]\,dp\right], \tag{4.6}$$

where $L_p^{-1}L_s^{-1}$ indicates double inverse conformable fractional double Laplace transform.

The conformable fractional double Laplace transform decomposition method (CFDLDM) defines the solutions $u(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta})$ with the help of infinite series as follows:

$$u\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)=\sum_{n=0}^{\infty}u_n\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right). \tag{4.7}$$

By substituting Eq. (4.7) into Eq. (4.6), we obtain

$$\sum_{n=0}^{\infty}u_n\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)$$
$$=f_1\left(\frac{x^\alpha}{\alpha}\right)+\frac{t^\beta}{\beta}f_2\left(\frac{x^\alpha}{\alpha}\right)+L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}F_{\alpha,\beta}(p,s)\right]$$

$$
\begin{aligned}
&-L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x^\alpha L_t^\beta\left[\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}\left(\sum_{n=0}^\infty u_n\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)\right)\right)\right]dp\right]\\
&+L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x^\alpha L_t^\beta\left[\left(\frac{x^\alpha}{\alpha}a\left(\frac{x^\alpha}{\alpha}\right)\frac{\partial^\alpha}{\partial x^\alpha}\left(\sum_{n=0}^\infty u_n\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)\right)\right)\right]dp\right]\\
&+L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x^\alpha L_t^\beta\left[\left(\frac{x^\alpha}{\alpha}b\left(\frac{x^\alpha}{\alpha}\right)\frac{\partial^\alpha}{\partial x^\alpha}\left(\sum_{n=0}^\infty u_n\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)\right)\right)\right]dp\right].
\end{aligned}
\tag{4.8}
$$

By comparing both sides of Eq. (4.6), we get

$$
\begin{aligned}
u_0\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)&=f_1\left(\frac{x^\alpha}{\alpha}\right)+\frac{t^\beta}{\beta}f_2\left(\frac{x^\alpha}{\alpha}\right)+L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}F_{\alpha,\beta}(p,s)\right],\\
u_1\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)&=-L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x^\alpha L_t^\beta\left[\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha u_0}{\partial x^\alpha}\right)\right]dp\right]\\
&\quad+L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x^\alpha L_t^\beta\left[\left(\frac{x^\alpha}{\alpha}a\left(\frac{x^\alpha}{\alpha}\right)\frac{\partial^\alpha u_0}{\partial x^\alpha}\right.\right.\right.\\
&\quad\left.\left.\left.+\frac{x^\alpha}{\alpha}b\left(\frac{x^\alpha}{\alpha}\right)\frac{\partial^\alpha u_0}{\partial x^\alpha}\right)\right]dp\right].
\end{aligned}
\tag{4.9}
$$

In general, the remaining terms are given by

$$
\begin{aligned}
&u_n\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)\\
&=-L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x^\alpha L_t^\beta\left[\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}\left(\sum_{n=1}^\infty u_{n-1}\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)\right)\right)\right]dp\right]\\
&\quad+L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x^\alpha L_t^\beta\left[\left(\frac{x^\alpha}{\alpha}a\left(\frac{x^\alpha}{\alpha}\right)\frac{\partial^\alpha}{\partial x^\alpha}\left(\sum_{n=1}^\infty u_{n-1}\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)\right)\right)\right]dp\right]\\
&\quad+L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x^\alpha L_t^\beta\left[\left(\frac{x^\alpha}{\alpha}b\left(\frac{x^\alpha}{\alpha}\right)\frac{\partial^\alpha}{\partial x^\alpha}\left(\sum_{n=1}^\infty u_{n-1}\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)\right)\right)\right]dp\right],
\end{aligned}
\tag{4.10}
$$

where the inverse conformable fractional double Laplace transform is given by $L_p^{-1}L_s^{-1}$. To explain this method for solving the conformable fractional Boussinesq equation, we let $a(\frac{x^\alpha}{\alpha})=1$, $b(\frac{x^\alpha}{\alpha})=-1$, and $f(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta})=-(\frac{x^\alpha}{\alpha})^2\sin\frac{t^\beta}{\beta}-2\sin\frac{t^\beta}{\beta}$, in Eq. (4.1), we obtain the following example.

Example 4 Consider a singular conformable fractional Boussinesq equation in one dimension

$$
\frac{\partial^{2\beta}u}{\partial t^{2\beta}}-\frac{\alpha}{x^\alpha}\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha u}{\partial x^\alpha}\right)+\frac{\partial^{4\alpha}u}{\partial x^{4\alpha}}-\frac{\partial^{2\alpha+2\beta}u}{\partial x^{2\alpha}\partial t^{2\beta}}=-\left(\frac{x^\alpha}{\alpha}\right)^2\sin\left(\frac{t^\beta}{\beta}\right)-2\sin\left(\frac{t^\beta}{\beta}\right),
\tag{4.11}
$$

subject to the initial condition

$$
u\left(\frac{x^\alpha}{\alpha},0\right)=0,\qquad\frac{\partial u(\frac{x^\alpha}{\alpha},0)}{\partial t}=\left(\frac{x^\alpha}{\alpha}\right)^2.
\tag{4.12}
$$

