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A new class of $2m$-point binary non-stationary subdivision schemes

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Abstract

A new class of $2m$-point non-stationary subdivision schemes (SSs) is presented, including some of their important properties, such as continuity, curvature, torsion monotonicity, and convexity preservation. The multivariate analysis of subdivision schemes is extended to a class of non-stationary schemes which are asymptotically equivalent to converging stationary or non-stationary schemes. A comparison between the proposed schemes, their stationary counterparts and some existing non-stationary schemes has been depicted through examples. It is observed that the proposed SSs give better approximation and more effective results.

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1 Introduction

Subdivision schemes (SSs) have become one of the most essential tools for the generation of curve/surfaces and have been appreciated in many fields such as computer aided geometric design (CAGD), image processing, animation industry, computer graphics, etc.

Several univariate SSs studied in the literature are stationary. It seems that stationary SSs cannot generate circles; on the other hand, non-stationary SSs are capable of reproducing conic sections, spirals, trigonometric and hyperbolic functions of great interest in graphical and engineering applications. The non-stationary SSs were established for the first time by Dyn and Levin [17] in 1991. In 2002, Jena et al. [23] presented a scheme for trigonometric spline curves. Later on, in 2003 Jena et al. [24] also proposed a binary four-point interpolating non-stationary SSs which can generate C^1 limit curve. In 2007, Beccari et al. [4, 5] proposed a couple of four-point non-stationary C^2 SSs with tension control parameter. For a general treatment of SSs, the readers can refer to [3, 20, 21, 27–29]. Recent proposals of non-stationary SSs have been presented by Daniel and Shunmugaraj [10–12], Conti and Romani [8, 9], Siddiqi et al. [30, 31], Bari and Mustafa [2], and Tan et al. [32] who have constructed new attractive artifacts in the subdivision museum.

The property of shape preservation is of extraordinary significance and usually used in curve & surface modeling. Several research papers have been published on shape preservation in the last couple of years. In 1994, Méhauté and Utreras [26] introduced a new technique to solve the problem of shape preservation in interpolating SSs. In 1998, Kuijt and Damme [25] constructed local SSs that interpolate functional univariate data preserving convexity. Dyn et al. [16] examined the convexity preservation properties of 4-point binary interpolating SSs of Dyn et al. [19] in 1999. In 2009, Cai [6] discussed the convexity preservation of 4-point ternary stationary SSs. Recently, in 2017, Wang and Li [33] proposed a family of convexity preserving SSs and Akram et al. [1] deduced the shape preserving properties of binary 4-point non-stationary interpolating SSs.

The main objective of this research is to define a new class of $2m$-point binary approximating subdivision schemes by using the Lagrange interpolation method. For simplicity, we have analyzed and discussed only 2-point and 4-point non-stationary SSs. It is observed that our proposed SSs are asymptotically equivalent to existing famous Chaikin's scheme [7] and 4-point binary scheme of Siddiqi et al. [31] and Dyn et al. [15] for different choices of m, respectively. The results show that the binary approximating schemes developed by the proposed algorithm have the ability to reproduce or regenerate the conic sections and trigonometric polynomials as well. Some examples are considered, by choosing an appropriate tension parameter $0 < \alpha < \frac{\pi}{2}$, to show the usefulness. We also examine the shape preserving properties (monotonicity and convexity preservation) of SSs when applied to functional univariate strictly convex data. Furthermore, motivated by applications in computer graphics and animation, the curvature and torsion of the obtained curves are also presented in this paper.

The plan of this paper is as follows: Sect. 2 is for derivation of a new family of $2m$-point approximating non-stationary SSs. Section 3 is devoted for investigation of convergence and continuity of proposed SSs, and in Sect. 4 we deduce the shape preserving properties (monotonicity and convexity preservation) of binary 4-point approximating stationary scheme. Section 5 is devoted to results and discussion. Concluding remarks are presented in Sect. 6.

2 Binary 2m-point non-stationary schemes

In this section, a procedure for constructing a new family of $2n$-point binary non-stationary SSs is presented. The following is a general form of one subdivision level of the non-stationary SS:

$$q^{j+1}_{2i+\gamma} = \sum_{k=0}^{n} \alpha^{j}_{i+\gamma} q^{j}_{i+k}, \quad \gamma = 0, 1; i \in \mathbb{Z}, \tag{1}$$

where the finite set $a^j = \{a^j_i, i \in \mathbb{Z}\}$ is called the mask. The symbol of the scheme is defined by $a(z) = \sum_{i\in\mathbb{Z}} a_i z^i$.

Theorem 1 ([18]) *Let S be a convergent non-stationary SS with the mask $a^j_{i+\gamma}$, then*

$$\sum_{k=0}^{m} a^{j}_{\gamma,k} = 1, \quad \gamma = 0, 1. \tag{2}$$

Here we reformulated Lagrange interpolation polynomials and presented some basic identities which key role in this sections. Consider the Lagrange interpolation polynomials of degree $(2m-1)$:

$$L_n^{2m-1}(y) = \prod_{k=-(m-1),n\neq k}^{m} \frac{y-k}{n-k}, \quad n = -(m-1), -(m-2), \ldots, (m). \tag{3}$$

Lemma 2 *If $n = -(m-1), \ldots, (m)$, then following results holds*:

$$\prod_{k=-(m-1),k\neq n}^{m} (n-k) = (-1)^{m-n}(m+n-1)!(m-n)!. \tag{4}$$

Proof We derive this implication individually for each value of n. Now, for $n = -(m-1)$, we get

$$\prod_{k=-(m-1),k\neq n}^{m} (n-k) = (-1)(-2)(-3)\cdots(-2m+2)(-2m+1).$$

Therefore,

$$\prod_{k=-(m-1),k\neq n}^{m} (n-k) = 0!(-1)^{2m-1}(2m-1)!.$$

Since $n = -(m-1)$, the above identity can be composed as (4).

In the same manner for $n = -(m-2), \ldots, 0, \ldots, n$, we have (4), completing the proof. □

Lemma 3 *If $L_n^{2m-1}(x)$ is a Lagrange interpolation polynomial of degree $(2m-1)$, obtained in (3) analogously to the nodes $\{n\}_{-(m-1)}^{m}$, then we get*

$$V_n = L_n^{2m-1}\left(\frac{1}{4}\right) = \frac{(-1)^n(4m-1)(4m-3)!}{2^{6m-4}(1-4n)(2m-2)!(m+n-1)!(m-n)!}, \tag{5}$$

where $n = -(m-1), \ldots, (m)$.

