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# Stability analysis of a fractional-order SIS model on complex networks with linear treatment function

Na Liu<sup>1</sup>, Jie Fang<sup>1\*</sup>, Wei Deng<sup>1</sup> and Jun-wei Sun<sup>1</sup>

\*Correspondence:

[fang0511jie@126.com](mailto:fang0511jie@126.com)

<sup>1</sup>School of Electric and Information Engineering, Zhengzhou University of Light Industry, Zhengzhou, China

## Abstract

In recent years, many research works have been focusing on the propagation dynamics of infectious diseases in complex networks, and some interesting results have been obtained. The main purpose of this paper is to investigate the stability of a fractional SIS model on complex networks with linear treatment function. Based on the basic reproduction number, the stability of the disease-free equilibrium point and the endemic equilibrium point is analyzed in detail. That is, when  $R_0 \leq 1$ , the disease-free equilibrium point is globally asymptotically stable and the disease will die out ultimately; when  $R_0 > 1$ , there exists a unique endemic equilibrium point, and both the disease-free equilibrium point and the endemic equilibrium point are stable and the disease will not spread to all individuals. Finally, numerical simulations are presented to demonstrate the theoretical results. Moreover, the influence of the fractional-order parameter and the coefficient of the linear treatment function on the decay rate of the infectious is depicted separately.

**Keywords:** Fractional-order; Stability; Treatment function; Complex network

## 1 Introduction

Many physical systems exhibit fractional-order dynamical behaviors due to their special material and chemical properties. When the fractional-order model is used to describe the objects with fractional-order characteristics, the essential characteristics and behaviors of the objects can be better revealed. Compared with the integer-order model, the fractional-order model has clearer physical meaning and simpler expression when describing complex physical and mechanical problems. Fractional calculus has memory function, which ensures revealing the influences of historical information on present and future, and hence is beneficial to improving the quality of control. Fractional calculus has been used to model the real world problems [1–7]. It is playing a very important role in the field of science, engineering, finance, communication, epidemic, etc. In the branch of fractional calculus, fractional derivatives and fractional integrals are important aspects. Like that for the classical integral-order differential systems, stability analysis is an important research topic for fractional-order differential systems. The concepts of Lyapunov function and direct Lyapunov method have been extended to differential equations of fractional order [8–16]. Li et al. [8, 9] proposed the definition of Mittag-Leffler stability and introduced the fractional Lyapunov's second method. Liu et al. [10, 11] dealt with fractional nonlinear systems

and several algebraic criteria of Mittag-Leffler and asymptotical stability were obtained. Zhang et al. [12] studied the stability of nonlinear fractional differential systems with Caputo derivatives by using the comparison method. Wu et al. [16] analyzed the stability of nonlinear discrete fractional systems via the Lyapunov second direct method. Meanwhile, however, numerical algorithms for fractional differential equations are not mature enough. On the premise of the reliability and accuracy of calculations, there still exist some challenges to improve the computational efficiency, e.g., to lower the computational burdens and storage consumptions. To solve fractional differential equations, many numerical algorithms have been developed [2, 17, 18], including finite difference method, finite element method, polynomial configuration method, etc. Among these methods, the finite difference method is most commonly used.

As we all know, the classical models for disease dynamics are the Susceptible-Infected-Susceptible (SIS), Susceptible-Infected-Recovered (SIR), Susceptible-Infected-Recovered-Susceptible (SIRS), and Susceptible-Exposed-Infected-Recovered (SEIR) models, which represent different stages of infection for individuals in a population. Based on the memory function of a fractional-order system, many scholars use fractional-order differential equations to analyze the transmission dynamics of infectious diseases, which has more realistic significance than using the integer-order differential equation [19–23]. However, as the transmission dynamics of an infectious disease is greatly affected by inter-individual connections, also known as network topology, it has practical significance to analyze the transmission characteristics of fractional-order infectious diseases in complex networks. Specifically, infectious diseases spread out in population through certain types of inter-personal contacts. Such contacts naturally form into a complex social network, where a network node represents an individual, and the inter-personal contacts through which the disease can transmit are represented as edges. A large number of different network transmission models have been proposed (e.g., [24–28]). Zhu et al. [29] provided a typical mean-field modeling framework to describe the time-evolution dynamics and offer some theoretical analyses to study the spreading threshold and the global stability of the model. In Ref. [30], a general SIS model with infective vectors on complex networks was studied, and a new technique based on the basic reproduction matrix was introduced. Wang et al. [31] reviewed two node-based SIR models incorporating degree correlations and an edge-based SIR model without considering the degree correlation to predict the disease evolution on correlated networks. Huo [32] proposed a fractional SIR model with birth and death rates on heterogeneous complex networks and investigated the stability of a disease-free equilibrium point and an endemic equilibrium point.

