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Positive solutions for a class of two-term fractional differential equations with multipoint boundary value conditions

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Abstract

In this article, we study the existence of positive solutions to a class of two-term fractional nonlocal boundary value problems. The existence and multiplicity of positive solutions are established by means of fixed point index theory. The nonlinearity $f(t,x)$ permits a singularity at $t = 0, 1$ and $x = 0$.

Keywords: Two-term fractional differential equation; Singularity; Fixed point index; Positive solution

1 Introduction

In this paper, we consider the following two-term fractional differential equation boundary value problems:

$$\begin{cases} -D_{0+}^{\alpha}u(t) + bu(t) = f(t,u(t)), \quad 0 < t < 1, \\ u(0) = 0, \qquad u(1) = \sum_{i=1}^{m-2} \eta_i u(\xi_i), \end{cases} \tag{1.1}$$

where D_{0+}^{α} is the standard Riemann–Liouville derivative, $1 < \alpha \le 2$, $b > 0$, $\eta_i > 0$, $0 < \xi_1 < \cdots < \xi_{m-2} < 1$, $\sum_{i=1}^{m-2} \eta_i \xi_i^{\alpha-1} \le 1$, $f : (0,1) \times (0,+\infty) \to [0,+\infty)$ is continuous and may be singular at $t = 0, 1$ and $x = 0$.

Fractional differential equation boundary value problems (FBVPs) have attracted a great deal of attention during the past decades. The literature on boundary value problems of fractional differential equations is now much enriched (see [1–19]). Multi-term fractional differential equations appear in the mathematical models of many real world problems. For example, multi-term fractional differential equations have been used to model various types of visco-elastic damping ([20–22]). In [20], the authors introduced the Bagley–Torvik equation:

$$a_1 D^2 x(t) + a_2 D^{\frac{3}{2}} x(t) + a_3 x(t) = f(t),$$

to describe model for the motion of thin plate in Newtonian fluid. In [22], the authors investigated the endolymph equation:

$$D^2 x(t) + a_1 D x(t) + a_2 D^{\frac{1}{2}} x(t) + a_3 x(t) = -g(t),$$

which can be used to describe model for the response of the semicircular canals to the angular acceleration. The existing literature on multi-term fractional differential equations equipped with initial conditions is quite wide. However, the boundary value problems of multi-term fractional differential equations needs to be investigated. For some recent developments on Caputo type multi-term FBVPs, we mention the papers [23, 24] and the references cited therein.

In [9], by using the technique of [25], we rewrite the original resonant problems as an equivalent non-resonant two-term FBVPs. Combining with the properties of the Green's function we derived, the existence and uniqueness results of positive solutions are obtained by using of the fixed point index theory and iterative technique. It is well known that the suitable cone plays an important role in seeking positive solutions, which is usually depended on the positive properties of the Green function. However, there are much more difficulties in dealing with the Green functions of fractional-order boundary value problems than ordinary-order problems, especially for the case that $1 < \alpha < 2$. In [26], we established some new positive properties of the Green function for a class of two-term fractional differential equation with Dirichlet-type boundary value conditions. It should be noted that the properties of the Green function derived in [9] is not suitable for some methods of nonlinear analysis to be used. By employing height functions of the nonlinear term on special bounded sets together with Leggett–Williams and Krasnosel'skii fixed point theorems, Zhang and Zhong [27] established the existence of triple positive solutions for a class of fractional differential equations with integral conditions.

Motivated by the above work, in this paper we aim to establish the existence of positive solutions to the FBVP (1.1). Our work presented in this paper has the following features. Firstly, we consider few cases of Riemann–Liouville type two-term FBVPs which has been studied before. Secondly, some new properties of the Green function for the case that $1 < \alpha < 2$ have been discovered to deal with the difficulties related to the Green function for this case. Thirdly, the FBVP (1.1) possesses a singularity, that is, $f(t,x)$ may be singular at $t = 0, 1$ and $x = 0$.

2 Basic definitions and preliminaries

For the convenience of the reader, we present some preliminaries and lemmas.

Definition 2.1 ([28]) The fractional integral of order $\alpha > 0$ of a function $u : (0, +\infty) \to R$ is given by

$$I_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)\, ds$$

provided that the right-hand side is point-wise defined on $(0, +\infty)$.

Definition 2.2 ([28]) The Riemann–Liouville fractional derivative of order $\alpha > 0$ of a function $u : (0, +\infty) \to R$ is given by

$$D_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} u(s)\, ds,$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of number α, provided that the right-hand side is point-wise defined on $(0, +\infty)$.

In [9], we have proved that the function

$$g(t) =: \frac{\alpha - 2}{\Gamma(\alpha - 1)} + \sum_{k=1}^{+\infty} \frac{t^k}{\Gamma((k+1)\alpha - 2)},$$

has a unique positive root b^*. Throughout this paper, we assume that the following conditions hold:

(A_1) $b \in (0, b^*]$ is a constant.

