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Two competitive products diffusion in heterogeneous consumer social networks with repeat purchase

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Abstract

This paper studies the dynamics of two competitive products diffusion in heterogeneous consumer social networks with repeat purchase. We demonstrate a threshold for the diffusion rate above which a single product can persistently diffuse in heterogeneous consumer social networks without considering advertising strategy if the other product fails to diffuse, and there exists a unique positive equilibrium state where two competitive products coexist and persistently diffuse in heterogeneous consumer social networks without considering advertising strategy. We also prove that there exists at least one positive equilibrium state where two products coexist and persistently diffuse in heterogeneous consumer social networks if considering advertising strategy. The numerical simulations show that the higher the average degree of heterogeneous consumer social networks, the faster the two competitive products diffuse, and the shorter the time required to reach the stable state.

Keywords: Product diffusion; Competitive products; Consumer social networks; Dynamics; Heterogeneity

1 Introduction

Social network plays a significant role in product diffusion. Consumers use social networks to transmit product information to influence other consumers' decisions [1]. Consumer's decision-making is no longer a completely independent individual choice. The social network structure influences consumer decision-making behavior and the state of product diffusion system. Consumers use the offline interpersonal network or online social network to obtain product information, which forms consumer social networks. Many companies launch products to compete in the market around the same time over social networks, such as videogame consoles (Sony's Playstation vs. Mircosoft's X-Box [2]), smartphone (HTC's one vs Samsung's Galaxy S4 [3]). Studying the dynamics of two competitive products diffusion in consumer social networks is meaningful, which influences the enterprise revenue.

Classic product diffusion theory believes that external factors (such as advertisement) and internal factors (such as interpersonal word-of-mouth) influence potential consumers to buy product [4] and assumes that consumer social network is a fully connected regular network [5]. However, a large number of social network empirical studies show that the actual social network is not a regular network or a random network, and the structure of the consumer social network is heterogeneous [6]. Many scholars have studied the impact of network structure on the product diffusion speed and the number of adopters (the consumers who have purchased the product), such as the small world network [7], the scale-free network [8], and a network sampled from actual data [9]. However, most of the research assumes that a product could successfully diffuse in social networks. In practice, consumer social networks cannot only promote product diffusion, but also hinder product diffusion. Product may fail to diffuse or successfully diffuse. Therefore, the heterogeneity of the network structure should be considered when we study product diffusion in consumer social networks, and studying the dynamics of product diffusion in consumer social networks is an important complement to the existing product diffusion theory.

Many scholars have studied this topic and achieved many results. Wang et al. divided the process of consumer adoption into the awareness stage and the decision-making stage, proposed a multi-stage model of a single product diffusion, and obtained a threshold above which a single product could successfully diffuse when the adoption rate is bilinear and imitations are dominant [10]. However, this research has not considered the influence of consumer social networks. The classic product diffusion model is similar to the epidemic model and the rumor propagation model. The research ideas and analytical methods of the dynamics of epidemics or rumors spreading in complex networks can be used in researching product diffusion. Pastor-Satorras and Vespignani proposed a disease diffusion model in a scale-free network, and Nekovee et al. proposed a rumor diffusion model in complex networks [11, 12], which are the most influential diffusion models. Considering the influence of heterogeneous social networks, López-Pintado constructed a product diffusion model in complex social networks and obtained a threshold for the spreading rate above which the behavior spreads and becomes persistent in the population [13, 14]. Based on the work of Wang et al. [10], Li and Jin studied a single product diffusion in heterogeneous consumer social networks. They found that if mass media is neglected in the decision-making stage, there is a threshold whether the innovation diffusion is successful or not, or else it is proved that the network model has at least one positive equilibrium [15].

The research mentioned above focuses on a single product diffusion. Many works have been done on two or more products diffusion theory. The most influential model was proposed by Norton and Bass [16]. Different to Norton and Bass, Savin and Terwiesch believed that cross word-of-mouth effect should also be considered (Mahajan et al. labeled it as the brand competition effect) [17, 18]. For example, the market has two competitive products 1 and 2, and potential consumers still adopt product 1 after interacting with those who already adopted product 2. Regardless of the influence of consumer social networks structure, the dynamics of multiple products diffusion is studied. Yu et al. constructed three competitive products diffusion model to study the diffusion dynamics in social systems, obtained the equilibrium solutions, and proved the global stability of the equilibrium solutions [19].

Furthermore, the classic diffusion model also assumes that consumer only purchases one product during the product diffusion cycle. However, in reality some products have a long life cycle, and consumers may repeat purchase product for various reasons (such as damage caused by improper use, technical improvement, and aging [20]). Therefore, many scholars have expanded the research on product diffusion with considering consumer repeat purchase behavior and achieved many results (see Dodson and Muller [21], Lilien et al. [22], Rao and Yamada [23]).

It can be seen from previous research that few works focus on the multiple products diffusion in consumer social networks. Therefore, this paper considers the consumer repeat purchase behavior and the heterogeneity of consumer social networks structure in a multiple products diffusion framework and proposes a two competitive products diffusion model to analyze the dynamics of two competitive products diffusion in heterogeneous social networks.

The rest of the paper is organized as follows. In Sect. 2, we introduce a diffusion model of two competitive products in the heterogeneous consumer social network. In Sect. 3, we implement two cases to analyze the dynamics of two competitive products diffusion in the network: without considering advertising strategy and with considering advertising strategy. Section 4 verifies the conclusions by simulation. Finally, we give a brief conclusion in Sect. 5.

2 Model

2.1 State transition in the consumer social network

Assuming that the node size of the consumer social network is N, the undirected connection between nodes indicates the interaction between consumers. Each node is occupied by at most one consumer. In this paper, the degree distribution is used to describe the topological structure of the consumer social network. The degree distribution function is $p(k)$ which satisfies $\sum_{k\geq 1} p(k) = 1$, and k denotes the number of neighbors connecting to the node. The states of nodes in the network change as follows.

We assume that there are two products in the market competing for the same consumer population. At time t, one part of the potential consumers in the heterogeneous consumer social network is influenced by external factors such as advertisements or other mass media and converts into adopters of product 1 and product 2 with probability b_1 and b_2, respectively. At the same time, after interacting with the adopters of product 1 and product 2, another part of the potential consumers purchases the same product with probability a_1 and a_2, respectively. Due to the cross word-of-mouth effect, after interacting with the adopters of product j, another part of the potential consumers purchases product i with probability d_1 and d_2, respectively $(i,j = \{1,2\}, i \neq j)$. In these three cases, the node state changes from a potential consumer state to an adopter state. The rest of the potential consumers' state will remain unchanged. The adopters of product 1 and 2 transform into potential consumers with probability μ_1 and μ_2 due to repeat purchase, and the states also shift. The node states in the heterogeneous consumer social network change as shown in Fig. 1.

2.2 The mean-field dynamics

Assuming that consumers only buy one product at a time, consumers will replace the product with a certain probability, and the market potential will remain unchanged during the product diffusion cycle. Potential consumers can obtain product information through ad-

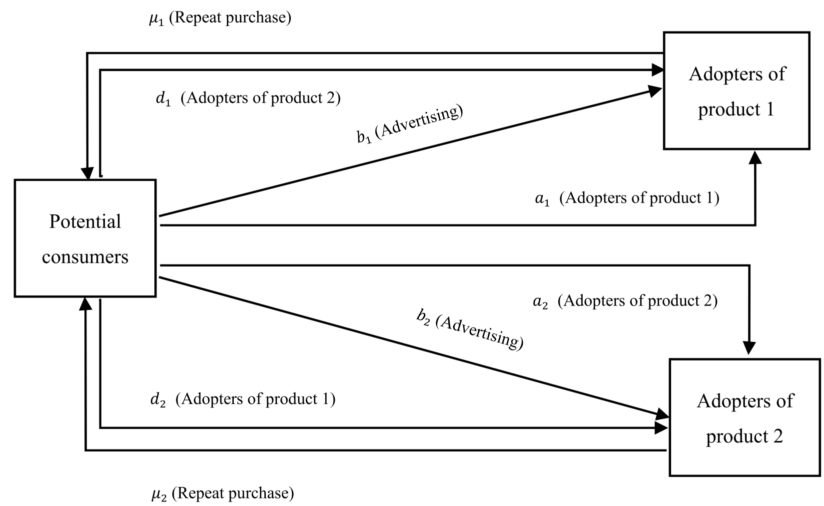

Figure 1 Node state transformation in the heterogeneous consumer social network

vertisement, or through word-of-mouth, or learning from some social networks (such as product evaluation posted in forums or product evaluation pages of e-commerce). The rules for two competitive products diffusion in the heterogeneous consumer social network are as follows.