Despite of these extensive efforts, only a few studies have investigated the fractional epidemic models on complex networks under disease control. By adopting some control strategies, the disease may only be able to spread in a finite population, or even disappear ultimately. We shall use a linear proportion control named linear treatment to control the disease propagation. The main objective is to analyze the stability of the disease-free equilibrium point and the endemic equilibrium point for a fractional-order SIS network model with linear treatment function, based on the reproduction number.

The organization of the rest part of this manuscript is as follows. The definition of Caputo fractional derivative and some of its important properties are given in Sect. 2. In Sect. 3, the SIS fractional disease model with treatment function is described and the basic reproduction number is derived. In Sect. 4, the stability of the disease-free equilibrium

point and the endemic equilibrium point is discussed in detail. In Sect. 5, an example is presented to verify the effectiveness of the theoretical results. Finally, Sect. 6 concludes the whole report.

## 2 Preliminaries

Let us recall some basic definitions on Caputo differential operator of fractional calculus. Firstly, we introduce the definition of Caputo fractional derivative.

**Definition 1** ([14]) Suppose that  $\alpha > 0, t > t_0, \alpha, t_0, t \in R$ . The Caputo fractional derivative for a smooth function  $f = f(t)$  is given by

$${}^c D_t^\alpha f(t) = \frac{1}{\Gamma(n - \alpha)} \int_0^t \frac{f^n(\zeta)}{(t - \zeta)^{\alpha + 1 - n}} d\zeta, \quad n - 1 < \alpha < n, n \in \mathbb{N}.$$

where  $\Gamma$  denotes the gamma function.

If  $n = 1$ , then  $0 < \alpha < 1$  and

$${}^c D_t^\alpha f(t) = \frac{1}{\Gamma(1 - \alpha)} \int_0^t \frac{f^n(\zeta)}{(t - \zeta)^\alpha} d\zeta.$$

**Lemma 1** Assume that  $0 < \alpha \leq 1, {}_{t_0} D_t^\alpha x(t) \geq {}_{t_0} D_t^\alpha y(t), x(t_0) = x_0 \geq y(t_0) = y_0$ , then  $x(t) \geq y(t)$ .

For simplicity, the Caputo fractional derivative  ${}^c D_t^\alpha$  is always rewritten as  $D^\alpha$ .

Considering the following general type of fractional differential equations involving Caputo derivative:

$$D^\alpha x(t) = f(x, t), \tag{1}$$

with the initial condition  $x_0 = x(t_0)$ , we have the following definition.

**Definition 2** The constant  $x^*$  is an equilibrium point of the Caputo fractional dynamic system (1) if and only if  $f(t, x^*) = 0$ .

When  $0 < \alpha < 1$ , the Caputo fractional-order system (1) has the same equilibrium points as the integer-order system  $\frac{dx}{dt} = f(t, x)$ .

## 3 Model description and basic reproduction number

Based on the Caputo derivative, we propose a fractional SIS complex network model as follows:

$$\begin{cases} D^\alpha S_k(t) = b(S_k + I_k) - \beta S_k - \lambda(k)S_k \Theta(t) + \gamma I_k, \\ D^\alpha I_k(t) = \lambda(k)S_k \Theta(t) - (\gamma + \beta + \mu)I_k, \quad k = 1, 2, \dots, n, \end{cases} \tag{2}$$

where  $D^\alpha$  is the Caputo derivative,  $\alpha$  ( $0 < \alpha < 1$ ) is the order of the differential operator for system (2).  $S_k(t), I_k(t)$  are respectively the densities of the susceptible and infected nodes (individuals) with the degree  $k$  ( $k = 1, 2, \dots, n$ ) at time  $t$ ,  $b$  is the birth rate,  $\beta$  is the natural death rate,  $\mu$  is the death rate due to illness,  $\gamma$  is the recovery rate of the infected nodes,

