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# Representation of solution for a linear fractional delay differential equation of Hadamard type

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**Abstract**
This paper is devoted to seeking the representation of solutions to a linear fractional delay differential equation of Hadamard type. By introducing the Mittag-Leffler delay matrix functions with logarithmic functions and analyzing their properties, we derive the representation of solutions via the constant variation method.

**Keywords:** Hadamard; Linear fractional delay differential equation; Mittag-Leffler delay matrix functions; Representation of solutions

## 1 Introduction

In the recent decades, fractional differential equations have been applied in engineering, physics, finance, and signal analysis. The researchers focused on the investigation of the existence, asymptotic stability, and finite-time stability of solutions of fractional linear and non-linear differential equations of Caputo type, Riemann–Liouville type, and Hadamard type [1–11].

Recently, the representation of solutions to delay differential equations has been considered. Klusainov and Shukin [12], Diblik and Klusainov [13, 14] derived the exact expressions of solutions of linear time invariant continuous and discrete delay equations by proposing the concepts of delay matrix functions. Next, stability and controllability problems of linear delay differential equations were studied extensively in [15–17]. For the literature on the related topic of linear fractional delay equations of Caputo and Riemann–Liouville type, we refer the reader to [18–25]. However, we find that there exists very limited work on the representation of solutions of fractional order delay differential equations of Hadamard type, even for linear case.

Motivated by the above-mentioned works, we try to introduce a new concept on fractional delay matrix function with a logarithmic function and use it to study the following linear fractional delay differential equations of Hadamard type:

$$\begin{cases} ({}_{H}\mathbb{D}^{\alpha}_{1^+}y)(x) = By(x-\tau), & B \in R^{n\times n}, x \in (\tau, T], \tau > 0, \\ y(x) = \varphi(x), & \varphi(x) \in R^{n}, 1 < x \leq \tau, \\ ({}_{H}\mathbb{I}^{1-\alpha}_{1^+}y)(1^+) = b, & b \in R^{n}, \end{cases} \tag{1}$$

where ${}_H\mathbb{D}^{\alpha}_{1^+}y$ denotes the $\alpha$ order Hadamard derivative, ${}_H\mathbb{I}^{1-\alpha}_{1^+}y$ denotes $1-\alpha$ order Hadamard fractional integral, $\alpha \in (0,1)$. $T = k^*\tau$, $k^* \in N^+ = \{1,2,\ldots\}$, $\tau$ is a fixed moment. $\varphi(\cdot)$ is an arbitrary Hadamard differentiable function, i.e., ${}_H\mathbb{D}^{\alpha}_{1^+}\varphi$ exists.

We use the idea from [19] and introduce a fractional delay matrix function with a logarithmic function that is used to seek a representation of solution of (1) by utilizing the constant variation method.

## 2 Preliminaries

Let $a,b \in R$, $a < b$, and $C((a,b],R^n)$ denotes a Banach space composed of continuous vector-valued functions. $\Theta$ denotes zero matrix, $I$ denotes the standard identity matrix.

**Definition 2.1** (see [1]) For a function $y:(a,b) \to R^n$, the $\alpha$ order Hadamard integral of $y$ is defined by

$$\left({}_H\mathbb{I}^{\alpha}_{a^+}y\right)(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \left(\ln\frac{x}{t}\right)^{\alpha-1} y(t)\frac{dt}{t}, \quad \alpha \in (0,1).$$

**Definition 2.2** (see [1]) For a function $y:(a,b) \to R^n$, the $\alpha$ order Hadamard derivative of $y$ is defined by

$$\left({}_H\mathbb{D}^{\alpha}_{a^+}y\right)(x) = \frac{1}{\Gamma(1-\alpha)}\left(x\frac{d}{dx}\right)\int_a^x \left(\ln\frac{x}{t}\right)^{-\alpha} y(t)\frac{dt}{t}, \quad \alpha \in (0,1).$$

Now we propose a new concept of fractional delay matrix function with logarithmic function.