$x_k(t)$, $y_k(t)$, and $z_k(t)$ are the population density of potential consumers, the population density of adopters of product 1, and the population density of adopters of product 2 with k neighbors at time t, respectively, then $x_k(t) + y_k(t) + z_k(t) = 1$. At time t, due to the influence of advertising, potential consumers adopt product 1 and product 2 with probability b_1 and b_2, respectively. Therefore, the numbers of new adopters of product 1 and product 2 influenced by advertising are $b_1 x_k(t)$ and $b_2 x_k(t)$, respectively. Due to the influence of word-of-mouth and cross word-of-mouth, the conditional probability that any potential consumer with node degree k connects with the adopter node with node degree k' is $p(k'|k)$, then the probabilities of potential consumers with node degree k adopting product 1 are $ka_1 \sum_{k'=1}^{M} p(k'|k) y_{k'}(t)$ and $kd_1 \sum_{k'=1}^{M} p(k'|k) z_{k'}(t)$, respectively. The probabilities of potential consumers with node degree k adopting product 2 are $ka_2 \sum_{k'=1}^{M} p(k'|k) z_{k'}(t)$ and $kd_2 \sum_{k'=1}^{M} p(k'|k) y_{k'}(t)$, respectively. The population density of potential consumers with node degree k is $x_k(t)$, so the number of new adopters of product 1 per unit time is $kx_k(t)[a_1 \sum_{k'=1}^{M} p(k'|k) y_{k'}(t) + d_1 \sum_{k'=1}^{M} p(k'|k) z_{k'}(t)]$, and the number of new adopters of product 2 per unit time is $kx_k(t)[a_2 \sum_{k'=1}^{M} p(k'|k) z_{k'}(t) + d_2 \sum_{k'=1}^{M} p(k'|k) y_{k'}(t)]$. Therefore, at time t in the heterogeneous consumer social network, the total number of potential consumers is $x(t) = \sum_{k=1}^{N} x_k(t) p(k)$, the total number of adopters of product 1 is $y(t) = \sum_{k=1}^{N} y_k(t) p(k)$, and the total number of adopters of product 2 is $z(t) = \sum_{k=1}^{N} z_k(t) p(k)$.

Therefore, considering the consumer repeat purchase behavior and the heterogeneity of the structure of consumer social networks, the model of two competitive products diffusion in heterogeneous consumer social networks is as follows:

$$\begin{cases} \frac{dx_k(t)}{dt} = -x_k(t)[b_1 + ka_1 \sum_{k'=1}^{M} p(k'|k)y_{k'}(t) + kd_1 \sum_{k'=1}^{M} p(k'|k)z_{k'}(t)] - x_k(t) \\ \qquad \times [b_2 + ka_2 \sum_{k'=1}^{M} p(k'|k)z_{k'}(t) \\ \qquad + kd_2 \sum_{k'=1}^{M} p(k'|k)y_{k'}(t)] + \rho_1 y_k(t) + \rho_2 z_k(t), \\ \frac{dy_k(t)}{dt} = x_k(t)[b_1 + ka_1 \sum_{k'=1}^{M} p(k'|k)y_{k'}(t) \\ \qquad + kd_1 \sum_{k'=1}^{M} p(k'|k)z_{k'}(t)] - \rho_1 y_k(t), \\ \frac{dz_k(t)}{dt} = x_k(t)[b_2 + ka_2 \sum_{k'=1}^{M} p(k'|k)z_{k'}(t) \\ \qquad + kd_2 \sum_{k'=1}^{M} p(k'|k)y_{k'}(t)] - \rho_2 z_k(t). \end{cases} \tag{1}$$

When the degree of the network is irrelevant, we obtain

$$\begin{cases} \theta_1 = \frac{\sum_k kp(k)y_k(t)}{\langle k \rangle}, \\ \theta_2 = \frac{\sum_k kp(k)z_k(t)}{\langle k \rangle}. \end{cases} \tag{2}$$

Then model (1) can be simplified as

$$\begin{cases} \frac{dx_k(t)}{dt} = -x_k(t)[b_1 + a_1 k\theta_1 + d_1 k\theta_2] - x_k(t)[b_2 + a_2 k\theta_2 + d_2 k\theta_1] \\ \qquad + \rho_1 y_k(t) + \rho_2 z_k(t), \\ \frac{dy_k(t)}{dt} = x_k(t)[b_1 + a_1 k\theta_1 + d_1 k\theta_2] - \rho_1 y_k(t), \\ \frac{dz_k(t)}{dt} = x_k(t)[b_2 + a_2 k\theta_2 + d_2 k\theta_1] - \rho_2 z_k(t), \end{cases} \tag{3}$$

where $\langle k \rangle = \sum_{k=1}^{N} kp(k)$ denotes the average degree of the consumer social network.

3 Model analysis

This section analyzes the dynamics of product diffusion according to the differential equation theory. Based on the diffusion model proposed in the previous section, we consider two cases to analyze the dynamics of two competitive products diffusion in the heterogeneous consumer social network: without considering advertising strategy and with considering advertising strategy.

3.1 Without considering advertising strategy

Without considering advertising strategy, the advertising influence rate $a_1 = a_2 = 0$. The ideal situation is that there is only one product or two products coexisting in the final market. Therefore, it can be divided into two cases.

(1) If product 2 fails to diffuse and product 1 exists in the market ($z_k(t) = 0$).

Theorem 1 *When product* 2 *fails to compete and exits the market, without considering advertising strategy, if* $a_1 > \frac{\rho_1 \langle k \rangle}{\langle k^2 \rangle}$, *there is a unique positive equilibrium solution for system* (1).

Proof Substituting $x_k(t) = 1 - y_k(t) - z_k(t)$ into formula (1), we obtain

$$\begin{cases} \frac{dy_k(t)}{dt} = (a_1 k\theta_1 + d_1 k\theta_2)(1 - y_k(t) - z_k(t)) - \rho_1 y_k(t), \\ \frac{dz_k(t)}{dt} = (a_2 k\theta_2 + d_2 k\theta_1)(1 - y_k(t) - z_k(t)) - \rho_2 z_k(t). \end{cases} \tag{4}$$

Setting the right sides of two equations of system (4) equal to zero, there is

$$\begin{cases} (a_1 k\theta_1 + d_1 k\theta_2)(1 - y_k(t) - z_k(t)) - \mu_1 y_k(t) = 0, \\ (a_2 k\theta_2 + d_2 k\theta_1)(1 - y_k(t) - z_k(t)) - \mu_2 z_k(t) = 0. \end{cases} \tag{5}$$

Then the equilibrium solutions of system (5) are

$$\begin{cases} y_k(t) = 0, \\ z_k(t) = 0, \end{cases} \tag{6}$$

$$\begin{cases} y_k(t) = \frac{\rho_2(a_1 k\theta_1 + d_1 k\theta_2)}{(a_1\rho_2 k\theta_1 + a_2\rho_1 k\theta_2 + d_1\rho_2 k\theta_2 + d_2\rho_1 k\theta_1 + \rho_1\rho_2)}, \\ z_k(t) = \frac{\rho_1(a_2 k\theta_2 + d_2 k\theta_1)}{(a_1\rho_2 k\theta_1 + a_2\rho_1 k\theta_2 + d_1\rho_2 k\theta_2 + d_2\rho_1 k\theta_1 + \rho_1\rho_2)}. \end{cases} \tag{7}$$