$\lambda(k)$  is the infection rate and satisfies  $c \leq \lambda(k) \leq d$ ,  $\Theta(t) = \langle k \rangle^{-1} \sum_{k'=1}^n \varphi(k')P(k')I_{k'}$  is the probability that a randomly chosen edge emanating from a node of degree  $k$  points to an infected node of degree  $k'$ .  $\varphi(k')$  is the density of node  $k'$ ,  $P(k')$  is the degree distribution of node  $k'$ , and  $\langle k \rangle$  is the average degree of the network. It can be seen that the spread of disease networks is influenced by the topological structure of social networks.

The effective management of infectious diseases is treatment. In this paper the linear treatment function  $T(I_k) = aI_k$  is introduced. Then system (2) is rewritten as follows:

$$\begin{cases} D^\alpha S_k(t) = b(S_k + I_k) - \beta S_k - \lambda(k)S_k\Theta(t) + \gamma I_k + aI_k, \\ D^\alpha I_k(t) = \lambda(k)S_k\Theta(t) - (\gamma + \beta + \mu)I_k - aI_k, \quad k = 1, 2, \dots, n. \end{cases} \tag{3}$$

For system (3), the equilibrium points should satisfy

$$\begin{cases} b(S_k + I_k) - \beta S_k - \lambda(k)S_k\Theta(t) + \gamma I_k + aI_k = 0, \\ \lambda(k)S_k\Theta(t) - (\gamma + \beta + \mu + a)I_k = 0, \quad k = 1, 2, \dots, n. \end{cases} \tag{4}$$

The disease-free equilibrium point  $E_0$  of system (3) corresponds to  $I_k = 0$  ( $k = 1, 2, \dots, n$ ). Substituting it into Eq. (4), since  $S_k + I_k \equiv 1$ , we have

$$\begin{cases} S_k^0 = \frac{b}{\beta}, \\ I_k^0 = 0, \quad k = 1, 2, \dots, n. \end{cases}$$

The endemic equilibrium point of system (3) corresponds to the case where the disease persists in crowd  $I_k \neq 0$  ( $k = 1, 2, \dots, n$ ). So, the equilibrium point  $E^* = (S_k^*, I_k^*)$  has the form

$$\begin{aligned} I_k^* &= \frac{\lambda(k)\Theta(t)}{\lambda(k)\Theta(t) + \gamma + \mu + \beta + a}, \\ \Theta(t) &= \langle k \rangle^{-1} \frac{\sum_{k'=1}^n \varphi(k')P(k')\lambda(k')\Theta(t)}{\lambda(k)\Theta(t) + \gamma + \mu + \beta + a}. \end{aligned} \tag{5}$$

From Eq. (5) we can obtain that  $\Theta(t)(1 - \langle k \rangle^{-1} \frac{\sum_{k'=1}^n \varphi(k')P(k')\lambda(k')}{\lambda(k)\Theta(t) + \gamma + \mu + \beta + a}) = 0$ , let  $f(\Theta) = 1 - \langle k \rangle^{-1} \frac{\sum_{k'=1}^n \varphi(k')P(k')\lambda(k')}{\lambda(k)\Theta(t) + \gamma + \mu + \beta + a}$ , obviously  $f'(\Theta) > 0$ ,  $\lim_{\Theta \rightarrow \infty} f(\Theta) = 1$ .

Since  $\Theta \geq 0$ , there exists a unique positive solution for equation  $f(\Theta) = 0$  if and only if  $f(0) < 0$ .

Choose the basic reproduction number

$$R_0 = \langle k \rangle^{-1} \frac{\sum_{k'=1}^n \varphi(k')P(k')\lambda(k')}{\gamma + \mu + \beta + a}.$$

When  $R_0 > 1$ , there exists a unique endemic equilibrium point  $E^* = (S_k^*, I_k^*)$ . When  $R_0 < 1$ , system (3) only has a disease-free equilibrium point  $E_0 = (S_k^0, I_k^0)$ .

*Remark 1* For the basic reproduction number  $R_0$ , when  $\varphi(k') = k'$  and  $\lambda(k') = \lambda k'$ ,  $R_0$  can be simplified to  $R_0 = \frac{\langle k^2 \rangle}{\langle k \rangle} \frac{\lambda}{\gamma + \mu + \beta + a}$ .

