### RESEARCH

#### **Open Access**

# Upon solutions to the transonic plane-parallel gas flows



Yufeng Zhang<sup>1\*</sup>, Chao Yue<sup>2</sup> and Xiangzhi Zhang<sup>1</sup>

<sup>1</sup>Correspondence: zyfxz@cumt.edu.cn <sup>1</sup>College of Mathematics, China University of Mining and Technology, Xuzhou, P.R. China Full list of author information is available at the end of the article

#### Abstract

Some similarity solutions of the time-fractional stationary transonic plane-parallel gas flows (STPPGF) and their generalized space-fractional nonlinear system are obtained by using scalar similarity transformations, including traveling-wave similarity solutions. Two approximated solution formulas of the obtained ordinary fractional differential equations for the generalized space-fractional nonlinear system are generated as well. Finally, a class of approximated solutions of the time-fractional nonlinear system of STPPGF with initial-boundary-value conditions are produced by applying the separated variable method.

PACS Codes: 05.45.Yv; 02.30.Jr; 02.30.lk

Keywords: Similarity solution; Fractional derivative; Nonlinear equation

#### 1 Introduction

Fractional differential equations (FDEs) have extensive applications in scientific fields, such as porous media, fractals, acoustics, control theory, and signal processing, and so on, while Lie-group analysis method is a powerful tool for studying symmetries of ordinary and partial differential equations. Recently, this method has been extended to investigating fractional partial differential equations (FPDEs) and obtaining efficient calculation formulas of infinitesimal operators, symmetries, and invariant solutions, see the works in Refs. [1–7]. In Ref. [7] Djordjevic and Atanackovic introduced a type of similarity transformations to obtain similarity solutions of a generalized heat conduction equation with time-fractional derivatives. Dorjgotov et al. [8] adopted the Lie symmetry analysis method to study different infinitesimal operators, invariant solutions, and classification of a generalized nonlinear model of STPPGF. In the paper we want to study similarity solutions of the following nonlinear model of STPPGF:

$$\begin{cases} \frac{\partial^{\alpha} u}{\partial t^{\alpha}} = v_x, \\ \frac{\partial^{\alpha} v}{\partial t^{\alpha}} = -uu_x, \quad 0 < \alpha < 1, \end{cases}$$

(1)



© The Author(s) 2019. This article is distributed under the terms of the Creative Commons Attribution 4.0 International License (http://creativecommons.org/licenses/by/4.0/), which permits unrestricted use, distribution, and reproduction in any medium, provided you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made.

and further we study similarity solutions of a generalized nonlinear model of STPPGF with space-fractional derivatives as follows:

$$\begin{cases} \frac{\partial u}{\partial t} - \nu_x \nu^\rho = \frac{\partial^\beta u}{\partial x^\beta}, \\ \frac{\partial v}{\partial t} + u u_x = \frac{\partial^\beta v}{\partial x^\beta}, \quad 0 \le \beta \le 1. \end{cases}$$
(2)

In Ref. [9], the authors proposed many possible definitions for fractional derivatives and integrals and provided a fundamental connection with classical fractional calculus by writing these general fractional operators in terms of the original Riemann-Liouville fractional integral operator. In Ref. [10], some solutions of the density-dependent diffusion Nagumo equation were obtained by using a new approach, the Lie symmetry group-preserving scheme. In Ref. [11], a hot topic which finds the symmetries of a given fractional differential equation in the field of fractional differentiation was presented; and in the manuscript, the Lie symmetries of the time fractional gas dynamics (TFGD) equation were analyzed and new exact solutions were obtained. Hashemi and Baleanu [12] derived the Lie point symmetries of the time fractional Fisher (TFF) equation using a systematic investigation. Further, they used the obtained Lie point symmetries, TFF equation has been transformed into a nonlinear fractional ordinary differential equation with the EK fractional derivative. Our method is different from the general Lie-group analysis method, which implies that a deformed transformation of similarity transformation given in [7] is applied to Eqs. (1) and (2) so that various similarity solutions, including traveling wave solutions, are obtained, which enriches and supplements the results in [8]. Specially, by applying Volterra integral equation, system (1) with its initial values transforms to a Volterra integral equation such that a type of separated variable solutions is produced with the help of separated variable method. First of all, we recall a few associated notations. For any  $0 < \alpha \le 1$  and an absolutely continuous function f(t), the left Riemann–Liouville fractional derivative of order  $\alpha$  is defined as

$$\frac{d^{\alpha}f}{dt^{\alpha}} = \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial t} \int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha}} d\tau = \frac{1}{\Gamma(1-\alpha)} \left[ \frac{f(0)}{t^{\alpha}} + \int_0^t \frac{f^{(1)}(\tau)}{(t-\tau)^{\alpha}} d\tau \right],\tag{3}$$

where  $\Gamma$  is the Euler gamma function

$$\Gamma(\alpha) = \int_0^\infty e^{-z} z^{\alpha-1} \, dz.$$

For  $p - 1 < \beta \le p$ , the higher-order fractional derivative of function f(t) is defined as

$$\frac{d^{\beta}f}{dt^{\beta}} = \frac{1}{\Gamma(p-\beta)} \frac{\partial^{p}}{\partial t^{p}} \int_{0}^{t} \frac{f(\tau)}{(t-\tau)^{\beta+1-p}} d\tau.$$
(4)

#### 2 Similarity solutions of Eq. (1)

Consider a similarity transformation by introducing new independent and dependent variables [7]:

$$t = \lambda \tilde{t}, \qquad x = \lambda^p \tilde{x}, \qquad T = \lambda^q \tilde{T}(\tilde{x}, \tilde{t}),$$
(5)

from which we have

$$\frac{t}{x^{1/p}} = \frac{\tilde{t}}{\tilde{x}^{1/p}}, \qquad \frac{T}{\tilde{T}} = \left(\frac{x}{\tilde{x}}\right)^{q/p}.$$
(6)

By using (6), a generalized heat conduction equation was transformed to a time-fractional ordinary differential equation whose similarity solutions were obtained. Again by rewriting (5) as follows:

$$\frac{x}{t_p} = \frac{\tilde{x}}{\tilde{t}^p}, \qquad \frac{\phi}{\tilde{\phi}} = \left(\frac{t}{\tilde{t}}\right)^q,\tag{7}$$

a generalized Burgers/Korteweg–de Vries equation was transformed to a space-fractional differential equation from which a type of similarity solutions was obtained. However, in the paper we write (5) as follows:

$$\tilde{t} = \lambda^{-1}t, \qquad \tilde{x} = \lambda^{-p}x, \qquad \tilde{u}(\tilde{x}, \tilde{t}) = \lambda^{-q}u(x, t).$$
(8)

In terms of the method for seeking infinitesimal operators of ordinary or partial differential equations, we have

$$\frac{dt}{t} = \frac{dx}{px} = \frac{du}{qu}.$$
(9)

Based on the idea as above, assume that

$$t = \lambda \tilde{t}, \qquad x = \lambda^p \tilde{x}, \qquad u = \lambda^q \tilde{u}(\tilde{x}, \tilde{t}), \qquad v = \lambda^r \tilde{v}(\tilde{x}, \tilde{t}),$$
(10)

and substitute (10) into Eq. (1), one infers that

$$\begin{cases} \lambda^{q-\alpha} \frac{\partial^{\alpha} \tilde{u}}{\partial \tilde{t}^{\alpha}} = \lambda^{r-p} \frac{\partial \tilde{v}}{\partial \tilde{x}}, \\ \lambda^{r-\alpha} \frac{\partial^{\alpha} \tilde{v}}{\partial \tilde{t}^{\alpha}} = -\lambda^{2q-p} \tilde{u} \frac{\partial \tilde{u}}{\partial \tilde{x}}. \end{cases}$$
(11)

For Eq. (1) to be invariant under the transformation (10), it is necessary to require that

$$q-\alpha=r-p, \qquad r-\alpha=2q-p,$$

that is,

$$q = \frac{2}{3}r, \qquad p = \alpha + \frac{1}{3}r,$$
 (12)

where r is an arbitrary constant. In terms of Eq. (9), transformation (10) leads to a characteristic equation

$$\frac{dt}{t} = \frac{dx}{px} = \frac{du}{qu} = \frac{dv}{rv},\tag{13}$$

which gives

$$\xi = x^{-\frac{1}{p}}t, \qquad u = x^{\frac{q}{p}}U(\xi), \qquad v = x^{\frac{r}{p}}V(\xi), \tag{14}$$

where  $U(\xi)$ ,  $V(\xi)$  are arbitrary invariant functions with respect to the invariant variable  $\xi$ . Besides, it is easy to calculate that

$$\begin{split} \frac{\partial^{\alpha} u}{\partial t^{\alpha}} &= \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial t} \int_{0}^{t} \frac{x^{\frac{q}{p}} U(x^{-\frac{1}{p}}\tau)}{(t-\tau)^{\alpha}} d\tau \\ &= \frac{1}{\Gamma(1-\alpha)} x^{-\frac{1}{p}} \frac{\partial}{\partial \xi} \int_{0}^{\xi} \frac{x^{\frac{q}{p}} U(y)}{x^{\frac{p}{p}} (\xi-y)^{\alpha}} x^{\frac{1}{p}} dy = x^{\frac{q-\alpha}{p}} \frac{d^{\alpha} U(\xi)}{d\xi^{\alpha}}, \\ \frac{\partial^{\alpha} v}{\partial t^{\alpha}} &= x^{\frac{r-\alpha}{p}} \frac{d^{\alpha} V(\xi)}{d\xi^{\alpha}}, \\ \frac{\partial v}{\partial x} &= x^{\frac{r-p}{p}} \left[ \frac{r}{p} V(\xi) - \frac{1}{p} \xi V'(\xi) \right], \\ \frac{\partial u}{\partial x} &= x^{\frac{q-p}{p}} \left[ \frac{q}{p} U(\xi) - \frac{1}{p} \xi U'(\xi) \right]. \end{split}$$

