

RESEARCH

Open Access



Soliton, breather, lump and their interaction solutions of the $(2 + 1)$ -dimensional asymmetrical Nizhnik–Novikov–Veselov equation

Yaqing Liu¹ and Xiao-Yong Wen^{1*}

*Correspondence:

xiaoyongwen@163.com

¹School of Applied Science, Beijing Information Science and Technology University, Beijing, China

Abstract

In this work, the $(2 + 1)$ -dimensional asymmetrical Nizhnik–Novikov–Veselov equation is investigated. Hirota's bilinear method is used to determine the N -soliton solutions for this equation, from which the M -lump solutions are obtained by using long wave limit when N is even (i.e., $N = 2M$). Then, taking $N = 5$ as an example, we discuss some novel mixed lump-soliton and lump-soliton-breather solutions by using long wave limit and choosing special conjugate complex parameters from the five-soliton solution. Figures are plotted to reveal the dynamical features of such obtained lump and mixed interaction solutions. These results may be useful for understanding the propagation phenomena of nonlinear localized waves.

Keywords: Hirota bilinear method; NNV equation; N -soliton solution; Interaction solution

1 Introduction

Nonlinear evolution equations are well used to describe various significant nonlinear phenomena in nature, which display significant prosperities as the soliton solution, infinite number of conservation laws, symmetries, and Hamiltonian structures. Searching for exact solutions of nonlinear evolution equations is important in scientific and engineering applications because it offers rich knowledge on the mechanism of the complicated physical phenomena. A set of systematic methods have been used in the literature to obtain reliable treatments of nonlinear evolution equations. So far, researchers have established several methods to find the exact solutions, including the inverse scattering transform [1], the Bäcklund transformation [2–5], the Darboux transformation [6–14], the Riemann–Hilbert approach [15–17] and Hirota's bilinear method [18–28], Jacobian elliptic function method and modified tanh-function method [29–33]. Each of these approaches has its features, Hirota's bilinear method is widely popular due to its simplicity and directness. In Refs. [34, 35], some lump solutions and interaction solutions of Hirota–Satsuma–Ito equation are computed via Hirota's bilinear form through conducting symbolic computations. In Ref. [36], two kinds of lump solutions are constructed explicitly through Hirota's bilinear method. Hirota's bilinear method can be used usually to construct the exact lo-

calized wave solutions such as soliton, breather (alias periodic soliton), and lump. Soliton has the property of stability caused by the balance of nonlinear and dispersive effects on the medium [31, 32, 37]. Breather is the partially localized breathing waves with a periodic structure in a certain direction [38–40]. Lump is a kind of rational function solutions in all space directions, which have some physical applications in shallow water wave, plasma, optic media, and Bose–Einstein condensate [41, 42].

In this paper, we consider the following (2 + 1)-dimensional asymmetrical Nizhnik–Novikov–Veselov (ANNV) equation:

$$u_t + u_{xxx} + 3 \left(u \int u_x dy \right)_x = 0, \tag{1}$$

where the subscripts respectively denote the partial derivatives with respect to the two scaled space coordinates x, y and time t , u is the functions of x, y , and t , and u is the physical field. The ANNV equation, which is an isotropic Lax integrable extension of the KdV equation, has been proposed in the modern string theory and theory of biological membranes. Many papers focus on analyzing the exact solutions of Eq. (1). Boiti et al. [43] have first derived Eq. (1) and solved it by the inverse scattering transformation. Dai et al. [44] have derived the variable separation solutions of Eq. (1) by using extended tanh-function method. Wazwaz [45] has investigated the multiple soliton solutions for generalized, asymmetric, and modified NNV equation with the help of a simplified form of Hirota’s bilinear method. Fan [46] has investigated the quasi-periodic wave solutions and established the relations between the quasi-periodic wave solutions and soliton solutions of Eq. (1) based on a multi-dimensional Riemann theta function and Hirota’s bilinear method. Zhao et al. [47] have presented the lump stripe solution of Eq. (1) by using bilinear form. As far as we know, the M -lump solutions and different types of localized wave interaction solutions including lump-soliton and lump-soliton-breather solutions have not been reported before.

The rest of this paper is arranged as follows. In Sect. 2, we firstly present the N -soliton solutions of Eq. (1) by using Hirota’s bilinear method. Section 3 is devoted to the M -lump solutions by using long wave limit to even N -soliton solutions. In Sect. 4, we take odd five-soliton solution as an example and give some mixed lump-soliton and lump-soliton-breather solutions by using long wave limit and choosing special parameters. Some conclusions are given in the last section.