Proof Since

$$\prod_{k=-(m-1)}^{m}\left(\frac{1}{4}-k\right) = \left(\frac{1}{4}\right)^{2m}\bigl\{(4m-3)(4m-7)(4m-11)\cdots(5)(1)(-3)\cdots$$
$$\times(-4m+13)(-4m+9)(-4m+5)(-4m+1)\bigr\},$$

we get

$$\prod_{k=-(m-1),k\neq n}^{m}\left(\frac{1}{4}-k\right) = \frac{1}{4^{2m-1}(1-4n)} \prod_{m=-m+1}^{n} (1-4n).$$

This leads to

$$\prod_{k=-(m-1),k\neq n}^{m}\left(\frac{1}{4}-k\right)$$

$$
\begin{aligned}
&= \frac{(-1)^m}{4^{2m-1}(1-4n)} \biggl\{ (4m-3)\frac{(4m-4)}{(4m-4)}\frac{(4m-5)}{(4m-5)}\frac{(4m-6)}{(4m-6)} \\
&\quad \times (4m-7)\frac{(4m-8)}{(4m-8)}\frac{(4m-9)}{(4m-9)}\frac{(4m-10)}{(4m-10)}(4m-11)\frac{(4m-12)}{(4m-12)} \\
&\quad \times \cdots \left(\frac{8}{8}\right)\left(\frac{7}{7}\right)\left(\frac{6}{6}\right)(5)\left(\frac{4}{4}\right)\left(\frac{3}{3}\right)\left(\frac{2}{2}\right)(1)\left(\frac{2}{2}\right)(3) \\
&\quad \times \left(\frac{4}{4}\right)\left(\frac{5}{5}\right)\left(\frac{6}{6}\right)(7)\cdots(4m-13)\frac{(4m-12)}{(4m-12)}\frac{(4m-11)}{(4m-11)} \\
&\quad \times \frac{(4m-10)}{(4m-10)}(4m-9)\frac{(4m-8)}{(4m-8)}\frac{(4m-7)}{(4m-7)}\frac{(4m-6)}{(4m-6)} \\
&\quad \times (4m-5)\frac{(4m-4)}{(4m-4)}\frac{(4m-3)}{(4m-3)}\frac{(4m-2)}{(4m-2)}(4m-1) \biggr\}.
\end{aligned}
$$

This implies

$$
\prod_{k=-(m-1),k\neq n}^{m} \left(\frac{1}{4}-k\right) = \frac{(-1)^m(4m-1)(4m-3)!}{2^{6m-4}(1-4n)(2m-2)!}.
$$

Applying (3)–(4) and $y = \frac{1}{4}$, we get (5). This completes the proof. □

Given $m \geq 0$, the mask of the following $2m$-point non-stationary SSs is:

$$
\begin{cases}
q_{2i}^{j+1} = \sum_{k=-(m-1)}^{m} \mu_k^j q_{i+k}^j, \\
q_{2i+1}^{j+1} = \sum_{k=-(m-1)}^{m} \mu_{-k+1}^j q_{i+k}^j,
\end{cases}
\tag{6}
$$

and also

$$
\mu_k^j = \frac{\sin(\frac{a}{2^{j+1}} U_m V_n)}{\sin(\frac{a}{2^{j+1}} U_m)},
$$

where $0 \leq a \leq \frac{\pi}{2}$, $U_m = m(4^{2m-1})$ while V_n is defined in Eq. (5).

2.1 Binary 2-point scheme

For $m = 1$ in (6), the 2-point SS is

$$
\begin{cases}
q_{2i}^{j+1} = \mu_1^j q_i^j + \mu_0^j q_{i+1}^j, \\
q_{2i+1}^{j+1} = \mu_0^k q_i^j + \mu_1^j q_{i+1}^j,
\end{cases}
\tag{7}
$$

where

$$
\mu_0^j = \frac{\sin(\frac{3a}{2^{j+1}})}{\sin(\frac{a}{2^{j-1}})}, \qquad \mu_1^j = \frac{\sin(\frac{a}{2^{j+1}})}{\sin(\frac{a}{2^{j-1}})}.
$$

Remark 2.1

- For $m = 1$, the proposed SS (6) becomes the two-point non-stationary SS [14].
- The two-point SS constructed in [23] for the generation of the trigonometric spline of order m, $m > 2$ also agrees with the proposed SS (6).

- Now for $m = 1$, we derive the normalized SS (corresponding to (7)). Note that

$$\begin{aligned}
\mu^j = \mu_0^j + \mu_1^j &= \frac{\sin(\frac{3a}{2^{j+1}})}{\sin(\frac{a}{2^{j-1}})} + \frac{\sin(\frac{a}{2^{j+1}})}{\sin(\frac{a}{2^{j-1}})} \\
&= \frac{1}{\sin(\frac{a}{2^{j-1}})}\left\{\sin\left(\frac{3a}{2^{j+1}}\right) + \sin\left(\frac{a}{2^{j+1}}\right)\right\} \\
&= \frac{1}{\sin(\frac{a}{2^{j-1}})}\left\{2\sin\left(\frac{2a}{2^{j+1}}\right)\cos\left(\frac{a}{2^{j+1}}\right)\right\} = \frac{\cos(\frac{a}{2^{j+1}})}{\cos(\frac{a}{2^{j}})}.
\end{aligned}$$

The corresponding normalized SS is obtained by dividing the stencil of the SS (7) at the jth refinement level by their sum:

$$\begin{aligned}
q_{2i}^{j+1} &= \eta_0^j q_i^j + \eta_1^j q_{i+1}^j, \\
q_{2i+1}^{j+1} &= \eta_1^j q_i^j + \eta_0^j q_{i+1}^j,
\end{aligned} \tag{8}$$

where

$$\eta_0^j = \frac{\cos(\frac{a}{2^j})}{\cos(\frac{a}{2^{j+1}})}\mu_0^j, \qquad \eta_1^j = \frac{\cos(\frac{a}{2^j})}{\cos(\frac{a}{2^{j+1}})}\mu_1^j.$$

Lemma 4 *If f is the limit function of the SS* (7), *then* $(\cos a)f(x)$ *is the limit function of the proposed normalized SS.*

Proof It is clear that

$$\begin{aligned}
\lim_{n\to\infty}\prod_{j=0}^{n}\frac{1}{\eta_0^j + \eta_1^j} &= \lim_{n\to\infty}\prod_{j=0}^{n}\frac{\cos(\frac{a}{2^j})}{\cos(\frac{a}{2^{j+1}})} \\
&= \lim_{n\to\infty}\frac{\cos a}{\cos(\frac{a}{2^{n+1}})} \\
&= \cos a.
\end{aligned}$$

□

2.2 Binary 4-point scheme

For $m = 2$ in (6), we get a new four-point symmetric binary approximating SS

$$\begin{aligned}
q_{2i}^{j+1} &= \mu_{-1}^j q_{i-1}^j + \mu_0^j q_i^j + \mu_1^j q_{i+1}^j + \mu_2^j q_{i+2}^j, \\
q_{2i+1}^{j+1} &= \mu_2^j q_{i-1}^j + \mu_1^j q_i^j + \mu_0^j q_{i+1}^j + \mu_{-1}^j q_{i+2}^j,
\end{aligned} \tag{9}$$

where

$$\mu_{-1}^j = \frac{\sin(-\frac{7a}{2^{j+1}})}{\sin(\frac{32a}{2^{j-1}})}, \qquad \mu_0^j = \frac{\sin(\frac{105a}{2^{j+1}})}{\sin(\frac{32a}{2^{j-1}})}, \qquad \mu_1^j = \frac{\sin(\frac{35a}{2^{j+1}})}{\sin(\frac{32a}{2^{j-1}})} \quad \text{and} \quad \mu_2^j = \frac{\sin(-\frac{5a}{2^{j+1}})}{\sin(\frac{32a}{2^{j-1}})}.$$

Similarly, the corresponding normalized SS is obtained by dividing the stencil of mask at the jth refinement level of the SS (9) by their sum:

$$\begin{aligned}
q_{2i}^{j+1} &= \lambda_{-1}^j q_{i-1}^j + \lambda_0^j q_i^j + \lambda_1^j q_{i+1}^j + \lambda_2^j q_{i+2}^j, \\
q_{2i+1}^{j+1} &= \lambda_2^j q_{i-1}^j + \lambda_1^j q_i^j + \lambda_0^j q_{i+1}^j + \lambda_{-1}^j q_{i+2}^j,
\end{aligned} \tag{10}$$

where

$$\lambda_k^j = \frac{\mu_k^j}{\mu^j}, \quad k = -1, 0, 1, 2.$$

The above normalized SS generates the function $q(x) = 1$ because $\sum \lambda_k^j = 1, k = -m + 1, \dots, m$.