Inserting the above results into Eq. (1) yields that

$$\begin{cases} \frac{d^{\alpha}U(\xi)}{d\xi^{\alpha}} = \frac{1}{p}[rV(\xi) - \xi V'(\xi)],\\ \frac{d^{\alpha}V(\xi)}{d\xi^{\alpha}} = -\frac{1}{p}U(\xi)[qU(\xi) - \xi U'(\xi)]. \end{cases}$$
(15)

Set the solutions of (15) to be as follows:

$$U(\xi) = U_1 \xi^{\beta}, \qquad V(\xi) = V_1 \xi^{\sigma},$$
 (16)

where  $U_1, V_1, \beta, \sigma$  are constants to be determined later. Substituting (16) into (15) gives that

$$\frac{\underline{U_{1B}(1-\alpha,1+\beta)}}{\Gamma(1-\alpha)}(1-\alpha+\beta)\xi^{\beta-\alpha} = \frac{1}{p}(rV_1-\sigma V_1)\xi^{\sigma},$$

$$\frac{\underline{V_{1B}(1-\alpha,1+\sigma)}}{\Gamma(1-\alpha)}(1-\alpha+\sigma)\xi^{\sigma-\alpha} = -\frac{1}{p}(qU_1-U_1\beta)\xi^{2\beta}.$$
(17)

Therefore, we get

$$\beta - \alpha = \sigma$$
,  $\sigma - \alpha = 2\beta$ ,

which implies that

$$\beta = -2\alpha, \qquad \sigma = -3\alpha. \tag{18}$$

Equation (17) reduces to

$$\begin{cases} \frac{U_{1B}(1-\alpha,1-2\alpha)}{\Gamma(1-\alpha)}(1-3\alpha) = \frac{1}{p}(r+3\alpha)V_1,\\ \frac{V_{1B}(1-\alpha,1-3\alpha)}{\Gamma(1-\alpha)}(1-4\alpha) = -\frac{1}{p}U_1^2(q+2\alpha), \end{cases}$$

from which we have

$$U_{1} = \frac{(3\alpha - 1)(1 - 4\alpha)p^{2}B(1 - \alpha, 1 - 2\alpha)B(1 - \alpha, 1 - 3\alpha)}{(r + 3\alpha)(q + 2\alpha)\Gamma^{2}(1 - \alpha)},$$

$$V_{1} = \frac{(1 - 3\alpha)^{2}(4\alpha - 1)p^{2}B^{2}(1 - \alpha, 1 - 2\alpha)B(1 - \alpha, 1 - 3\alpha)}{(q + 2\alpha)(r + 3\alpha)^{2}\Gamma^{3}(1 - \alpha)},$$
(19)

where q, r, p satisfy (12). Thus, we obtain the similarity solutions to Eq. (1):

$$u(x,t) = U_1 \xi^{-2\alpha}, \qquad v(x,t) = V_1 \xi^{-3\alpha},$$
(20)

where  $U_1$ ,  $V_1$  are presented in (19).

In what follows, we study the traveling-wave similarity solutions of Eq. (1). Hence, we set

$$u = x^{\frac{q}{p}} U(\xi), \qquad v = x^{\frac{r}{p}} V(\xi), \qquad \xi = \frac{ct}{x} - 1.$$
(21)

It is easy to see that

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial t} \int_{0}^{t} \frac{x^{\frac{q}{p}} U(\frac{c\tau}{x}-1)}{(t-\tau)^{\alpha}} d\tau.$$
(22)

Let  $y = \frac{c\tau}{x} - 1$ , then  $t - \tau = \frac{x}{c}(\xi - y)$ ,  $\frac{\partial}{\partial t} = \frac{c}{x}\frac{\partial}{\partial \xi}$ , Eq. (22) can be written as

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = \frac{1}{\Gamma(1-\alpha)} \frac{c}{x} \frac{\partial}{\partial \xi} \int_{-1}^{\xi} \frac{x^{\frac{q}{p}} U(y)}{(\frac{x}{c})^{\alpha} (\xi-y)^{\alpha}} \frac{x}{c} \, dy = c^{\alpha} x^{\frac{q-\alpha p}{p}} \frac{\overline{d}^{\alpha} U(\xi)}{\overline{d} \xi^{\alpha}}.$$

Similarly, one gets

$$\frac{\partial^{\alpha} v}{\partial t^{\alpha}} = c^{\alpha} x^{\frac{r-\alpha p}{p}} \frac{\bar{d}^{\alpha} V(\xi)}{\bar{d} \xi^{\alpha}},$$

where

$$\frac{\bar{d}^{\alpha}f(\xi)}{\bar{d}\xi^{\alpha}} = \frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial\xi}\int_{-1}^{\xi}\frac{f(y)}{(\xi-y)^{\alpha}}\,dy.$$

Again we have

$$u_{x} = x^{\frac{q-p}{p}} \left[ \frac{q}{p} U(\xi) - (\xi + 1)U'(\xi) \right],$$
  
$$v_{x} = x^{\frac{r-p}{p}} \left[ \frac{r}{p} V(\xi) - (\xi + 1)V'(\xi) \right].$$

Substituting the above consequences into Eq. (1) gives

$$\begin{cases} c^{\alpha} x^{\frac{q-\alpha p}{p}} \frac{\bar{d}^{\alpha} U(\xi)}{\bar{d}\xi^{\alpha}} = x^{\frac{r-p}{p}} [\frac{r}{p} V(\xi) - (\xi+1)V'(\xi)],\\ c^{\alpha} x^{\frac{r-\alpha p}{p}} \frac{\bar{d}^{\alpha} V(\xi)}{\bar{d}\xi^{\alpha}} = -x^{\frac{2q-p}{p}} U(\xi) [\frac{q}{p} U(\xi) - (\xi+1)U'(\xi)], \end{cases}$$

from which we get

$$q-\alpha p=r-p,$$
  $r-\alpha p=2q-p,$ 

that is,

$$q = 2p - 2\alpha p, \qquad r = 3p - 3\alpha p. \tag{23}$$

The corresponding ordinary time-fractional differential systems are as follows:

$$\begin{cases} \frac{\bar{d}^{\alpha}U(\xi)}{d\xi^{\alpha}} = \frac{r}{p}V(\xi) - (\xi+1)V'(\xi), \\ \frac{\bar{d}^{\alpha}V(\xi)}{d\xi^{\alpha}} = -U(\xi)[\frac{q}{p}U(\xi) - (\xi+1)U'(\xi)]. \end{cases}$$
(24)

Let us take special solutions of (24) in the forms

$$U(\xi) = \begin{cases} U_1 \xi^{\beta}, & \xi \ge 0, \\ 0, & \xi < 0, \end{cases} \qquad V(\xi) = \begin{cases} V_1 \xi^{\gamma}, & \xi \ge 0, \\ 0, & \xi < 0, \end{cases}$$
(25)

and insert into (24), we have that

$$\begin{cases} \frac{c^{\alpha} \mathcal{U}_{1} \Gamma(1+\beta)}{\Gamma(2-\alpha+\beta)} (1-\alpha+\beta) \xi^{\beta-\alpha} = V_{1} \xi^{\gamma-1} [\frac{r}{p} \xi - (\xi+1)\gamma],\\ \frac{c^{\alpha} V_{1} \Gamma(1+\gamma)}{\Gamma(2-\alpha+\gamma)} (1-\alpha+\gamma)^{\gamma-\alpha} = U_{1} \xi^{\beta} [U_{1} \beta(\xi+1) \xi^{\beta-1} - \frac{q}{p} U_{1} \xi^{\beta}]. \end{cases}$$

Set  $\beta - \alpha = \gamma - 1$ ,  $\gamma - \alpha = 2\beta - 1$ , one infers that

$$\beta = 2 - 2\alpha, \qquad \gamma = 3 - 3\alpha,$$

and

.

$$\begin{cases} \frac{U_{1}c^{\alpha}\Gamma(3-2\alpha)}{\Gamma(4-3\alpha)}(3-3\alpha) = V_{1}[\frac{r}{p}\xi - (\xi+1)(3-3\alpha)],\\ \frac{V_{1}c^{\alpha}\Gamma(4-3\alpha)}{\Gamma(4-3\alpha)}(4-4\alpha) = U_{1}^{2}[(2-2\alpha)(\xi+1) - \frac{q}{p}\xi]. \end{cases}$$
(26)