(A_2) $f : (0,1) \times (0,+\infty) \to [0,+\infty)$ is continuous. In addition, for any $R \geq r > 0$, there exists $\Psi_{r,R} \in L^1[0,1] \cap C(0,1)$ such that

$$f(t,x) \leq \Psi_{r,R}(t), \quad \forall t \in (0,1), x \in \left[rt^{\alpha-1}, R\right].$$

Lemma 2.1 *The unique solution of the problem*

$$\begin{cases} -D^{\alpha}_{0+}u(t) + bu(t) = y(t), & 0 < t < 1, \\ u(0) = 0, \qquad u(1) = \sum_{i=1}^{m-2} \eta_i u(\xi_i), \end{cases}$$

can be expressed by

$$u(t) = \int_0^1 K(t,s)y(s)\,ds,$$

where

$$K(t,s) = K_1(t,s) + t^{\alpha-1}E_{\alpha,\alpha}\left(bt^{\alpha}\right)q(s),$$

$$q(s) = \frac{\sum_{i=1}^{m-2} \eta_i K_1(\xi_i, s)}{E_{\alpha,\alpha}(b) - \sum_{i=1}^{m-2} \eta_i \xi_i^{\alpha-1} E_{\alpha,\alpha}(b\xi_i^{\alpha})},$$

$$K_1(t,s) = \begin{cases} \frac{(t-ts)^{\alpha-1}E_{\alpha,\alpha}(bt^{\alpha})E_{\alpha,\alpha}(b(1-s)^{\alpha})}{E_{\alpha,\alpha}(b)}, & 0 \leq t \leq s \leq 1, \\ \frac{(t-ts)^{\alpha-1}E_{\alpha,\alpha}(bt^{\alpha})E_{\alpha,\alpha}(b(1-s)^{\alpha})}{E_{\alpha,\alpha}(b)} \\ \quad - \frac{(t-s)^{\alpha-1}E_{\alpha,\alpha}(b(t-s)^{\alpha})E_{\alpha,\alpha}(b)}{E_{\alpha,\alpha}(b)}, & 0 \leq s \leq t \leq 1, \end{cases}$$

here

$$E_{\alpha,\alpha}(x) = \sum_{k=0}^{+\infty} \frac{x^k}{\Gamma((k+1)\alpha)}$$

is the Mittag-Leffler function.

Proof Noticing that $\eta_i > 0$, $0 < \xi_i < 1$ and $\sum_{i=1}^{m-2} \eta_i \xi_i^{\alpha-1} \leq 1$, we have

$$\begin{aligned} E_{\alpha,\alpha}(b) - \sum_{i=1}^{m-2} \eta_i \xi_i^{\alpha-1} E_{\alpha,\alpha}\left(b\xi_i^{\alpha}\right) &= \sum_{k=0}^{+\infty} \frac{b^k[1 - \sum_{i=1}^{m-2} \eta_i \xi_i^{(k+1)\alpha-1}]}{\Gamma((k+1)\alpha)} \\ &\geq \sum_{k=0}^{+\infty} \frac{b^k \sum_{i=1}^{m-2} \eta_i \xi_i^{\alpha-1}(1 - \xi_i^{k\alpha})}{\Gamma((k+1)\alpha)} > 0. \end{aligned}$$

The proof is similar to Lemma 2.1 in [9], we omit it here. □

Lemma 2.2 *The function $K(t,s)$ satisfies the following properties*:

(1) $K(t,s) > 0$, $\forall t,s \in (0,1)$;

(2) $h_2(s)t^{\alpha-1} \le K(t,s) \le h_1(s)t^{\alpha-1}$, $\forall t,s \in [0,1]$, *where*

$$h_1(s) = (1-s)^{\alpha-1}E_{\alpha,\alpha}\bigl(b(1-s)^{\alpha}\bigr) + E_{\alpha,\alpha}(b)q(s), \qquad h_2(s) = \frac{q(s)}{\Gamma(\alpha)}.$$

Proof The proof is similar to Lemma 2.2 in [9], we omit it here. □

Lemma 2.3 ([26]) *The function $K_1(t,s)$ has the following properties*:

(1) $K_1(t,s) > 0$, $\forall t,s \in (0,1)$;

(2) $K_1(t,s) = K_1(1-s,1-t)$, $\forall t,s \in [0,1]$;

(3) $K_1(t,s) \le E_{\alpha,\alpha}(b)s(1-s)^{\alpha-1}t^{\alpha-2}$, $\forall s \in [0,1]$, $t \in (0,1]$;

(4) $K_1(t,s) \ge M_1 s(1-s)^{\alpha-1}(1-t)t^{\alpha-1}$, $\forall t,s \in [0,1]$, *where*

$$M_1 = \min\left\{\frac{1}{[\Gamma(\alpha)]^2E_{\alpha,\alpha}(b)}, (\alpha-1)^2E_{\alpha,\alpha}(b)\right\}.$$

Lemma 2.4 *The function $K(t,s)$ has the following properties*:

(1) $K(t,s) \le t^{\alpha-2}E_{\alpha,\alpha}(b)[s(1-s)^{\alpha-1} + q(s)]$, $\forall s \in [0,1]$, $t \in (0,1]$;