2 The N -soliton solution of the (2 + 1)-dimensional ANNV equation

Taking the transformation

$$u = u_0 + 2(\ln f)_{xy}, \tag{2}$$

Equation (1) is converted into the following bilinear formulism:

$$(D_y D_t + D_x^3 D_y + 3u_0 D_x^2) f \cdot f = 0, \tag{3}$$

where the bilinear differential operator D is defined [48] by

$$D_x^m D_y^n D_t^p (a \cdot b) = (\partial_x - \partial_{x'})^m (\partial_y - \partial_{y'})^n (\partial_t - \partial_{t'})^p$$

$$\times a(x, y, t)b(x', y', t')|_{x=x', y=y', t=t'}, \tag{4}$$

then Eq. (3) is equivalent to

$$2f_{yt} - 2f_t f_y + 2f_{xxx} f - 2f_{xxx} f_y - 6f_{xxy} f_x + 6f_{xx} f_{xy} + 6u_0 f_{xx} f - 6u_0 f_x^2 = 0. \tag{5}$$

Based on Hirota’s bilinear method, the N -soliton solution of Eq. (1) is obtained by substituting

$$f = 1 + \sum_{s=1}^N \exp(\eta_s) + \sum_{s<j}^N A_{sj} \exp(\eta_s + \eta_j) + \sum_{s<j<k}^N A_{sj} A_{sk} A_{jk} \times \exp(\eta_s + \eta_j + \eta_k) + \dots + \left(\prod_{s<j} A_{sj} \right) \exp \left\{ \sum_{s=1}^N \eta_s \right\} \tag{6}$$

into Eq. (2) with

$$\eta_s = a_s x + b_s y + c_s t + \eta_{0s}, \quad c_s = -\frac{a_s^3 b_s + 3u_0 a_s^2}{b_s}, \tag{7}$$

$$A_{sj} = -\frac{(a_s - a_j)^3 (b_s - b_j) + 3u_0 (a_s - a_j)^2 + (b_s - b_j)(c_s - c_j)}{(a_s + a_j)^3 (b_s + b_j) + 3u_0 (a_s + a_j)^2 + (b_s + b_j)(c_s + c_j)},$$

$$s, j = 1, 2, \dots, N, \tag{8}$$

where the parameters $a_s, b_s,$ and η_{0s} are constants related to the amplitude and phase of the N th soliton, respectively. Motivated by the work of [9, 21], we have the following theorem.

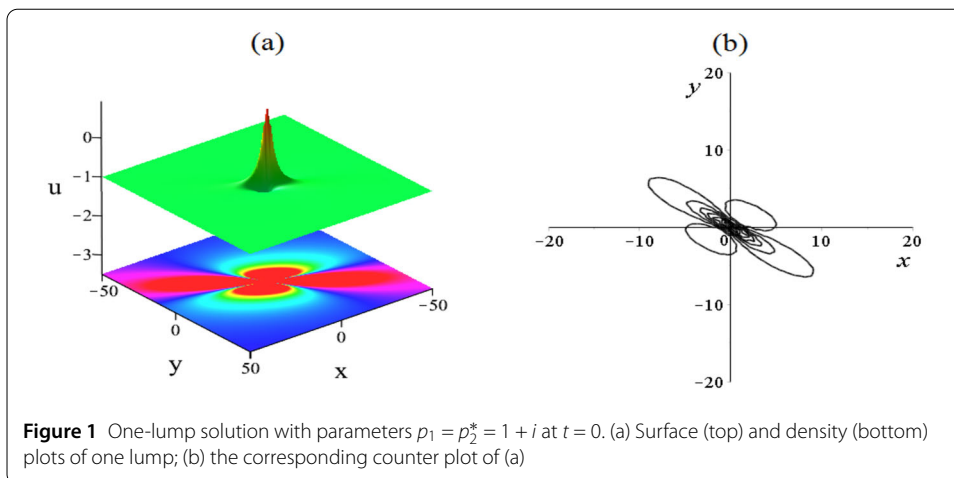
Theorem 1 *Let $b_s = p_s a_s$ ($s = 1, \dots, N$), $a_j = l_j \epsilon$, $\exp(\eta_{0j}) = -1$ ($j = 1, \dots, 2M$), $p_n = p_{n+M}^*$ ($n = 1, \dots, M$), $a_{2M+l} = a_{2M+P+l}^*$ ($l = 1, \dots, P$), and $a_{2M+2P+k}$ ($k = 1, \dots, Q$) be real constants. When $\epsilon \rightarrow 0$, the N -soliton solution of Eq. (1) can reduce to the interaction solutions of M -lump, P -breather, and Q -soliton, where $N = 2M + 2P + Q$, in which M, P, Q are nonnegative integers and represent the numbers of lump, breather, and soliton, respectively.*

Next, we will apply the above conclusion in Theorem 1 to give the M -lump, mixed lump-soliton, and lump-soliton-breather solutions of Eq. (1).