Obviously, equilibrium solution (6) satisfies system (4), but the two products could not successfully diffuse in the heterogeneous consumer social network. For equilibrium solution (7), substituting formula (7) into the first equation of formula (2), we obtain the self-consistent equation

$$\theta_1 = \frac{1}{\langle k\rangle} \sum_{k'=1}^{M} k' p(k') \frac{(a_1 k'\theta_1 + d_1 k'\theta_2)\rho_2}{(a_1 k'\theta_1 + d_1 k'\theta_2)\rho_2 + (a_2 k'\theta_2 + d_2 k'\theta_1)\rho_1 + \rho_1\rho_2}. \tag{8}$$

Define

$$F(\theta_1) = \frac{1}{\langle k\rangle} \sum_{k'=1}^{M} k' p(k') \frac{(a_1 k'\theta_1 + d_1 k'\theta_2)\rho_2}{(a_1 k'\theta_1 + d_1 k'\theta_2)\rho_2 + (a_2 k'\theta_2 + d_2 k'\theta_1)\rho_1 + \rho_1\rho_2} - \theta_1. \tag{9}$$

Substituting $z_k(t) = 0$ into formulas (2) and (9), we have

$$F(\theta_1) = \frac{1}{\langle k\rangle} \sum_{k'=1}^{M} k' p(k') \frac{a_1 k'\theta_1\rho_2}{a_1 k'\theta_1\rho_2 + d_2 k'\theta_1\rho_1 + \rho_1\rho_2} - \theta_1. \tag{10}$$

Calculating first-order derivative and second-order derivative of $F(\theta_1)$, we obtain

$$\frac{dF(\theta_1)}{d\theta_1} = \frac{1}{\langle k\rangle} \sum_{k'=1}^{M} k' p(k') \frac{a_1 k'\rho_1\rho_2^2}{(a_1 k'\theta_1\rho_2 + d_2 k'\theta_1\rho_1 + \rho_1\rho_2)^2} - 1, \tag{11}$$

$$\frac{d^2 F(\theta_1)}{d\theta_1^2} = -\frac{1}{\langle k\rangle} \sum_{k'=1}^{M} k' p(k') \frac{2a_1 k'^2\theta\rho_1\rho_2^2(a_1\rho_2 + d_2\rho_1)}{(a_1 k'\theta_1\rho_2 + d_2 k'\theta_1\rho_1 + \rho_1\rho_2)^3}. \tag{12}$$

Since $a_1, d_1, d_2, \rho_1, \rho_2$ change within the interval $(0, 1)$, and $\theta_1 \in [0, 1]$, then $\frac{d^2 F(\theta_1)}{d\theta_1^2} \le 0$. So $F(\theta_1)$ is a convex function in the interval $0 \le \theta_1 \le 1$. And substituting $\theta = 0$ and $\theta = 1$ into formula (10), we have $F(0) = 0$ and $F(1) < 0$. Therefore, the condition for system (5) to have a unique positive equilibrium solution is

$$\left.\frac{dF(\theta_1)}{d\theta_1}\right|_{\theta_1=0} = \frac{1}{\langle k\rangle} \sum_{k'=1}^{M} k' p(k') \left.\frac{a_1 k'\rho_1\rho_2^2}{(a_1 k'\theta_1\rho_2 + d_2 k'\theta_1\rho_1 + \rho_1\rho_2)^2}\right|_{\theta_1=0} - 1 > 0. \tag{13}$$

Then the threshold for product 1 persistently diffusing in the heterogeneous consumer social network is

$$a_1 > \frac{\rho_1 \langle k \rangle}{\langle k^2 \rangle}, \tag{14}$$

where $\langle k^2 \rangle = \sum_{k=1}^{N} k^2 p(k)$ denotes the second-order moments of the consumer social network. Therefore, if $a_1 > \frac{\rho_1 \langle k \rangle}{\langle k^2 \rangle}$, system (1) has a unique positive equilibrium solution. □

Theorem 1 shows that, when consumers are only influenced by internal word-of-mouth and there is only product 1 diffusing in the market, if $a_1 > \frac{\rho_1 \langle k \rangle}{\langle k^2 \rangle}$, which means the probability of potential consumer adopting product 1 influenced by the internal word-of-mouth is larger than a threshold, product 1 will persistently diffuse in the heterogeneous consumer social network. This threshold depends on the network characteristics of consumer social network (the degree distribution $p(k)$ and the size of the network N) and the repeat purchase rate of adopters ρ_1.

Then we introduce Lemma 1 to prove the stability of equilibrium solutions in Theorem 1.

Lemma 1 ([24]) *Consider the system*

$$\frac{dy}{dt} = Ay + N(y), \tag{15}$$

where A is an $n \times n$ matrix, and $N(y)$ is continuously differentiable in a region $D \in R^n$. Assume that the system simultaneously satisfies the following.

1. *Compact convex set $S \subset D$ is positively invariant with respect to system (15), and $0 \in S$;*
2. $\lim_{y \to 0} \|N(y)\| / \|y\| = 0$;
3. *For all $y \in S$, there exist $r > 0$ and a real eigenvector w such that $\omega y \geq r\|y\|$;*
4. *For all $y \in S$, $\omega N(y) \leq 0$;*
5. *In the set $H = \{y \in S | (\omega \cdot N(y)) = 0\}$, the largest positively invariant set is $y = 0$;*

Then either $y = 0$ is globally asymptotically stable in S or, for any $y_0 \in S - \{0\}$, the solution $\phi(t, y_0)$ of system (15) satisfies $\lim_{t \to \infty} \inf \|\phi(t, y_0)\| \geq m$, where m is independent of y_0. Moreover, system (15) has a constant solution y^* with $y^* \in S - \{0\}$.

Theorem 2 *The system satisfies Lemma 1, and product 2 fails to diffuse. Define $R_0 = \frac{a_1 \langle k^2 \rangle}{\rho_1 \langle k \rangle}$.*

1. *If $R_0 < 1$, then system (1) is globally asymptotically stable at equilibrium solution $E_0 = (1, 0, 0)$.*

2. *If $R_0 > 1$, then system (1) has a unique positive equilibrium solution which guarantees that product 1 could persistently diffuse in the heterogeneous consumer social network. That is, at the condition $x_k(0) > 0$, $y_k(0) > 0$, $y_k(0) > 0$, as to any solution of system (1), there exists a real number $\varepsilon > 0$ which satisfies $\lim_{t \to \infty} \inf\{x_k(t), y_k(t), z_k(t)\}_{k=1}^{N} > \varepsilon$.*

Proof Let $\Omega = \{(y_1, z_1, \ldots, y_k, z_k) \in R^{2N}, y_k \geq 0, z_k \geq 0, y_k + z_k \leq 1, 1 \leq k \leq N\}$. It can be verified that region Ω is positively invariant and satisfies Lemma 1 [25].