(2) $K(t,s) \ge Mt^{\alpha-1}[s(1-s)^{\alpha-1} + q(s)]$, $\forall t,s \in [0,1]$, *where*

$$M = \min\left\{\frac{1}{2\Gamma(\alpha)}, M_1, \frac{M_1}{2\Gamma(\alpha)} \times \frac{\sum_{i=1}^{m-2}\eta_i\xi_i^{\alpha-1} - \sum_{i=1}^{m-2}\eta_i\xi_i^{\alpha}}{E_{\alpha,\alpha}(b) - \sum_{i=1}^{m-2}\eta_i\xi_i^{\alpha-1}E_{\alpha,\alpha}(b\xi_i^{\alpha})}\right\}.$$

Proof From (3) of Lemma 2.3, we have

$$\begin{aligned} K(t,s) &= K_1(t,s) + t^{\alpha-1}E_{\alpha,\alpha}\bigl(bt^{\alpha}\bigr)q(s) \\ &\le t^{\alpha-2}E_{\alpha,\alpha}(b)s(1-s)^{\alpha-1} + t^{\alpha-1}E_{\alpha,\alpha}(b)q(s) \\ &\le t^{\alpha-2}E_{\alpha,\alpha}(b)\bigl[s(1-s)^{\alpha-1} + q(s)\bigr]. \end{aligned}$$

Therefore (1) holds.

On the other hand, it follows from (4) of Lemma 2.3 that

$$\begin{aligned} q(s) &= \frac{\sum_{i=1}^{m-2}\eta_iK_1(\xi_i,s)}{E_{\alpha,\alpha}(b) - \sum_{i=1}^{m-2}\eta_i\xi_i^{\alpha-1}E_{\alpha,\alpha}(b\xi_i^{\alpha})} \\ &\ge \frac{\sum_{i=1}^{m-2}\eta_i(1-\xi_i)\xi_i^{\alpha-1}}{E_{\alpha,\alpha}(b) - \sum_{i=1}^{m-2}\eta_i\xi_i^{\alpha-1}E_{\alpha,\alpha}(b\xi_i^{\alpha})}M_1s(1-s)^{\alpha-1} \\ &= \frac{\sum_{i=1}^{m-2}\eta_i\xi_i^{\alpha-1} - \sum_{i=1}^{m-2}\eta_i\xi_i^{\alpha}}{E_{\alpha,\alpha}(b) - \sum_{i=1}^{m-2}\eta_i\xi_i^{\alpha-1}E_{\alpha,\alpha}(b\xi_i^{\alpha})}M_1s(1-s)^{\alpha-1}. \end{aligned} \tag{2.1}$$

Denote

$$M_2 = \min\left\{\frac{M_1}{2\Gamma(\alpha)} \times \frac{\sum_{i=1}^{m-2}\eta_i\xi_i^{\alpha-1} - \sum_{i=1}^{m-2}\eta_i\xi_i^{\alpha}}{E_{\alpha,\alpha}(b) - \sum_{i=1}^{m-2}\eta_i\xi_i^{\alpha-1}E_{\alpha,\alpha}(b\xi_i^{\alpha})}, M_1\right\}. \tag{2.2}$$

From (2.1) and (2.2), one has

$$\frac{t^{\alpha-1}}{2\Gamma(\alpha)}q(s) - M_2 s(1-s)^{\alpha-1}t^{\alpha} \geq \frac{t^{\alpha-1}}{2\Gamma(\alpha)}q(s) - M_2 s(1-s)^{\alpha-1}t^{\alpha-1} \geq 0.$$

Then

$$\begin{aligned}
K(t,s) &= K_1(t,s) + t^{\alpha-1}E_{\alpha,\alpha}\left(bt^{\alpha}\right)q(s) \\
&\geq M_1 s(1-s)^{\alpha-1}(1-t)t^{\alpha-1} + t^{\alpha-1}E_{\alpha,\alpha}\left(bt^{\alpha}\right)q(s) \\
&\geq M_1 s(1-s)^{\alpha-1}(1-t)t^{\alpha-1} + t^{\alpha-1}\frac{1}{\Gamma(\alpha)}q(s) \\
&\geq M_2 s(1-s)^{\alpha-1}(1-t)t^{\alpha-1} + \frac{t^{\alpha-1}}{\Gamma(\alpha)}q(s) \\
&= M_2 s(1-s)^{\alpha-1}t^{\alpha-1} + \frac{t^{\alpha-1}}{2\Gamma(\alpha)}q(s) + \frac{t^{\alpha-1}}{2\Gamma(\alpha)}q(s) - M_2 s(1-s)^{\alpha-1}t^{\alpha} \\
&\geq M_2 s(1-s)^{\alpha-1}t^{\alpha-1} + \frac{t^{\alpha-1}}{2\Gamma(\alpha)}q(s) \\
&\geq \min\left\{\frac{1}{2\Gamma(\alpha)}, M_2\right\}t^{\alpha-1}\left[s(1-s)^{\alpha-1} + q(s)\right] \\
&= Mt^{\alpha-1}\left[s(1-s)^{\alpha-1} + q(s)\right].
\end{aligned}$$

So (2) holds. □

By Lemma 2.2 and Lemma 2.4, we have the following lemma.