3 M -lump solutions

In this section, we let $P = Q = 0$ (i.e., $N = 2M$) in Theorem 1, we can obtain M -lump solutions of Eq. (1). By choosing parameters $b_s = p_s a_s, a_s = l_s \epsilon$ in Eq. (6) with the provision $\exp(\eta_{0s}) = -1, s = 1, 2, \dots, N$ (N is an even number), and then taking long wave limit as $\epsilon \rightarrow 0$, the function f in Eq. (6) is translated into pure rational function. Therefore, the general higher-order rational functions of Eq. (1) can be presented as [49, 50]

$$f_N = \prod_{s=1}^N \omega_s + \frac{1}{2} \prod_{sj}^N B_{sj} \sum_{l \neq s, j}^N \omega_l + \dots$$



$$+ \frac{1}{M!2^M} \sum_{s,k,\dots,m,n}^N \overbrace{B_{sj}B_{kl} \cdots B_{mn}}^M \prod_{p \neq s,j,k,l,\dots,m,n}^N \omega_p + \cdots, \tag{9}$$

where

$$B_{sj} = \frac{2p_s p_j (p_s + p_j)}{u_0 (p_s - p_j)^2}, \quad \omega_j = x + p_j y - \frac{3u_0}{p_j} t, \quad j = 1, 2, \dots, N. \tag{10}$$

If we choose $p_n = p_{n+M}^*$ ($n = 1, 2, \dots, M$) for $N = 2M$ with the condition $B_{sj} > 0$, we can get a class of nonsingular M -lump solutions.

(i) Setting $N = 2$ in Eq. (9), we have

$$f = \omega_1 \omega_2 + B_{12}. \tag{11}$$

Substituting Eq. (11) into the bilinear transformation in Eq. (2), we can obtain one-lump solutions of Eq. (1). Figure 1 shows one lump with one peak and two valleys at $t = 0$ if parameters $u_0 = -1, p_1 = p_2^* = 1 + i$. In the following, we always set the parameter $u_0 = -1$.