Multiplying both sides of Eq. (4.11) by $\frac{x^\alpha}{\alpha}$ and applying the definition of partial derivatives of the conformable fractional double Laplace transform, single conformable fractional transform for Eq. (4.12) respectively, we have

$$\begin{aligned}\frac{dU_{\alpha,\beta}(p,s)}{dp} &= -\frac{1}{s^2}L_x^\alpha L_t^\beta\left[\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha u}{\partial x^\alpha}\right) - \frac{x^\alpha}{\alpha}\frac{\partial^{4\alpha} u}{\partial x^{4\alpha}} + \frac{x^\alpha}{\alpha}\frac{\partial^{2\alpha+2\beta} u}{\partial x^{2\alpha}\partial t^{2\beta}}\right] \\ &\quad + \frac{3!}{p^4 s^2(s^2+1)} + \frac{1}{p^2 s^2(s^2+1)} - \frac{6}{p^4 s^2}. \qquad (4.13)\end{aligned}$$

Applying the integral for Eq. (4.13), from 0 to p with respect to p, we get

$$\begin{aligned}U_{\alpha,\beta}(p,s) &= -\frac{1}{s^2}\int_0^p L_x^\alpha L_t^\beta\left[\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha u}{\partial x^\alpha}\right) - \frac{x^\alpha}{\alpha}\frac{\partial^{4\alpha} u}{\partial x^{4\alpha}} + \frac{x^\alpha}{\alpha}\frac{\partial^{2\alpha+2\beta} u}{\partial x^{2\alpha}\partial t^{2\beta}}\right]dp \\ &\quad - \frac{2}{p^3 s^2(s^2+1)} - \frac{2}{p s^2(s^2+1)} + \frac{2!}{p^3 s^2}. \qquad (4.14)\end{aligned}$$

By implementing the inverse Laplace transform on both sides of Eq. (4.14), we have

$$\begin{aligned}u\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right) &= -L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x^\alpha L_t^\beta\left[\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha u}{\partial x^\alpha}\right) - \frac{x^\alpha}{\alpha}\frac{\partial^{4\alpha} u}{\partial x^{4\alpha}} + \frac{\partial^{2\alpha+2\beta} u}{\partial x^{2\alpha}\partial t^{2\beta}}\right]dp\right] \\ &\quad + \left(\frac{x^\alpha}{\alpha}\right)^2\sin\left(\frac{t^\beta}{\beta}\right) + 2\sin\left(\frac{t^\beta}{\beta}\right) - 2\left(\frac{t^\beta}{\beta}\right). \qquad (4.15)\end{aligned}$$

Substituting Eq. (4.7) into Eq. (4.15), we have

$$\begin{aligned}\sum_{n=0}^{\infty}u_n\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right) &= -L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x^\alpha L_t^\beta\left[\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}\sum_{n=0}^{\infty}u_n\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)\right)\right]dp\right] \\ &\quad + L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x^\alpha L_t^\beta\left[\frac{x^\alpha}{\alpha}\frac{\partial^{4\alpha}}{\partial x^{4\alpha}}\left(\sum_{n=0}^{\infty}u_n\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)\right)\right]dp\right] \\ &\quad - L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x^\alpha L_t^\beta\left[\frac{x^\alpha}{\alpha}\frac{\partial^{2\alpha+2\beta}}{\partial x^{2\alpha}\partial t^{2\beta}}\left(\sum_{n=0}^{\infty}u_n\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)\right)\right]dp\right] \\ &\quad + \left(\frac{x^\alpha}{\alpha}\right)^2\sin\left(\frac{t^\beta}{\beta}\right) + 2\sin\left(\frac{t^\beta}{\beta}\right) - 2\left(\frac{t^\beta}{\beta}\right). \qquad (4.16)\end{aligned}$$

By applying the conformable fractional double Laplace transform decomposition method, we obtain

$$u_0 = \left(\frac{x^\alpha}{\alpha}\right)^2\sin\left(\frac{t^\beta}{\beta}\right) + 2\sin\left(\frac{t^\beta}{\beta}\right) - 2\left(\frac{t^\beta}{\beta}\right)$$

and

$$\begin{aligned}&u_{n+1}\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right) \\ &\quad = -L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x^\alpha L_t^\beta\left[\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}\sum_{n=0}^{\infty}u_n\left(\frac{x^\alpha}{\alpha},t\right)\right)\right]dp\right]\end{aligned}$$

$$
\begin{aligned}
&+ L_p^{-1} L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x^{\alpha} L_t^{\beta}\left[\frac{x^{\alpha}}{\alpha}\frac{\partial^{4\alpha}}{\partial x^{4\alpha}}\left(\sum_{n=0}^{\infty} u_n\left(\frac{x^{\alpha}}{\alpha},\frac{t^{\beta}}{\beta}\right)\right)\right]dp\right] \\
&- L_p^{-1} L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x^{\alpha} L_t^{\beta}\left[\frac{x^{\alpha}}{\alpha}\frac{\partial^{2\alpha+2\beta}}{\partial x^{2\alpha}\partial t^{2\beta}}\left(\sum_{n=0}^{\infty} u_n\left(\frac{x^{\alpha}}{\alpha},\frac{t^{\beta}}{\beta}\right)\right)\right]dp\right],
\end{aligned}
\tag{4.17}
$$