Lemma 5 *Let $j \geq 0$ and $m > 0$ be fixed integers. If $q_i^j = \cos\{(2i)\frac{a}{2^j}\}$ then for $-1 \leq i \leq 2^j m$, we have*

$$q_{2i}^{j+1} = \cos\left\{\left(2i + \frac{1}{2}\right)\frac{a}{2^j}\right\} \quad \textit{and} \quad q_{2i+1}^{j+1} = \cos\left\{\left(2i + \frac{3}{2}\right)\frac{a}{2^j}\right\}.$$

Similarly, if $q_i^j = \sin\{(2i)\frac{a}{2^j}\}$ then for $-1 \leq i \leq 2^j n$ we have

$$q_{2i}^{j+1} = \sin\left\{\left(2i + \frac{1}{2}\right)\frac{a}{2^j}\right\} \quad \textit{and} \quad q_{2i+1}^{j+1} = \sin\left\{\left(2i + \frac{3}{2}\right)\frac{a}{2^j}\right\}.$$

Proof Here we prove the first part. Let $q_i^0 = \cos(2ia)$. In the first step of the SS (7), we get

$$\begin{aligned} q_{2i}^1 &= \eta_0^0 \cos(2ia) + \eta_1^0 \cos\big((2i+2)a\big) = \frac{\sin(\frac{3a}{2})}{\sin(2a)}\cos(2ia) + \frac{\sin(\frac{a}{2})}{\sin(2a)}\cos\big((2i+2)a\big) \\ &= \frac{\sin(2a - \frac{a}{2})}{\sin(2a)}\cos(2ia) + \frac{\sin(\frac{a}{2})}{\sin(2a)}\cos\big((2i+2)a\big) \\ &= \cos\left(\frac{a}{2}\right)\cos(2ia) - \sin\left(\frac{a}{2}\right)\sin(2ia) = \cos\left(\left(2i + \frac{1}{2}\right)a\right). \end{aligned}$$

At the jth step of the SS, we get

$$\begin{aligned} q_{2i}^{j+1} &= \eta_0^j \cos\left(2i\frac{a}{2^j}\right) + \eta_1^j \cos\left((2i+2)\frac{a}{2^j}\right) \\ &= \frac{\sin(\frac{3a}{2^{j+1}})}{\sin(\frac{a}{2^{j-1}})}\cos\left((2i)\frac{a}{2^j}\right) + \frac{\sin(\frac{a}{2^{j+1}})}{\sin(\frac{a}{2^{j-1}})}\cos\left((2i+2)\frac{a}{2^j}\right) \\ &= \frac{\sin(\frac{a}{2^{j-1}} - \frac{a}{2^{j+1}})}{\sin(\frac{a}{2^{j-1}})}\cos\left(2i\frac{a}{2^j}\right) + \frac{\sin(\frac{a}{2^{j+1}})}{\sin(\frac{a}{2^{j-1}})}\cos\left((2i+2)\frac{a}{2^j}\right) \\ &= \cos\left(\frac{a}{2^{j+1}}\right)\cos\left(2i\frac{a}{2^j}\right) - \sin\left(\frac{a}{2^{j+1}}\right)\sin\left(2i\frac{a}{2^j}\right) \\ &= \cos\left(\left(2i + \frac{1}{2}\right)\frac{a}{2^j}\right). \end{aligned}$$

Similarly, we can show that

$$q_{2i+1}^j = \cos\left(\left(2i + \frac{3}{2}\right)\frac{a}{2^j}\right).$$

The proof of the other part is similar. Analogously, we can prove that SS (9) also generates functions $\cos(ax)$ and $\sin(ax)$. □

3 Convergence analysis

In this section, we use the asymptotic equivalence method to find the smoothness of the normalized SSs (8) and (10).

Definition 1 ([18]) Two binary SSs, $\{S_{\alpha_j}\}$ and $\{S_{\beta_j}\}$, are asymptotically equivalent if

$$\sum_{j=1}^{\infty} \|S_{\alpha_j} - S_{\beta_j}\| < \infty,$$

where $\|S_{\alpha_j}\|_\infty = \max\{\sum_{i\in\mathbb{Z}} |\alpha_{2i}^{(j)}|, |\alpha_{2i+1}^{(j)}|\}$.

Theorem 6 ([18]) *Assume that $\{S_{\alpha_j}\}$ is a non-stationary SS and $\{S_{\beta_j}\}$ is a stationary SS. Let $\{S_{\alpha_j}\}$ and $\{S_{\beta_j}\}$ be two asymptotically equivalent SS having finite masks of the same support. If $\{S_{\beta_j}\}$ is C^m and $\sum_{j=0}^{\infty} 2^{mj}\|S_{\alpha_j} - S_{\beta_j}\| < \infty$, then the non-stationary SS $\{S_{a_j}\}$ is C^m.*

Some estimates of stencils $\eta_k^j, k = 0,1$ and $\lambda_k^j, k = -1,0,1,2$, are required to find the smoothness of the proposed schemes which are given in the following lemmas.

Lemma 7 *The following inequalities hold*:

(a) $\dfrac{1}{4} \le \lambda_1^j \le \dfrac{1}{4}\dfrac{1}{\cos(\frac{a}{2^{j-1}})}$,

(b) $\dfrac{3}{4} \le \lambda_0^j \le \dfrac{3}{4}\dfrac{1}{\cos(\frac{a}{2^{j-1}})}$.

Proof We give the proof of (*a*). Note that

$$\lambda_1^j = \frac{\cos(\frac{a}{2^j})\sin(\frac{a}{2^{j+1}})}{\cos(\frac{a}{2^{j+1}})\sin(\frac{a}{2^{j-1}})} \ge \frac{\frac{a}{2^{j+1}}}{\frac{a}{2^{j-1}}} = \left(\frac{1}{4}\right).$$

Also

$$\lambda_1^j = \frac{\cos(\frac{a}{2^j})(\sin(\frac{a}{2^{j+1}}))}{\cos(\frac{a}{2^{j+1}})\sin(\frac{a}{2^{j-1}})} \le \frac{\frac{\sin(\frac{a}{2^j})}{\frac{a}{2^j}}\sin(\frac{a}{2^{j+1}})}{\frac{\sin(\frac{a}{2^{j+1}})}{\frac{a}{2^{j+1}}}\sin(\frac{a}{2^{j-1}})} = \frac{\sin(\frac{a}{2^j})}{2\sin(\frac{a}{2^{j-1}})}$$

$$\le \frac{\frac{a}{2^j}}{2\frac{a}{2^{j-1}}\cos(\frac{a}{2^{j-1}})} \le \frac{\frac{a}{2^{j+1}}}{\frac{a}{2^{j-1}}\cos(\frac{a}{2^{j-1}})} = \frac{1}{4}\frac{1}{\cos(\frac{a}{2^{j-1}})}.$$

This completes the proof of (a). The proof of (b) is obtained similarly. □

Now, by Lemma 7, we have the following result.