From (23), we see that

$$\frac{q}{p} = 2 - 2\alpha, \qquad \frac{r}{p} = 3 - 3\alpha.$$

Thus, Eq. (26) becomes

$$\begin{cases} \frac{U_1 c^{\alpha} \Gamma(3-2\alpha)}{\Gamma(4-3\alpha)} = (3-3\alpha) V_1, \\ 2c^{\alpha} V_1 = U_1^2, \end{cases}$$

from which we have that

$$\begin{split} \mathcal{U}_1 &= V_0, \\ \left\{ \begin{aligned} & \mathcal{U}_1 = \frac{2c^{2\alpha}}{3\alpha - 3} \frac{\Gamma(3 - 2\alpha)}{\Gamma(4 - 3\alpha)}, \\ & V_1 = \frac{2c^{3\alpha}}{(3\alpha - 3)^2} \frac{\Gamma^2(3 - 2\alpha)}{\Gamma^2(4 - 3\alpha)}. \end{aligned} \right. \end{split}$$

Hence, when  $\xi > 0$ , we obtain the traveling-wave similarity solutions to Eq. (1) as follows:

$$\begin{cases} u(x,t) = x^{\frac{q}{p}} U_1 \xi^{\beta} = U_1 (ct-x)^{2-2\alpha} = \frac{2c^{2\alpha}}{3\alpha-3} \frac{\Gamma(3-2\alpha)}{\Gamma(4-3\alpha)} (ct-x)^{2-2\alpha}, \\ v(x,t) = x^{\frac{r}{p}} V_1 \xi^{\gamma} = V_1 (ct-x)^{3-3\alpha} = \frac{2c^{3\alpha}}{(3\alpha-3)^2} \frac{\Gamma^2(3-2\alpha)}{\Gamma^2(4-3\alpha)} (ct-x)^{3-3\alpha}. \end{cases}$$

#### 3 Discussion on solutions to Eq. (2)

In the section we shall discuss similarity solutions to Eq. (2) by using the deformed similarity transformation (8). Applying (8), Eq. (2) is transformed to

$$\begin{cases} \lambda^{q-1} \frac{\partial \tilde{u}}{\partial \tilde{t}} - \lambda^{r-p+\rho r} \tilde{\nu}^{\rho} \frac{\partial \tilde{v}}{\partial \tilde{x}} = \lambda^{q-\beta p} \frac{\partial^{\beta} \tilde{u}}{\partial \tilde{x}^{\beta}}, \\ \lambda^{r-1} \frac{\partial \tilde{\nu}}{\partial \tilde{t}} + \lambda^{2q-p} \tilde{u} \frac{\partial \tilde{u}}{\partial \tilde{x}} = \lambda^{r-\beta p} \frac{\partial^{\beta} \tilde{\nu}}{\partial \tilde{x}^{\beta}}. \end{cases}$$
(27)

In order to keep invariant solutions to Eq. (2), we require that

$$q-1=r-p+\rho r=q-\beta p, \qquad r-1=2q-p=r-\beta p,$$

which gives that

$$q = \frac{2p - 2 + \rho(p - 1)}{1 + 2\rho}, \qquad r = \frac{3p - 3}{1 + 2\rho}, \qquad p = \frac{1}{\beta}.$$
(28)

According to the characteristic equation (9), we get that

$$\xi = t^{-p}x, \qquad u = t^{q}U(\xi), \qquad v = t^{p}V(\xi).$$
 (29)

For  $n - 1 \le \beta \le n$ , by using definition (4), it is easy to compute that

$$\frac{d^{\beta}\phi(x,t)}{dx^{\beta}} = \frac{1}{\Gamma(n-\beta)} \frac{\partial^{n}}{\partial x^{n}} \int_{0}^{x} \frac{\phi(\tilde{x},t)}{(x-\tilde{x})^{\beta+1-n}} d\tilde{x} =: \frac{\partial^{n}}{\partial x^{n}} I(x,t),$$

where

$$I(x,t)=\frac{1}{\Gamma(n-\beta)}\int_0^x\frac{\phi(\tilde{x},t)}{(x-\tilde{x})^{\beta+1-n}}\,d\tilde{x}.$$

For  $\phi(x, t) = u(x, t)$ , we have

$$\frac{\partial^n I(x,t)}{\partial x^n} = \frac{1}{\Gamma(n-\beta)} \frac{\partial^n}{\partial x^n} \int_0^x \frac{t^q U(t^{-p} \tilde{x})}{(x-\tilde{x})^{\beta+1-n}} d\tilde{x}.$$
(30)

Set  $\eta = t^{-p} \tilde{x}$ , then we find that

$$x-\tilde{x}=t^p(\xi-\eta),\qquad rac{\partial^n}{\partial x^n}=t^{-np}rac{\partial^n}{\partial \xi^n}.$$

Inserting the above calculations into (30) yields

$$\frac{d^{\beta}\phi(x,t)}{dx^{\beta}} = \frac{1}{\Gamma(n-\beta)}t^{-np}\frac{\partial^{n}}{\partial\xi^{n}}\int_{0}^{\xi}\frac{t^{q}U(\eta)t^{p}}{t^{p(\beta+1-n)}(\xi-\eta)^{\beta+1-n}}\,d\eta = t^{q-\beta p}\frac{d^{\beta}U(\xi)}{d\xi^{\beta}}.$$

Similarly, we get

$$\frac{d^{\beta}v}{dx^{\beta}} = t^{r-\beta p} \frac{d^{\beta}V(\xi)}{d\xi^{\beta}}.$$

Substituting the above results into Eq. (2), we have

$$\begin{cases} t^{q-1}[qU(\xi) - p\xi U'(\xi)] - t^{r+\rho r-p}(V(\xi))^{\rho} V'(\xi) = t^{q-\beta p} \frac{d^{\beta} U(\xi)}{d\xi^{\beta}}, \\ t^{r-1}[rV(\xi) - p\xi V'(\xi)] + t^{2q-p} U(\xi) U'(\xi) = t^{r-\beta p} \frac{d^{\beta} V(\xi)}{d\xi^{\beta}}, \end{cases}$$
(31)

which requires that

$$\begin{cases} q-1=r+\rho r-p=q-\beta p,\\ r-1=2q-p=r-\beta p, \end{cases}$$

which is equivalent to (28). Thus, a space-fractional ordinary differential system is given by

$$\begin{cases} qU(\xi) - p\xi U'(\xi) - V^{\rho}(\xi)V'(\xi) = \frac{d^{\beta}U(\xi)}{d\xi^{\beta}}, \\ rV(\xi) - p\xi V'(\xi) + U(\xi)U'(\xi) = \frac{d^{\beta}V(\xi)}{d\xi^{\beta}}. \end{cases}$$
(32)

Next we seek similarity solutions in the form

$$U(\xi) = U_1 \xi^{\sigma}, \qquad V(\xi) = V_1 \xi^{s},$$
(33)

where  $U_1, V_1, \sigma, s$  are constants to be determined. Substituting (33) into (32) gives

$$\begin{cases} qU_{1}\xi^{\sigma} - pU_{1}\sigma\xi^{\sigma} - V_{1}^{\rho+1}s\xi^{s\rho+s-1} = \frac{U_{1}B(1-\beta,1+\sigma)}{\Gamma(1-\beta)}(1-\beta+\sigma)\xi^{\sigma-\beta}, \\ (rV_{1} - pV_{1}s)\xi^{s} + U_{1}^{2}\sigma\xi^{2\sigma-1} = \frac{V_{1}B(1-\beta,1+s)}{\Gamma(1-\beta)}(1-\beta+s)\xi^{s-\beta}. \end{cases}$$
(34)

Let

$$\begin{cases} \sigma = s\rho + s - 1 = \sigma - \beta, \\ s = 2\sigma - 1 = s - \beta, \end{cases}$$

then it is easy to get

$$\beta = 0, \qquad \sigma = \frac{2+\rho}{1+2\rho}, \qquad s = \frac{3}{1+2\rho}.$$

Thus, Eq. (34) reduces to

$$\begin{cases} (q - p\sigma)U_1 - sV_1^{\rho+1} = (1 + \sigma)B(1, 1 + \sigma)U_1, \\ (r - ps)V_1 + \sigma U_1^2 = (1 + s)B(1, 1 + s)V_1, \end{cases}$$
(35)

which has solution as follows:

$$\begin{cases} V_1 = \left(\frac{[(1+\sigma)B(1,1+\sigma)-q+p\sigma]^2[(1+s)B(1,1+s)-r+ps]}{\sigma s^2}\right)^{\frac{1}{2\rho+1}},\\ U_1 = \pm \left(\frac{[(1+s)B(1,1+s)-r+ps]}{\sigma}V_1\right)^{\frac{1}{2}}. \end{cases}$$

Hence, we only obtain the solutions like (33) of the reduced system (32). That is to say, when  $\beta \neq 0$ , the space-fractional differential system (32) does not have the similarity solution (33). Therefore, the numerical solutions of (32) could be considered by some numerical methods.