Lemma 2.5 *The function $K^*(t,s) =: t^{2-\alpha}K(t,s)$ satisfies*:

(1) $K^*(t,s) > 0$, $\forall t,s \in (0,1)$;

(2) $K^*(t,s) \leq h_1(s)t$, $\forall t,s \in [0,1]$;

(3) $K^*(t,s) \leq E_{\alpha,\alpha}(b)[s(1-s)^{\alpha-1} + q(s)]$, $\forall s \in [0,1]$, $t \in (0,1]$;

(4) $K^*(t,s) \geq Mt[s(1-s)^{\alpha-1} + q(s)]$, $\forall t,s \in [0,1]$.

Let $E = C[0,1]$ be endowed with the maximum norm $\|u\| = \max_{0\leq t\leq 1}|u(t)|$, θ is the zero element of E, $B_r = \{u \in E : \|u\| < r\}$. Define a cone P by

$$P = \left\{u \in E : \exists l_u > 0, \text{ such that } l_u t \geq u(t) \geq \frac{M\|u\|}{E_{\alpha,\alpha}(b)}t, t \in [0,1]\right\}.$$

Define the height functions as follows:

$$\psi(t,r) = \min\left\{f\left(t,t^{\alpha-2}x\right) : \frac{Mr}{E_{\alpha,\alpha}(b)}t \leq x \leq r\right\};$$

$$\Psi(t,r) = \max\left\{f\left(t,t^{\alpha-2}x\right) : \frac{Mr}{E_{\alpha,\alpha}(b)}t \leq x \leq r\right\}.$$

For convenience, we list here some assumptions to be used later:

(H_1) there exist $r_1 > 0$ and a nonnegative function $b_1 \in L^1[0,1]$ with $\int_0^1 b_1(s)\,ds > 0$, such that

$$f\left(t,t^{\alpha-2}x\right) \geq b_1(t)x, \quad \forall (t,x) \in (0,1) \times (0,r_1];$$

(H_2) there exist $r_2 > 0$ and a nonnegative function $b_2 \in L^1[0,1]$ with $\int_0^1 b_2(s)\,ds > 0$, such that

$$f(t, t^{\alpha-2}x) \leq b_2(t)x, \quad \forall (t,x) \in (0,1) \times [r_2, +\infty);$$

(H_3) there exists $r_3 > 0$ such that

$$\int_0^1 \bigl[s(1-s)^{\alpha-1} + q(s)\bigr]\psi(t, r_3) > \frac{r_3}{M};$$

(H_4) there exist $r_4 > 0$ such that

$$\int_0^1 \bigl[s(1-s)^{\alpha-1} + q(s)\bigr]\Psi(t, r_4) < \frac{r_4}{E_{\alpha,\alpha}(b)}.$$

Define operators A, L_1 and L_2 as follows:

$$A(u)(t) = \int_0^1 K^*(t,s) f\bigl(s, s^{\alpha-2}u(s)\bigr)\,ds,$$
$$L_i u(t) = \int_0^1 K^*(t,s) b_i(s) u(s)\,ds, \quad i = 1, 2.$$

Lemma 2.6 *For any $r > 0$, $A : P \setminus B_r \to P$ is completely continuous.*

Proof The proof is similar to Lemma 2.3 in [13], we omit it here. □

By the extension theorem of a completely continuous operator (see Theorem 2.7 of [29]), for any $r > 0$, there exists an extension operator $\widetilde{A} : P \to P$, which is still completely continuous. Without loss of the generality, we still write it as A. By virtue of the Krein–Rutmann theorem and Lemma 2.5, we have the following lemma.

Lemma 2.7 *$L_i : P \to P$ $(i = 1, 2)$ are completely continuous linear operator. Moreover, the spectral radius $r(L_i) > 0$ and L_i has a positive eigenfunction φ_i corresponding to its first eigenvalue $(r(L_i))^{-1}$, that is, $L_i \varphi_i = r(L_i)\varphi_i$.*

Lemma 2.8 ([29]) *Let P be a cone in a Banach space E, and Ω be a bounded open set in E. Suppose that $A : \overline{\Omega} \cap P \to P$ is a completely continuous operator. If there exists $u_0 \in P$ with $u_0 \neq \theta$ such that*

$$u - Au \neq \lambda u_0, \quad \forall \lambda \geq 0, x \in \partial\Omega \cap P,$$

then $i(A, \Omega \cap P, P) = 0$.

Lemma 2.9 ([29]) *Let P be a cone in a Banach space E, and Ω be a bounded open set in E. Suppose that $A : \overline{\Omega} \cap P \to P$ is a completely continuous operator. If*

$$Au \neq \lambda u, \quad \forall \lambda \geq 1, u \in \partial\Omega \cap P,$$

then $i(A, \Omega \cap P, P) = 1$.

Lemma 2.10 ([29]) *Let P be a cone in a Banach space E, and Ω be a bounded open set in E. Suppose that $A : \overline{\Omega} \cap P \to P$ is a completely continuous operator. If*

$$\inf_{u\in\partial\Omega\cap P} \|Au\| > 0,$$
$$Au \neq \lambda u, \quad \forall\lambda \in (0,1], u \in \partial\Omega \cap P,$$

then $i(A, \Omega \cap P, P) = 0$.

3 Main results

Theorem 3.1 *Assume that there exist $r_2 > r_1 > 0$ such that (H_1) and (H_2) hold, and*

$$0 < r(L_2) < 1 \le r(L_1).$$

Then FBVP (1.1) has at least one positive solution.