(ii) Setting $N = 4$ in Eq. (9), we have

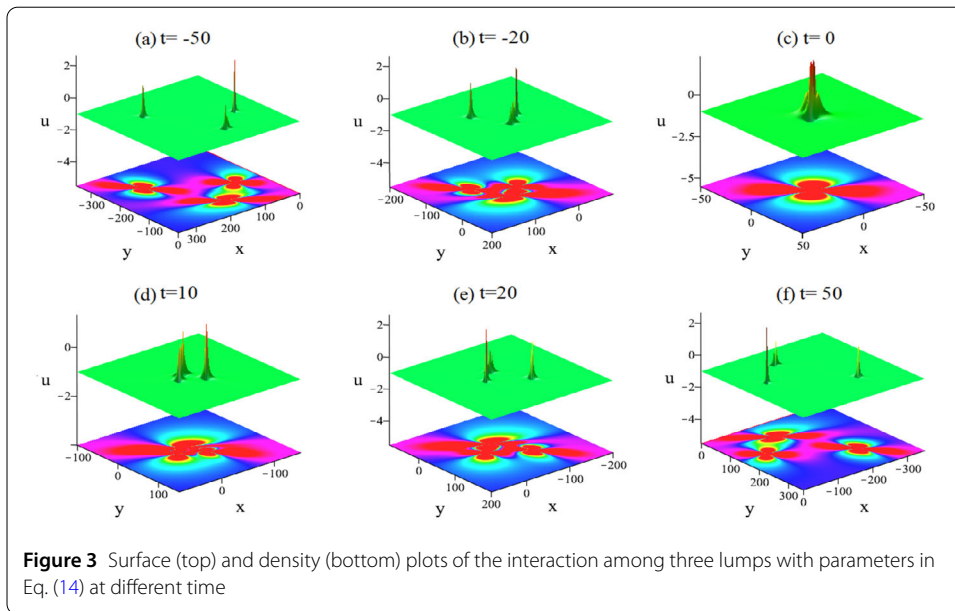
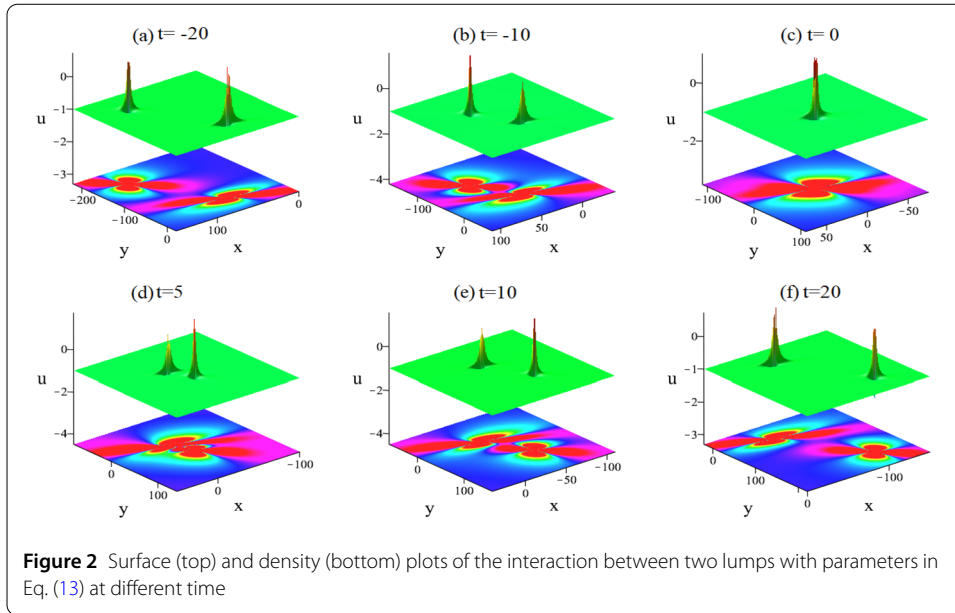
$$f = \omega_1 \omega_2 \omega_3 \omega_4 + B_{34} \omega_1 \omega_2 + B_{24} \omega_1 \omega_3 + B_{23} \omega_1 \omega_4 + B_{14} \omega_2 \omega_3 + B_{13} \omega_2 \omega_4 + B_{12} \omega_3 \omega_4 + B_{12} B_{34} + B_{13} B_{24} + B_{14} B_{23}. \tag{12}$$

Substituting Eq. (12) into the bilinear transformation in Eq. (2), we can obtain two lumps of Eq. (1). Figure 2 shows the elastic interaction between two lumps at different time with parameters

$$p_1 = p_2^* = 1 + i, \quad p_3 = p_4^* = \frac{1}{3} + \frac{1}{2}i. \tag{13}$$

From Fig. 2, we can clearly see that two lumps grow closer with time increasing, they collide at $t = 0$, then separate again, after the interaction, two lumps keep their shapes and amplitudes invariant, so their interaction is elastic.

(iii) Setting $N = 6$ in Eq. (9), we can obtain three-lump solution. In this case, there are many soliton parameters and the solution is very complicated, so we omit its expression



here. Figure 3 shows the elastic interaction among three lumps at different time with parameters

$$p_1 = p_2^* = \frac{1}{4} + i, \quad p_3 = p_4^* = 1 + i, \quad p_5 = p_6^* = \frac{1}{2} + \frac{3}{5}i. \tag{14}$$

From Fig. 3, we can clearly see that three lumps array a triangle structure at $t = -50$ and grow closer with time increasing, they interact at $t = 0$, after the interaction, then separate and rearrange a triangle. As time goes on, they are farther and farther away, and their shapes and amplitudes remain the same as before.

4 The lump interacts with soliton or breather

In the previous section, we discussed the M -lump solutions of Eq. (1) by use of long wave limit. In this section, we will consider the interaction solutions of different localized waves such as the interaction of lump and soliton or breather by using long wave limit and choosing conjugate spectral parameters. Here, we take $N = 5$ in Eq. (6) as an example.