hence

$$
\begin{aligned}
u_1 &= -L_p^{-1} L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x^{\alpha} L_t^{\beta}\left[\frac{\partial^{\alpha}}{\partial x^{\alpha}}\left(\frac{x^{\alpha}}{\alpha}\frac{\partial^{\alpha}}{\partial x^{\alpha}}u_0\left(\frac{x^{\alpha}}{\alpha},\frac{t^{\beta}}{\beta}\right)\right) - \frac{x^{\alpha}}{\alpha}\frac{\partial^{4\alpha}}{\partial x^{4\alpha}}u_0\left(\frac{x^{\alpha}}{\alpha},\frac{t^{\beta}}{\beta}\right)\right]dp\right] \\
&\quad - L_p^{-1} L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x^{\alpha} L_t^{\beta}\left[\frac{x^{\alpha}}{\alpha}\frac{\partial^{2\alpha+2\beta}}{\partial x^{2\alpha}\partial t^{2\beta}}u_0\left(\frac{x^{\alpha}}{\alpha},\frac{t^{\beta}}{\beta}\right)\right]dp\right], \\
u_1 &= -L_p^{-1} L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x^{\alpha} L_t^{\beta}\left[2\left(\frac{x^{\alpha}}{\alpha}\right)\sin\left(\frac{t^{\beta}}{\beta}\right)\right]dp\right] = -L_p^{-1} L_s^{-1}\left[\frac{-2}{p^2 s^2 (s^2+1)}\right], \\
&= 2\left(\frac{t^{\beta}}{\beta}\right) - 2\sin\left(\frac{t^{\beta}}{\beta}\right)
\end{aligned}
$$

and

$$
u_2 = -L_p^{-1} L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x^{\alpha} L_t^{\beta}[0]\,dp\right] = 0.
$$

By applying Eq. (4.7), we get

$$
\begin{aligned}
\sum_{n=0}^{\infty} u_n\left(\frac{x^{\alpha}}{\alpha},t\right) &= u_0 + u_1 + u_2 + \cdots \\
&= \left(\frac{x^{\alpha}}{\alpha}\right)^2 \sin\left(\frac{t^{\beta}}{\beta}\right) + 2\sin\left(\frac{t^{\beta}}{\beta}\right) \\
&\quad - 2\left(\frac{t^{\beta}}{\beta}\right) + 2\left(\frac{t^{\beta}}{\beta}\right) - 2\sin\left(\frac{t^{\beta}}{\beta}\right) + 0.
\end{aligned}
$$

Therefore, the solution is denoted by

$$
u\left(\frac{x^{\alpha}}{\alpha},\frac{t^{\beta}}{\beta}\right) = \left(\frac{x^{\alpha}}{\alpha}\right)^2 \sin\left(\frac{t^{\beta}}{\beta}\right).
$$

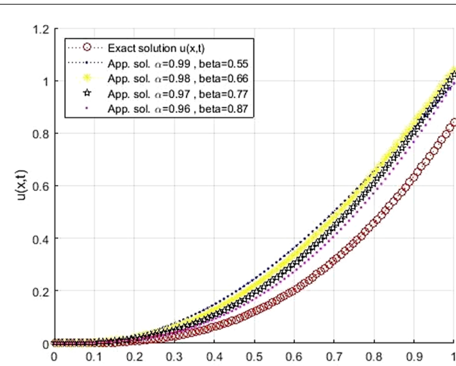


Figure 1 The exact and approximate solutions of u(x,t), for Example 4, we take different values of fractional orders of α and β: $\alpha = 0.99$, $\beta = 0.55$, $\alpha = 0.98$, $\beta = 0.66$, $\alpha = 0.97$, $\beta = 0.77$, and $\alpha = 0.96$, $\beta = 0.87$

Second problem: Consider the following general form of the nonlinear singular Boussinesq equation in one dimension:

$$\begin{aligned} &\frac{\partial^{2\beta} u}{\partial t^{2\beta}} - \frac{\alpha}{x^\alpha}\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha u}{\partial x^\alpha}\right) + a\left(\frac{x^\alpha}{\alpha}\right)\frac{\partial^{4\alpha} u}{\partial x^{4\alpha}} \\ &\quad - b\left(\frac{x^\alpha}{\alpha}\right)\frac{\partial^{2\alpha+2\beta} u}{\partial x^{2\alpha}\partial t^{2\beta}} + c\left(\frac{x^\alpha}{\alpha}\right)\frac{\partial^\beta u}{\partial t^\beta}\frac{\partial^{2\alpha} u}{\partial x^{2\alpha}} + d\left(\frac{x^\alpha}{\alpha}\right)\frac{\partial^\alpha u}{\partial x^\alpha}\frac{\partial^{\alpha+\beta} u}{\partial x^\alpha \partial t^\beta} \\ &\quad = f\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right), \end{aligned} \tag{4.18}$$

with the initial condition

$$u\left(\frac{x^\alpha}{\alpha}, 0\right) = g_1\left(\frac{x^\alpha}{\alpha}\right), \qquad \frac{\partial^\beta u(\frac{x^\alpha}{\alpha}, 0)}{\partial t^\beta} = g_2\left(\frac{x^\alpha}{\alpha}\right), \tag{4.19}$$