**Definition 2.3** Let $\alpha \in (0,1)$. Fractional delay matrix function $\mathbb{E}^{B(\ln x)^\alpha}_{\tau,\alpha}$ with logarithmic function is defined by

$$\mathbb{E}^{B(\ln x)^\alpha}_{\tau,\alpha} = \begin{cases} \Theta, & -\infty < x \le 1, \\ I\frac{(\ln x)^{\alpha-1}}{\Gamma(\alpha)}, & 1 < x \le \tau, \\ I\frac{(\ln x)^{\alpha-1}}{\Gamma(\alpha)} + B\frac{(\ln x - \ln\tau)^{2\alpha-1}}{\Gamma(2\alpha)} + \cdots & \\ \quad + B^k\frac{(\ln x - \ln k\tau)^{(k+1)\alpha-1}}{\Gamma((k+1)\alpha)}, & k\tau < x \le (k+1)\tau, k \in N^+. \end{cases}$$

**Lemma 2.4** *For $k\tau < x \le (k+1)\tau$, $k \in N^+$, one has*

$$\int_{k\tau}^x (\ln x - \ln t)^{-\alpha}(\ln t - \ln k\tau)^{(k+1)\alpha-1}\frac{dt}{t} = (\ln x - \ln k\tau)^{k\alpha}\mathbb{B}\left[1-\alpha,(k+1)\alpha\right],$$

*where $\mathbb{B}[\xi,\eta] = \int_0^1 s^{\xi-1}(1-s)^{\eta-1}\,ds$ is a beta function.*

*Proof* Using the formula of integration by parts, we can obtain

$$\begin{aligned}&\int_{k\tau}^x (\ln x - \ln t)^{-\alpha}(\ln t - \ln k\tau)^{(k+1)\alpha-1}\frac{dt}{t} \\ &\quad = \int_0^{\ln x - k\tau} (\ln x - z - \ln k\tau)^{-\alpha} z^{(k+1)\alpha-1}\,dz\end{aligned}$$

$$= \int_0^{\ln x - k\tau} (\ln x - \ln k\tau)^{-\alpha} \left(1 - \frac{z}{\ln x - \ln k\tau}\right)^{-\alpha} z^{(k+1)\alpha-1}\, dz$$
$$= (\ln x - \ln k\tau)^{k\alpha} \mathbb{B}\big[1-\alpha, (k+1)\alpha\big].$$

The proof is completed. □

## 3 Representation of the solutions

In the section, we adopt the general method of solving linear fractional differential equations to seek the exact solutions by using the notation of $\mathbb{E}_{\tau,\alpha}^{B(\ln\cdot)^\alpha}$.

We establish the following fundamental result.

**Theorem 3.1** *If* $\mathbb{E}_{\tau,\alpha}^{B(\ln\cdot)^\alpha} : (k\tau, (k+1)\tau] \longrightarrow R^{n\times n}$, $k \in N^+$ *satisfies*

$$\big({}_H\mathbb{D}_{1^+}^\alpha \mathbb{E}_{\tau,\alpha}^{B(\ln t)^\alpha}\big)(x) = B\mathbb{E}_{\tau,\alpha}^{B(\ln x - \ln k\tau)^\alpha}, \tag{2}$$

*then* $\mathbb{E}_{\tau,\alpha}^{B(\ln x)^\alpha}$ *is a solution of* $({}_H\mathbb{D}_{1^+}^\alpha y)(x) = By(x-\tau)$ *with the initial value* $\mathbb{E}_{\tau,\alpha}^{B(\ln x)^\alpha} = I\frac{(\ln x)^{\alpha-1}}{\Gamma(\alpha)}$, $1 < x \le \tau$.

*Proof* If $x \in (-\infty, 1]$, according to Definition 2.3, we have $\mathbb{E}_{\tau,\alpha}^{B(\ln x)^\alpha} = \Theta$, obviously, (2) holds. Next, for $x \in (k\tau, (k+1)\tau]$, $k \in N^+$, we use mathematical induction to prove that the conclusion is also valid.