Lemma 8 *The following inequalities hold*:

(a) $\left|\lambda_1^j - \dfrac{1}{4}\right| \le C_0\dfrac{1}{2^{2j}}$,

(b) $\left|\lambda_0^j - \dfrac{3}{4}\right| \le C_1\dfrac{1}{2^{2j}}$,

where C_0 and C_1 are constants independent of j.

Proof We present the proof of (a). By Lemma 7(a), we get

$$\begin{aligned}\left|\lambda_1^j-\frac{1}{4}\right| &\leq \left(\frac{1}{4}\right)\left(\frac{1-\cos(\frac{a}{2^{j-1}})}{\cos(\frac{a}{2^{j-1}})}\right) \leq 2\frac{1}{4\cos(\frac{a}{2^{j-1}})}\sin^2\left(\frac{a}{2^j}\right)\\ &\leq \frac{1}{2\cos(\frac{a}{2^{j-1}})}\frac{a^2}{2^{2j}} \leq C_0\frac{1}{2^{2j}}.\end{aligned}$$

This complete the proof of (a). The proof of (b) is obtained similarly. □

Remark 3.1 The the normalized SS (8) is a non-stationary counterpart of the following stationary SS [7]:

$$\begin{cases} q_{2i}^{j+1} = \frac{3}{4}q_i^j + \frac{1}{4}q_{i+1}^j, \\ q_{2i+1}^{j+1} = \frac{1}{4}q_i^j + \frac{3}{4}q_{i+1}^j \end{cases} \tag{11}$$

because the stencils of the normalized SS (8) converge to the stencils of (11): $\lambda_0^j \to (\frac{3}{4})$ and $\lambda_1^j \to (\frac{1}{4})$ as $j \to \infty$. The proof of convergence follows from Lemma 8.

Lemma 9 *Suppose that the Laurent polynomial $a(z)$ of the stationary SS (16) can be written as*

$$a(z) = \left\{\left(\frac{1}{4}\right) + \left(\frac{3}{4}\right)z^1 + \left(\frac{3}{4}\right)z^2 + \left(\frac{1}{4}\right)z^3\right\},$$

then SS S_a corresponding to the Laurent polynomial $a(z)$ is C^1.

Proof To find the smoothness of the stationary scheme S_α, we consider $a(z)$,

$$a(z) = \frac{1}{4}\left(1+3z+3z^2+z^3\right).$$

If

$$c(z) = \frac{4a(z)}{(1+z)^2} = (1+z),$$

then

$$\left\|\frac{1}{2}S_c\right\| = \frac{1}{2}\max\left\{\sum_{k\in\mathbb{Z}}|c_{2k}|, \sum_{k\in\mathbb{Z}}|c_{2k+1}|\right\} = \max\left\{\frac{1}{2}, \frac{1}{2}\right\} < 1.$$

Hence by [18, Corollary 4.11], the SS S_a is C^1. □

Lemma 10 *The Laurent polynomial $a^j(z)$ of the jth refinement level of the stationary SS (10) can be written as $a^j(z) = (\frac{1+z}{2})b^j(z)$ where*

$$b^j(z) = 2\left\{\lambda_1^j + \left(\lambda_0^j - \lambda_1^j\right)z + \lambda_1^j z^2\right\}.$$

Proof Observe that

$$a^j(z) = \lambda_1^j + (\lambda_0^j)z + (\lambda_0^j)z^2 + (\lambda_1^j)z^3.$$

It can be easily proved that $a^j(z) = (\frac{1+z}{2})b^j(z)$. □

Theorem 11 *The stationary SSs* (8) *and* (11) *are asymptotically equivalent, that is,*

$$\sum_{j=0}^{\infty} \|S_{a^j} - S_a\|_\infty < \infty.$$

Proof From the stationary SSs (8) and (11), we get

$$\sum_{j=0}^{\infty} \|S_{a^j} - S_a\|_\infty = \sum_{j=0}^{\infty} \left\{ \left| \lambda_0^j - \frac{3}{4} \right| + \left| \lambda_1^j - \frac{1}{4} \right| \right\}.$$

From Lemma 8(a), it follows that

$$\sum_{j=0}^{\infty} \left| \lambda_1^j - \frac{1}{4} \right| \le \sum_{j=0}^{\infty} C_0 \frac{1}{2^{2j}} < \infty.$$

Similarly from Lemma 8(b) we obtain

$$\sum_{j=0}^{\infty} \left| \lambda_0^j - \frac{3}{4} \right| < \infty.$$

Hence

$$\sum_{j=0}^{\infty} \|S_{a^j} - S_a\|_\infty < \infty.$$

□

Theorem 12 *The non-stationary SS* (8) *is* C^1.

Proof Since S_a is C^1 by Lemma 9 and also the stationary SSs (8) and (11) are asymptotically equivalent by Theorem 11, by [18, Theorem 8(a)], it is sufficient to prove that

$$\sum_{j=0}^{\infty} 2^j \|S_{a^j} - S_a\|_\infty < \infty,$$

where

$$\begin{aligned} \|S_{a^j} - S_a\|_\infty &= \max\left\{ \sum_{k\in\mathbb{Z}} \left| a_{2k}^j - a_{2j} \right|, \sum_{k\in\mathbb{Z}} \left| a_{2k+1}^j - a_{2j+1} \right| \right\} \\ &= \sum_{j=0}^{\infty} \left\{ 2\left| \lambda_0^j - \frac{3}{4} \right| + 2\left| \lambda_1^j - \frac{1}{4} \right| \right\}. \end{aligned}$$

Note that

$$\left|\lambda_0^j + \lambda_1^j\right| \leq \left|\lambda_0^j - \frac{3}{4}\right| + \left|\lambda_1^j - \frac{1}{4}\right|.$$

Since

$$\sum_{j=0}^{\infty} 2^j \left|\lambda_0^j - \frac{3}{4}\right| < \infty \quad \text{and} \quad \sum_{j=0}^{\infty} 2^j \left|\lambda_1^j - \frac{1}{4}\right| < \infty,$$

by Lemma 8(a)–(b), it follows that

$$\sum_{j=0}^{\infty} 2^j \left|\lambda_0^j + \lambda_1^j - 1\right| < \infty.$$

Hence

$$\sum_{j=0}^{\infty} 2^j \|S_{a^j} - S_a\|_\infty < \infty. \qquad \square$$

Now we discuss the procedure for checking the smoothness of four-point non-stationary SS (9). The proofs of the following lemmas are similar to those of Lemmas 7 and 8.

Lemma 13 *The following inequalities hold*:

(a) $-\frac{7}{128} \leq \lambda_{-1}^j \leq -\frac{7}{128}$,

(b) $\frac{105}{128} \leq \lambda_0^j \leq \frac{105}{128\cos(a)}$,

(c) $\frac{35}{128} \leq \lambda_1^j \leq \frac{35}{128\cos(a)}$,

(d) $-\frac{5}{128} \leq \lambda_2^j \leq -\frac{5}{128}$.

Using Lemma 13, we get following result.

Lemma 14 *The following inequalities hold*:

(a) $\left|\lambda_{-1}^j - \left(-\frac{7}{128}\right)\right| \leq D_0 \frac{1}{2^{2j}}$,

(b) $\left|\lambda_0^j - \left(\frac{105}{128}\right)\right| \leq D_1 \frac{1}{2^{2j}}$,

(c) $\left|\lambda_1^j - \left(\frac{35}{128}\right)\right| \leq D_2 \frac{1}{2^{2j}}$,

(d) $\left|\lambda_2^j - \left(-\frac{5}{128}\right)\right| \leq D_3 \frac{1}{2^{2j}}$,

where D_0, D_1, D_2, and D_3 are some constants independent of j.