Case 1. One lump interacts with soliton or breather. Putting $b_s = p_s a_s$ ($s = 1, 2, 3, 4, 5$), $a_1 = l_1 \epsilon$, $a_2 = l_2 \epsilon$, $\eta_{01} = \eta_{02}^* = i\pi$, $\eta_{03} = \eta_{04} = \eta_{05} = 0$ and taking $\epsilon \rightarrow 0$, then the function f in Eq. (6) can be rewritten as

$$\begin{aligned}
 f = & (B_{12} + \omega_1 \omega_2) l_1 l_2 \epsilon^2 + (\omega_1 \omega_2 + B_{23} \omega_1 + B_{13} \omega_2 + B_{12} + B_{13} B_{23}) \exp(\eta_3) \\
 & \times l_1 l_2 \epsilon^2 + (\omega_1 \omega_2 + B_{24} \omega_1 + B_{14} \omega_2 + B_{12} + B_{14} B_{24}) \exp(\eta_4) l_1 l_2 \epsilon^2 \\
 & + B_{34} [\omega_1 \omega_2 + (B_{23} + B_{24}) \omega_1 + (B_{13} + B_{14}) \omega_2 + B_{12} + (B_{13} + B_{14}) \\
 & \times (B_{23} + B_{24})] \exp(\eta_3 + \eta_4) l_1 l_2 \epsilon^2 + (\omega_1 \omega_2 + B_{25} \omega_1 + B_{15} \omega_2 + B_{12} \\
 & + B_{15} B_{25}) \exp(\eta_5) l_1 l_2 \epsilon^2 + B_{35} [\omega_1 \omega_2 + (B_{23} + B_{25}) \omega_1 + (B_{13} + B_{15}) \omega_2 \\
 & + B_{12} + (B_{13} + B_{15})(B_{23} + B_{25})] \exp(\eta_3 + \eta_5) l_1 l_2 \epsilon^2 + B_{45} [\omega_1 \omega_2 \\
 & + (B_{24} + B_{25}) \omega_1 + (B_{14} + B_{15}) \omega_2 + B_{12} + (B_{14} + B_{15})(B_{24} + B_{25})] \\
 & \times \exp(\eta_4 + \eta_5) l_1 l_2 \epsilon^2 + B_{34} B_{35} B_{45} [\omega_1 \omega_2 + (B_{23} + B_{24} + B_{25}) \omega_1 \\
 & + (B_{13} + B_{14} + B_{15}) \omega_2 + B_{12} + (B_{13} + B_{14} + B_{15})(B_{23} + B_{24} \\
 & + B_{25})] \exp(\eta_3 + \eta_4 + \eta_5) l_1 l_2 \epsilon^2 + O(\epsilon^3), \tag{15}
 \end{aligned}$$

where

$$B_{12} = \frac{2p_1 p_2 (p_1 + p_2)}{u_0 (p_1 - p_2)^2}, \quad \omega_j = x + p_j y - \frac{3u_0}{p_j} t, \quad j = 1, 2, \tag{16}$$

$$B_{sj} = \frac{2p_s a_j p_j (p_s + p_j)}{u_0 (p_s - p_j)^2 - p_s p_j^2 a_j^2} \quad (s = 1, 2, j = 3, 4, 5), \tag{17}$$