(i) When $k = 1$, $\tau < x \le 2\tau$, we have

$$y(x) = \mathbb{E}_{\tau,\alpha}^{B(\ln x)^\alpha} = I\frac{(\ln x)^{\alpha-1}}{\Gamma(\alpha)} + B\frac{(\ln x - \ln\tau)^{2\alpha-1}}{\Gamma(2\alpha)}. \tag{3}$$

According to (3) and Lemma 2.4, we can get

$$\begin{aligned}
&\big({}_H\mathbb{D}_{1^+}^\alpha \mathbb{E}_{\tau,\alpha}^{B(\ln t)^\alpha}\big)(x) \\
&= \frac{1}{\Gamma(1-\alpha)}\left(x\frac{d}{dx}\right)\int_1^x \left(\ln\frac{x}{t}\right)^{-\alpha} y(t)\frac{dt}{t} \\
&= \frac{1}{\Gamma(1-\alpha)}\left(x\frac{d}{dx}\right)\left(\frac{I}{\Gamma(\alpha)}\int_1^x \left(\ln\frac{x}{t}\right)^{-\alpha}(\ln t)^{\alpha-1}\frac{dt}{t}\right. \\
&\quad \left. + \frac{B}{\Gamma(2\alpha)}\int_\tau^x \left(\ln\frac{x}{t}\right)^{-\alpha}(\ln t - \ln\tau)^{2\alpha-1}\frac{dt}{t}\right) \\
&= \frac{1}{\Gamma(1-\alpha)}\left(x\frac{d}{dx}\right)\left(\mathbb{B}[1-\alpha,\alpha] + \frac{B}{\Gamma(2\alpha)}(\ln x - \ln\tau)\mathbb{B}[1-\alpha, 2\alpha]\right) \\
&= B\frac{(\ln x - \ln\tau)^{\alpha-1}}{\Gamma(\alpha)}.
\end{aligned}$$

(ii) When $k = 2$, $2\tau < x \le 3\tau$, we have

$$y(x) = \mathbb{E}_{\tau,\alpha}^{B(\ln x)^\alpha} = I\frac{(\ln x)^{\alpha-1}}{\Gamma(\alpha)} + B\frac{(\ln x - \ln\tau)^{2\alpha-1}}{\Gamma(2\alpha)} + B^2\frac{(\ln x - \ln 2\tau)^{3\alpha-1}}{\Gamma(3\alpha)}. \tag{4}$$

According to (4) and Lemma 2.4, we can get

$$\begin{aligned}
&\left({}_H\mathbb{D}^{\alpha}_{1^+}\mathbb{E}^{B(\ln t)^{\alpha}}_{\tau,\alpha}\right)(x) \\
&= \frac{1}{\Gamma(1-\alpha)}\left(x\frac{d}{dx}\right)\left(\int_1^{2\tau}(\ln x-\ln t)^{-\alpha}y(t)\frac{dt}{t}+\int_{2\tau}^{x}(\ln x-\ln t)^{-\alpha}y(t)\frac{dt}{t}\right) \\
&= \frac{1}{\Gamma(1-\alpha)}\left(x\frac{d}{dx}\right)\left(\frac{I}{\Gamma(\alpha)}\int_1^{x}\left(\ln\frac{x}{t}\right)^{-\alpha}(\ln t)^{\alpha-1}\frac{dt}{t}\right. \\
&\quad+\frac{B}{\Gamma(2\alpha)}\int_{\tau}^{x}(\ln x-\ln t)^{-\alpha}(\ln t-\ln\tau)^{2\alpha-1}\frac{dt}{t} \\
&\quad\left.+\frac{B^2}{\Gamma(3\alpha)}\int_{2\tau}^{x}(\ln x-\ln t)^{-\alpha}(\ln x-\ln 2\tau)^{3\alpha-1}\frac{dt}{t}\right) \\
&= B\frac{(\ln x-\ln t)^{\alpha-1}}{\Gamma(\alpha)}+\frac{B^2}{\Gamma(3\alpha)}\frac{1}{\Gamma(1-\alpha)}\left(x\frac{d}{dx}\right)\int_{2\tau}^{x}(\ln x-\ln t)^{-\alpha}(\ln x-\ln 2\tau)^{3\alpha-1}\frac{dt}{t} \\
&= B\frac{(\ln x-\ln\tau)^{\alpha-1}}{\Gamma(\alpha)}+B^2\frac{(\ln x-\ln 2\tau)^{2\alpha-1}}{\Gamma(2\alpha)}.
\end{aligned}$$