### 4 The stability of the equilibrium points

In this section, we prove the stability of  $E_0$  and  $E^*$ , which is one of the most important topics in the study of mathematical biology.

**Theorem 1** *For system (3), when  $R_0 \leq 1$ , the disease-free equilibrium point is asymptotically stable, and the disease will die out ultimately; when  $R_0 > 1$ , there exists a unique endemic equilibrium point, and the disease-free equilibrium point and the endemic equilibrium point are both stable, which means that the disease will not spread to all individuals.*

*Proof* Since  $c \leq \lambda(k) \leq d$ ,

$$\begin{aligned}
 D^\alpha \Theta(t) &= \langle k \rangle^{-1} \sum_{i=1}^n \varphi(i)P(i)D^\alpha I_i \\
 &= \langle k \rangle^{-1} \sum_{i=1}^n \varphi(i)P(i) [\lambda(i)(1 - I_i)\Theta(t) - (\gamma + \mu + \beta)I_i - aI_i] \\
 &= \langle k \rangle^{-1} \left[ \sum_{i=1}^n \varphi(i)P(i)\lambda(i)(1 - I_i)\Theta(t) - \sum_{i=1}^n \varphi(i)P(i)(\gamma + \mu + \beta + a)I_i \right] \\
 &= \Theta(t) \left[ -(\gamma + \mu + \beta + a) + \langle k \rangle^{-1} \sum_{i=1}^n \varphi(i)P(i)\lambda(i)(1 - I_i) \right]. \tag{6}
 \end{aligned}$$

Let  $r = -(\gamma + \mu + \beta + a) + \langle k \rangle^{-1} \sum_{i=1}^n \varphi(i)P(i)\lambda(i)$ , then

$$R_0 - 1 = \frac{\langle k \rangle^{-1} \sum_{i=1}^n \varphi(i)P(i)\lambda(i) - (\gamma + \mu + \beta + a)}{\gamma + \mu + \beta + a}.$$

(1) For Eq. (6), when  $R_0 < 1$  ( $r < 0$ ),  $D^\alpha \Theta(t) \leq r\Theta(t) - c\Theta^2(t)$ .

First, let  $D^\alpha \Theta_1(t) = r\Theta_1(t)$ ,  $\Theta_1(t) = rE_{\alpha,1} \lfloor rt^\alpha \rfloor$ . Since  $r < 0$ ,  $\lim_{t \rightarrow \infty} \Theta_1(t) = 0$ , we have  $\lim_{t \rightarrow \infty} I_k(t) = 0$ .

Secondly, let  $D^\alpha \Theta_2(t) \leq -c\Theta_2^2(t)$ . Consider the comparing system

$$D^\alpha x(t) = -cx^2(t), \quad x(0) = \Theta(0), x(t) > 0,$$

and construct a linear fractional-order equation  $D^\alpha x(t) = x(0)x(t)$  its solution is  $x(t) = x(0)E_{\alpha,1} \lfloor x(0)t^\alpha \rfloor$ .

For equation  $D^\alpha x(t) = x^2(t)$ ,  $x(0) = \Theta(0) > 0$ , we can obtain that

$$D^{-\alpha} D^\alpha x(t) = D^{-\alpha} x^2(t),$$

that is to say,  $x(t) - x(0) = D^{-\alpha} x^2(t) > 0$ .

Therefore,  $x^2(t) > x(0)x(t)$  and  $-cx^2(t) < -cx(0)x(t)$ .

Since  $-cx(0) < 0$ , applying Lemma 1,  $\lim_{t \rightarrow \infty} x(t) = 0$ ,  $\lim_{t \rightarrow \infty} \Theta_2(t) \leq \lim_{t \rightarrow \infty} x(t) = 0$ .

Combining with the above analysis, we can summarize that when  $r < 0$ ,  $\lim_{t \rightarrow \infty} \Theta(t) = 0$ .

Hence,  $\lim_{t \rightarrow \infty} I_k(t) = 0$ .

(2) For Eq. (6), when  $R_0 = 1$  ( $r = 0$ ),  $D^\alpha \Theta(t) \leq -c\Theta^2(t)$ . Similarly,  $\lim_{t \rightarrow \infty} \Theta(t) = 0$ , namely  $\lim_{t \rightarrow \infty} I_k(t) = 0$ .