and

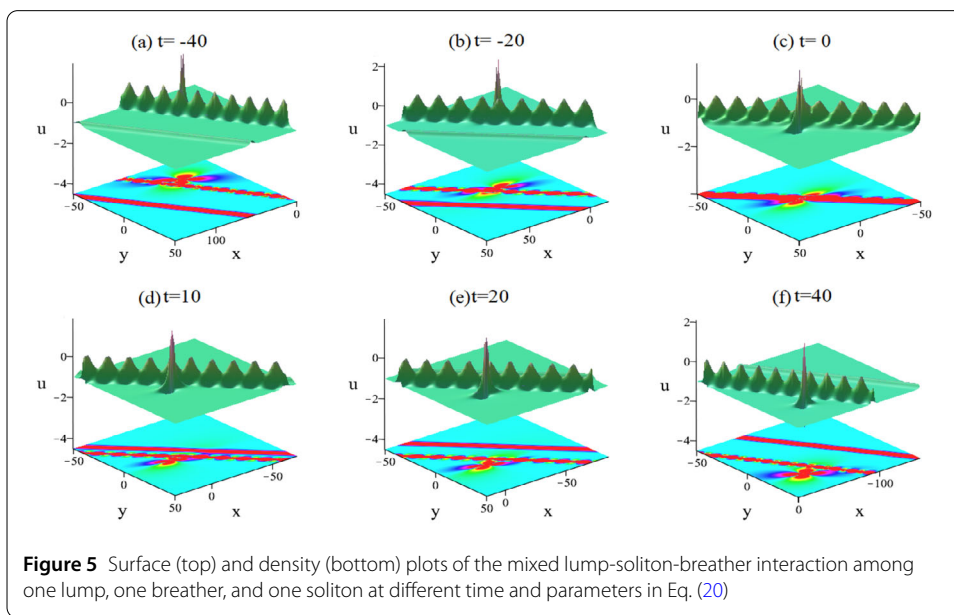
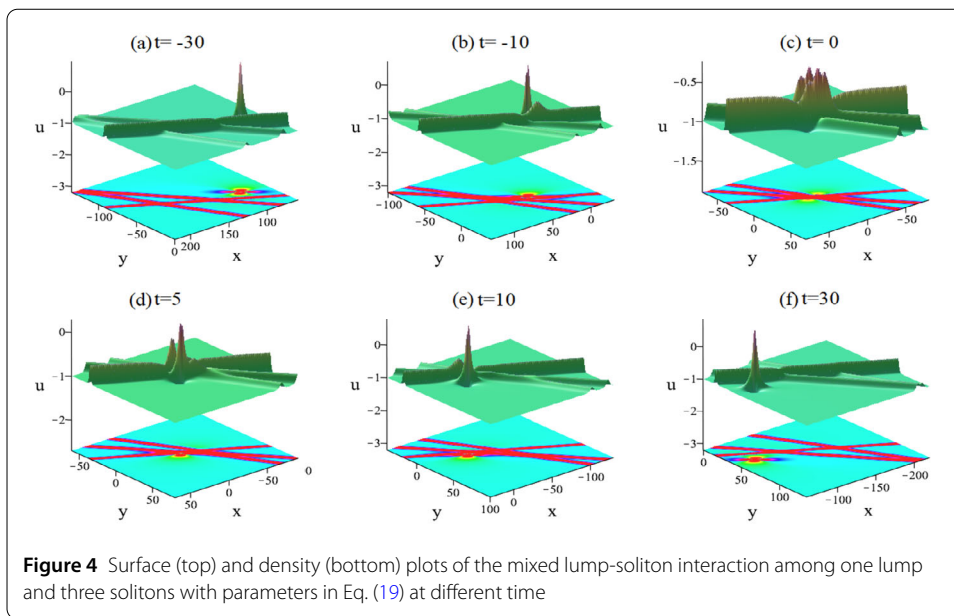
$$B_{sj} = \frac{u_0 (p_s - p_j)^2 - p_s p_j (a_s - a_j) (p_s a_s - p_j a_j)}{u_0 (p_s - p_j)^2 - p_s p_j (a_s + a_j) (p_s a_s + p_j a_j)}, \quad (3 \leq s < j \leq 5). \tag{18}$$

The solution u given by Eq. (15) expresses the interaction solution of lump soliton and breather or line solitons. Here, we will discuss two cases:

(i) When $M = 1, P = 0, Q = 3$ in Theorem 1, we can derive the interaction solution among one lump and three solitons. Considering

$$p_1 = p_2^* = 1 + i, \quad a_3 = a_4 = \frac{2}{3}, \quad p_3 = 1, \quad p_4 = 2, \quad a_5 = \frac{2}{3}, \quad p_5 = \frac{2}{3}, \tag{19}$$

when $\epsilon \rightarrow 0$, the solution u given by Eq. (15) expresses the elastic interaction among one lump and three solitons as shown in Fig. 4.



(ii) When $M = P = Q = 1$ in Theorem 1, we can derive the interaction solution among one lump, one breather, and one soliton. Taking

$$p_1 = p_2^* = 1 + 2i, \quad a_3 = a_4 = \frac{2}{3}, \quad p_3 = p_4^* = 1 + i, \quad a_5 = \frac{2}{3}, \quad p_5 = 1, \quad (20)$$

when $\epsilon \rightarrow 0$, the solution u given by Eq. (15) expresses the elastic interaction among one lump, one breather, and one soliton as shown in Fig. 5.

Case 2. Two lumps interact with one soliton. When $M = 2, P = 0, Q = 1$ in Theorem 1, we can derive the interaction solution among two lumps and one soliton. Putting $b_j = p_j a_j, a_j = l_j \epsilon$ ($j = 1, 2, 3, 4$), $b_5 = p_5 a_5, \eta_{01} = \eta_{02}^* = \eta_{03} = \eta_{04}^* = i\pi, \eta_{05} = 0$ and taking $\epsilon \rightarrow 0$,