Remark 3.2 The four-point stationary SS (9) is a non-stationary counterpart of following stationary SS [15]:

$$\begin{cases} q_{2i}^{j+1} = (-\frac{7}{128})q_{i-1}^j + (\frac{105}{128})q_i^j + (\frac{35}{128})q_{i+1}^j + (-\frac{5}{128})q_{i+2}^j, \\ q_{2i+1}^{j+1} = (-\frac{5}{128})q_{i-1}^j + (\frac{35}{128})q_i^j + (\frac{105}{128})q_{i+1}^j + (-\frac{7}{128})q_{i+2}^j \end{cases} \tag{12}$$

because the stencils of the normalized SS (9) converge to the stencils of the stationary SS (12): $\lambda_{-1}^j \to -\frac{7}{128}$, $\lambda_o^j \to \frac{105}{128}$, $\lambda_1^j \to \frac{35}{128}$ and $\lambda_2^j \to -\frac{5}{128}$ as $j \to \infty$. The proof of these facts follows from Lemma 14.

Theorem 15 *The stationary SSs* (9) *and* (12) *are asymptotically equivalent, that is,*

$$\sum_{j=0}^{\infty} \|S_{a^j} - S_a\|_\infty < \infty.$$

Proof From (9) and (12), we have

$$\begin{aligned}\sum_{j=0}^{\infty} \|S_{a^j} - S_a\|_\infty &= \sum_{j=0}^{\infty} \left\{ \left|\lambda_{-1}^j - \left(-\frac{7}{128}\right)\right| + \left|\lambda_0^j - \left(\frac{105}{128}\right)\right| \right. \\ &\quad \left. + \left|\lambda_1^j - \left(\frac{35}{128}\right)\right| + \left|\lambda_2^j - \left(\frac{-5}{128}\right)\right| \right\}.\end{aligned}$$

From Lemma 14(a), it follows that

$$\sum_{j=0}^{\infty} \left|\lambda_{-1}^j - \left(-\frac{7}{128}\right)\right| \le \sum_{j=0}^{\infty} D_0 \frac{1}{2^{2j}} < \infty.$$

Similarly, from Lemma 14(b)–(d) we obtain

$$\begin{aligned} &\left|\lambda_0^j - \left(\frac{105}{128}\right)\right| < \infty, \qquad \left|\lambda_1^j - \left(\frac{35}{128}\right)\right| < \infty, \\ &\left|\lambda_2^j - \left(\frac{-5}{128}\right)\right| < \infty. \end{aligned}$$

Hence

$$\sum_{j=0}^{\infty} \|S_{a^j} - S_a\|_\infty < \infty.$$

□

Theorem 16 *The non-stationary SS* (9) *is* C^2.

The proof of above theorem is similar to that of Theorem 12.

4 Shape preservation of binary four-point SS

In this section, we will check what axiom should be applied on the control points so that the limit curve achieved by binary 4-point subdivision scheme (9) is both monotonicity and convexity preserving.

4.1 Monotonicity preservation

Lemma 17 *Consider the control points* $\{q_i^0\}_{i\in\mathbb{Z}}$,

$$\cdots < q_{-2}^0 < q_{-1}^0 < q_0^0 < q_1^0 < \cdots < q_{n-1}^0 < q_n^0 < \cdots .$$

Define first order divided difference by $D_i^j = q_{i+1}^j - q_i^j$, *taking*

$$q_i^j = \frac{D_{i+1}^j}{D_i^j}, \qquad Q^j = \max_i \left\{ q_i^j, \frac{1}{q_i^j} \right\}, \quad \forall j \geq 0, i, j \in \mathbb{Z}.$$

Furthermore, consider $\frac{29-\sqrt{801}}{4} \leq \rho \leq 1$, $\rho \in \mathbb{R}$.
If $\frac{1}{\rho} \leq Q^0 \leq \rho$ *and* $\{p_i^j\}$ *is given by the SS* (9), *then*

$$D_i^j > 0, \qquad \frac{1}{\rho} \leq Q^j \leq \rho, \quad \forall j \geq 0, i, j \in \mathbb{Z}. \tag{13}$$

Proof To prove Lemma 17, we use mathematical induction on j.

(I) By hypothesis, when $j = 0$, $D_i^0 = q_{i+1}^0 - q_i^0 > 0$, $\frac{1}{\rho} \leq Q^0 \leq \rho$, then (13) is satisfied.

(II) Suppose that (13) is satisfied for some $j \geq 1$, then we have to prove that it is true for $j+1$.

We first prove $D_i^j > 0$, $\forall j \geq 0, i, j \in \mathbb{Z}$.

Assume that $D_i^j > 0$, $\forall i \in \mathbb{Z}$, is true for some $j \geq 1$. Then $\forall i \in \mathbb{Z}$, we have

$$\begin{aligned}
D_{2i}^{j+1} &= q_{2i+1}^{j+1} - q_{2i}^{j+1} \\
&= \frac{1}{128}\left[-2\left(q_i^j - q_{i-1}^j\right) + 58\left(q_{i+1}^j - q_i^j\right) - 2\left(q_{i+2}^j - q_{i+1}^j\right)\right] \\
&= \frac{1}{128}\left[-2D_{i-1}^j + 58D_i^j - 2D_{i+1}^j\right] \\
&= \frac{D_i^j}{128}\left[\frac{-2}{q_{i-1}^j} + 58 - 2q_i^j\right] \\
&= \frac{D_i^j}{128}\left[\frac{-2}{\rho} + 58 - 2\rho\right] > 0
\end{aligned} \tag{14}$$

and

$$\begin{aligned}
D_{2i+1}^{j+1} &= q_{2i+2}^{j+1} - q_{2i+1}^{j+1} \\
&= \frac{1}{128}\left[-5D_{i-1}^j + 37D_i^j + 37D_{i+1}^j - 5D_{i+2}^j\right] \\
&= \frac{D_i^j}{128}\left[-\frac{5}{q_{i-1}^j} + 37 + 37q_i^j - 5q_{i+1}^j q_i^j\right] \\
&= \frac{D_i^j}{128}\left[-5\frac{1}{\rho} + 37 + \left(37 - 5\frac{1}{\rho}\right)q_i^j\right] \\
&\geq \frac{D_i^j}{128}\left[-5\frac{1}{\rho} + 37 + \left(37 - 5\frac{1}{\rho}\right)\rho\right]
\end{aligned}$$

$$= \frac{D_i^j}{128\rho}\big[37\rho^2 + 32\rho - 5\big] > 0, \tag{15}$$

which implies that $D_i^{j+1} > 0$, $\forall i \in \mathbb{Z}$.

Therefore, by induction, $D_i^j > 0$, $\forall j \geq 0, i \in \mathbb{Z}, j \in \mathbb{Z}$.

(III) We now prove that $\frac{1}{\rho} \leq Q^j \leq \rho$, $\forall j \geq 0, j \in \mathbb{Z}$.