Proof For any $u \in \partial B_{r_1} \cap P$, it follows from (H_1) that

$$Au(t) = \int_0^1 K^*(t,s)f\bigl(s, s^{\alpha-2}u(s)\bigr)\,ds \ge \int_0^1 K^*(t,s)b_1(s)u(s)\,ds = L_1u(t).$$

Suppose that A has no fixed points on $\partial B_{r_1} \cap P$ (otherwise, the proof is finished). In the following, we will show that

$$u - Au \neq \mu\varphi_1, \quad \forall u \in \partial B_{r_1} \cap P, \mu > 0, \tag{3.1}$$

in which φ_1 is the positive eigenfunction of L_1 satisfying $L_1\varphi_1 = r(L_1)\varphi_1$. If otherwise, there exist $\mu_0 > 0$ and $u_1 \in \partial B_{r_1} \cap P$ such that

$$u_1 - Au_1 = \mu_0\varphi_1.$$

Therefore,

$$u_1 = Au_1 + \mu_0\varphi_1 \ge \mu_0\varphi_1.$$

Set

$$\mu^* = \sup\{\mu : u_1 \ge \mu\varphi_1\}.$$

It is clear that $\mu^* \ge \mu_0$, and $u_1 \ge \mu^*\varphi_1$. Since L_1 is nondecreasing linear operator, one has

$$L_1u_1 \ge \mu^*L_1\varphi_1 = \mu^*r(L_1)\varphi_1 \ge \mu^*\varphi_1.$$

Then

$$u_1 = Au_1 + \mu_0\varphi_1 \ge L_1u_1 + \mu_0\varphi_1 \ge \bigl(\mu^* + \mu_0\bigr)\varphi_1,$$

which contradicts the definition of μ^*. So (3.1) holds. It follows from Lemma 2.8 that

$$i(A, B_{r_1} \cap P, P) = 0. \tag{3.2}$$

Denote

$$W = \{u \in P \backslash B_{r_1} \mid u = \mu Au, 0 \leq \mu \leq 1\}.$$

Next, we will prove that W is bounded.

For any $u \in W$, one has

$$f\bigl(t, t^{\alpha-2}u(t)\bigr) \leq b_2(t)u(t) + f\bigl(t, t^{\alpha-2}\tilde{u}(t)\bigr),$$

in which $\tilde{u}(t) = \min\{u(t), r_2\}$. It is easy to see that

$$\frac{Mr_1}{E_{\alpha,\alpha}(b)}t \leq \tilde{u}(t) \leq u(t) \leq l_u t.$$

Therefore,

$$l_u \geq l_u t^{\alpha-1} \geq t^{\alpha-2}\tilde{u}(t) \geq \frac{Mr_1}{E_{\alpha,\alpha}(b)}t^{\alpha-1} =: r_0 t^{\alpha-1}.$$

Then

$$u(t) = \mu Au(t) \leq Au(t) \leq L_2 u(t) + A\tilde{u}(t) \leq L_2 u(t) + M_3,$$

here

$$M_3 = \int_0^1 h_1(s)\Psi_{r_0,l_u}(s)\,ds.$$

Then

$$(I - L_2)u(t) \leq M_3, t \in [0, 1]. \tag{3.3}$$

By $r(L_2) < 1$, the inverse operator of $(I - L_2)$ can be expressed by

$$(I - L_2)^{-1} = I + L_2 + L_2^2 + \cdots + L_2^n + \cdots.$$

Therefore, (3.3) yields $u(t) \leq (I - L_2)^{-1}M_3 \leq M_3\|(I - L_2)^{-1}\|$, $t \in [0, 1]$, so W is bounded.

Choose $R > \max\{r_2, M_3\|(I - L_2)^{-1}\|\}$. From Lemma 2.9, one has

$$i(A, B_R \cap P, P) = 1. \tag{3.4}$$

It follows from (3.2) and (3.4) that

$$i\bigl(A, (B_R \backslash \bar{B}_{r_1}) \cap P, P\bigr) = i(A, B_R \cap P, P) - i(A, B_{r_1} \cap P, P) = 1.$$

Then A has a fixed point $u^* \in (B_R \backslash \bar{B}_{r_1}) \cap P$, that is,

$$u^*(t) = Au^*(t) = \int_0^1 K^*(t,s) f\big(s, s^{\alpha-2} u^*(s)\big)\, ds.$$

It is easy to check that $t^{\alpha-2} u^*(t)$ is a positive solution of FBVP (1.1). □

Theorem 3.2 *Assume that there exist $r_4 > r_3 > 0$ such that (H_3) and (H_4) hold. Then FBVP (1.1) has at least one positive solution.*

Proof For any $u \in \partial B_{r_3} \cap P$, it follows from (H_3) and Lemma 2.5 that

$$\begin{aligned} Au &\geq Mt \int_0^1 \big[s(1-s)^{\alpha-1} + q(s)\big] f\big(s, s^{\alpha-2} u(s)\big)\, ds \\ &\geq Mt \int_0^1 \big[s(1-s)^{\alpha-1} + q(s)\big] \psi(t, r_3)\, ds > r_3 t. \end{aligned} \tag{3.5}$$