The Jacobin matrix of system (1) at $E_0 = (1, 0, 0)$ can be written as follows:

$$J = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix}, \tag{16}$$

where

$$B = \begin{pmatrix} -\rho_2 & 0 & \cdots & 0 \\ 0 & -\rho_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -\rho_2 \end{pmatrix},$$

$$C = \begin{pmatrix} -\frac{(a_1+d_2)p(1)}{\langle k\rangle} - \rho_1 & -2\frac{(a_1+d_2)p(2)}{\langle k\rangle} & \cdots & -n\frac{(a_1+d_2)p(n)}{\langle k\rangle} \\ -2\frac{(a_1+d_2)p(1)}{\langle k\rangle} & -2^2\frac{(a_1+d_2)p(2)}{\langle k\rangle} - \rho_1 & \cdots & -2n\frac{(a_1+d_2)p(n)}{\langle k\rangle} \\ \vdots & \vdots & \ddots & \vdots \\ -n\frac{(a_1+d_2)p(1)}{\langle k\rangle} & -2n\frac{(a_1+d_2)p(n)}{\langle k\rangle} & \cdots & -n^2\frac{(a_1+d_2)p(n)}{\langle k\rangle} - \rho_1 \end{pmatrix},$$

$$D = \begin{pmatrix} \frac{a_1p(1)}{\langle k\rangle} - \rho_1 & 2\frac{a_1p(2)}{\langle k\rangle} & \cdots & n\frac{a_1p(n)}{\langle k\rangle} \\ 2\frac{a_1p(1)}{\langle k\rangle} & 2^2\frac{a_1p(2)}{\langle k\rangle} - \rho_1 & \cdots & 2n\frac{a_1p(n)}{\langle k\rangle} \\ \vdots & \vdots & \ddots & \vdots \\ n\frac{a_1p(1)}{\langle k\rangle} & 2n\frac{a_1p(2)}{\langle k\rangle} & \cdots & n^2\frac{a_1p(n)}{\langle k\rangle} - \rho_1 \end{pmatrix}.$$

The eigenvalues of formula (16) are

$$\begin{aligned} &(\lambda + \rho_2)^N(\lambda + \rho_1)^{N-1}\left(\lambda - \rho_1\left(\frac{a_1}{\rho_1\langle k\rangle}\sum_{k=1}^{N} k^2p(k) - 1\right)\right) \\ &\quad = (\lambda + \rho_2)^N(\lambda + \rho_1)^{N-1}\left(\lambda - \rho_1\left(\frac{a_1\langle k^2\rangle}{\rho_1\langle k\rangle} - 1\right)\right). \end{aligned} \tag{17}$$

Therefore, if $R_0 = \frac{a_1\langle k^2\rangle}{\rho_1\langle k\rangle} > 1$, there exists only one positive eigenvalue, which means that system (1) has a unique positive equilibrium solution which guarantees that product 1 persistently diffuses in the heterogeneous consumer social network. According to Perron–Frobenius theorem, the maximal real part of all eigenvalues of (16) is positive if and only if $R_0 > 1$. Then, from Lemma 1, we complete the proof of Theorem 2. □

R_0 is called the basic productive number in biomathematics. Theorem 2 shows that, when product 2 fails to diffuse and if $R_0 < 1$, product 1 fails to diffuse without considering the influence of the advertising strategy. Because the number of the new adopters of product 1 is less than the number of the adopters transforming into potential consumers, the number of the adopters becomes smaller and smaller. If $R_0 > 1$, which means the probability that potential consumers adopt the product 1 satisfies the threshold condition, product 1 that diffuses in the heterogeneous consumer social network eventually evolves to a positive steady state, and can spread for a long time at this steady state point.

If there is only product 2 in the final market, the analysis is similar to the above process and will not be described again.

(2) If two products coexist in the final market.

Theorem 3 *Without considering advertising strategy, two competitive products have a unique positive equilibrium solution which guarantees that two products coexist and persistently diffuse in the heterogeneous consumer social network system* (1).

Proof Without considering advertising strategy, system (1) could be simplified as follows:

$$\begin{cases} \frac{dx_k(t)}{dt} = -x_k(t)[a_1k\theta_1 + d_1k\theta_2] - x_k(t)[a_2k\theta_2 + d_2k\theta_1] + \rho_1 y_k(t) + \rho_2 z_k(t), \\ \frac{dy_k(t)}{dt} = x_k(t)[a_1k\theta_1 + d_1k\theta_2] - \rho_1 y_k(t), \\ \frac{dz_k(t)}{dt} = x_k(t)[a_2k\theta_2 + d_2k\theta_1] - \rho_2 z_k(t). \end{cases} \tag{18}$$

Setting the right sides of three equations of system (18) equal to zero, we obtain the two equilibrium solutions of system (18):

$$\begin{cases} x = 1, \\ y = 0, \\ z = 0, \end{cases} \tag{19}$$

$$\begin{cases} x_k(t) = \frac{\rho_1\rho_2}{(a_1k\theta_1+d_1k\theta_2)\rho_2+(a_2k\theta_2+d_2k\theta_1)\rho_1+\rho_1\rho_2}, \\ y_k(t) = \frac{(a_1k\theta_1+d_2k\theta_2)\rho_2}{(a_1k\theta_1+d_1k\theta_2)\rho_2+(a_2k\theta_2+d_2k\theta_1)\rho_1+\rho_1\rho_2}, \\ z_k(t) = \frac{(a_2k\theta_2+d_1k\theta_1)\rho_1}{(a_1k\theta_1+d_1k\theta_2)\rho_2+(a_2k\theta_2+d_2k\theta_1)\rho_1+\rho_1\rho_2}. \end{cases} \tag{20}$$

Substituting $y_k(t)$ and $z_k(t)$ of formula (20) into formula (2), we obtain the self-consistent equations about θ_1 and θ_2:

$$\begin{cases} \theta_1 = \frac{1}{\langle k\rangle}\sum_{k'=1}^{M} k'p(k')\frac{(a_1k'\theta_1+d_1k'\theta_2)\rho_2}{(a_1k'\theta_1+d_1k'\theta_2)\rho_2+(a_2k'\theta_2+d_2k'\theta_1)\rho_1+\rho_1\rho_2}, \\ \theta_2 = \frac{1}{\langle k\rangle}\sum_{k'=1}^{M} k'p(k')\frac{(a_1k'\theta_1+d_2k'\theta_2)\rho_2}{(a_1k'\theta_1+d_1k'\theta_2)\rho_2+(a_2k'\theta_2+d_2k'\theta_1)\rho_1+\rho_1\rho_2}. \end{cases} \tag{21}$$

Using the identity transformation, we construct the function

$$F(\theta_1,\theta_2) = \frac{1}{\langle k\rangle}\sum_{k'=1}^{M} k'p(k')\frac{(a_1\theta_1+d_1\theta_2)k'\rho_2(1-\theta_1) - \theta_1[(a_2k'\theta_2+d_2k'\theta_1)\rho_1+\rho_1\rho_2]}{(a_1k'\theta_1+d_1k'\theta_2)\rho_2+(a_2k'\theta_2+d_2k'\theta_1)\rho_1+\rho_1\rho_2}. \tag{22}$$

Since $0 \le \theta_1 \le 1$, $0 \le \theta_2 \le 1$, there is

$$F_1(0,\theta_2) = \frac{1}{\langle k\rangle}\sum_{k'=1}^{M} k'p(k')\frac{d_1k'\theta_2\rho_2}{d_1k'\theta_2\rho_2 + a_2k'\theta_2\rho_1 + \rho_1\rho_2} > 0, \tag{23}$$

$$F_1(1,\theta_2) = \frac{1}{\langle k\rangle}\sum_{k'=1}^{M} k'p(k')\frac{-[(a_2k'\theta_2+d_2k')\rho_1+\rho_1\rho_2]}{(a_1k'+d_1k'\theta_2)\rho_2+(a_2k'\theta_2+d_2k')\rho_1+\rho_1\rho_2} < 0. \tag{24}$$

According to the interval value theorem of binary functions, there exists at least one point $\tilde{E} = (\tilde{\theta}_1,\tilde{\theta}_2)$ in the region $\Omega \triangleq [0,1]\times[0,1]$ such that $F(\tilde{\theta}_1,\tilde{\theta}_2) = 0$. It is easy to demonstrate that $F(\theta_1,\theta_2)$ has a continuous partial derivative in the region Ω. Then, according to the implicit function theorem, the equation $F(\theta_1,\theta_2) = 0$ could establish a continuous implicit function in the neighbor domain $\Omega_2(\subset \Omega)$.