Since

$$\begin{aligned}
q_{2i}^j &= \frac{D_{2i+1}^{j+1}}{D_{2i}^{j+1}} = \frac{\frac{D_i^j}{128}[-\frac{5}{q_{i-1}^j} + 37 + 37q_i^j - 5q_{i+1}^j q_i^j]}{\frac{D_i^j}{128}[\frac{-2}{q_{i-1}^j} + 68 - 2q_i^j]}, \\
q_{2i}^j - \rho &= \frac{[-\frac{5}{q_{i-1}^j} + 37 + 37q_i^j - 5q_{i+1}^j q_i^j] - \rho[\frac{-2}{q_{i-1}^j} + 68 - 2q_i^j]}{[\frac{-2}{q_{i-1}^j} + 68 - 2q_i^j]}, \\
q_{2i}^j - \rho &= \frac{-\frac{5}{q_{i-1}^j} + 37 + 37q_i^j - 5q_{i+1}^j q_i^j + \frac{2\rho}{q_{i-1}^j} - 68\rho + 2\rho q_i^j}{\frac{-2}{q_{i-1}^j} + 68 - 2q_i^j}, \\
q_{2i}^j - \rho &= \frac{N}{D},
\end{aligned} \tag{16}$$

and as the denominator in (16) is positive, i.e., $D > 0$, the numerator satisfies:

$$\begin{aligned}
N &\leq -\frac{5}{q_{i-1}^j} + 37 + 37q_i^j - 5q_{i+1}^j q_i^j + \frac{2\rho}{q_{i-1}^j} - 68\rho + 2\rho q_i^j \\
&= \left(\frac{-5}{\rho} + 37 + 2\rho\right) q_i^j + 37 - \frac{5}{q_{i-1}^j} + 2\rho\frac{1}{q_{i-1}^j} - 68\rho \\
&= \frac{1}{\rho}\big(2\rho^3 - 31\rho^2 + 34\rho - 5\big) \\
&\leq \frac{1}{\rho}(\rho - 1)\big(2\rho^2 - 29\rho + 5\big) \leq 0.
\end{aligned}$$

Therefore, $q_{2i}^j \leq \rho$.

Similarly, we can get $q_{2i+1}^j \leq \rho$, $\frac{1}{q_{2i}^j} \leq \rho$ and $\frac{1}{q_{2i+1}^j} \leq \rho$, which implies $\frac{1}{\rho} \leq Q^{j+1} \leq \rho$.

Therefore, by induction, we have $\frac{1}{\rho} \leq Q^j \leq \rho$, $\forall j \geq 0, j \in \mathbb{Z}$, completing the proof. □

A direct consequence of Lemma 17 is Theorem 18.

Theorem 18 *Suppose the control points $\{q_i^0\}_{i\in\mathbb{Z}}$ with $q_i^0 = (x_i^0, f_i^0)$ are strictly monotone decreasing (strictly monotone increasing). Denote*

$$X^0 = \max_i \left\{\frac{x_{i+2}^0 - x_{i+1}^0}{x_{i+1}^0 - x_i^0}, \frac{x_{i+1}^0 - x_i^0}{x_{i+2}^0 - x_{i+1}^0}\right\}, \qquad Q^0 = \max_i \left\{q_i^0, \frac{1}{q_i^0}\right\}.$$

Then, for $\frac{1}{\rho} \leq X^0 \leq \rho$ and $\frac{1}{\rho} \leq Q^0 \leq \rho$, we have

$$\frac{29 - \sqrt{801}}{4} \leq \rho \leq 1, \quad \rho \in \mathbb{R},$$

and the limit functions obtained by the SS (9) *are strictly monotone decreasing* (*strictly monotone increasing*).

4.2 Convexity preservation

Definition 2 Consider that data points $\{q_i^0\}_{i\in\mathbb{Z}}$ with $q_i^0 = (x_i^0, q_i^0)$ are strictly convex, where $\{x_i^0\}_{i\in\mathbb{Z}}$ are equidistant. For convenience, we let $\Delta x_i^0 = x_{i+1}^0 - x_i^0 = 1$. By SS (9), we have $\Delta x_i^{j+1} = x_{i+1}^{j+1} - x_i^{j+1} = \frac{1}{2}\Delta x_i^j = \frac{1}{2^{j+1}}$.

Definition 3 Let $d_i^j = 2^j(q_{i-1}^j - 2q_i^j + q_{i+1}^j)$ denote the 2nd order divided differences. In the following, we will prove $d_i^0 > 0$, $\forall j \geq 0, j, i \in \mathbb{Z}$. The SS (9) can thus be written in terms of 2nd order divided differences as follows:

$$d_{2i}^{j+1} = \frac{1}{32}\left[-5d_{i-1}^j + 34d_i^j + 3d_{i+1}^j\right],$$
$$d_{2i+1}^{j+1} = \frac{1}{32}\left[3d_i^j + 34d_{i+1}^j - 5d_{i+2}^j\right].$$

Theorem 19 *Consider the control points* $\{q_i^0\}_{i\in\mathbb{Z}}$, $q_i^0 = (x_i^0, q_i^0)$, *which are strictly convex, i.e.*, $d_i^0 > 0$, $\forall i \in \mathbb{Z}$. *Let* $\Gamma^j = \max_i\{r_i^j, \frac{1}{r_i^j}\}$, *where* $r_i^j = \frac{d_{i+1}^j}{d_i^j}$, $\forall j \geq 0, j \in \mathbb{Z}$.

Furthermore, consider $\frac{17-\sqrt{274}}{3} \leq \lambda \leq 1$, $\lambda \in \mathbb{R}$. *Then for* $\frac{1}{\lambda} \leq \Gamma^0 \leq \lambda$, *we get*

$$d_i^0 > 0, \qquad \frac{1}{\lambda} \geq \Gamma^j < \lambda, \quad \forall j \geq 0, i \in \mathbb{Z}, j \in \mathbb{Z}. \tag{17}$$

In particular, the limit functions generated by the four-point binary approximating stationary SS defined in (9) *preserve convexity.*

Proof To verify Theorem 19, we use mathematical induction on j.

(I) By hypothesis, (17) holds true for $j = 0$, as is easily seen to be true: $d_i^0 > 0$, $\frac{1}{\lambda} \leq \Gamma^0 < \lambda$.

(II) Suppose that if (17) true for some $j \geq 1$. It must then be shown that (17) holds true for $j+1$. To achieve this, we first prove that $d_i^j > 0$, $\forall j \geq 0, i, j \in \mathbb{Z}$. From the assumption that $d_i^j > 0$, $\forall i \in \mathbb{Z}$, it follows $\forall i \in \mathbb{Z}$ that

$$\begin{aligned}
d_{2i}^{j+1} &= \frac{1}{32}\left[-5d_{i-1}^j + 34d_i^j + 3d_{i+1}^j\right] \\
&= \frac{d_i^j}{32}\left[-5\frac{d_{i-1}^j}{d_i^j} + 34 + 3\frac{d_{i+1}^j}{d_i^j}\right] \\
&= \frac{d_i^j}{32}\left[-5\frac{1}{r_{i-1}^j} + 34 + 3r_i^j\right] \\
&\geq \frac{d_i^j}{32\lambda}\left[-5\lambda^2 + 34\lambda + 3\right] \\
&\geq 0,
\end{aligned}$$

and

$$d_{2i+1}^{j+1} = \frac{1}{32}\left[3d_i^j + 34d_{i+1}^j - 5d_{i+2}^j\right]$$

$$
\begin{aligned}
&= \frac{d_i^j}{32}\left[3 + 34\frac{d_{i+1}^j}{d_i^j} - 5\frac{d_{i+2}^j}{d_i^j}\right] \\
&= \frac{d_i^j}{32}\left[3 + (34 - 5\lambda)r_i^j\right] \\
&\geq \frac{d_i^j}{32\lambda}[-2\lambda + 34] \\
&\geq 0,
\end{aligned}
$$

which implies that $d_i^{j+1} > 0$, $\forall i \in \mathbb{Z}$.