(iii) Assume $k=n$, $n\tau<x\le(n+1)\tau$, the following equality holds:

$$\left({}_H\mathbb{D}^{\alpha}_{1^+}\mathbb{E}^{B(\ln t)^{\alpha}}_{\tau,\alpha}\right)(x)=B\frac{(\ln x)^{\alpha-1}}{\Gamma(\alpha)}+B^2\frac{(\ln x-\ln\tau)^{2\alpha-1}}{\Gamma(2\alpha)}+\cdots+B^n\frac{(\ln x-\ln n\tau)^{n\alpha-1}}{\Gamma(n\alpha)}.$$

For $k=n+1$, $(n+1)\tau<x\le(n+2)\tau$, we can get

$$\begin{aligned}
y(x)&=\mathbb{E}^{B(\ln x)^{\alpha}}_{\tau,\alpha} \\
&=I\frac{(\ln x)^{\alpha-1}}{\Gamma(\alpha)}+B\frac{(\ln x-\ln\tau)^{2\alpha-1}}{\Gamma(2\alpha)}+\cdots+B^{n+1}\frac{(\ln x-\ln(n+1)\tau)^{(n+2)\alpha-1}}{\Gamma((n+1)\alpha)+\alpha}.
\end{aligned}\tag{5}$$

According to (5) and Lemma 2.4, we can get

$$\begin{aligned}
&\left({}_H\mathbb{D}^{\alpha}_{1^+}\mathbb{E}^{B(\ln t)^{\alpha}}_{\tau,\alpha}\right)(x) \\
&= \frac{1}{\Gamma(1-\alpha)}\left(x\frac{d}{dx}\right)\left(\int_1^{\tau}(\ln x-\ln t)^{-\alpha}y(t)\frac{dt}{t}+\int_{\tau}^{2\tau}(\ln x-\ln t)^{-\alpha}y(t)\frac{dt}{t}+\cdots\right. \\
&\quad\left.+\int_{(n+1)\tau}^{x}(\ln x-\ln t)^{-\alpha}y(t)\frac{dt}{t}\right) \\
&= B\frac{(\ln x-\ln\tau)^{\alpha-1}}{\Gamma(\alpha)}+B^2\frac{(\ln x-\ln 2\tau)^{2\alpha-1}}{\Gamma(2\alpha)}+\cdots+B^n\frac{(\ln x-\ln n\tau)^{n\alpha-1}}{\Gamma(n\alpha)} \\
&\quad+\frac{1}{\Gamma(1-\alpha)}\left(x\frac{d}{dx}\right) \\
&\quad\times\left(\frac{B^{n+1}}{\Gamma((n+1)\alpha)+\alpha}\int_{n\tau}^{x}(\ln x-\ln t)^{-\alpha}\left(\ln x-\ln(n+1)\tau\right)^{(n+2)\alpha-1}\frac{dt}{t}\right) \\
&= B\frac{(\ln x-\ln\tau)^{\alpha-1}}{\Gamma(\alpha)}+B^2\frac{(\ln x-\ln 2\tau)^{2\alpha-1}}{\Gamma(2\alpha)}+\cdots+B^{n+1}\frac{(\ln x-\ln(n+1)\tau)^{(n+1)\alpha-1}}{\Gamma((n+1)\alpha)}.
\end{aligned}$$

Then, for $\forall k \in N^+$, $k\tau < x \leq (k+1)\tau$,

$$\begin{aligned}
&\left({}_H\mathbb{D}^{\alpha}_{1^+}\mathbb{E}^{B(\ln t)^{\alpha}}_{\tau,\alpha}\right)(x)\\
&= B\left(\frac{(\ln x - \ln\tau)^{\alpha-1}}{\Gamma(\alpha)} + B\frac{(\ln x - \ln 2\tau)^{2\alpha-1}}{\Gamma(2\alpha)} + \cdots + B^{k-1}\frac{(\ln x - \ln k\tau)^{k\alpha-1}}{\Gamma(k\alpha)}\right)\\
&= B\mathbb{E}^{B(\ln x - \ln k\tau)^{\alpha}}_{\tau,\alpha}.
\end{aligned}$$

The proof is completed. □

In what follows, we give the main result of this paper.