$$\theta_1 = g(\theta_2), \quad (\theta_1,\theta_2) \in \Omega_2. \tag{25}$$

Substituting formula (25) into the second equation of formula (21), we have

$$h(\theta_2) - \theta_2 = 0, \tag{26}$$

where

$$h(\theta_2) \triangleq \frac{1}{\langle k\rangle}\sum_{k'=1}^{M} k'p(k')\frac{(a_1g(\theta_2) + d_2\theta_2)k'\rho_2}{(a_1k'g(\theta_2) + d_1k'\theta_2)\rho_2 + (a_2k'\theta_2 + d_2k'g(\theta_2))\rho_1 + \rho_1\rho_2}. \tag{27}$$

Define $H(\theta_2) \triangleq h(\theta_2) - \theta_2$, $g(\theta_2)$ is continuous within the interval $(0,1)$. Calculating a second-order derivative of $H(\theta_2)$, we know that $\frac{d^2H(\theta_2)}{d\theta_2^2} < 0$, so $H(\theta_2)$ is a convex function in the interval $[0,1]$. Obviously, $h(0) - 0 = 0$ $h(1) - 1 < 0$. Therefore, there exists at least one point $\tilde{\theta}_2 \in (0,1)$ satisfying formula (26). Then, substituting $\tilde{\theta}_2$ into formula (25), we obtain $\tilde{\theta}_1 = \tilde{g}(\tilde{\theta}_2)$. After that, substituting $(\tilde{\theta}_1,\tilde{\theta}_2)$ into formula (20), we obtain a positive equilibrium solution of system (18), which is denoted as $\tilde{E}(\tilde{y}_1,\tilde{y}_2,\dots,\tilde{y}_n,\tilde{z}_1,\tilde{z}_2,\dots,\tilde{z}_n)$.

Next we will show that system (18) has a unique positive equilibrium solution by using reductio ad absurdum.

Assume that system (18) has another positive equilibrium solution $\bar{E}(\bar{y}_1,\bar{y}_2,\dots,\bar{y}_n,\bar{z}_1,\bar{z}_2,\dots,\bar{z}_n)$, and $\tilde{E} \neq \bar{E}$. Denote $\tilde{y}_k = \tilde{m}_k$, $\bar{y}_k = \bar{p}_k$, $\tilde{z}_k = \tilde{m}_{n+k}$, $\bar{z}_k = \bar{p}_{n+k}$, $k = 1,2,\dots,n$, then

$$\tilde{m} = (\tilde{m}_1,\tilde{m}_2,\dots,\tilde{m}_n,\tilde{m}_{n+1},\tilde{m}_{n+2},\dots,\tilde{m}_{2n}) = (\tilde{y}_1,\tilde{y}_2,\dots,\tilde{y}_n,\tilde{z}_1,\tilde{z}_2,\dots,\tilde{z}_n), \tag{28}$$

$$\bar{p} = (\bar{p}_1,\bar{p}_2,\dots,\bar{p}_n,\bar{p}_{n+1},\bar{p}_{n+2},\dots,\bar{p}_{2n}) = (\bar{y}_1,\bar{y}_2,\dots,\bar{y}_n,\bar{z}_1,\bar{z}_2,\dots,\bar{z}_n). \tag{29}$$

Since $\tilde{m} \neq \bar{p}$, there exists a point i such that $\tilde{m}_i \neq \tilde{p}_i$, $i = (1,2,\dots,2n)$. Without loss of generality, assuming $\tilde{m} > \bar{p}$ and $\frac{\tilde{m}_i}{\bar{p}_i} > \frac{\tilde{m}_j}{\bar{p}_j}$ $(j = 1,2,\dots,n)$, if $1 \leq i \leq n$, substituting the two positive equilibrium solutions into system (18), we obtain

$$\begin{aligned}
&\bigl(a_1i\theta_1(\tilde{m}) + d_1i\theta_2(\tilde{m})\bigr)(1 - \tilde{m}_i - \tilde{m}_{n+i}) - \rho_1\tilde{m}_i \\
&\quad = \bigl(a_1i\theta_1(\bar{p}) + d_1i\theta_2(\bar{p})\bigr)(1 - \bar{p}_i - \bar{p}_{n+i}) - \rho_1\bar{p}_i \\
&\quad = 0,
\end{aligned} \tag{30}$$

where

$$\begin{cases}
\theta_1(\tilde{m}) = \frac{\sum_k kp(k)\tilde{m}_k(t)}{\langle k\rangle}, \\
\theta_2(\tilde{m}) = \frac{\sum_k kp(k)\tilde{m}_{n+k}(t)}{\langle k\rangle},
\end{cases} \tag{31}$$

$$\begin{cases}
\theta_1(\bar{n}) = \frac{\sum_k kp(k)\bar{n}_k(t)}{\langle k\rangle}, \\
\theta_2(\bar{n}) = \frac{\sum_k kp(k)\bar{n}_{n+k}(t)}{\langle k\rangle}.
\end{cases} \tag{32}$$

After identical transformation, formula (30) is simplified as follows:

$$\begin{aligned}\frac{\bar{p}_i}{\tilde{m}_i}&\big(a_1 i\theta_1(\tilde{m})+d_1 i\theta_2(\tilde{m})\big)(1-\tilde{m}_k-\tilde{m}_{n+i})-\rho_1\tilde{m}_i\\&=\big(a_1 i\theta_1(\bar{p})+d_1 i\theta_2(\bar{p})\big)(1-\bar{p}_k-\bar{p}_{n+i})-\rho_1\bar{p}_i\\&=0.\end{aligned}\tag{33}$$

Since $\tilde{m}>\bar{p}$ and $\frac{\tilde{m}_i}{\bar{p}_i}>\frac{\tilde{m}_j}{\bar{p}_j}$ $(j=1,2,\dots,n)$, from formula (32), there is

$$\begin{aligned}\frac{\bar{p}_i}{\tilde{m}_i}&\big(a_1 i\theta_1(\tilde{m})+d_1 i\theta_2(\tilde{m})\big)(1-\tilde{m}_k-\tilde{m}_{n+i})-\rho_1\tilde{m}_i\\&<\big(a_1 i\theta_1(\bar{p})+d_1 i\theta_2(\bar{p})\big)(1-\bar{p}_k-\bar{p}_{n+i})-\rho_1\bar{p}_i.\end{aligned}\tag{34}$$

Obviously, formula (34) contradicts with formula (30). When $n+1\le i\le 2n$, we could obtain the same conclusion. Therefore, system (1) has a unique positive equilibrium solution. □

Theorem 3 shows that, without considering the influence of advertising strategy, there exists a unique stable state where two competitive products coexist and persistently diffuse in the heterogeneous consumer social network in a long term.