(3) For Eq. (6), when  $R_0 > 1$  ( $r > 0$ ),  $D^\alpha \Theta(t) \leq r\Theta(t) - c\Theta^2(t)$ .

Considering the comparing system

$$D^\alpha x(t) = rx(t) - cx^2(t) = -c \left[ x(t) - \frac{r}{2c} \right] + \frac{r^2}{4c}, \tag{7}$$

from the above analysis, we can derive that the solution of Eq. (7) is stable. Applying Lemma 1, we have that when  $r > 0$ , no matter the equilibrium point is disease free or endemic,  $\Theta(t)$  is stable. In other words, the disease will not spread to all individuals.

The proof is completed. □

*Remark 2* For Eq. (7), let the Lyapunov function  $V(t) = (x(t) - \frac{r}{c})^2$ , then

$$\begin{aligned} D^\alpha V(t) &= D^\alpha \left( x(t) - \frac{r}{c} \right)^2 \\ &\leq \left( x(t) - \frac{r}{c} \right) D^\alpha \left( x(t) - \frac{r}{c} \right) \\ &= \left( x(t) - \frac{r}{c} \right) [rx(t) - cx^2(t)] \\ &= -cx(t) \left( x(t) - \frac{r}{c} \right)^2. \end{aligned}$$

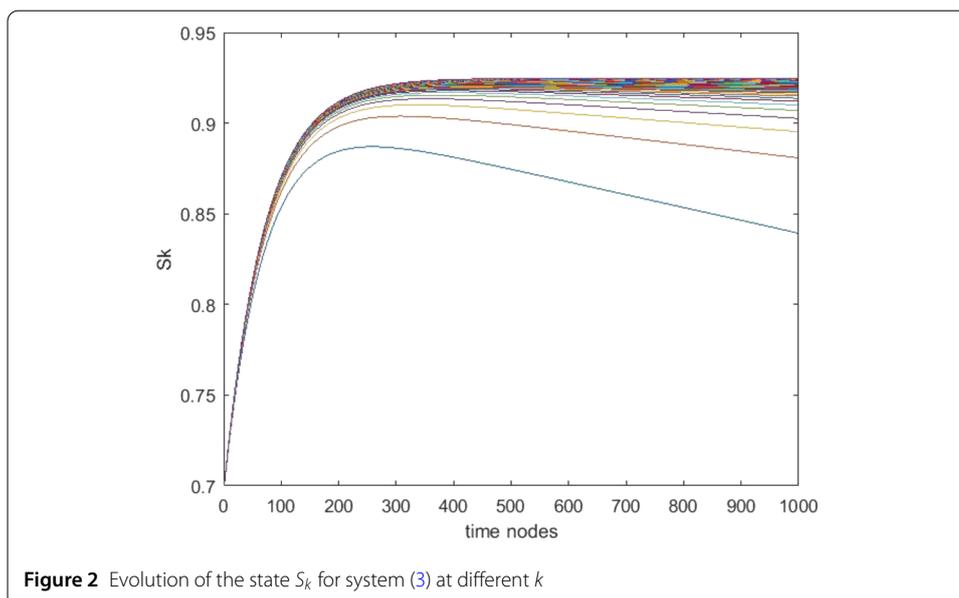
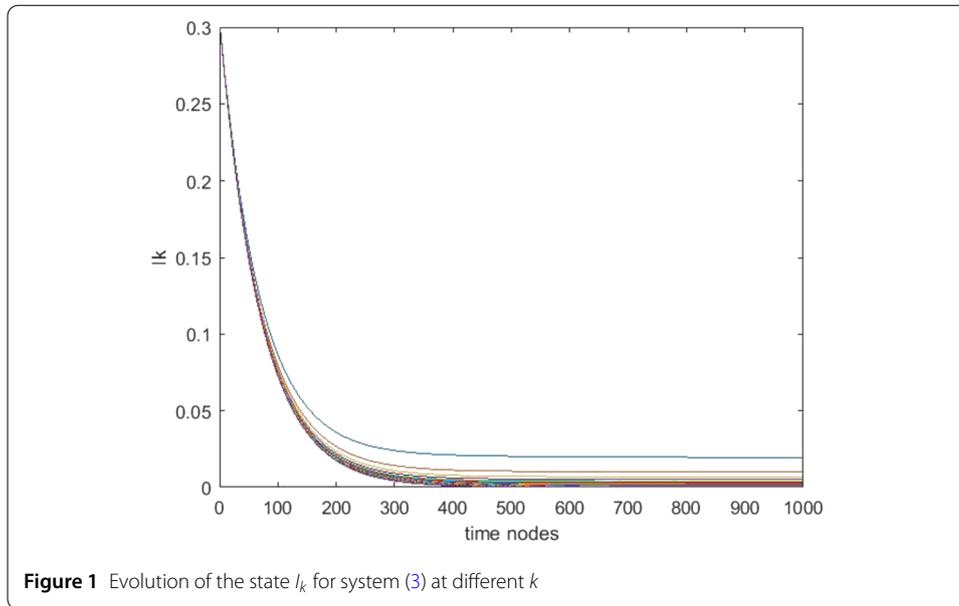
Since  $c > 0$ , then  $D^\alpha V(t) \leq 0$ , and the equilibrium points are stable.