## 4 Approximated solution formulas for solving ordinary fractional differential equations

Based on the works in [13-20], we want to present two approximated formulas for solving Eq. (32) so that the corresponding numerical solutions to Eq. (2) could be generated. The approach also suits for the ordinary fractional differential equations (15) and (24). From the binomial expansion formula

$$(1-z)^{-\alpha} = 1 + \sum_{p=1}^{\infty} \frac{\Gamma(\alpha+p)}{\Gamma(\alpha)p!} z^p = 1 + \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)} z + o(z^2),$$

we find that

$$\frac{d^{\alpha}\varphi(\xi)}{d\xi^{\alpha}} = \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial\xi} \int_{0}^{\xi} \frac{\varphi(y)}{(\xi-y)^{\alpha}} dy \approx \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial\xi} \int_{0}^{\xi} \frac{1}{\xi^{\alpha}} \left(1+\alpha\frac{y}{\xi}\right) \varphi(y) dy$$
$$= \frac{\alpha\xi^{-\alpha}}{\Gamma(1-\alpha)} \left[\frac{1}{\xi} \int_{0}^{\xi} \varphi(y) dy - (\alpha+1)\xi^{-2} \int_{0}^{\xi} y\varphi(y) dy - \frac{\alpha+1}{\alpha}\varphi(y)\right].$$
(36)

It follows from (36) that Eq. (32) reduces to the following integer-order ordinary differential equations:

$$\begin{cases} \frac{\beta\xi^{-\alpha}}{\Gamma(1-\alpha)} [\frac{1}{\xi} V_0(\xi) - (\alpha+1)\xi^{-2} V_1(\xi) - \frac{\alpha+1}{\alpha} U(\xi)] \\ = q U(\xi) - p\xi U'(\xi) - V^{\rho}(\xi) V'(\xi), \\ \frac{\beta\xi^{-\alpha}}{\Gamma(1-\alpha)} [\frac{1}{\xi} w_0(\xi) - (\alpha+1)\xi^{-2} w_1(\xi) - \frac{\alpha+1}{\alpha} V(\xi)] \\ = r V(\xi) - p\xi V'(\xi) + U(\xi) U'(\xi), \end{cases}$$
(37)

where

$$V_i(\xi) = \int_0^{\xi} \tau^i U(\tau) \, d\tau, \qquad w_i(\xi) = \int_0^{\xi} \tau^i V(\tau) \, d\tau, \quad i = 0, 1, \dots.$$

Taking  $x_1 = x_1(\xi) = U(\xi)$ ,  $y_1 = y_1(\xi) = V(\xi)$ , then (37) becomes

$$\begin{cases} p\xi x_1' + y_1^{\rho}y_1' = qx_1 - \frac{\beta\xi^{-\alpha}}{\Gamma(1-\alpha)} [\frac{1}{\xi}V_0(\xi) - (\alpha+1)\xi^{-2}V_1(\xi) - \frac{\alpha+1}{\alpha}x_1], \\ p\xi y_1' - x_1x_1' = ry_1 - \frac{\beta\xi^{-\alpha}}{\Gamma(1-\alpha)} [\frac{1}{\xi}w_0(\xi) - (\alpha+1)\xi^{-2}w_1(\xi) - \frac{\alpha+1}{\alpha}y_1], \end{cases}$$
(38)

subject to the initial conditions

$$x_1(0) = x_0, \qquad y_1(0) = y_0, \qquad V_0(0) = V_1(0) = 0, \qquad w_0(0) = w_1(0) = 0.$$
 (39)

By utilizing (38) and (39) we could obtain some numerical solutions of Eq. (2) through following the works in [10–15]. Here we only generate the approximated formula of Eq. (32).

For an open and boundary set  $G \subset R$ , any absolutely continuous function  $\phi(x) : G \to R$ , the left Riemann–Liouville fractional derivative of order  $\alpha$  (0 <  $\alpha \le 1$ ) can be written as

$$\frac{\partial^{\alpha}\phi(\xi)}{\partial\xi^{\alpha}} = \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial\xi} \int_{0}^{\xi} \frac{\phi(\tau)}{(\xi-\tau)^{\alpha}} d\tau = \frac{1}{\Gamma(1-\alpha)} \left(\frac{\phi(0)}{\xi^{\alpha}} + \int_{0}^{\xi} (\xi-\tau)^{-\alpha} \phi^{(1)}(\tau) d\tau\right)$$
$$= \frac{1}{\xi^{\alpha}} \frac{1}{\Gamma(1-\alpha)} \int_{0}^{\xi} \phi^{(1)}(\tau) \left(1 + \sum_{i=1}^{\infty} \frac{\Gamma(\alpha+i)}{\Gamma(\alpha)i!} \left(\frac{\tau}{\xi}\right)^{i}\right) + \frac{\phi(0)}{\xi^{\alpha}\Gamma(1-\alpha)}$$
$$= \frac{1}{\xi^{\alpha}} \int_{0}^{\xi} \frac{\phi^{(1)}(\tau)}{\Gamma(1-\alpha)} d\tau + \frac{\phi(0)}{\xi^{\alpha}\Gamma(1-\alpha)} + \sum_{i=1}^{\infty} \frac{1}{\xi^{\alpha}} \int_{0}^{\xi} \frac{\Gamma(\alpha+i)(\frac{\tau}{\xi})^{i}}{\Gamma(\alpha)\Gamma(1-\alpha)i!} d\tau$$
$$= \frac{1}{\xi^{\alpha}} \frac{1}{\Gamma(1-\alpha)} \phi(\xi) + \frac{1}{\xi^{\alpha}} \frac{\sin(\pi\alpha)}{\pi} \sum_{i=1}^{\infty} \frac{\Gamma(i+\alpha)}{i!} \int_{0}^{\xi} \phi^{(1)}(\tau) \left(\frac{\tau}{\xi}\right)^{i} d\tau. \tag{40}$$

Since

$$\int_0^{\xi} \phi^{(1)}(\tau) \left(\frac{\tau}{\xi}\right)^i d\tau = \xi^{-i} \int_0^{\xi} d\phi(\tau) = \phi(\xi) - i\xi^{-\alpha} \int_0^{\xi} \tau^{i-1} \phi(\tau) d\tau,$$

(40) can be written as

$$\frac{\partial^{\alpha}\phi(\xi)}{\partial\xi^{\alpha}} = \frac{\phi(\xi)}{\Gamma(1-\alpha)\xi^{\alpha}} \left[ 1 + \frac{\sin(\pi\alpha)}{\pi} \sum_{i=1}^{\infty} \frac{\Gamma(i+\alpha)}{i!} \right] 
- \frac{\sin(\pi\alpha)}{\pi\xi^{\alpha}} \sum_{i=1}^{\infty} \frac{\Gamma(i+\alpha)}{(i-1)!} \xi^{-i} \int_{0}^{\xi} \tau^{i-1}\phi(\tau) d\tau 
= \frac{\phi(\xi)}{\Gamma(1-\alpha)\xi^{\alpha}} \left[ 1 + \frac{\sin(\pi\alpha)}{\pi} \sum_{i=1}^{m} \frac{\Gamma(i+\alpha)}{i!} \right] 
- \frac{\sin(\pi\alpha)}{\pi\xi^{\alpha}} \sum_{i=1}^{m} \frac{\Gamma(i+\alpha)}{(i-1)!} \xi^{-i} \int_{0}^{\xi} \tau^{i-1}\phi(\tau) d\tau 
+ R_{m+1}(\xi),$$
(41)

where

$$R_{m+1}(\xi) = \frac{\phi(\xi)}{\Gamma(1-\alpha)\xi^{\alpha}} \sum_{i=m+1}^{\infty} \frac{\Gamma(i+\alpha)}{i!} - \frac{\sin(\pi\alpha)}{\pi\xi^{\alpha}} \sum_{i=m+1}^{\infty} \frac{\Gamma(i+\alpha)}{(i-1)!} \xi^{-i} \int_{0}^{\xi} \tau^{i-1} \phi(\tau) d\tau.$$

When  $i \to \infty$ ,  $i \in N$ ,  $\alpha \in R$ ,  $|\alpha| \to \infty$ , we have

$$\frac{\Gamma(i+\alpha)}{i!} \approx \frac{i+1}{i^{2-\alpha}},$$
$$\left|\sum_{i=m+1}^{\infty} \frac{\Gamma(i+\alpha)}{i!}\right| \le \left|\sum_{i=m+1}^{\infty} \frac{i+1}{i^{2-\alpha}}\right| \le 2\sum_{i=m+1} \frac{1}{i^{2-\alpha}} \frac{1}{m+1}.$$

Choose  $2 - \alpha - \alpha_1 > 1$ , then the series  $\sum_{i=m+1}^{\infty} \frac{\Gamma(i+\alpha)}{i!}$  is convergent and tends to zero when  $m \to \infty$ . Since  $\phi(t)$  is absolutely continuous, there exists a positive constant M such that

 $|\phi(t)| \le M, t \in [0, \xi]$ . For  $\xi \in [0, T]$ , it holds that

$$\left|\sum_{i=m+1}^{\infty} \frac{\Gamma(i+\alpha)}{(i-1)!} \xi^{-i} \int_0^{\xi} \tau^{i-1} \phi(\tau) \, d\tau\right| \le M \sum_{i=m+1}^{\infty} \left|\frac{\Gamma(i+\alpha)}{i!}\right|.$$

When  $m \to \infty$ ,  $|R_{m+1}(\xi)| \to 0$ . Thus, an approximated formula of system (32) is given by

$$\begin{cases} \frac{U(\xi)}{\Gamma(1-\alpha)\xi^{\alpha}} \left[1 + \frac{\sin(\pi\alpha)}{\pi} \sum_{i=1}^{m} \frac{\Gamma(i+\alpha)}{i!}\right] - \frac{\sin(\pi\alpha)}{\pi\xi^{\alpha}} \sum_{i=1}^{m} \frac{\Gamma(i+\alpha)}{(i-1)!} \xi^{-i} \int_{0}^{\xi} \tau^{i-1} U(\tau) d\tau \\ = qU(\xi) - p\xi U'(\xi) - V^{\rho}(\xi) V'(\xi), \\ \frac{V(\xi)}{\Gamma(1-\alpha)\xi^{\alpha}} \left[1 + \frac{\sin(\pi\alpha)}{\pi} \sum_{i=1}^{m} \frac{\Gamma(i+\alpha)}{i!}\right] - \frac{\sin(\pi\alpha)}{\pi\xi^{\alpha}} \sum_{i=1}^{m} \frac{\Gamma(i+\alpha)}{(i-1)!} \xi^{-i} \int_{0}^{\xi} \tau^{i-1} V(\tau) d\tau \\ = rV(\xi) - p\xi V'(\xi) + U(\xi) U'(\xi). \end{cases}$$
(42)