Therefore, by mathematical induction, we have $d_i^j > 0$, $\forall j \geq 0, i, j \in \mathbb{Z}$.

(III) Now we prove that $\frac{1}{\lambda} \geq \Gamma^{j+1} < \lambda$, $j \geq 0$, $i \in \mathbb{Z}, j \in \mathbb{Z}$.

Since

$$
r_{2i}^{j+1} = \frac{d_{2i+1}^{j+1}}{d_{2i}^{j+1}} = \frac{\frac{d_i^j}{32}[3 + 34r_i^j - 5r_i^j r_{i+1}^j]}{\frac{d_i^j}{32}[-5\frac{1}{r_{i-1}^j} + 34 + 3r_i^j]} = \frac{3 + 34r_i^j - 5r_i^j r_{i+1}^j}{-5\frac{1}{r_{i-1}^j} + 34 + 3r_i^j},
$$

we get

$$
r_{2i}^{j+1} - \lambda = \frac{3 + 34r_i^j - 5r_i^j r_{i+1}^j + 5\lambda\frac{1}{r_{i-1}^j} - 34\lambda - 3\lambda r_i^j}{-5\frac{1}{r_{i-1}^j} + 34 + 3r_i^j}.
$$

Since $d_{2i}^{j+1} \geq 0$, the numerator of the above expression satisfies:

$$
\begin{aligned}
\text{Numerator} &\leq 3 + 34r_i^j - 5r_i^j r_{i+1}^j + 5\lambda\frac{1}{r_{i-1}^j} - 5 - 3\lambda r_i^j \\
&= \left(34 - 5\frac{1}{\lambda} - 3\lambda\right)r_i^j + 3 + 5\lambda\frac{1}{r_{i-1}^j} - 34\lambda \\
&= \left(34 - 5\frac{1}{\lambda} - 3\lambda\right)\lambda + 3 + 5\lambda^2 - 34\lambda \\
&= 2\lambda^2 - 2 \\
&= 2(\lambda - 1)(\lambda + 1) \\
&\leq 0,
\end{aligned}
$$

therefore $r_{2i}^{j+1} \leq \lambda$.

Similarly, we get $r_{2i+1}^{j+1} \leq \lambda$, $\frac{1}{r_{2i}^{j+1}} \leq \lambda$, and $\frac{1}{r_{2i+1}^{j+1}} \leq \lambda$, which implies $\frac{1}{\lambda} \geq \Gamma^{j+1} < \lambda$.

Therefore, by mathematical induction, we have $\frac{1}{\lambda} \geq \Gamma^j < \lambda$, $\forall j \geq 0, j \in \mathbb{Z}$, completing the proof. □

5 Results and discussion

Now, we compare the proposed SSs (8) and (10) with some known existing ASS [4, 12–14, 22, 23] and illustrate through their smooth curves helix, curvature, and torsion plots. The curves in the figures of this section are drawn after the fifth subdivision level.

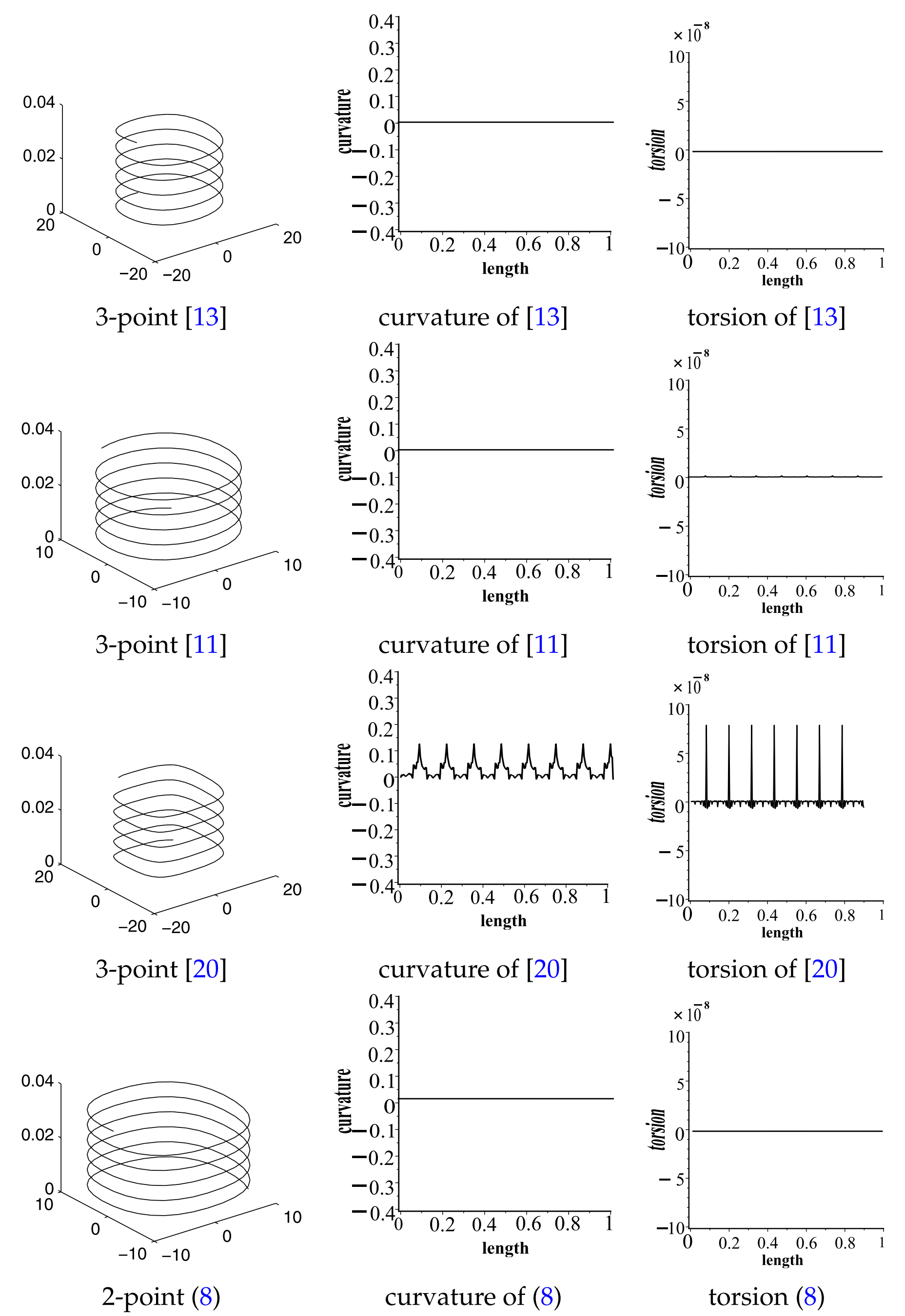

Figure 1 Limit curves obtained after the fifth iteration (left), the corresponding curvature (center) and torsion (right)

In Fig. 1, we first compare the helix, curvature and torsion plots of the 3-point schemes [11, 13, 23] and the 2-point proposed scheme (8). Similarly, in Fig. 2, we compare the helix, curvature and torsion plots of the 4-point schemes [14, 22] and the proposed scheme (10).

The limit curves generated by existing SSs [4, 12–14, 22, 23] and proposed schemes (8) and (10), along with their curvature plots, are illustrated in Fig. 3.