Therefore,

$$\inf_{u \in \partial B_{r_3} \cap P} \|Au\| \geq r_3 > 0.$$

$\forall \lambda \in (0,1]$, $u \in \partial B_{r_3} \cap P$, we have $\lambda u \leq u \leq r_3$. This with (3.5) implies

$$Au \neq \lambda u, \quad \forall \lambda \in (0,1], u \in \partial B_{r_3} \cap P.$$

It follows from Lemma 2.10 that

$$i(A, B_{r_3} \cap P, P) = 0. \tag{3.6}$$

Next, we prove that

$$Au \neq \mu u, \quad \forall u \in \partial B_{r_4} \cap P, \mu \geq 1.$$

If otherwise, there exists $u_1 \in \partial B_{r_4} \cap P$, $\mu_0 \geq 1$ such that $Au_1 = \mu_0 u_1$. From (H_4) and Lemma 2.5, we have

$$\begin{aligned} u_1 \leq \mu_0 u_1 = Au_1 &\leq E_{\alpha,\alpha}(b) \int_0^1 \big[s(1-s)^{\alpha-1} + q(s)\big] f\big(s, s^{\alpha-2} u_1(s)\big)\, ds \\ &\leq E_{\alpha,\alpha}(b) \int_0^1 \big[s(1-s)^{\alpha-1} + q(s)\big] \psi(t, r_4)\, ds < r_4, \end{aligned}$$

which contradicts $\|u_1\| = r_4$. Then, by Lemma 2.9, we have

$$i(A, B_{r_4} \cap P, P) = 1. \tag{3.7}$$

Equations (3.6) and (3.7) yield

$$i\big(A, (B_{r_4} \backslash \bar{B}_{r_3}) \cap P, P\big) = i(A, B_{r_4} \cap P, P) - i(A, B_{r_3} \cap P, P) = 1.$$

Then A has a fixed point $u^* \in (B_R \backslash \bar{B}_{r_1}) \cap P$. Clearly, $t^{\alpha-2}u^*(t)$ is a positive solution of FBVP (1.1). □

Theorem 3.3 *Assume that there exist $r_3 > r_4 > r_1 > 0$ such that (H_1), (H_3) and (H_4) hold with $r(L_1) \geq 1$. Then FBVP (1.1) has at least two positive solutions.*

Proof Suppose that $\exists r_1' \in (0, r_1)$ such that A has no fixed points on $\partial B_{r_1'} \cap P$ (otherwise, the proof is finished). By the proof of Theorem 3.1 and Theorem 3.2, we have

$$i(A, B_{r_1'} \cap P, P) = 0, \qquad i(A, B_{r_4} \cap P, P) = 1, \qquad i(A, B_{r_3} \cap Q, Q) = 0.$$

Therefore,

$$\begin{aligned} &i\bigl(A, (B_{r_4} \backslash \bar{B}_{r_1'}) \cap P, P\bigr) = i(A, B_{r_4} \cap P, P) - i(A, B_{r_1'} \cap P, P) = 1, \\ &i\bigl(A, (B_{r_3} \backslash \bar{B}_{r_4}) \cap P, P\bigr) = i(A, B_{r_3} \cap P, P) - i(A, B_{r_4} \cap P, P) = -1. \end{aligned}$$

Then FBVP (1.1) has at least two positive solutions. □

Theorem 3.4 *Assume that there exist $r_2 > r_3 > r_4 > r_1 > 0$ such that (H_1)–(H_4) hold with $r(L_1) \geq 1 > r(L_2) > 0$. Then FBVP (1.1) has at least three positive solutions.*

Proof By Theorem 3.3 and (3.4), we get

$$\begin{aligned} &i(A, B_{r_1'} \cap P, P) = 0, \qquad i(A, B_{r_4} \cap P, P) = 1, \\ &i(A, B_{r_3} \cap Q, Q) = 0, \qquad i(A, B_R \cap P, P) = 1. \end{aligned}$$

Therefore,

$$\begin{aligned} &i\bigl(A, (B_{r_4} \backslash \bar{B}_{r_1'}) \cap P, P\bigr) = i(A, B_{r_4} \cap P, P) - i(A, B_{r_1'} \cap P, P) = 1, \\ &i\bigl(A, (B_{r_3} \backslash \bar{B}_{r_4}) \cap P, P\bigr) = i(A, B_{r_3} \cap P, P) - i(A, B_{r_4} \cap P, P) = -1, \\ &i\bigl(A, (B_R \backslash \bar{B}_{r_3}) \cap P, P\bigr) = i(A, B_R \cap P, P) - i(A, B_{r_3} \cap P, P) = 1. \end{aligned}$$

Then FBVP (1.1) has at least three positive solutions. □

4 Example

Example 4.1 Consider the following problem:

$$\begin{cases} -D_{0+}^{\frac{3}{2}}u(t) + \frac{1}{5}u(t) = f(t, u(t)), \quad 0 < t < 1, \\ u(0) = 0, \qquad u(1) = 3u(\frac{1}{9}), \end{cases} \tag{4.1}$$

where

$$f(t, x) = \frac{(t - \frac{1}{2})^2}{\sqrt{t(1-t)}}\bigl[x^{\frac{1}{2}} + x^{-\frac{1}{2}}\bigr].$$