Taking  $x = x(\xi) = U(\xi)$ ,  $y = y(\xi) = V(\xi)$ , system (42) is rewritten as

$$\begin{cases} p\xi x' + V^{\rho}(\xi)y' \\ = qx + \frac{x}{\Gamma(1-\alpha)\xi^{\alpha}} \left[1 + \frac{\sin(\pi\alpha)}{\pi} \sum_{i=1}^{m} \frac{\Gamma(i+\alpha)}{i!}\right] - \frac{\sin(\pi\alpha)}{\pi\xi^{\alpha}} \sum_{i=1}^{m} \frac{\Gamma(i+\alpha)}{(i-1)!} \xi^{-i} V_{i}(\xi), \\ p\xi y' - xx' \\ = ry - \frac{x}{\Gamma(1-\alpha)\xi^{\alpha}} \left[1 + \frac{\sin(\pi\alpha)}{\pi} \sum_{i=1}^{m} \frac{\Gamma(i+\alpha)}{i!}\right] + \frac{\sin(\pi\alpha)}{\pi\xi^{\alpha}} \sum_{i=1}^{m} \frac{\gamma(i+\alpha)}{(i-1)!} \xi^{-i} w_{i}(\xi), \end{cases}$$
(43)

where

$$V_i(\xi) = \int_0^{\xi} \tau^{i-1} U(\tau) \, d\tau, \qquad w_i(\xi) = \int_0^{\xi} \tau^{i-1} V(\tau) \, d\tau,$$

subject to the initial conditions

$$x(0) = x_0,$$
  $y(0) = y_0,$   
 $V_0(0) = V_1(0) = 0,$  ...,  $w_0(0) = w_1(0) = 0,$  ....

Similarly, with the help of (43) and its initial values, we could generate some numerical solutions of system (2) just like those presented in [10-15], here we also skip them.

#### **5** Separated variable solutions

Luchko [16] introduced a generalized time-fractional diffusion equation

$$(D_t^{\alpha})u(t) = -L(u) + F(x,t), \quad 0 < \alpha \le 1, (x,t) \in G \times (0,T), G \subset \mathbb{R}^2,$$
(44)

where

$$L(u) = -\operatorname{div}(p(x)\operatorname{grad} u) + q(x)u, \quad p(x) \ge 0, q(x) \ge 0, x \in \overline{G}.$$

 $D^{\alpha}_t$  is the Caputo–Dzherbashyan fractional derivative. An initial-boundary-value problem is introduced by

$$u|_{t=0} = u_0(x), \quad x \in \bar{G},$$
(45)

$$u|_{S} = v(x,t), \quad (x,t) \in S \times [0,T],$$
(46)

here *S* stands for the surface of  $\overline{G}$ . A classical solution of (44)–(46) u = u(x,t) is defined in the domain  $\overline{\Omega}_T =: \overline{G} \times [0, T]$  that belongs to the space  $C(\overline{\Omega}_T) \cap W'_t((0, T]) \cap C^2_x(G)$ , where  $W'_t((0, T])$  represents a space of the functions  $f \in C'((0, T])$  so that  $f' \in L((0, T)), u_0, v$  belong to the spaces  $C(\overline{\Omega}), C(S \times [0, T])$ , respectively. Luchko presented two results of classical solutions to (44)–(46) as follows.

**Theorem 1** If u is a classical solution of Eqs. (44)–(46), and  $F \in C(\overline{\Omega}_T)$  with the norm  $M = ||F||_{C_{\overline{\Omega}_T}}$ , then the estimate

$$\|u\|_{C(\bar{\Omega}_T)} \le \max\{M_0, M_1\} + \frac{T^{\alpha}}{\Gamma(1+\alpha)}M$$

holds true, where  $M_0 = ||u_0||_{C(\bar{G})}, M_1 = ||v||_{C(S \times [0,T])}$ .

**Theorem 2** The initial-boundary-value problem (45), (46) possesses at most one classical solution, which continuously depends on the data given in problem (44)-(46).

**Definition** ([16]) Assume  $F_k \in C(\overline{\Omega}_T)$ ,  $u_{0k} \in C(\overline{G})$ ,  $v_k \in (S \times [0, T])$ , k = 1, 2, ..., satisfy the following items:

(1) There exist the functions F,  $u_0$ , and v such that

 $\begin{aligned} \|F_k - F\|_{C(\bar{\Omega}_T)} &\to 0, \text{ as } k \to \infty, \\ \|u_{0k} - u_0\|_{C(\bar{G})} &\to 0, \text{ as } k \to \infty, \\ \|v_k - v\|_{C(S \times [0,T])} &\to 0, \text{ as } k \to \infty. \end{aligned}$ 

(2) For any k = 1, 2, ..., there exist the classical solutions  $u_k$  of the

initial-boundary-value problem

 $u_k|_{t=0} = u_{0k}(x), x\bar{G},$ 

$$u_k|_S = v_k(x, t), (x, t) \in S \times [0, T],$$

for the generalized time-fractional diffusion equation

 $(D_t^{\alpha}u_k)(t) = -L(u_k) + F_k(x,t).$ 

If there exists a function  $u \in C(\overline{\Omega}_T)$  such that  $||u_k - u||_{C(\overline{G})} \to 0$ , as  $k \to \infty$ , the function u is called a generalized solution of problem (44)–(46).

Based on the definition, Luchko showed us the following result.

**Theorem 3** *Problem* (44)–(46) *possesses at most one generalized solution. If it exists, then it continuously depends on the data given in the problem in the sense of Theorem 2.* 

It is easy to find that the Caputo–Dzhershyan fractional derivative is a special case of the left Riemann–Liouville fractional derivative (3), that is, when f(0) = 0, the Riemann–Liouville fractional derivative (3) is reduced to the Caputo–Dzherbashyan fractional derivative. For the nonlinear model of STPPGF (1), we assume  $u(x, t), v(x, t) \in G_T \subset G \times [0, T]$ , here  $G_T$  is an open and boundary subset,  $G \subset R$  is an open interval of R. Given

an initial-boundary-value condition of system (1):

$$u|_{t=0} = u_0(x), \qquad v|_{t=0} = v_0(x), \qquad u|_S = \bar{u}(x,t),$$

$$v|_S = \bar{v}(x,t), \qquad (x,t) \in S \times [0,T].$$
(47)

If there exist positive constants  $M_1$  and  $M_2$  such that

$$0 \le u_0(x) \le M_1 t^{lpha}$$
,  $0 \le v_0(x) \le M_2 t^{lpha}$ ,

and u(x, t), v(x, t) at  $t_0 \in [0, T]$  attain their maximums, then the Riemann–Liouville fractional derivative satisfies

$$\frac{\partial^{\alpha} u}{\partial t_{0}^{\alpha}} \ge 0, \qquad \frac{\partial^{\alpha} v}{\partial t_{0}^{\alpha}} \ge 0.$$
(48)

Actually, according to definition (3), we find that

$$\frac{\partial^{\alpha} u}{\partial t_{0}^{\alpha}} = \frac{1}{\Gamma(1-\alpha)} \left[ \frac{u_{0}(x)}{t^{\alpha}} + \int_{0}^{t} \frac{u(x,\tau)}{(t-\tau)^{\alpha}} d\tau \right] = \frac{u_{0}(x)}{\Gamma(1-\alpha)t^{\alpha}} + D_{t_{0}}^{\alpha} u(x,\tau).$$

By introducing an auxiliary function [16]

$$U(\tau) = u(x, t_0) - u(x, \tau), \qquad V(\tau) = v(x, t_0) - v(x, \tau), \quad \tau \in [0, T],$$

it is easy to see for the fixed x that

$$U(\tau) \ge 0, \tag{49}$$

$$D^{\alpha}U(t) = -(D^{\alpha}u)(x,t), \quad t \in [0,T],$$
(50)

$$D^{\alpha}V(t) = -(D^{\alpha}v)(x,t), \quad t \in [0,T],$$
(51)

$$\left| U(\tau) \right| \le c_1 |t_0 - \tau|, \qquad \left| V(\tau) \right| \le c_2 |t_0 - \tau|, \tag{52}$$

where  $c_1 = c_1(x, \epsilon)$ ,  $c_2 = c_2(x, \epsilon)$  are constants with respect to variable *t* for the fixed *x*. (50) and (51) indicate that (48) holds true. According to Theorems 1 and 2, we can prove the following.