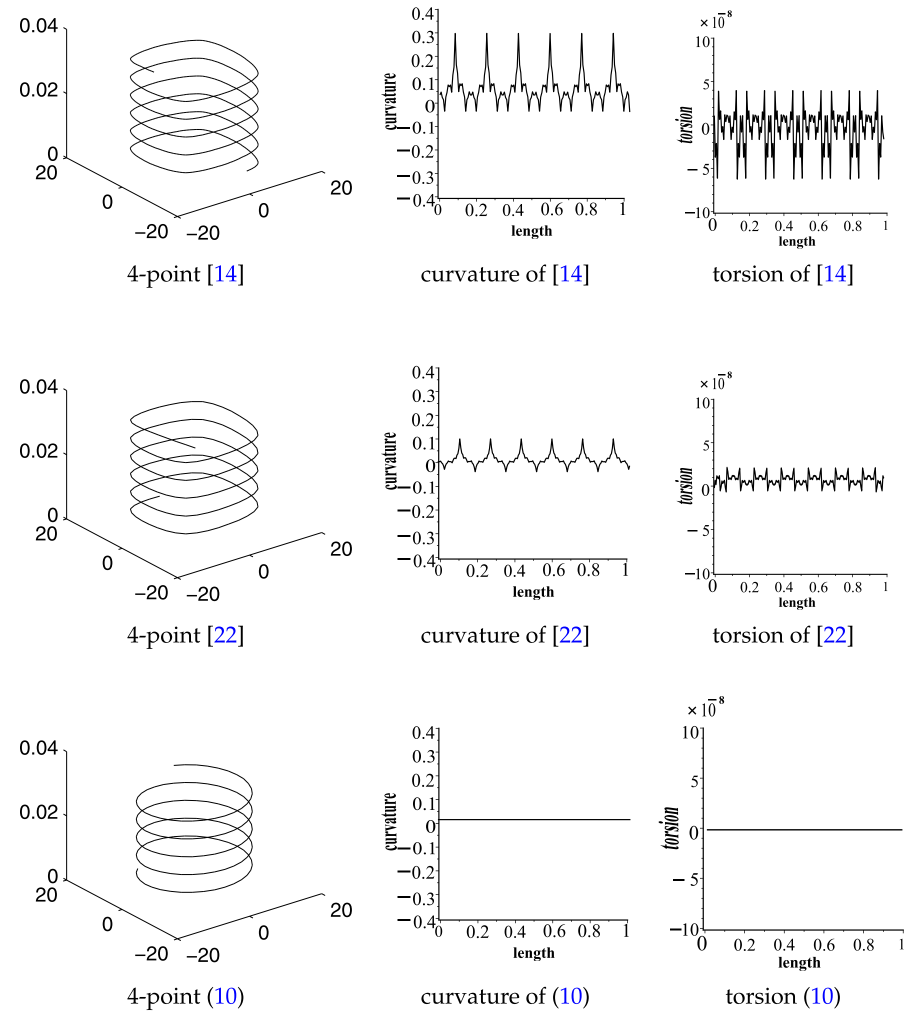

Figure 2 Limit curves obtained after the fifth iteration, the corresponding curvature (center) and torsion (right)

6 Conclusion

In this paper, we have constructed a simple and efficient algorithm to generate binary $2m$-point approximating non-stationary SS for any integer $m \geq 2$. The proposed 2-point (8) and 4-point (10) SSs have been assumed as non-stationary counterparts of the stationary SSs [7] and [15, 31], respectively. The constructions of the SSs (8) and (10) have been associated with trigonometric polynomials that reproduce the functions. It has been proved that our schemes have the ability to reconstruct the conics, especially circles. The asymptotic equivalence method is applied to investigate the smoothness of our SSs. A comparison of our SSs with the existing non-stationary SSs has been depicted by their helix, curvature and torsion plots. It is clear that the proposed SSs give better approximation and are more effective with the control polygons. Also the shape preserving properties of the binary 4-point ASS (9) generating C^2-continuous limit curves have been derived.

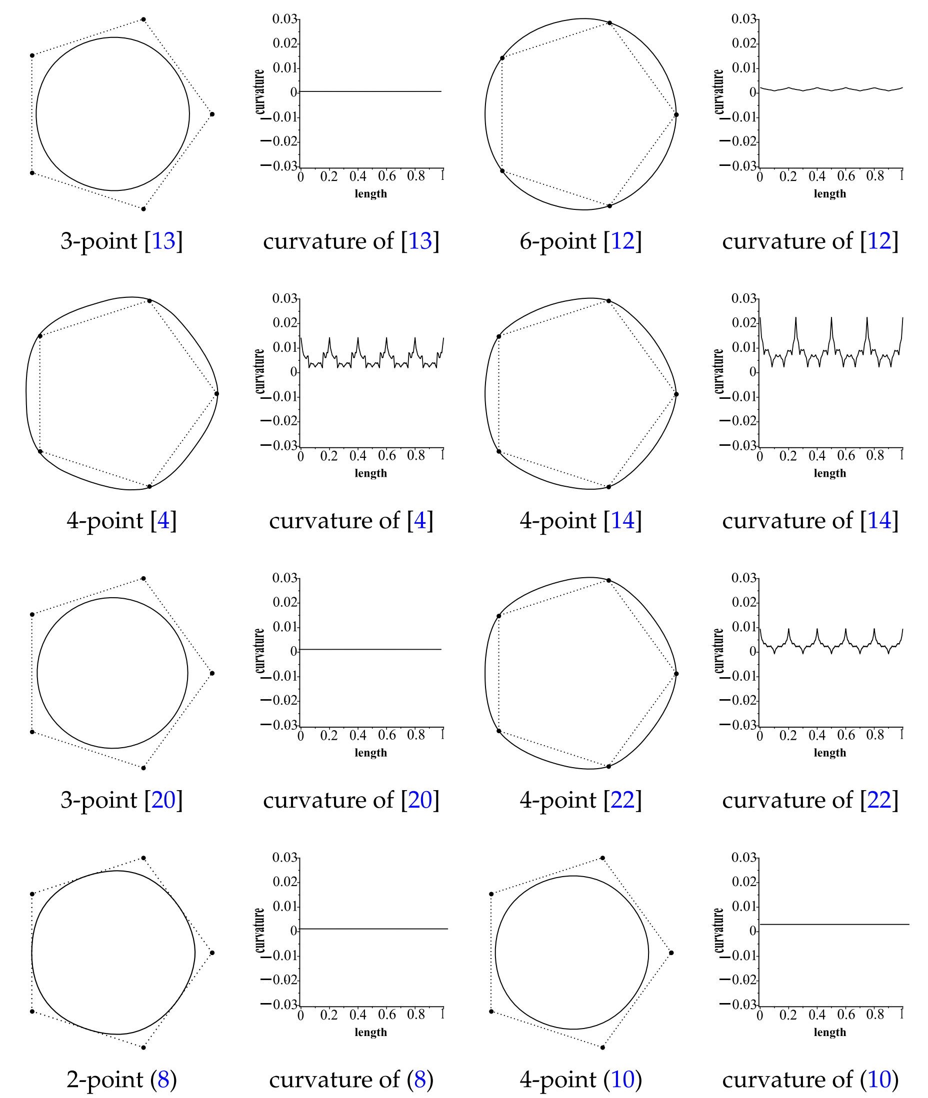

Figure 3 Comparison of the existing [4, 12–14, 22, 23] and proposed schemes (8) and (10) when five initial control points are sampled from a circle

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