3.2 With considering advertising strategy

Theorem 4 *When considering the influence of advertising strategy, there exists at least one positive equilibrium solution in system* (1) *which guarantees that two competitive products could coexist and persistently diffuse in the heterogeneous consumer social network.*

Proof System (1) has a positive equilibrium solution as follows:

$$\begin{cases}x_k(t)=\frac{\rho_1\rho_2}{(a_1k\theta_1+d_1k\theta_2)\rho_2+(a_2k\theta_2+d_2k\theta_1)\rho_1+b_1\rho_2+b_2\rho_1+\rho_1\rho_2},\\ y_k(t)=\frac{(a_1k\theta_1+d_1k\theta_2+b_1)\rho_2}{(a_1k\theta_1+d_1k\theta_2)\rho_2+(a_2k\theta_2+d_2k\theta_1)\rho_1+b_1\rho_2+b_2\rho_1+\rho_1\rho_2},\\ z_k(t)=\frac{(a_2k\theta_2+d_1k\theta_1+b_2)\rho_1}{(a_1k\theta_1+d_1k\theta_2)\rho_2+(a_2k\theta_2+d_2k\theta_1)\rho_1+b_1\rho_2+b_2\rho_1+\rho_1\rho_2}.\end{cases}\tag{35}$$

Substituting the second and third equations into formula (2), we obtain the self-consistent equations

$$\begin{cases}\theta_1=\frac{1}{\langle k\rangle}\sum_{k'=1}^M k'p(k')\frac{(a_1k'\theta_1+d_1k'\theta_2+b_1)\rho_2}{(a_1k'\theta_1+d_1k'\theta_2)\rho_2+(a_2k'\theta_2+d_2k'\theta_1)\rho_1+b_1\rho_2+b_2\rho_1+\rho_1\rho_2},\\ \theta_2=\frac{1}{\langle k\rangle}\sum_{k'=1}^M k'p(k')\frac{(a_2k'\theta_2+d_2k'\theta_1+b_2)\rho_1}{(a_1k'\theta_1+d_1k'\theta_2)\rho_2+(a_2k'\theta_2+d_2k'\theta_1)\rho_1+b_1\rho_2+b_2\rho_1+\rho_1\rho_2}.\end{cases}\tag{36}$$

Using identity transformation, we construct the function

$$\begin{aligned}F(\theta_1,\theta_2)=\frac{1}{\langle k\rangle}\sum_{k'=1}^M&k'p(k')\big[\big(a_1k'\theta_1+d_1k'\theta_2+b_1\big)\rho_2(1-\theta_1)\\&-\theta_1\big[(a_2\theta_2+d_2\theta_1)k'\rho_1+b_1\rho_2\\&+b_2\rho_1+\rho_1\rho_2\big]\times\big[\big(a_1k'\theta_1+d_1k'\theta_2\big)\rho_2\end{aligned}$$

$$+ \left(a_2k'\theta_2 + d_2k'\theta_1\right)\rho_1 + b_1\rho_2 + b_2\rho_1 + \rho_1\rho_2\big]^{-1}. \tag{37}$$

Since $0 \le \theta_1 \le 1$, $0 \le \theta_2 \le 1$, then

$$F(0,\theta_2) = \frac{1}{\langle k\rangle}\sum_{k'=1}^{M} k'p(k')\frac{(d_1k'\theta_2 + b_1)\rho_2}{a_1k'\theta_1\rho_2 + d_2k'\theta_1\rho_1 + b_1\rho_2 + b_2\rho_1 + \rho_1\rho_2} > 0, \tag{38}$$

$$\begin{aligned} &F(1,\theta_2)\\ &\quad = \frac{1}{\langle k\rangle}\sum_{k'=1}^{M} k'p(k')\frac{-[(a_2k'\theta_2 + d_2k'\theta_1)\rho_1 + b_1\rho_2 + b_2\rho_1 + \rho_1\rho_2]}{(a_1k'\theta_1 + d_1k')\rho_2 + (a_2k' + d_2k'\theta_1)\rho_1 + b_1\rho_2 + b_2\rho_1 + \rho_1\rho_2}\\ &\quad < 0. \end{aligned} \tag{39}$$

According to the interval value theorem of binary functions, there exists at least one point $E' = (\theta_1'', \theta_2'')$ in the region $\Omega \triangleq [0,1] \times [0,1]$ such that $F_1(\theta_1'', \theta_2'') = 0$. It is easy to demonstrate that $F(\theta_1, \theta_2)$ has a continuous partial derivative in the region Ω. Then, according to the implicit function theorem, the equation $F(\theta_1, \theta_2) = 0$ could establish a continuous implicit function in the neighbor domain $\Omega_2(\subset \Omega)$.

$$\theta_1 = g(\theta_2), \quad (\theta_1, \theta_2) \in \Omega_2. \tag{40}$$

Substituting formula (40) into the second equation of formula (36), we obtain

$$h(\theta_2) - \theta_2 = 0. \tag{41}$$

Define

$$h(\theta_2) \triangleq \frac{1}{\langle k\rangle}\sum_{k'=1}^{M} k'p(k')\frac{(a_2g(\theta_2) + d_1\theta_2 + b_2)k'\rho_1}{(a_1k'g(\theta_2) + d_1k'\theta_2)\rho_2 + (a_2k'\theta_2 + d_2k'g(\theta_2))\rho_1 + b_1\rho_2 + b_2\rho_1 + \rho_1\rho_2}. \tag{42}$$

Since $h(0) - 0 > 0$, $h(1) - 1 < 0$, so there exists at least one point $\hat{\theta}_2 \in (0,1)$ which satisfies formula (41). Then, substituting $\hat{\theta}_2$ into formula (40), we obtain $\hat{\theta}_1 = \hat{g}(\hat{\theta}_2)$. After that, substituting $(\hat{\theta}_1, \hat{\theta}_2)$ into formula (36), we obtain a positive equilibrium solution of system (1), which is denoted as $E^*\{(x_k^*, y_k^*, z_k^*)\}_{k=1}^{n}$. □

Theorem 4 shows that, considering consumer repeat purchase behavior, when potential consumers are influenced by advertising strategy and word-of-mouth, the stable state of two competitive products diffusion in the heterogeneous consumer social network is not unique. In the steady state, the two competitive products could coexist and persistently diffuse in a long time.

Theorem 5 *Considering the influence of advertising strategy, suppose that $\rho_1 = \rho_2$, the positive equilibrium solution $\{(x_k^*, y_k^*, z_k^*)\}_{k=1}^{n}$ in Theorem 4 is locally asymptotically stable within Ω.*

Proof Define $q(k) = \frac{kp(k)}{\langle k\rangle}$, where $\langle k\rangle = \sum_{k=1} kp(k)$, $k = 1, 2, \dots, N$. If $\rho_1 = \rho_2 = \rho$, system (1) has a positive equilibrium solution as follows:

$$x_k^* = \frac{\rho}{\rho + W}, \qquad y_k^* = \frac{b_1 + W_1}{\rho + W}, \qquad z_k^* = \frac{b_2 + W_2}{\rho + W}, \tag{43}$$

where $W = \sum_{k=1} q(k)[(a_1 + d_2)y_k^* + (a_2 + d_1)z_k^*]$, $W_1 = \sum_{k=1} q(k)(a_1 y_k^* + d_1 z_k^*)$, and $W_2 = \sum_{k=1} q(k)(a_2 z_k^* + d_2 y_k^*)$.

Consider the linearized dynamics of system (1) at $\{(x_k^*, y_k^*, z_k^*)\}_{k=1}^N$:

$$\begin{cases} \frac{d\tilde{x}_k(t)}{dt} = -(b_1 + b_2)\tilde{x}_k(t) - k(x_k^* \tilde{W} + W\tilde{x}_k(t)) + \rho\tilde{y}_k(t) + \rho\tilde{z}_k(t), \\ \frac{d\tilde{y}_k(t)}{dt} = k(x_k^* \sum_{k=1} q(k)(a_1\tilde{y}_k(t) + d_1\tilde{z}_k(t)) + W_1\tilde{x}_k(t)) - \rho\tilde{y}_k(t), \\ \frac{d\tilde{z}_k(t)}{dt} = k(x_k^* \sum_{k=1} q(k)(a_2\tilde{z}_k(t) + d_2\tilde{y}_k(t)) + W_2\tilde{x}_k(t)) - \rho\tilde{z}_k(t), \end{cases} \tag{44}$$

where $\tilde{W} = \sum_{k=1} q(k)((a_1 + d_2)\tilde{y}_k(t) + (a_2 + d_1)\tilde{z}_k(t))$.