where the functions $a(\frac{x^\alpha}{\alpha})$, $b(\frac{x^\alpha}{\alpha})$, $c(\frac{x^\alpha}{\alpha})$, and $d(\frac{x^\alpha}{\alpha})$ are arbitrary. In order to obtain the solution of Eq. (4.18), first, by multiplying Eq. (4.18) by $\frac{x^\alpha}{\alpha}$ and taking conformable fractional double Laplace transform, we have

$$\begin{aligned} &L_x^\alpha L_t^\beta\left[\frac{x^\alpha}{\alpha}\frac{\partial^{2\beta} u}{\partial t^{2\beta}}\right] \\ &\quad = -L_x^\alpha L_t^\beta\left[-\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha u}{\partial x^\alpha}\right) + \frac{x^\alpha}{\alpha} a\left(\frac{x^\alpha}{\alpha}\right)\frac{\partial^{4\alpha} u}{\partial x^{4\alpha}} - \frac{x^\alpha}{\alpha} b\left(\frac{x^\alpha}{\alpha}\right)\frac{\partial^{2\alpha+2\beta} u}{\partial x^{2\alpha}\partial t^{2\beta}}\right] \\ &\qquad - L_x^\alpha L_t^\beta\left[\frac{x^\alpha}{\alpha} c\left(\frac{x^\alpha}{\alpha}\right)\frac{\partial^\beta u}{\partial t^\beta}\frac{\partial^{2\alpha} u}{\partial x^{2\alpha}} + \frac{x^\alpha}{\alpha} d\left(\frac{x^\alpha}{\alpha}\right)\frac{\partial^\alpha u}{\partial x^\alpha}\frac{\partial^{\alpha+\beta} u}{\partial x^\alpha \partial t^\beta}\right] \\ &\qquad + L_x^\alpha L_t^\beta\left[\frac{x^\alpha}{\alpha} f\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right)\right]. \end{aligned} \tag{4.20}$$

Second, applying Eq. (3.9), Eq. (3.10), Eq. (3.8) and the initial condition given in Eq. (4.19), one can get that

$$\frac{d}{dp}\left[U_{\alpha,\beta}(p,s)\right] = \frac{1}{s}\frac{d}{dp} g_1(p,0) + \frac{1}{s^2}\frac{d}{dp} g_2(p,0) - \frac{1}{s^2} L_{\frac{x^\alpha}{\alpha}}^\alpha L_t^\beta[\Psi] + \frac{1}{s^2}\frac{d}{dp} F_{\alpha,\beta}(p,s), \tag{4.21}$$

where

$$\begin{aligned} \Psi &= \frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha u}{\partial x^\alpha}\right) - \frac{x^\alpha}{\alpha} a\left(\frac{x^\alpha}{\alpha}\right)\frac{\partial^{4\alpha} u}{\partial x^{4\alpha}} + \frac{x^\alpha}{\alpha} b\left(\frac{x^\alpha}{\alpha}\right)\frac{\partial^{2\alpha+2\beta} u}{\partial x^{2\alpha}\partial t^{2\beta}} \\ &\quad - \frac{x^\alpha}{\alpha} c\left(\frac{x^\alpha}{\alpha}\right)\frac{\partial^\beta u}{\partial t^\beta}\frac{\partial^{2\alpha} u}{\partial x^{2\alpha}} - \frac{x^\alpha}{\alpha} d\left(\frac{x^\alpha}{\alpha}\right)\frac{\partial^\alpha u}{\partial x^\alpha}\frac{\partial^{\alpha+\beta} u}{\partial x^\alpha \partial t^\beta}. \end{aligned}$$

By taking the integral for Eq. (4.21), from 0 to p with respect to p, we have

$$U_{\alpha,\beta}(p,s) = \frac{G_1(p,0)}{s} + \frac{\nu G_2(p,0)}{s^2} - \frac{1}{s^2}\int_0^p L_x^\alpha L_t^\beta[\Psi]\, dp + F_{\alpha,\beta}(p,s). \tag{4.22}$$

The third step, using (CFDLDM), the solution can be written in infinite series as in Eq. (4.7). Applying the inverse transform for Eq. (4.22), we obtain

$$\begin{aligned} u\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right) &= g_1\left(\frac{x^\alpha}{\alpha}\right)+\frac{t^\beta}{\beta}g_2\left(\frac{x^\alpha}{\alpha}\right)+L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}F_{\alpha,\beta}(p,s)\right] \\ &\quad - L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x^\alpha L_t^\beta[\Psi]\,dp\right]. \end{aligned} \tag{4.23}$$

Furthermore, the nonlinear terms $\frac{\partial^\beta u}{\partial t^\beta}\frac{\partial^{2\alpha} u}{\partial x^{2\alpha}}$ and $\frac{\partial^\alpha u}{\partial x^\alpha}\frac{\partial^{\alpha+\beta} u}{\partial x^\alpha \partial t^\beta}$ can be defined by

$$\frac{\partial^\beta u}{\partial t^\beta}\frac{\partial^{2\alpha} u}{\partial x^{2\alpha}} = N_1 = \sum_{n=0}^{\infty} A_n, \qquad \frac{\partial^\alpha u}{\partial x^\alpha}\frac{\partial^{\alpha+\beta} u}{\partial x^\alpha \partial t^\beta} = N_2 = \sum_{n=0}^{\infty} B_n. \tag{4.24}$$