**Proposition** If  $u, \tilde{u}$  are the classical solutions of (1) and (47), the initial conditions  $u_0, \tilde{u}_0$  and the boundary conditions v and  $\tilde{v}$  satisfy

$$\begin{split} \|u_{0} - \tilde{u}_{0}\|_{C(\tilde{G})} &\leq \epsilon_{0}, \qquad \|v_{0} - \tilde{v}_{0}\|_{C(\tilde{G})} \leq \epsilon_{1}, \\ \|\bar{u} - \tilde{u}\|_{C(S \times [0,T])} &\leq \epsilon_{2}, \qquad \|\bar{v} - \tilde{v}\|_{C(S \times [0,T])} \leq \epsilon_{3}, \end{split}$$

then the normal estimates hold:

$$\|u - \tilde{u}\|_{C(\bar{\Omega}_T)} \le \max\{\epsilon_0, \epsilon_2\},$$
$$\|v - \tilde{v}\|_{C(\bar{\Omega}_T)} \le \max\{\epsilon_1, \epsilon_3\}.$$

For the generalized solutions to system (2) with (47), the above proposition still holds true.

In what follows, we apply the variable separation method to investigate generalized solutions of system (1) with (47). Set

$$u(x,t) = f(x)g(t), \quad v(x,t) = p(x)r(t),$$
(53)

and substitute into system (1), we have

$$\begin{cases} f(x)\frac{\partial^{\alpha}g(t)}{\partial t^{\alpha}} = p'(x)r(t), \\ p(x)\frac{\partial^{\alpha}r(t)}{\partial t^{\alpha}} = -g^{2}(t)f(x)f'(x), \end{cases}$$

which can be written as

$$\frac{\frac{\partial^{\alpha}g(t)}{\partial t^{\alpha}}}{r(t)} = \frac{p'(x)}{f(x)} = \lambda,$$
(54)

$$\frac{\frac{\partial^{\alpha} r(t)}{\partial t^{\alpha}}}{g^2(t)} = -\frac{f(x)f'(x)}{p(x)} = \mu.$$
(55)

(54) and (55) can be written as again

$$\frac{\partial^{\alpha}}{\partial t^{\alpha}} \left( \frac{\partial^{\alpha}}{\partial t^{\alpha}} g \right)(t) = \lambda \mu g^{2}(t), \tag{56}$$

$$(f'(x))^{2} + f(x)f''(x) = -\lambda \mu f(x),$$
(57)

or

$$p'(x)p''(x) = -\mu\lambda^2 p(x), \tag{58}$$

subject to the initial values

$$u_0(x) = f(x)g(0), \qquad v_0(x) = p(x)r(0),$$
  
$$u|_S = \bar{u}(x,t) = f(x)g(t)|_S, \qquad v|_S = \bar{v}(x,t) = p(x)r(t)|_S.$$

Assume that

$$g(0) = g_0, \qquad \frac{\partial^{\alpha - 1}}{\partial t^{\alpha - 1}} g(0) = x_0, \qquad p(0) = 0, \qquad p'(0) = 0, \tag{59}$$

then Eq. (58) can be transformed to

$$y^{2} \frac{dy}{dp} = -\lambda^{2} \mu p =: \frac{K}{2} (p^{2}(x))',$$

$$\frac{dp}{dx} = y = \left(\frac{3K}{2} p^{2}\right)^{\frac{1}{3}},$$
(60)

where y(x) = p'(x), and we have taken the integral constant to be zero. A solution of (60) is given by

$$p(x) = \frac{1}{27} \left(\frac{3K}{2}\right)^{\frac{1}{3}} x^3,\tag{61}$$

where we also took integral constant zero. For the different  $K_i$  (i = 1, 2, ...), we assume that  $K_1 < K_2 < \cdots$ , then Eq. (61) can be written as

$$p_i(x) = \frac{1}{27} \left(\frac{3K_i}{2}\right)^{\frac{1}{3}} x^3, \quad K_i = -\lambda_i^2 \mu_i, i = 1, 2, \dots$$
 (62)

In terms of Ref. [13], we find that Eq. (56) with its initial value conditions (59) can be written as

$$\frac{\partial^{\alpha}}{\partial t^{\alpha}}\left(\frac{\partial^{\alpha}}{\partial t^{\alpha}}g(t)\right) = \frac{\partial^{2\alpha}}{\partial t^{2\alpha}}g(t) - \left(\frac{\partial^{\alpha-1}}{\partial t^{\alpha-1}}g(t)|_{t=0}\right)\frac{t^{-\alpha}}{\Gamma(1-\alpha)} = \frac{\partial^{2\alpha}}{\partial t^{2\alpha}}g(t) - x_0\frac{t^{-\alpha}}{\Gamma(1-\alpha)}.$$

Thus, we have

$$\frac{\partial^{2\alpha}g(t)}{\partial t^{2\alpha}} = \frac{x_0}{\Gamma(1-\alpha)}t^{-\alpha} + \lambda\mu g^2(t) =: G(t,g(t)).$$
(63)

In terms of definition of the Riemann–Liouville type of fractional integral of order  $\alpha > 0$  of a function  $f : (0, \infty) \rightarrow R$ :

$$I^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}f(\tau)\,d\tau,$$

where  $0 < \alpha \leq 1$ , it holds

$$I^{\alpha} \left( D^{\alpha} f \right)(t) = f(t). \tag{64}$$

Hence, Eq. (63) can be expressed by the following Volterra fractional integral equation:

$$g(t) = g_0 + \frac{1}{\Gamma(2\alpha)} \int_0^t (t-\tau)^{2\alpha-1} G(\tau, g(\tau)) d\tau - I^{2\alpha} \left(\frac{g_0 t^{-2\alpha}}{\Gamma(1-2\alpha)}\right).$$
(65)

Demirci and Ozalp [18] discussed the initial value problem (IVP) with Caputo type FDE as follows:

$$D^{\alpha}x(t) = f(t, x(t)), \qquad x(0) = x_0, \quad 0 < \alpha \le 1,$$
(66)

and transformed (66) into a Volterra fractional integral equation. Then, based on an existence theorem presented in [18], a resulting extended result was obtained which was given by the following theorem.

**Theorem 4** Set  $\|\cdot\|$  denoting a convenient norm on  $\mathbb{R}^n$ . If  $f \in C[\mathbb{R}_0]$ ,  $\mathbb{R}_0 = \{(t,x) : 0 \le t \le a, \|X - X_0\| \le b\}$ ,  $f = (f_1, ..., f_n)^T$ ,  $X = (x_1, ..., x_n)T$ , and  $\|f(t, x)\| \le M$  on  $\mathbb{R}_0$ , then there exists

at least one solution for the system of FDE given by

$$D^{\alpha}X(t) = f(t, X(t)), \qquad X(0) = X_0, \tag{67}$$

on  $0 \le t \le \beta, 0 < \alpha < 1, \beta = \min(a, [\frac{b}{M}\Gamma(1+\alpha)]^{\frac{1}{\alpha}})$ . The solutions of the initial problem (67) are given by the following theorem.

#### Theorem 5 Let

$$g(\nu, X_*(\nu)) = f\left(t - \left(t^\alpha - \nu \Gamma(\alpha+1)\right)^{\frac{1}{\alpha}}, X\left(t - \left(t^\alpha - \nu \Gamma(\alpha+1)\right)^{\frac{1}{\alpha}}\right)\right),$$

and assume that the conditions of Theorem 4 hold. Then a solution of (67) can be given by

$$X(t) = X_* \left(\frac{t^{\alpha}}{\Gamma(\alpha+1)}\right),\tag{68}$$

where  $X_*(v)$  is a solution of the system of integer order differential equations

$$\frac{d(X_{*}(\nu))}{d\nu} = g(\nu, X_{*}(\nu)),$$
(69)

with the initial conditions

$$X_*(0) = X_0. (70)$$

For the Riemann–Liouville type derivative, Theorems 4 and 5 suit the IVP in the form

$$\frac{\partial^{\alpha}}{\partial t^{\alpha}} (X(t) - X_0) = f(t, X(t)), \qquad X(0) = X_0.$$
(71)

Therefore, to apply the variable separation method to IVP (71), one sets

$$H(t,X(t)) = f(t,X(t)) - \frac{X_0 t^{-\alpha}}{\Gamma(1-\alpha)},$$

and solves the initial value problem

$$D^{\alpha}(X(t) - X_0) = H(t, X(t)), \qquad X(0) = X_0.$$
(72)

In order to apply Theorem 5 to solve (72), we only consider the following equation:

$$D^{\alpha}(X(t) - X_0) = H(t, X(t)), \tag{73}$$

where  $D^{\alpha}$  stands for the Caputo fractional derivative,  $0 < \alpha < 1$ . Actually, when  $0 < \alpha < 1$ , the Caputo fractional derivative is in essence the Caputo–Dzherbashyan fractional derivative. The Volterra fractional integral equation (65) was generated by (64) and (71)–(73). Since

$$I^{2\alpha}\left(\frac{g_0t^{-2\alpha}}{\Gamma(1-2\alpha)}\right) = \frac{1}{\Gamma(2\alpha)}\int_0^t \frac{g_0\tau^{-2\alpha}}{\Gamma(1-2\alpha)}(t-\tau)^{2\alpha-1}\,d\tau$$

$$=\frac{g_0}{\Gamma(2\alpha)\Gamma(1-2\alpha)}\int_0^t \tau^{-2\alpha}t^{2\alpha-1}\left(1-\frac{\tau}{t}\right)^{2\alpha-1}d\tau.$$
 (74)