Then we set N is odd. The case where N is even is similar. From equations (43) and (44), we obtain

$$\frac{d\tilde{W}(t)}{dt} = \frac{W}{\langle k\rangle} \sum_{k=1} k^2 p(k)\tilde{x}_k(t). \tag{45}$$

Consider the linear dynamics

$$\frac{d}{dt}\begin{pmatrix} \tilde{W}(t) \\ \tilde{x}_1(t) \\ \vdots \\ \tilde{x}_k(t) \end{pmatrix} = A \begin{pmatrix} \tilde{W}(t) \\ \tilde{x}_1(t) \\ \vdots \\ \tilde{x}_k(t) \end{pmatrix}, \tag{46}$$

where

$$A = \begin{pmatrix} 0 & Wq(1) & \cdots & Wkq(k) & \cdots & WNq(N) \\ -x_1^* & -c - W & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ -kx_k^* & 0 & \cdots & -c - kW & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ -Nx_M^* & 0 & \cdots & 0 & \cdots & -c - NW \end{pmatrix},$$

and $c = \rho + b_1 + b_2$.

Then we will prove that the real parts of all eigenvalues of A are negative. The characteristic equation $F_N(\lambda)$ of A is

$$\begin{aligned} F_N(\lambda) = {} & \lambda(\lambda + c + W)\cdots(\lambda + c + NW) \\ & + g_1(\lambda + c + 2W)\cdots(\lambda + c + NW) \\ & + g_2(\lambda + c + W)(\lambda + c + 3W)\cdots(\lambda + c + NW) \\ & + \cdots \end{aligned}$$

$$+ g_N(\lambda + c + W)\cdots\bigl(\lambda + c + (N-1)W\bigr), \tag{47}$$

where $g_k = k^2 x_k^* Wq(k)$.

Equation (47) shows that $F_N(0) > 0$, $\lim_{\lambda\to\infty} F_N(\lambda) = \infty$, and the coefficient of degree k is

$$\sum_{k=1}(c + kW). \tag{48}$$

From equation (47), we also could obtain $F_N(-c-kW)F_N(-c-(k-1)W) < 0, 2 \le k \le N$, there exists at least one root $-\omega_k$ of the equation $F_N(\lambda) = 0$ in $(-c-kW, -c-(k-1)W)$. Then we could rewrite equation (47) into $F_N(\lambda) = (\lambda + \omega_1)\cdots(\lambda + \omega_N)(\lambda^2 + \alpha\lambda + \beta)$, where α and β are real numbers. Then the coefficient of degree k is

$$\sum_{k=1}(\omega_k + \alpha). \tag{49}$$

From equations (48) and (49) together with $w_k < c + kw$, it follows that $\alpha > 0$. So we could find the real part $-\frac{\alpha}{2}$ of solutions of the equation $\lambda^2 + \alpha\lambda + \beta = 0$ are negative. Therefore, the real parts of all eigenvalues of A are negative. (x_k^*, y_k^*, z_k^*) is locally asymptotically stable within Ω. □

Theorem 5 shows that, considering the influence of the advertising strategy, when $\rho_1 = \rho_2$, which means the repeat purchase ratio of adopters of product 1 is equal to that of adopters of product 2, the two competitive products could coexist and diffuse in the heterogeneous consumer social network. The diffusion of the two products eventually evolves to a positive steady state and could spread for a long time at this steady state point.

4 Numerical simulation

This section uses the numerical simulation to verify the theorems. Firstly, in the two cases, without considering advertising strategy and with considering advertising strategy, we compare and analyze the figures of two competitive products coexisting and persistently diffusing in the heterogeneous consumer social network or failing to diffuse. Then, we adjust relevant parameters to analyze the figure of a single product persistently diffusing in the market. Finally, we analyze the impact of the structural characteristics of the network on the two competitive products diffusion.

The relevant parameters are as follows. The size of the network is $N = 200$. The node degree of the heterogeneous consumer social network follows power law distribution $p(k) = mk^{-r}$, $\sum_k mk^{-r} = 1$, $r = 2.5$. The adoption parameters influenced by advertising are $b_1 = b_2 = 0.001$. The adoption parameters influenced by word-of-mouth are $a_1 = 0.1$ and $a_2 = 0.01$ (assuming that product 1 is more competitive than product 2), respectively. The adoption parameters influenced by cross word-of-mouth from adopters of a competitive product are $d_1 = 0.002$ and $d_2 = 0.001$, respectively. The repeat purchase rates of adopters of product 1 and 2 are $\rho_1 = \rho_2 = 0.01$ (assuming that after the adopters of product 1 and 2 become potential consumers, they have no preference for product 1 and 2). In the comparative analysis of network structural characteristics, we choose $r_1 = 3$ and $r_2 = 2$. We select the same periods for simulation.

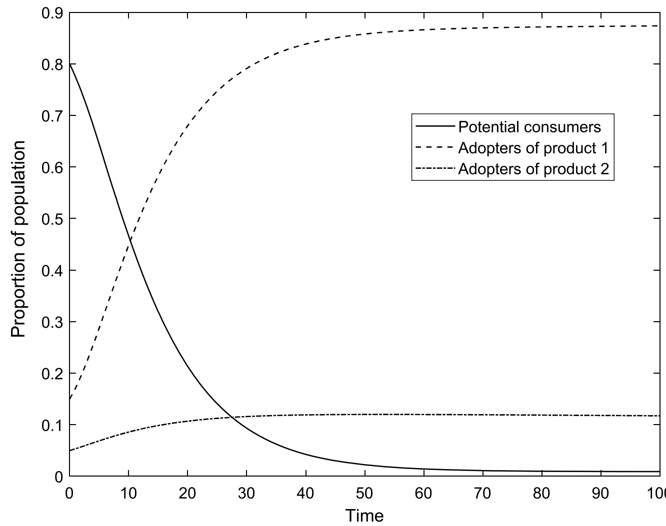

Figure 2 Two competitive products coexist and persistently diffuse without considering advertising strategy

(1) Simulation analysis of two competitive products diffusion without considering advertising strategy

Without considering advertising strategy, Fig. 2 shows that two competitive products coexist and persistently diffuse in the heterogeneous consumer social network. It can be seen that the proportion of potential consumers in the network continuously declines, and the proportion of adopters of product 1 and 2 continuously rises. But the trend gradually slows down and eventually reaches a steady state. Diffusion curve is similar to the S-curve of the classical product diffusion.

Then, reducing the internal influence coefficient of product 2 to the 1/1000 of its initial value, Fig. 3 shows that only product 1 persistently diffuses in the heterogeneous consumer social network. It can be seen that the proportion of adopters of product 2 finally declines to zero. This is due to the number of new adopters of product 2 influenced by an adopter of product 1 and 2 being smaller than the number of repeat purchase adopters of product 2. Therefore, the proportion of adopters of product 2 finally tends to zero, and the potential consumers basically transform into adopters of product 1.

Finally, we continue to reduce the internal influence coefficient of product 1 to the 1/1000 of its initial value, two competitive products fail to diffuse in the heterogeneous consumer social network, which is shown in Fig. 4. It can be seen from Fig. 4 that the proportion of adopters of product 1 and 2 gradually reduces. The reason is that the adoption parameters of product 1 and 2 are too small, and the numbers of new adopters of two products are smaller than the numbers of repeat purchase adopters. Therefore, the proportions of adopters of product 1 and 2 eventually tend to zero. The proportion of potential consumers which transform from adopters increases and reaches a stable state.