We have a few terms of the Adomian polynomials for A_n and B_n that are denoted by

$$A_n = \frac{1}{n!}\left(\frac{d^n}{d\lambda^n}\left[N_1\sum_{i=0}^{\infty}(\lambda^n u_n)\right]\right)_{\lambda=0} \tag{4.25}$$

and

$$B_n = \frac{1}{n!}\left(\frac{d^n}{d\lambda^n}\left[N_2\sum_{i=0}^{\infty}(\lambda^n u_n)\right]\right)_{\lambda=0}, \tag{4.26}$$

where $n = 0,1,2,\ldots$. By putting Eq. (4.25), Eq. (4.26), and Eq. (4.24) into Eq. (4.23), we get

$$\begin{aligned} &\sum_{n=0}^{\infty} u_n\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right) \\ &\quad = f_1\left(\frac{x^\alpha}{\alpha}\right)+\frac{t^\beta}{\beta}f_2\left(\frac{x^\alpha}{\alpha}\right)+L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}F_{\alpha,\beta}(p,s)\right] \\ &\qquad - L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x^\alpha L_t^\beta\left[\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}\left(\sum_{n=0}^{\infty}u_n\right)\right)\right]dp\right] \\ &\qquad + L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x^\alpha L_t^\beta\left[\frac{x^\alpha}{\alpha}a\left(\frac{x^\alpha}{\alpha}\right)\frac{\partial^{4\alpha}}{\partial x^{4\alpha}}\left(\sum_{n=0}^{\infty}u_n\right)\right]dp\right] \\ &\qquad - L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x^\alpha L_t^\beta\left[\frac{x^\alpha}{\alpha}b\left(\frac{x^\alpha}{\alpha}\right)\frac{\partial^{2\alpha+2\beta}}{\partial x^{2\alpha}\partial t^{2\beta}}\left(\sum_{n=0}^{\infty}u_n\right)\right]dp\right] \\ &\qquad + L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x^\alpha L_t^\beta\left[\frac{x^\alpha}{\alpha}c\left(\frac{x^\alpha}{\alpha}\right)\sum_{n=0}^{\infty}A_n+\frac{x^\alpha}{\alpha}d\left(\frac{x^\alpha}{\alpha}\right)\sum_{n=0}^{\infty}B_n\right]dp\right], \end{aligned} \tag{4.27}$$

where A_n and B_n are defined as

$$\begin{aligned}
A_0 &= \frac{\partial^\beta u_0}{\partial t^\beta}\frac{\partial^{2\alpha} u_0}{\partial x^{2\alpha}},\\
A_1 &= \frac{\partial^\beta u_0}{\partial t^\beta}\frac{\partial^{2\alpha} u_1}{\partial x^{2\alpha}} + \frac{\partial^\beta u_1}{\partial t^\beta}\frac{\partial^{2\alpha} u_0}{\partial x^{2\alpha}},\\
A_2 &= \frac{\partial^\beta u_0}{\partial t^\beta}\frac{\partial^{2\alpha} u_2}{\partial x^{2\alpha}} + \frac{\partial^\beta u_1}{\partial t^\beta}\frac{\partial^{2\alpha} u_1}{\partial x^{2\alpha}} + \frac{\partial^\beta u_2}{\partial t^\beta}\frac{\partial^{2\alpha} u_0}{\partial x^{2\alpha}},\\
A_3 &= \frac{\partial^\beta u_0}{\partial t^\beta}\frac{\partial^{2\alpha} u_3}{\partial x^{2\alpha}} + \frac{\partial^\beta u_1}{\partial t^\beta}\frac{\partial^{2\alpha} u_2}{\partial x^{2\alpha}} + \frac{\partial^\beta u_2}{\partial t^\beta}\frac{\partial^{2\alpha} u_1}{\partial x^{2\alpha}} + \frac{\partial^\beta u_3}{\partial t^\beta}\frac{\partial^{2\alpha} u_0}{\partial x^{2\alpha}}
\end{aligned} \tag{4.28}$$