Set  $y = \frac{\tau}{t}$ ,  $\tau = 0 \rightarrow y = 0$ ;  $\tau = t \rightarrow y = 1$ . Hence, (74) can be written as

$$I^{2\alpha}\left(\frac{g_0 t^{-2\alpha}}{\Gamma(1-2\alpha)}\right) = \frac{g_0}{\Gamma(2\alpha)\Gamma(1-2\alpha)} \int_0^1 y^{-2\alpha} (1-y)^{1-2\alpha} dy$$
$$= \frac{g_0}{\Gamma(2\alpha)\Gamma(1-2\alpha)} B(1-2\alpha,2-2\alpha), \tag{75}$$

where  $B(1-2\alpha, 2-2\alpha)$  is the beta function with parameter  $\alpha$ . Thus, Eq. (65) can be written as

$$g(t) = g_0 + \frac{1}{\Gamma(2\alpha)} \int_0^t (t-\tau)^{2\alpha-1} G(\tau, g(\tau)) d\tau - \frac{g_0}{\Gamma(2\alpha)\Gamma(1-2\alpha)} B(1-2\alpha, 2-2\alpha),$$
(76)

where G(t,g(t)) is given by (63). Set  $(t-\tau)^{2\alpha} = t^{2\alpha} - \nu\Gamma(1+2\alpha)$ , then  $t = \tau + (t^{2\alpha} - \nu\Gamma(1+2\alpha))^{\frac{1}{2\alpha}}$ . When  $\tau = 0 \rightarrow \nu = 0$ ;  $\tau = t \rightarrow \nu = \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}$ . Hence, Eq. (76) becomes

$$g(t) = \left(1 - \frac{B(1 - 2\alpha, 2 - 2\alpha)}{\Gamma(2\alpha)\Gamma(1 - 2\alpha)}\right)g_0 + \frac{1}{\Gamma(2\alpha)}\int_0^t (t - \tau)^{2\alpha - 1}G(\tau, g(\tau))\,d\tau$$
$$=: mg_0 + \frac{1}{\Gamma(2\alpha)}\int_0^t (t - \tau)^{2\alpha - 1}G(\tau, g(\tau))\,d\tau.$$
(77)

Let us consider the integer order differential equation with initial value based on Theorem 5:

$$\begin{cases} \frac{dg_*(v)}{dv} = w(v, g_*(v)), \\ g_*(0) = mg_0, \end{cases}$$
(78)

where

$$w(v,g_*(v)) =: w(v) = G\left[t - (t^{2\alpha} - v\Gamma(1+2\alpha))^{\frac{1}{2\alpha}}, g(t - (t^{2\alpha} - v\Gamma(1+2\alpha))^{\frac{1}{2\alpha}})\right].$$

(77) can be written as

$$g(t) = mg_0 + \int_0^{\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}} G\left[t - \left(t^{2\alpha} - \nu\Gamma(1+2\alpha)\right)^{\frac{1}{2\alpha}}, g\left(t - \left(t^{2\alpha} - \nu\Gamma(1+2\alpha)\right)^{\frac{1}{2\alpha}}\right)\right] d\nu$$
  
=:  $mg_0 + \int_0^{\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}} w(\nu, g_*(\nu)) d\nu.$  (79)

Every solution of IVP (78) is given by

$$g_*(v) = mg_0 + \int_0^v w(s, g_*(s)) \, ds. \tag{80}$$

Therefore, when  $v = \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}$ , we find that  $g_*(\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}) = g(t)$ . For a given  $\alpha$ , we can get explicit solutions of Eq. (79) by using (80) and (63). For example, let  $\alpha = \frac{1}{4}$ , then  $w(v) = G[t - (t^{\frac{1}{2}} - v\Gamma(\frac{3}{2}))^2, g(t - (t^{\frac{1}{2}} - v\Gamma(\frac{3}{2}))^2)] = \frac{x_0}{\Gamma(\frac{3}{2})}[t - (t^{\frac{1}{2}} - v\Gamma(\frac{3}{2}))^2]^{-\frac{1}{4}} + \frac{1}{2}$ 

$$\begin{split} \lambda \mu g^2(t - (t^{\frac{1}{2}} - \nu \Gamma(\frac{3}{2})^2) &= \frac{x_0}{\Gamma(\frac{3}{2})} (2\nu \Gamma(\frac{3}{2})t^{\frac{1}{2}} - \nu^2 \Gamma^2(\frac{3}{2}))^{-\frac{1}{4}} + \lambda \mu g^2 [2\nu \Gamma(\frac{3}{2})t^{\frac{1}{2}} - \nu^2 \Gamma^2(\frac{3}{2})] &=: \\ \frac{x_0}{\Gamma(\frac{3}{2})} [\Gamma(\frac{1}{2})\nu t^{\frac{1}{2}} - \nu^2 \Gamma^2(\frac{3}{2})]^{-\frac{1}{4}} + \lambda \mu g^2(\nu). \\ \text{Thus, we have} \end{split}$$

$$\frac{dg_1(\nu)}{d\nu} = \frac{x_0}{\Gamma(\frac{3}{2})} \left[ \Gamma\left(\frac{1}{2}\right) t^{\frac{1}{2}} \nu - \Gamma^2\left(\frac{3}{2}\right) \nu^2 \right]^{-\frac{1}{4}} + \lambda \mu g_1^2(\nu) =: \lambda \mu g_1^2(\nu) + P(\nu), \tag{81}$$

with initial value  $g_1(0) = mg_0$ , where  $P(\nu) = \frac{x_0}{\Gamma(\frac{3}{2})} [\Gamma(\frac{1}{2})t^{\frac{1}{2}}\nu - \Gamma^2(\frac{3}{2})\nu^2]^{-\frac{1}{4}}$ . Equation (81) possesses the general solution as follows:

$$g_1(v) = \frac{2P(v) - cP(v)(\int P(v) \, dv)^3}{\lambda \mu [\int P(v) \, dv + c(\int P(v) \, dv)^4]},$$

where *c* is an arbitrary constant, with the constraint  $\lambda \mu = 2(\frac{1}{\int P(v) dv})'$ . It is easy to see that

$$\int P(v) \, dv = \frac{2}{\lambda \mu v}, \qquad \lim_{\nu \to 0} \frac{2P(v) - cP(v)(\int P(v) \, dv)^3}{\lambda \mu [\int P(v) \, dv + c(\int P(v) \, dv)^4]} = \frac{1}{\lambda \mu}.$$

Therefore, under the constrained conditions

$$\left(1 - \frac{B(\frac{1}{2}, \frac{3}{2})}{\Gamma^{2}(\frac{1}{2})}\right)g_{0} = \frac{1}{\lambda\mu}, \qquad \frac{x_{0}}{\Gamma(\frac{3}{2})}\left[\Gamma\left(\frac{1}{2}\right)t^{\frac{1}{2}}v - \Gamma^{2}\left(\frac{3}{2}\right)v^{2}\right]^{-\frac{1}{4}} = -\frac{2}{\lambda\mu\nu^{2}},$$

the above solution  $g_1(\nu)$  can be expressed by

$$g_1(\nu) = \frac{8 - 2(\lambda\mu)^3 \nu^3}{\lambda\mu [8 + (\lambda\mu)^3 \nu^3]}.$$

Thus, we get that

$$g_i(t) = g_1\left(\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right) = g_1\left(\frac{t^{\frac{1}{2}}}{\Gamma(\frac{3}{2})}\right) = \frac{8\Gamma^3(\frac{3}{2}) - 2(\lambda\mu)^3 t^{\frac{3}{2}}}{\lambda_i\mu_i[8\Gamma^3(\frac{3}{2}) + (\lambda_i\mu_i)^3 t^{\frac{3}{2}}]}, \quad i = 1, 2, \dots.$$
(82)

From (54), we see that

$$f_i(x) = \frac{1}{\lambda_i} p_i'(x), \tag{83}$$

that is,

$$f_i(x) = \frac{1}{9\lambda_i} \left(\frac{3}{2}K\right)^{\frac{1}{3}} x^2, \quad i = 1, 2, \dots$$
 (84)

In order to seek r(t), we have in terms of (55) that

$$\frac{\partial^{\alpha} r(t)}{\partial t^{\alpha}} = \mu g^{2}(t) = \frac{1}{\lambda^{2} \mu} \left( \frac{8\Gamma^{3}(\frac{3}{2}) - 2(\lambda\mu)^{3} t^{\frac{3}{2}}}{8\Gamma^{3}(\frac{3}{2}) + (\lambda\mu)^{3} t^{\frac{3}{2}}} \right)^{2}.$$
(85)

For the Riemann-Liouville fractional derivative, we usually have the initial value problem

$$\frac{\partial^{\alpha} r(t)}{\partial t^{\alpha}} = \mu g^2(t) - \frac{r_0 t^{-\alpha}}{\Gamma(1-\alpha)}, \qquad r(0) = r_0,$$
(86)

which can be transformed to a Volterra integral equation

$$\frac{\partial^{\alpha} r(t)}{\partial t^{\alpha}} = r_0 + \frac{\mu}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} g^2(\tau) \, d\tau - \frac{r_0}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \frac{\tau^{-\alpha}}{\Gamma(1-\alpha)} \, d\tau. \tag{87}$$

To calculate conveniently, we set  $r_0 = 0$ . Equation (86) reduces to

$$\frac{\partial^{\alpha} r(t)}{\partial t^{\alpha}} = \mu g^2(t), \qquad r(0) = 0.$$
(88)