For any $t \in [0,+\infty)$, noticing the function $\Gamma(\cdot)$ is strictly increasing on $[2,+\infty)$, we have

$$\begin{aligned} g(t) &= -\frac{1}{2\sqrt{\pi}} + t + \sum_{k=2}^{+\infty} \frac{t^k}{\Gamma(\frac{3}{2}k - \frac{1}{2})} \\ &\le -\frac{1}{2\sqrt{\pi}} + t + \sum_{k=2}^{+\infty} \frac{t^k}{\Gamma(k)} = -\frac{1}{2\sqrt{\pi}} + t\left[1 + \sum_{k=1}^{+\infty} \frac{t^k}{k!}\right] \\ &= -\frac{1}{2\sqrt{\pi}} + te^t. \end{aligned}$$

Therefore $g(\frac{1}{5}) \le -\frac{1}{2\sqrt{\pi}} + \frac{1}{5}e^{\frac{1}{5}} \approx -0.282 + 0.243 = -0.039 < 0$, which implies $\frac{1}{5} < b^*$.

For any $R \ge r > 0$, $\forall t \in (0,1)$, $x \in [r\sqrt{t}, R]$, we have

$$f(t,x) \le \frac{(t-\frac{1}{2})^2}{\sqrt{t(1-t)}}\bigl[\sqrt{R} + r^{-\frac{1}{2}} t^{-\frac{1}{4}}\bigr].$$

It is clear that

$$f\bigl(t, t^{-\frac{1}{2}}x\bigr) = \frac{(t-\frac{1}{2})^2}{\sqrt{t(1-t)}}\bigl[t^{-\frac{1}{4}} x^{\frac{1}{2}} + t^{\frac{1}{4}} x^{-\frac{1}{2}}\bigr], \quad \forall (t,x) \in (0,1) \times (0,+\infty).$$

Therefore, we have

$$\begin{aligned} &f\bigl(t, t^{-\frac{1}{2}}x\bigr) \ge (1-t)^{-\frac{1}{2}} \left(t - \frac{1}{2}\right)^2 t^{-\frac{1}{4}} x^{-\frac{1}{2}}, \quad \forall (t,x) \in (0,1) \times (0,1]; \\ &f\bigl(t, t^{-\frac{1}{2}}x\bigr) \le \frac{(t-\frac{1}{2})^2}{\sqrt{t(1-t)}}\bigl[t^{-\frac{1}{4}} + t^{\frac{1}{4}}\bigr] x^{\frac{1}{2}}, \quad \forall (t,x) \in (0,1) \times [1,+\infty). \end{aligned}$$

Denote

$$b_3(t) = (1-t)^{-\frac{1}{2}} \left(t - \frac{1}{2}\right)^2 t^{-\frac{1}{4}}, \qquad b_4(t) = \frac{(t-\frac{1}{2})^2}{\sqrt{t(1-t)}}\bigl(t^{-\frac{1}{4}} + t^{\frac{1}{4}}\bigr).$$

Let

$$\begin{aligned} L_3 u(t) &= \int_0^1 K^*(t,s) b_3(s) u(s)\,ds, \\ L_4 u(t) &= \int_0^1 K^*(t,s) b_4(s) u(s)\,ds. \end{aligned}$$

It follows from Lemma 2.7 that $r(L_3), r(L_4) > 0$.

Set

$$r_1 = \min\bigl\{\bigl[r(L_3)\bigr]^{\frac{2}{3}}, 1\bigr\}, \qquad r_2 = 1 + \bigl[r(L_4)\bigr]^2,$$

and

$$b_1(t) = r_1^{-\frac{3}{2}} b_3(t), \qquad b_2(t) = r_2^{-\frac{1}{2}} b_4(t).$$

Then

$$f\left(t,t^{-\frac{1}{2}}x\right) \geq b_1(t)x, \quad \forall (t,x) \in (0,1) \times \left(0,r_1\right];$$
$$f\left(t,t^{-\frac{1}{2}}x\right) \leq b_2(t)x, \quad \forall (t,x) \in (0,1) \times \left[r_2,+\infty\right).$$

It is easy to see that

$$r(L_1) = r_1^{-\frac{3}{2}} r(L_3) \geq 1,$$
$$0 < r(L_2) = r_2^{-\frac{1}{2}} r(L_4) < 1.$$

So the assumptions of Theorem 3.1 are satisfied. Thus Theorem 3.1 ensures that FBVP (4.1) has at least one positive solution.

5 Conclusions

In this article, we consider a class of Riemann–Liouville type two-term fractional nonlocal boundary value problems for the case that $1 < \alpha < 2$. Some new properties of the Green function have been discovered to construct an exact cone. By using fixed point index theory on the exact cone, the existence and multiplicity of positive solutions are established. The nonlinearity $f(t,x)$ permits a singularity at $t = 0, 1$ and $x = 0$.

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Abbreviations
FBVP, Fractional differential equations boundary value problems.