### 5 Numerical simulations

In this section, numerical simulations are presented to illustrate the above-mentioned theoretical results. There are many numerical methods for solving the fractional equation. In this manuscript, the finite difference method is used, and the simulation results are presented as follows.

For system (3) on a BA random scale-free network, the number of nodes on the networks is  $n = 200$ , the degree distribution  $P(k)$  satisfies  $P(k) = 2m^2k^{-3}$ . For simplicity, let  $\varphi(k) = k$ ,  $\lambda(k) = \lambda k$ ,  $b = 0.01$ ,  $\lambda = 0.03$ ,  $\beta = 0.01$ ,  $\gamma = 0.02$ ,  $\mu = 0.01$ , and the initial condition is  $(0.7, 0.3)$ . At this moment, there is a disease-free equilibrium point  $E_0 = (1, 0)$  and an endemic equilibrium point  $E^* = (0.93, 0.02)$  for system (3), which are both stable. By calculation, the basic reproduction number is  $R_0 = 1.3238 > 1$ . Figure 1 and Fig. 2 show time series plots for the infectious nodes  $I_k$  and the susceptible nodes  $S_k$  respectively, when degree  $k$  changes from 1 to 200. From Fig. 1 and Fig. 2 we can observe that the disease will not spread to all individuals. Besides, we can find that the larger the degree of node is, the quicker the disease will decay.