and

$$\begin{aligned}
B_0 &= \frac{\partial^\alpha u_0}{\partial x^\alpha}\frac{\partial^{\alpha+\beta} u_0}{\partial x^\alpha \partial t^\beta},\\
B_1 &= \frac{\partial^\alpha u_0}{\partial x^\alpha}\frac{\partial^{\alpha+\beta} u_1}{\partial x^\alpha \partial t^\beta} + \frac{\partial^\alpha u_1}{\partial x^\alpha}\frac{\partial^{\alpha+\beta} u_0}{\partial x^\alpha \partial t^\beta},\\
B_2 &= \frac{\partial^\alpha u_0}{\partial x^\alpha}\frac{\partial^{\alpha+\beta} u_2}{\partial x^\alpha \partial t^\beta} + \frac{\partial^\alpha u_1}{\partial x^\alpha}\frac{\partial^{\alpha+\beta} u_1}{\partial x^\alpha \partial t^\beta} + \frac{\partial^\alpha u_2}{\partial x^\alpha}\frac{\partial^{\alpha+\beta} u_0}{\partial x^\alpha \partial t^\beta},\\
B_3 &= \frac{\partial^\alpha u_0}{\partial x^\alpha}\frac{\partial^{\alpha+\beta} u_3}{\partial x^\alpha \partial t^\beta} + \frac{\partial^\alpha u_1}{\partial x^\alpha}\frac{\partial^{\alpha+\beta} u_2}{\partial x^\alpha \partial t^\beta} + \frac{\partial^\alpha u_2}{\partial x^\alpha}\frac{\partial^{\alpha+\beta} u_1}{\partial x^\alpha \partial t^\beta} + \frac{\partial^\alpha u_3}{\partial x^\alpha}\frac{\partial^{\alpha+\beta} u_0}{\partial x^\alpha \partial t^\beta}.
\end{aligned} \tag{4.29}$$

Hence, from Eq. (4.27) above, we have

$$u_0\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right) = f_1\left(\frac{x^\alpha}{\alpha}\right) + \frac{t^\beta}{\beta} f_2\left(\frac{x^\alpha}{\alpha}\right) + L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}F_{\alpha,\beta}(p,s)\right]$$

and

$$\begin{aligned}
&u_n\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right)\\
&\quad = -L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x^\alpha L_t^\beta\left[\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}\left(\sum_{n=0}^\infty u_{n-1}\right)\right)\right]dp\right]\\
&\qquad + L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x^\alpha L_t^\beta\left[\frac{x^\alpha}{\alpha}a\left(\frac{x^\alpha}{\alpha}\right)\frac{\partial^{4\alpha}}{\partial x^{4\alpha}}\left(\sum_{n=0}^\infty u_{n-1}\right)\right]dp\right]\\
&\qquad - L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x^\alpha L_t^\beta\left[\frac{x^\alpha}{\alpha}b\left(\frac{x^\alpha}{\alpha}\right)\frac{\partial^{2\alpha+2\beta}}{\partial x^{2\alpha}\partial t^{2\beta}}\left(\sum_{n=0}^\infty u_{n-1}\right)\right]dp\right]\\
&\qquad + L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x^\alpha L_t^\beta\left[\frac{x^\alpha}{\alpha}c\left(\frac{x^\alpha}{\alpha}\right)\sum_{n=0}^\infty A_{n-1} + \frac{x^\alpha}{\alpha}d\left(\frac{x^\alpha}{\alpha}\right)\sum_{n=0}^\infty B_{n-1}\right]dp\right].
\end{aligned} \tag{4.30}$$

In the next example, the proposed method is applied in order to obtain the solution of the nonlinear singular Boussinesq equation.

Example 5 Consider that nonlinear singular Boussinesq equation in one dimension is governed by

$$\begin{aligned} &\frac{\partial^{2\beta}u}{\partial t^{2\beta}}-\frac{\alpha}{x^\alpha}\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha u}{\partial x^\alpha}\right)+\frac{\partial^{4\alpha}u}{\partial x^{4\alpha}}-\frac{\partial^{2\alpha+2\beta}u}{\partial x^{2\alpha}\partial t^{2\beta}}-4\frac{\partial^\beta u}{\partial t^\beta}\frac{\partial^{2\alpha}u}{\partial x^{2\alpha}}+2\frac{\partial^\alpha u}{\partial x^\alpha}\frac{\partial^{\alpha+\beta}u}{\partial x^\alpha\partial t^\beta} \\ &\quad=-4\left(\frac{t^\beta}{\beta}\right), \end{aligned} \tag{4.31}$$

subject to the initial condition

$$u\left(\frac{x^\alpha}{\alpha},0\right)=0,\qquad \frac{\partial^\beta u(\frac{x^\alpha}{\alpha},0)}{\partial t^\beta}=\left(\frac{x^\alpha}{\alpha}\right)^2. \tag{4.32}$$

In order to implement our method for Eq. (4.31), we have

$$\psi_0\left(\frac{x^\alpha}{\alpha},t\right)=\frac{x^{\alpha\,2}}{\alpha}\left(\frac{t^\beta}{\beta}\right)-\frac{2}{3}\left(\frac{t^\beta}{\beta}\right)^3$$

and

$$\begin{aligned} &u_{n+1}\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right) \\ &\quad=-L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x^\alpha L_t^\beta\left[\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha}{\partial x^\alpha}\left(\sum_{n=0}^\infty u_n\right)\right)\right]dp\right] \\ &\qquad-L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x^\alpha L_t^\beta\left[\frac{x^\alpha}{\alpha}\frac{\partial^{4\alpha}}{\partial x^{4\alpha}}\left(\sum_{n=0}^\infty u_n\right)-\frac{x^\alpha}{\alpha}\frac{\partial^{2\alpha+2\beta}}{\partial x^{2\alpha}\partial t^{2\beta}}\left(\sum_{n=0}^\infty u_n\right)\right]dp\right] \\ &\qquad-L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x^\alpha L_t^\beta\left[4\frac{x^\alpha}{\alpha}\sum_{n=0}^\infty A_n-2\frac{x^\alpha}{\alpha}\sum_{n=0}^\infty B_n\right]dp\right]. \end{aligned}$$