**Theorem 3.2** *For $k\tau < x \leq (k+1)\tau$, $k \in N^+$, the solution $y \in C(X, R^n)$ of (1) can be written to*

$$y(x) = \mathbb{E}^{B(\ln x)^{\alpha}}_{\tau,\alpha} b + \int_1^{\tau} \mathbb{E}^{B(\ln x - \ln s)^{\alpha}}_{\tau,\alpha}\left({}_H\mathbb{D}^{\alpha}_{1^+}\varphi\right)(s)\frac{ds}{s}, \tag{6}$$

*where $X = (k\tau, (k+1)\tau] \cap (1, (k+1)\tau]$, $0 < \alpha < \frac{1}{n+1}$ or $X = [k\tau, (k+1)\tau] \cap [1, (k+1)\tau]$, $\alpha \geq \frac{1}{n+1}$.*

*Proof* Assume that $Y_0(x) = B\mathbb{E}^{B(\ln x)^{\alpha}}_{\tau,\alpha}$ satisfies Theorem 3.1, and the solution of (1) is given by

$$y(x) = B\mathbb{E}^{B(\ln x)^{\alpha}}_{\tau,\alpha} C + \int_1^{\tau} \mathbb{E}^{B(\ln x - \ln s)^{\alpha}}_{\tau,\alpha} z(s)\frac{ds}{s}, \tag{7}$$

where $C \in R^n$ is an unknown constant vector, $z(\cdot)$ is an unknown Hadamard differentiable function. Since $Y_0(x)$ is the solution of Equation (1) and $\mathbb{E}^{B(\ln x)^{\alpha}}_{\tau,\alpha} = I\frac{(\ln x)^{\alpha-1}}{\Gamma(\alpha)}$, $1 < x \leq \tau$, thus we can choose $C$ such that $({}_H\mathbb{I}^{1-\alpha}_{1^+}y)(1^+) = b$.

Let $x \to 1^+$, by Definition 2.3, we have $\mathbb{E}^{B(-\ln\tau - \ln s)^{\alpha}}_{\tau,\alpha} = \Theta$, $1 < s \leq \tau$. For $1 < x \leq \tau$, we obtain

$$\begin{aligned}
b &= \left({}_H\mathbb{I}^{1-\alpha}_{1^+}y\right)(1^+) = \lim_{x\to 1^+}\left({}_H\mathbb{I}^{1-\alpha}_{1^+}y\right)(x)\\
&= \lim_{x\to 1^+}\left(\frac{1}{\Gamma(1-\alpha)}\int_1^x (\ln x - \ln t)^{-\alpha} Y_0(t) C\frac{dt}{t}\right)\\
&= \frac{C}{\Gamma(1-\alpha)}\lim_{x\to 1^+}\left(\int_1^x (\ln x - \ln t)^{-\alpha}(\ln t)^{\alpha-1}\frac{dt}{t}\right)\\
&= \lim_{x\to 1^+} C = C.
\end{aligned}$$

It indicates that (7) has the form

$$y(x) = B\mathbb{E}^{B(\ln x)^{\alpha}}_{\tau,\alpha} b + \int_1^{\tau} \mathbb{E}^{B(\ln x - \ln s)^{\alpha}}_{\tau,\alpha} z(s)\frac{ds}{s}.$$

By Definition 2.3, we divide $(0, \tau]$ into two subintervals, we can get:

(i) For $1 < s \leq x$ and $0 \leq \ln x - \ln s \leq \ln x$, we have

$$\mathbb{E}^{B(\ln x - \ln s)^{\alpha}}_{\tau,\alpha} = I\frac{(\ln x - \ln s)^{\alpha-1}}{\Gamma(\alpha)}.$$

(ii) For $x < s \le \tau$ and $\ln x - \ln\tau \le \ln x - \ln s \le 0$, then $\mathbb{E}_{\tau,\alpha}^{B(-\ln\tau-\ln s)^\alpha} = \Theta$. Thus, for $1 < x \le \tau$, we have

$$\varphi(x) = I\frac{(\ln x)^{\alpha-1}}{\Gamma(\alpha)}b + \int_1^\tau \mathbb{E}_{\tau,\alpha}^{B(\ln x-\ln s)^\alpha} z(s)\frac{ds}{s}. \tag{8}$$