Specially, when  $\alpha = \frac{1}{4}$ , in terms of (86), we find that

$$r(t) = \frac{1}{\Gamma(\alpha)} \int_{0}^{t} (t-\tau)^{\alpha-1} \mu g^{2}(\tau) d\tau = \frac{\mu}{\Gamma(\alpha)} \int^{t} (t-\tau)^{\alpha-1} g^{2}(\tau) d\tau$$
$$= \frac{1}{\lambda^{2} \Gamma(\frac{1}{4})} \int_{0}^{t} (t-\tau)^{-\frac{3}{4}} \left( \frac{8\Gamma^{3}(\frac{3}{2}) - 2(\lambda\mu)^{3}t^{\frac{3}{2}}}{8\Gamma^{3}(\frac{3}{2}) + (\lambda\mu)^{3}t^{\frac{3}{2}}} \right)^{2} d\tau$$
$$= \frac{t^{\frac{3}{4}}}{\lambda^{2} \Gamma(\frac{1}{4})} \int_{0}^{t} \left( \frac{A - 2B\tau^{\frac{3}{2}}}{A + B\tau^{\frac{3}{2}}} \right)^{2} \frac{1}{(1-\frac{\tau}{t})^{\frac{3}{4}}} d\tau$$
$$= \frac{t^{\frac{7}{4}}}{\lambda^{2} \Gamma(\frac{1}{4})} \int_{0}^{1} \left[ 1 - \frac{6Bt^{\frac{3}{2}}y^{\frac{3}{2}}}{A + Bt^{\frac{3}{2}}y^{\frac{3}{2}}} + \frac{9B^{2}t^{3}y^{3}}{(A + Bt^{\frac{3}{2}}y^{\frac{3}{2}})^{2}} \right] (1-y)^{-\frac{3}{4}} dy, \tag{89}$$

where  $A = 8\Gamma^3(\frac{3}{2}), B = (\lambda \mu)^3$ . Obviously, it is difficult to compute integral (89). However, when  $A + Bt^{\frac{3}{2}}y^{\frac{3}{2}} \gg 3Bt^{\frac{3}{2}}y^{\frac{3}{2}}$ , i.e.,  $2Bt^{\frac{3}{2}}y^{\frac{3}{2}} \ll A$ , we can get an approximated solution of (89) as follows:

$$r(t) = \frac{t^{\frac{7}{4}}}{\lambda^2 \Gamma(\frac{1}{4})} \int_0^1 (1-y)^{-\frac{3}{4}} dy = \frac{4t^{\frac{7}{4}}}{\lambda^2 \Gamma(\frac{1}{4})}.$$
(90)

Thus, substituting (82)–(84) and (90) into (53), we can obtain the special approximated solutions of system (1) with (47) when  $\alpha = \frac{1}{4}$ . Of course, if  $\alpha = \frac{1}{2}, \frac{3}{4}, \ldots$ , we could generate different special solutions (53), here we omit the further discussions in the paper.

It was remarked that the solutions of (81) are not satisfied because there exists a case where a constrained condition on the solutions appears. In addition, we still have another approach for obtaining solutions of Eq. (81). Actually, if  $y_1(v)$ ,  $y_2(v)$  are two linear independent solutions to the following equation

$$y''(\nu) + \lambda \mu P(\nu) e^{-\lambda \mu \int y(\nu) d\nu} = 0, \tag{91}$$

then we can prove that a general solution to Eq. (81) is given by

$$g_1(\nu) = \frac{c_1 y_1' + c_2 y_2'}{\lambda \mu (c_1 y_1 + c_2 y_2)},\tag{92}$$

where  $c_1$ ,  $c_2$  are arbitrary constants. Although there is not a constrained condition on the solutions (92) of Eq. (81), it is not easy to solve Eq. (91). For this problem, we shall discuss it in another paper. For the separated variable solution to system (2) by the separated variable method, similar considerations as above can be investigated, here we omit them again.

#### 6 Conclusions

We have investigated some different solutions of the time-fractional stationary transonic plane-parallel gas flows (STPPGF) and its generalized space-fractional nonlinear system. Ma and Zhou [21, 22] have explored the existence of diverse lump and interaction solutions to linear partial differential equations in both (2 + 1)-dimension and (3 + 1)-dimension. The remarkable richness of exact solutions to a class of linear partial differential equations yielded exact lump, lump-periodic, and lump-soliton solutions. They also analyzed a class of lump solutions, generated from quadratic functions, to nonlinear partial differential equations based on the Hirota bilinear formulations. Can we follow the approaches to study the lump solutions and the rational solutions to the STPPGF and its generalized space-fractional nonlinear system? This will be further discussed in the future.

Acknowledgements

Not applicable.

#### Funding

This work is supported by the Fundamental Research Funds for the Central University (No. 2017XKZD11).

#### **Competing interests**

The authors declare that they have no competing interests.

#### Authors' contributions

YZ investigated the similarity solutions and traveling-wave solutions of the system and its generalized space-fractional nonlinear system. Two approximated solution formulas were also given by him. CY obtained a class of approximated solutions of the time-fractional nonlinear system. XZ checked the results presented in the paper by using the Maple; furthermore, he modified the version suggested by the reviewers. All authors read and approved the final manuscript.

#### Author details

<sup>1</sup>College of Mathematics, China University of Mining and Technology, Xuzhou, P.R. China. <sup>2</sup>College of Medical Information Engineering, Shandong First Medical University and Shandong Academy of Medical Sciences, Taian, China.

#### **Publisher's Note**

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

#### Received: 8 October 2018 Accepted: 16 July 2019 Published online: 25 July 2019

#### References

- 1. Bagley, R., Torvik, P.: A theoretical basis for the application of fractional calculus to viscoelasticity. J. Rheol. 27, 201–210 (1983)
- 2. Carpinteri, A., Mainardi, F.: Fractals and Fractional Calculus in Continuum Mechanics. Springer, Wien (1997)
- 3. Gazizov, R.K., Kasatkin, A.A., Lukashchuk, S.Y.: Symmetry properties of fractional diffusion equation. Phys. Scr. T 136, 014016 (2009)
- Huang, Q., Zhdanov, R.: Symmetries and exact solutions of the time fractional Harry–Dym equation with Riemann–Liouville derivative. Physica A 409, 110–118 (2014)
- 5. Huang, Q., Shen, S.F.: Lie symmetries and group classification of a class of time fractional evolution systems. J. Math. Phys. 56, 123504 (2015)
- Wang, G.W., Xu, T.Z.: Invariant analysis and explicit solutions of the time fractional nonlinear perturbed Burgers equation. Nonlinear Anal., Model. Control 20, 570–584 (2015)
- Djordjevic, V.D., Atanackovic, T.M.: Similarity solutions to nonlinear heat conduction and Burgers–Korteweg–de Vries fractional equations. J. Comput. Appl. Math. 222, 701–714 (2008)
- Dorjgotov, K., et al.: Lie symmetry analysis of a class of time fractional nonlinear evolution systems. Appl. Math. Comput. 329, 105–117 (2018)
- 9. Fernandez, A., Allözarslan, M., Baleanu, D.: On fractional calculus with general analytic kernels. Appl. Math. Comput. 354, 248–265 (2019)

- 10. Hashemi, M.S., Darvishi, E.: Baleanu, D.: A geometric approach for solving the density-dependent diffusion Nagumo equation. Adv. Differ. Equ., 2016, Article ID 89 (2016)
- 11. Hashemia, M.S., Baleanu, D.: Lie symmetry analysis and exact solutions of the time fractional gas dynamics equation. J. Optoelectron. Adv. Mater. 18, 383–388 (2016)
- 12. Hashemi, M.S., Baleanu, D.: On the time fractional generalized Fisher equation: group similarities and analytical solutions. Commun. Theor. Phys. 65, 11–16 (2016)
- Yuan, L., Agrawal, O.P.: A numerical scheme for dynamic systems containing fractional derivatives. J. Vib. Acoust. 14, 321–324 (2002)
- 14. Atanackovic, T.M., et al.: Expansion formula for fractional derivatives in variational problems. J. Math. Anal. Appl. 409, 911–924 (2014)
- Atanackovic, T.M., Stankovic, B.: On a numerical scheme for solving differential equations of fractional order. Mech. Res. Commun. 35, 429–438 (2008)
- Luchko, Y.: Some uniqueness and existence results for the initial-boundary-value problems for the generalized time-fractional diffusion equation. Comput. Math. Appl. 59, 1766–1772 (2010)
- 17. Podlubny, I.: Fractional Differential Equations Mathematics in Science and Engineering 198. Academic Press, San Diego (1999)
- Demirci, E., Ozalp, N.: A method for solving differential equations of fractional order. J. Comput. Appl. Math. 236, 2754–2762 (2012)
- Lakshmikantham, V., Vatsala, A.S.: Theory of fractional differential inequalities and applications. Commun. Appl. Anal. 11, 395–402 (2007)
- 20. Zhang, Y., et al.: Symmetry properties and explicit solutions of some nonlinear differential and fractional equations. Appl. Math. Comput. **337**, 408–418 (2018)
- Ma, W.X.: Abundant lumps and their interaction solutions of (3 + 1)-dimensional linear PDEs. J. Geom. Phys. 133, 10–16 (2018)
- 22. Ma, W.X., Zhou, Y.: Lump solutions to nonlinear partial differential equations via Hirota bilinear forms. J. Differ. Equ. 264, 2633–2659 (2018)

## Submit your manuscript to a SpringerOpen<sup>®</sup> journal and benefit from:

- Convenient online submission
- ► Rigorous peer review
- ► Open access: articles freely available online
- ► High visibility within the field
- ▶ Retaining the copyright to your article

Submit your next manuscript at > springeropen.com