Table 1 Mixed interaction solutions from five-soliton

<i>N</i> -soliton	Mixed lump-soliton and lump-soliton-breather interaction solutions	Parameters
<i>N</i> = 5	one lump + three solitons	$b_s = p_s a_s$ ($s = 1, 2, 3, 4, 5$), $a_1 = l_1 \varepsilon$, $a_2 = l_2 \varepsilon$, $a_3 = a_4 = \kappa_1$, $p_1 = p_2^* = \gamma_1 + i\sigma_1$, $p_3 = \gamma_2$, $p_4 = \gamma_3$, $a_5 = \kappa_2$, $p_5 = \gamma_4$, $\eta_{01} = \eta_{02}^* = i\pi$, $\eta_{03} = \eta_{04} = \eta_{05} = 0$, $\varepsilon \rightarrow 0$
	one lump + one breather + one soliton	$b_s = p_s a_s$ ($s = 1, 2, 3, 4, 5$), $a_1 = l_1 \varepsilon$, $a_2 = l_2 \varepsilon$, $a_3 = a_4 = \kappa_1$, $p_1 = p_2^* = \gamma_1 + i\sigma_1$, $p_3 = p_4^* = \gamma_2 + i\sigma_2$, $a_5 = \kappa_2$, $p_5 = \gamma_3$, $\eta_{01} = \eta_{02}^* = i\pi$, $\eta_{03} = \eta_{04} = \eta_{05} = 0$, $\varepsilon \rightarrow 0$
	two lumps+one soliton	$b_s = p_s a_s$, $a_s = l_s \varepsilon$ ($s = 1, 2, 3, 4$), $b_5 = p_5 a_5$, $p_1 = p_2^* = \gamma_1 + i\sigma_1$, $p_3 = p_4^* = \gamma_2 + i\sigma_2$, $a_5 = \kappa_2$, $p_5 = \gamma_3$, $\eta_{01} = \eta_{02}^* = \eta_{03} = \eta_{04}^* = i\pi$, $\eta_{05} = 0$, $\varepsilon \rightarrow 0$

Note: $\kappa_{1,2}, \gamma_{1,2,3,4}, \sigma_{1,2}$ are nonzero real constants.

then the function f in Eq. (6) can be rewritten as

$$\begin{aligned}
 f = & (\omega_1 \omega_2 \omega_3 \omega_4 + B_{34} \omega_1 \omega_2 + B_{24} \omega_1 \omega_3 + B_{23} \omega_1 \omega_4 + B_{14} \omega_2 \omega_3 + B_{13} \omega_2 \omega_4 \\
 & + B_{12} \omega_3 \omega_4 + B_{12} B_{34} + B_{13} B_{24} + B_{14} B_{23}) l_1 l_2 l_3 l_4 \varepsilon^4 + \{ \omega_1 \omega_2 \omega_3 \omega_4 \\
 & + B_{45} \omega_1 \omega_2 \omega_3 + B_{35} \omega_1 \omega_2 \omega_4 + B_{25} \omega_1 \omega_3 \omega_4 + B_{15} \omega_2 \omega_3 \omega_4 + \omega_1 [(B_{35} B_{45} \\
 & + B_{34}) \omega_2 + (B_{25} B_{45} + B_{24}) \omega_3 + (B_{25} B_{35} + B_{23}) \omega_4] + \omega_2 [(B_{15} B_{45} \\
 & + B_{14}) \omega_3 + (B_{15} B_{35} + B_{13}) \omega_4] + (B_{15} B_{25} + B_{12}) \omega_3 \omega_4 + \omega_1 [(B_{25} B_{35} \\
 & + B_{23}) B_{45} + B_{24} B_{35} + B_{25} B_{34}] + \omega_2 [(B_{15} B_{35} + B_{13}) B_{45} + B_{14} B_{35} \\
 & + B_{15} B_{34}] + \omega_3 [(B_{15} B_{25} + B_{12}) B_{45} + B_{14} B_{25} + B_{15} B_{24}] + \omega_4 \\
 & \times [(B_{15} B_{25} + B_{12}) B_{35} + B_{13} B_{25} + B_{15} B_{23}] + B_{13} B_{24} + B_{23} B_{14} \\
 & + B_{34} (B_{15} B_{25} + B_{12}) + B_{35} (B_{14} B_{25} + B_{15} B_{24}) + B_{45} [(B_{15} B_{25} \\
 & + B_{12}) B_{35} + B_{13} B_{25} + B_{15} B_{23}] \} \exp(\eta_5) l_1 l_2 l_3 l_4 \varepsilon^4 + O(\varepsilon^5), \tag{21}
 \end{aligned}$$