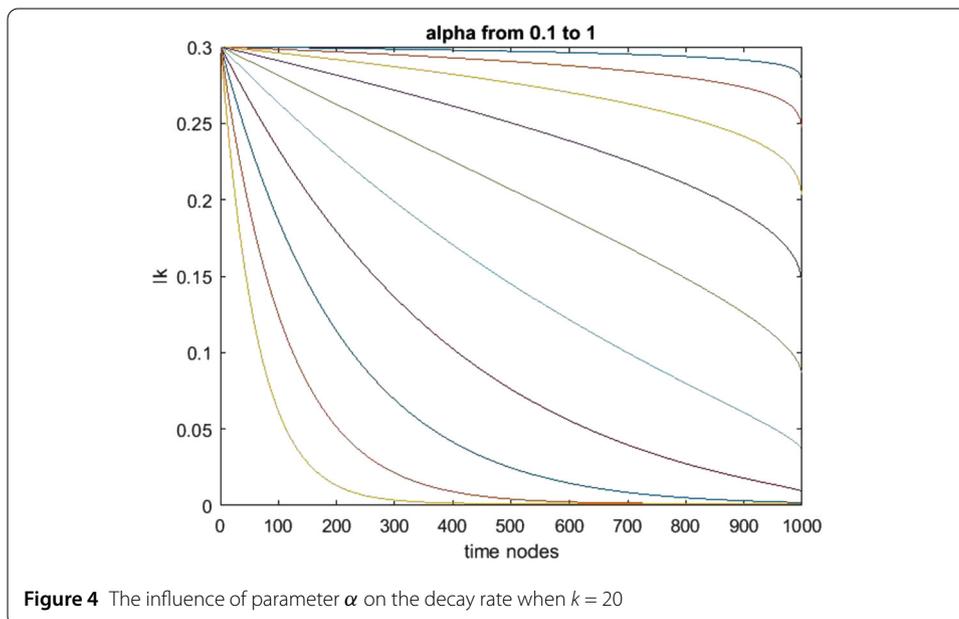
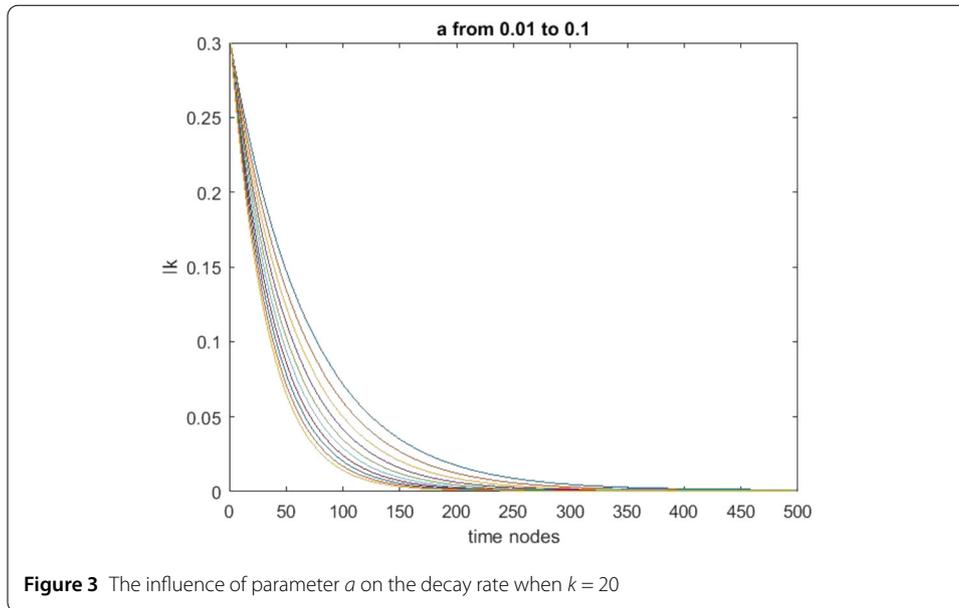
From Fig. 3 we can observe that, with the help of the linear treatment function, the infectious scale would decrease much quicker. The larger the parameter  $a$  is, the smaller the basic reproduction number is and the faster the disease decreases. For example, when  $a = 0.02$ ,  $R_0 = 0.8825 < 1$ ; when  $a = 0.03$ ,  $R_0 = 0.7565 < 1$ . But no matter how small the parameter  $a$  is, the disease will die out ultimately. Additionally, we also find that the fractional-order parameter has a strong impact on the decay rate of the infectious scale, which is presented in Fig. 4. The larger the parameter  $\alpha$  is, the faster the disease dies out.



From Figs. 1–4, we can observe that without control, the disease may continue to spread over a small area and would not be extinct. By contrast, with treatment function the disease will decrease more quickly and be extinct ultimately. Moreover, it also takes much less time to reach a steady state.

## 6 Conclusions

In this article, the stability of a fractional-order SIS complex network model with linear treatment function has been investigated. The expression of the basic reproduction number  $R_0$  was provided and some theoretical results have been obtained. When  $R_0 \leq 1$ , the disease-free equilibrium point is asymptotically stable, and the disease will become extinct ultimately regardless of the initial density of the infected individuals. When  $R_0 > 1$ , there exists a unique endemic equilibrium point. Whether the system ends up at a disease-free



equilibrium point or an endemic equilibrium point, both states are stable and the disease will not spread to all individuals. Finally, numerical simulations were presented to demonstrate the effectiveness of the theoretical results and the influence of treatment function and fractional order to the spreading dynamics. Note that the linear treatment function is not suitable for some infectious disease, for example, cholera, flu, etc., because they have complex transmission characteristics and are affected by many factors. More effective control methods will be developed and evaluated in our future studies. Besides, the feasibility of solving the fractional equations with other numerical methods, e.g., the methods proposed in Refs. [33, 34], shall also be an interesting issue to be investigated in our future research.

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### Competing interests

The authors declare that they have no competing interests.

### Authors' contributions

All authors worked together to produce the results, read and approved the final manuscript.

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