The first iteration is given by

$$\begin{aligned} u_1\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)&=-L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_{\frac{x^\alpha}{\alpha}}^\alpha L_t^\beta\left[\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha u_0}{\partial x^\alpha}\right)\right]dp\right] \\ &\quad-L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x^\alpha L_t^\beta\left[\frac{x^\alpha}{\alpha}\frac{\partial^{4\alpha}u_0}{\partial x^{4\alpha}}-\frac{x^\alpha}{\alpha}\frac{\partial^{2\alpha+2\beta}u_0}{\partial x^{2\alpha}\partial t^{2\beta}}\right]dp\right] \\ &\quad-L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_{\frac{x^\alpha}{\alpha}}^\alpha L_t^\beta\left[4\frac{x^\alpha}{\alpha}A_0-2\frac{x^\alpha}{\alpha}B_0\right]dp\right], \end{aligned}$$

where

$$\begin{aligned} &\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha u_0}{\partial x^\alpha}\right)=4\frac{x^\alpha}{\alpha}\frac{t^\beta}{\beta},\qquad \frac{x^\alpha}{\alpha}\frac{\partial^{4\alpha}u_0}{\partial x^{4\alpha}}=0,\qquad \frac{x^\alpha}{\alpha}\frac{\partial^{2\alpha+2\beta}u_0}{\partial x^{2\alpha}\partial t^{2\beta}}=0, \\ &A_0=\frac{\partial^\beta u_0}{\partial t^\beta}\frac{\partial^{2\alpha}u_0}{\partial x^{2\alpha}}=2\left(\frac{x^\alpha}{\alpha}\right)^2\frac{t^\beta}{\beta}-4\left(\frac{t^\beta}{\beta}\right)^3,\qquad B_0=\frac{\partial^\alpha u_0}{\partial x^\alpha}\frac{\partial^{\alpha+\beta}u_0}{\partial x^\alpha\partial t^\beta}=4\left(\frac{x^\alpha}{\alpha}\right)^2\frac{t^\beta}{\beta}. \end{aligned}$$

Therefore,

$$\begin{aligned}
u_1\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right) &= -L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x^\alpha L_t^\beta\left[4\frac{x^\alpha}{\alpha}\frac{t^\beta}{\beta}\right]dp\right] \\
&\quad - L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x^\alpha L_t^\beta[0-0]\,dp\right] \\
&\quad + L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x^\alpha L_t^\beta\left[16\frac{x^\alpha}{\alpha}\left(\frac{t^\beta}{\beta}\right)^3\right]dp\right], \\
u_1\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right) &= \frac{2}{3}\left(\frac{t^\beta}{\beta}\right)^3-\frac{4}{5}\left(\frac{t^\beta}{\beta}\right)^5.
\end{aligned}$$

In the same way,

$$\begin{aligned}
&\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha u_1}{\partial x^\alpha}\right)=0, \qquad \frac{x^\alpha}{\alpha}\frac{\partial^{4\alpha}u_1}{\partial x^{4\alpha}}=0, \qquad \frac{x^\alpha}{\alpha}\frac{\partial^{2\alpha+2\beta}u_1}{\partial x^{2\alpha}\partial t^{2\beta}}=0, \\
&A_1=\frac{\partial^\beta u_0}{\partial t^\beta}\frac{\partial^{2\alpha}u_1}{\partial x^{2\alpha}}+\frac{\partial^\beta u_1}{\partial t^\beta}\frac{\partial^{2\alpha}u_0}{\partial x^{2\alpha}}=4\left(\frac{t^\beta}{\beta}\right)^3-8\left(\frac{t^\beta}{\beta}\right)^5, \\
&B_0=\frac{\partial^\alpha u_0}{\partial x^\alpha}\frac{\partial^{\alpha+\beta}u_1}{\partial x^\alpha\partial t^\beta}+\frac{\partial^\alpha u_1}{\partial x^\alpha}\frac{\partial^{\alpha+\beta}u_0}{\partial x^\alpha\partial t^\beta}=0,
\end{aligned}$$

hence,

$$\begin{aligned}
u_2\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right) &= -L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x^\alpha L_t^\beta\left[16\frac{x^\alpha}{\alpha}\left(\frac{t^\beta}{\beta}\right)^3-32\frac{x^\alpha}{\alpha}\left(\frac{t^\beta}{\beta}\right)^5\right]dp\right] \\
&\quad - L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p\left[\frac{16\times 3!}{p^2s^4}-\frac{32\times 5!}{p^2s^6}\right]dp\right] \\
&= L_p^{-1}L_s^{-1}\left[\frac{16\times 3!}{ps^6}-\frac{32\times 5!}{ps^8}\right], \\
u_2\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right) &= \frac{4}{5}\left(\frac{t^\beta}{\beta}\right)^5-\frac{16}{21}\left(\frac{t^\beta}{\beta}\right)^7.
\end{aligned}$$