Availability of data and materials
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References

1. Cabada, A., Wang, G.: Positive solutions of nonlinear fractional differential equations with integral boundary value conditions. J. Math. Anal. Appl. **389**(1), 403–411 (2012)
2. Cabada, A., Hamdi, Z.: Nonlinear fractional differential equations with integral boundary value conditions. Appl. Math. Comput. **228**, 251–257 (2014)
3. Cui, Y.: Uniqueness of solution for boundary value problems for fractional differential equations. Appl. Math. Lett. **51**, 48–54 (2016)
4. Cui, Y., Ma, W., Sun, Q., Su, X.: New uniqueness results for boundary value problem of fractional differential equation. Nonlinear Anal., Model. Control **23**, 31–39 (2018)
5. Hao, X., Wang, H., Liu, L., Cui, Y.: Positive solutions for a system of nonlinear fractional nonlocal boundary value problems with parameters and p-Laplacian operator. Bound. Value Probl. **2017**, 182 (2017)
6. Henderson, J., Luca, R.: Existence of positive solutions for a singular fractional boundary value problem. Nonlinear Anal., Model. Control **22**(1), 99–114 (2017)

7. Wang, G., Ahmad, B., Zhang, L.: Existence results for nonlinear fractional differential equations with closed boundary conditions and impulses. Adv. Differ. Equ. **2012**, 169 (2012)
8. Wang, Y., Liu, L., Wu, Y.: Positive solutions for a nonlocal fractional differential equation. Nonlinear Anal. **74**(11), 3599–3605 (2011)
9. Wang, Y., Liu, L.: Positive solutions for a class of fractional 3-point boundary value problems at resonance. Adv. Differ. Equ. **2017**, 7 (2017)
10. Wu, J., Zhang, X., Liu, L., Cui, Y.: The convergence analysis and error estimation for unique solution of a p-Laplacian fractional differential equation with singular decreasing nonlinearity. Bound. Value Probl. **2018**, 82 (2018)
11. Zhang, X., Shao, Z., Zhong, Q.: Positive solutions for semipositone $(k, n-k)$ conjugate boundary value problems with singularities on space variables. Appl. Math. Lett. **72**, 50–57 (2017)
12. Wang, Y., Liu, L.: Positive solutions for a class of fractional infinite-point boundary value problems. Bound. Value Probl. **2018**, 118 (2018)
13. Wang, Y.: Existence and multiplicity of positive solutions for a class of singular fractional nonlocal boundary value problems. Bound. Value Probl. **2019**, 92 (2019)
14. Wang, Y.: Necessary conditions for the existence of positive solutions to fractional boundary value problems at resonance. Appl. Math. Lett. (2019). https://doi.org/10.1016/j.aml.2019.05.007
15. Zhang, X., Wang, L., Sun, Q.: Existence of positive solutions for a class of nonlinear fractional differential equations with integral boundary conditions and a parameter. Appl. Math. Comput. **226**, 708–718 (2014)
16. Zhang, X., Liu, L., Wu, Y.: The uniqueness of positive solution for a fractional order model of turbulent flow in a porous medium. Appl. Math. Lett. **37**, 26–33 (2014)
17. Zhang, X., Liu, L., Wu, Y., Cui, Y.: New result on the critical exponent for solution of an ordinary fractional differential problem. J. Funct. Spaces **2017**, Article ID 3976469 (2017)
18. Zou, Y., He, G.: On the uniqueness of solutions for a class of fractional differential equations. Appl. Math. Lett. **74**, 68–73 (2017)
19. Zuo, M., Hao, X., Liu, L., Cui, Y.: Existence results for impulsive fractional integro-differential equation of mixed type with constant coefficient and antiperiodic boundary conditions. Bound. Value Probl. **2017**, 161 (2017)
20. Torvik, P.J., Bagley, R.L.: On the appearance of the fractional derivative in the behavior of real materials. J. Appl. Mech. **51**(2), 294–298 (1984)
21. Metzler, R., Klafter, J.: Boundary value problems for fractional diffusion equations. Physica A **278**(1–2), 107–125 (2000)
22. Elshehawey, E.F., Elbarbary, E.M.A., Afifi, N.A.S., El-Shahed, M.: On the solution of the endolymph equation using fractional calculus. Appl. Math. Comput. **124**(3), 337–341 (2001)
23. Ahmad, B., Ntouyas, S.K.: Existence results for a coupled system of Caputo type sequential fractional differential equations with nonlocal integral boundary conditions. Appl. Math. Comput. **266**, 615–622 (2015)
24. Ahmad, B., Luca, R.: Existence of solutions for sequential fractional integro-differential equations and inclusions with nonlocal boundary conditions. Appl. Math. Comput. **339**, 516–534 (2018)
25. Webb, J.R.L., Zima, M.: Multiple positive solutions of resonant and non-resonant nonlocal boundary value problems. Nonlinear Anal. **71**(3–4), 1369–1378 (2009)
26. Wang, Y., Liu, L.: Positive properties of the Green function for two-term fractional differential equations and its application. J. Nonlinear Sci. Appl. **10**, 2094–2102 (2017)
27. Zhang, X., Zhong, Q.: Triple positive solutions for nonlocal fractional differential equations with singularities both on time and space variables. Appl. Math. Lett. **80**, 12–19 (2018)
28. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Differential Equations. Elsevier, Amsterdam (2006)
29. Guo, D.: Nonlinear Functional Analysis. Shandong Science and Technology Press, Jinan (1985) (in Chinese)