By calculating Hadamard type fractional order derivatives on both sides of (8), we can get

$$\begin{aligned}
&\left({}_H\mathbb{D}_{1^+}^{\alpha}\varphi\right)(x)\\
&= \frac{1}{\Gamma(1-\alpha)}\left(x\frac{d}{dx}\right)\int_1^x (\ln x - \ln t)^{-\alpha}\left(I\frac{(\ln t)^{\alpha-1}}{\Gamma(\alpha)}\varphi(1) + \int_1^x I\frac{(\ln t-\ln s)^{\alpha-1}}{\Gamma(\alpha)}z(s)\frac{ds}{s}\right)\frac{dt}{t}\\
&= \frac{1}{\Gamma(1-\alpha)}\left(x\frac{d}{dx}\right)\int_1^x \frac{z(s)}{\Gamma(\alpha)}\left(\int_s^x (\ln x-\ln t)^{-\alpha}(\ln t-\ln s)^{\alpha-1}\frac{dt}{t}\right)\frac{ds}{s}\\
&= x\frac{d}{dx}\int_{-\tau}^x z(s)\frac{ds}{s} = z(s).
\end{aligned}$$

The proof is completed. □

To end this paper, we give an example to illustrate the above theoretical result.

Let $\alpha = 0.3$, $\tau = 1.2$, $k^* = 4$. Consider

$$\begin{cases}
({}_H\mathbb{D}_{1^+}^{0.3}y)(x) = By(x-1.2), & x\in(1.2,6], \tau>0,\\
y(x) = (0.1,0.2), & 1 < x \le 1.2,\\
({}_H\mathbb{I}_{1^+}^{0.7}y)(1^+) = b,
\end{cases} \tag{9}$$

where $y(x) = (y_1(x), y_2(x))^T$, $b = (1,2)^T$, and

$$B = \begin{pmatrix} 2 & 1\\ 3 & 5\end{pmatrix}.$$

By Theorem 3.2, for every $x \in (1.2k, 1.2(k+1)]$, $k = \{0,1,2,3,4\}$, the solution of (9) can be represented by

$$y(x) = \mathbb{E}_{1,2,0.3}^{B(\ln x)^\alpha} b + \int_1^{1.2} \mathbb{E}_{1,2,0.3}^{B(\ln x-\ln s)^{0.3}} \left({}_H\mathbb{D}_{1^+}^{0.3}\varphi\right)(s)\frac{ds}{s},$$

where

$$\mathbb{E}_{1,2,0.3}^{B(\ln x)^\alpha} b = \begin{cases}
\mathbf{0}, & -\infty < x \le 1,\\
I\frac{(\ln x)^{-0.7}}{\Gamma(0.3)}(1,2)^T, & 1 < x \le 1.2,\\
(I\frac{(\ln x)^{-0.7}}{\Gamma(0.3)} + B\frac{(\ln x-\ln 1.2)^{-0.4}}{\Gamma(0.6)})(1,2)^T, & 1.2 < x \le 2.4,\\
(I\frac{(\ln x)^{-0.7}}{\Gamma(0.3)} + B\frac{(\ln x-\ln 1.2)^{-0.4}}{\Gamma(0.6)} + B^2\frac{(\ln x-\ln 2.4)^{-0.1}}{\Gamma(0.9)})(1,2)^T, & 2.4 < x \le 3.6,\\
(I\frac{(\ln x)^{-0.7}}{\Gamma(0.3)} + B\frac{(\ln x-\ln 1.2)^{-0.4}}{\Gamma(0.6)} + B^2\frac{(\ln x-\ln 2.4)^{-0.1}}{\Gamma(0.9)} & \\
\quad + B^3\frac{(\ln x-\ln 3.6)^{0.2}}{\Gamma(1.2)})(1,2)^T, & 3.6 < x \le 4.8,\\
(I\frac{(\ln x)^{-0.7}}{\Gamma(0.3)} + B\frac{(\ln x-\ln 1.2)^{-0.4}}{\Gamma(0.6)} + B^2\frac{(\ln x-\ln 2.4)^{-0.1}}{\Gamma(0.9)} & \\
\quad + B^3\frac{(\ln x-\ln 3.6)^{0.2}}{\Gamma(1.2)} + B^3\frac{(\ln x-\ln 3.6)^{0.5}}{\Gamma(1.5)})(1,2)^T, & 4.8 < x \le 6.
\end{cases}$$

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**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
All authors read and approved the final manuscript.

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