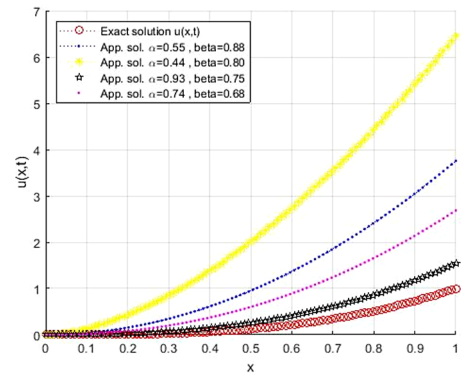
Similarly,

$$\begin{aligned}
u_3\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right) &= -L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x^\alpha L_t^\beta\left[\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha u_2}{\partial x^\alpha}\right)\right]dp\right] \\
&\quad - L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x^\alpha L_t^\beta\left[\frac{x^\alpha}{\alpha}\frac{\partial^{4\alpha}u_2}{\partial x^{4\alpha}}-\frac{x^\alpha}{\alpha}\frac{\partial^{2\alpha+2\beta}u_2}{\partial x^{2\alpha}\partial t^{2\beta}}\right]dp\right] \\
&\quad - L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x^\alpha L_t^\beta\left[4\frac{x^\alpha}{\alpha}A_2-2\frac{x^\alpha}{\alpha}B_2\right]dp\right].
\end{aligned}$$

Therefore,

$$\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{x^\alpha}{\alpha}\frac{\partial^\alpha u_2}{\partial x^\alpha}\right)=0, \qquad \frac{x^\alpha}{\alpha}\frac{\partial^\alpha u_2}{\partial x^\alpha}=0, \qquad \frac{x^\alpha}{\alpha}\frac{\partial^{2\alpha+2\beta}u_2}{\partial x^{2\alpha}\partial t^{2\beta}}=0,$$

Figure 2 The exact and approximate solutions of u(x,t), for Example 5, we take different values of fractional orders of α and β: $\alpha = 0.55$, $\beta = 0.88$, $\alpha = 0.44$, $\beta = 0.80$, $\alpha = 0.93$, $\beta = 0.75$, and $\alpha = 0.74$, $\beta = 0.68$



$$\begin{aligned} A_2 &= \frac{\partial^\beta u_0}{\partial t^\beta}\frac{\partial^{2\alpha} u_2}{\partial x^{2\alpha}} + \frac{\partial^\beta u_1}{\partial t^\beta}\frac{\partial^{2\alpha} u_1}{\partial x^{2\alpha}} + \frac{\partial^\beta u_2}{\partial t^\beta}\frac{\partial^{2\alpha} u_0}{\partial x^{2\alpha}} \\ &= 8\left(\frac{t^\beta}{\beta}\right)^5 - \frac{32}{3}\left(\frac{t^\beta}{\beta}\right)^7, \\ B_2 &= \frac{\partial^\alpha u_0}{\partial x^\alpha}\frac{\partial^{\alpha+\beta} u_2}{\partial x^\alpha \partial t^\beta} + \frac{\partial^\alpha u_1}{\partial x^\alpha}\frac{\partial^{\alpha+\beta} u_1}{\partial x^\alpha \partial t^\beta} + \frac{\partial^\alpha u_2}{\partial x^\alpha}\frac{\partial^{\alpha+\beta} u_0}{\partial x^\alpha \partial t^\beta} = 0. \end{aligned}$$

Then we have

$$u_3\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right) = \frac{16}{21}\left(\frac{t^\beta}{\beta}\right)^7 - \frac{16}{27}\left(\frac{t^\beta}{\beta}\right)^9.$$

By using Eq. (4.7) the series solutions is denoted by

$$\begin{aligned} \sum_{n=0}^{\infty} u_n\left(\frac{x^\alpha}{\alpha}, t\right) &= \psi_0 + \psi_1 + \psi_2 + \cdots \\ &= \left(\frac{x^\alpha}{\alpha}\right)^2 \frac{t^\beta}{\beta} - \frac{2}{3}\left(\frac{t^\beta}{\beta}\right)^3 + \frac{2}{3}\left(\frac{t^\beta}{\beta}\right)^3 - \frac{4}{5}\left(\frac{t^\beta}{\beta}\right)^5 + \frac{4}{5}\left(\frac{t^\beta}{\beta}\right)^5 \\ &\quad - \frac{16}{21}\left(\frac{t^\beta}{\beta}\right)^7 + \frac{16}{21}\left(\frac{t^\beta}{\beta}\right)^7 - \frac{16}{27}\left(\frac{t^\beta}{\beta}\right)^9 + \cdots, \end{aligned}$$

and hence the conformable solution is given by

$$u\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right) = \left(\frac{x^\alpha}{\alpha}\right)^2 \frac{t^\beta}{\beta}. \tag{4.33}$$

5 Conclusion

In the present research, we proposed a combination of conformable double Laplace transform and decomposition methods to solve the singular linear and nonlinear Boussinesq equations. The new method, developed in the current work, was tested on two examples. In addition, if we let $\alpha = 1$ and $\beta = 1$ in two examples, we obtain the solutions which are studied in [19].

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