(2) Simulation analysis of two competitive products diffusion with considering advertising strategy

Considering advertising strategy, Fig. 5 shows that two competitive products coexist and persistently diffuse in the heterogeneous consumer social network, and Fig. 6 shows that two competitive products fail to diffuse in the network. From the diffusion curve, it is similar to the diffusion curve without considering advertising strategy. Comparing

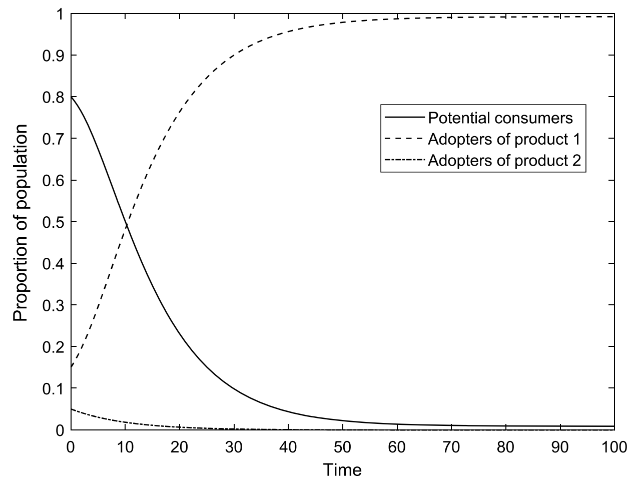

Figure 3 Only product 1 persistently diffuses without considering advertising strategy

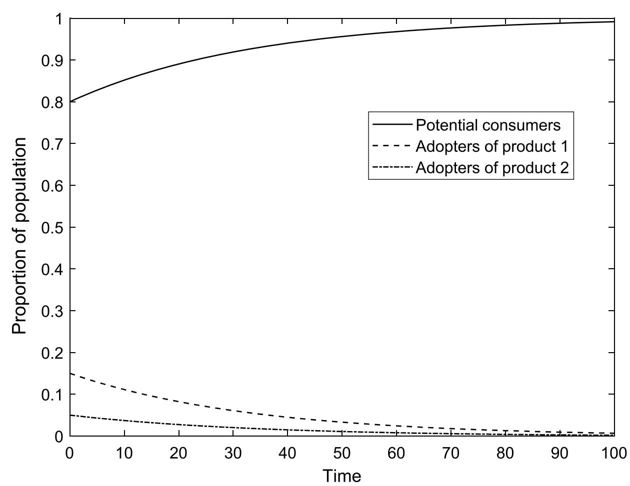

Figure 4 Two competitive products fail to diffuse without considering advertising strategy

the diffusion curve of product 1 in the two cases, we find that the time required to reach the stable state in Fig. 5 is less than the case in Fig. 2. The reason is that the influence of advertising increases the numbers of adopters of two products. However, the proportion of adopters of product 1 in a stable state with considering advertising strategy is higher than the case without considering advertising strategy, and the proportion of adopters of product 2 is reversed in these two cases.

(3) Simulation analysis of the influence of structural characteristic of the heterogeneous consumer social network

Let the parameters $r_1 = 3$ and $r_2 = 2$, Figs. 7(a) and 7(b) show the diffusion curve of two products coexisting and persistently diffusing in the heterogeneous consumer social network. It can be seen that the smaller r, the faster products diffuse and the shorter the time required to reach a stable state. The reason is that as r increases, the average degree of the network becomes larger. So the difference of network structure has stronger effect on

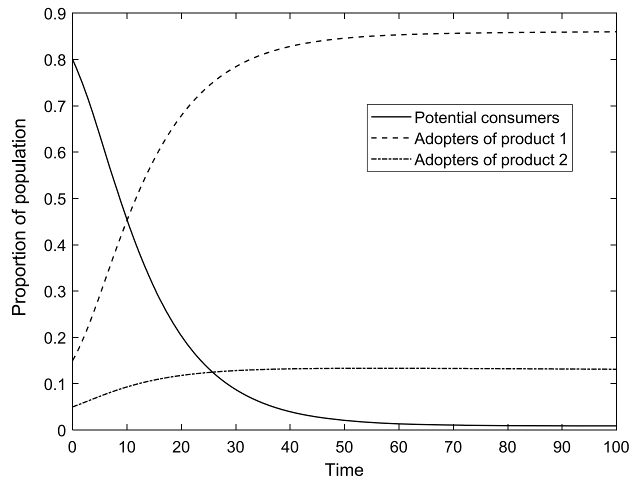

Figure 5 Two competitive products coexist and persistently diffuse with considering advertising strategy

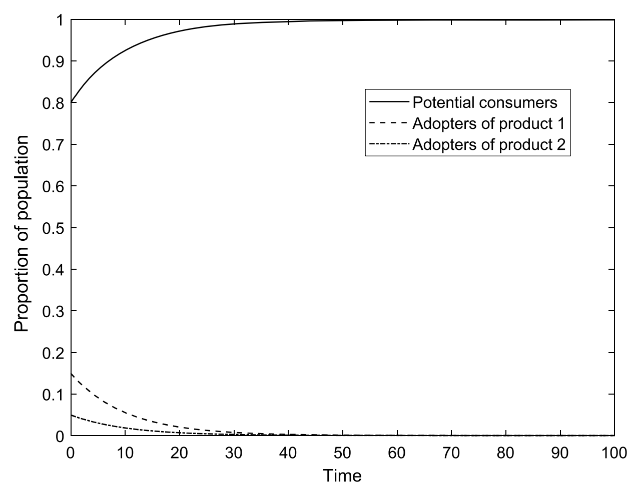

Figure 6 Two competitive products fail to diffuse with considering advertising strategy

product diffusion, resulting in a faster reduction rate of potential consumers and a faster increasing rate of new adopters.

5 Conclusion

Consumer social networks influence product diffusion. The heterogeneity of the consumer social network structure affects the interaction between consumers. Understanding the dynamics of two competitive products diffusion in heterogeneous consumer social networks is becoming an urgent need. Considering the consumer repeat purchase behavior, this paper studies the dynamics of two competitive products diffusion in the heterogeneous consumer social network without considering the influence of advertising strategy and with considering the influence of advertising strategy, respectively. This paper expands the model proposed by Savin and Terwiesch [18], which considers not only the word-of-mouth effect from adopters of the same product, but also the cross word-of-mouth effect from adopters of the competitive product. Considering the heterogeneity of

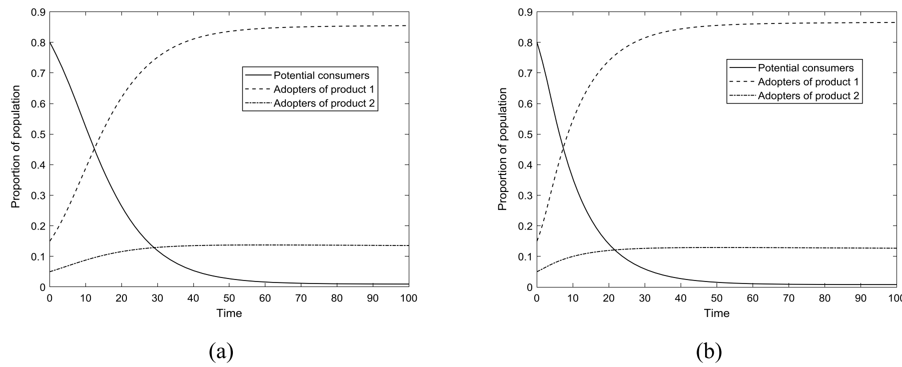

Figure 7 Diffusion curve of two products coexisting and persistently diffusing with considering advertising strategy when $r_1 = 3$ and $r_2 = 2$, respectively

the structure of consumer social networks, our model can better describe the diffusion mechanism of two competitive products in heterogeneous consumer social networks and the diffusion dynamics. In addition, studying the dynamics of two competitive products diffusion in heterogeneous consumer social networks cannot only enrich the theoretical study of product diffusion, but also guide the practice of enterprises.

It is worth noting that we only consider the positive influence of word-of-mouth. However, negative influence of word-of-mouth is also an important factor in product diffusion [26]. Furthermore, other structural characteristics of consumer social networks, such as clusters, path length, and so on, may influence the product diffusion. Furthermore, according to the information level about the product, the state of the consumer could be divided into multiple states. Studying the convergence of the product diffusion system may be a good research topic [27, 28]. In the future research, we will consider these factors.

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Authors' contributions
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