where

$$\omega_j = x + p_j y - \frac{3u_0}{p_j} t, \quad j = 1, 2, 3, 4, \tag{22}$$

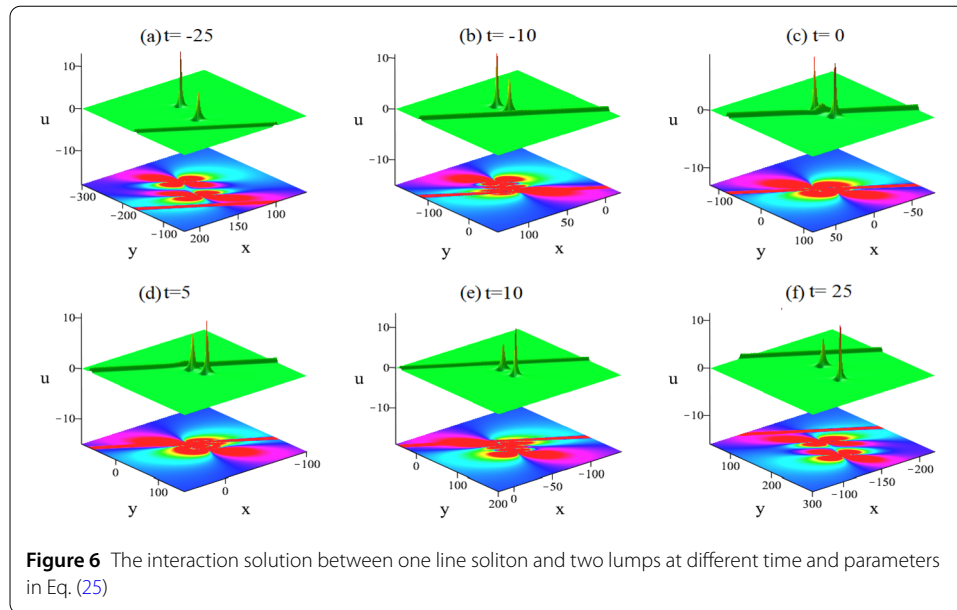
$$B_{sj} = \frac{2p_s p_j (p_s + p_j)}{u_0 (p_s - p_j)^2} \quad (1 \leq s < j \leq 4), \tag{23}$$

$$B_{s5} = \frac{2p_s a_5 p_5 (p_s + p_5)}{u_0 (p_s - p_5)^2 - p_s p_5^2 a_5^2} \quad (s = 1, 2, 3, 4). \tag{24}$$

Taking

$$p_1 = p_2^* = \frac{1}{2} + \frac{1}{2}i, \quad p_3 = p_4^* = \frac{1}{3} + \frac{1}{2}i, \quad a_5 = 1, \quad p_5 = 1, \tag{25}$$

the solution u given by Eq. (21) expresses the elastic interaction among one soliton and two lumps as shown in Fig. 6.



5 Conclusions

In this paper, based on Hirota's bilinear method, we have obtained N -soliton solution of Eq. (1). By using long wave limit and choosing special parameters to N -soliton solution, we have given a general conclusion to obtain mixed lump-breather-soliton interaction solutions. Especially, by using long wave limit to even N -soliton ($N = 2M$) solution under special parameters, M -lumps and their dynamic properties have been obtained and discussed in Figs. 1–3. In addition, we choose the case $N = 5$ as an example. The mixed lump-soliton interaction solution including one lump and three solitons (see Fig. 4), mixed lump-soliton-breather interaction solution including one lump, one breather, and one soliton (see Fig. 5), and mixed lump-soliton interaction solution including two lumps and one soliton (see Fig. 6) are derived by using long wave limit and choosing special conjugate complex parameters. Table 1 shows some mathematical features to obtain lump-soliton and lump-breather-soliton from five-soliton solutions of Eq. (1) on how to select appropriate parameters. The results given in this paper show that the long wave limit is a direct and powerful mathematical tool to construct mixed interaction solutions of different kinds of localized waves to nonlinear evolution equation, which would be used to investigate other nonlinear models in mathematics and physics.

Funding

This work is supported by the Scientific Research Common Program of Beijing Municipal Commission of Education under Grant No. KM201911232011, by the National Natural Science Foundation of China under Grant Nos. 11772063 and 11805114, the Beijing Natural Science Foundation under Grant No. 1182009, and the Beijing Great Wall Talents Cultivation Program under Grant No. CIT & TCD 20180325.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

YL performed the design of the study, the theory analysis, and carried out the computations. XW participated in the theory analysis and revised the manuscript. All authors have read and approved the final manuscript.

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 16 April 2019 Accepted: 29 July 2019 Published online: 08 August 2019

References

1. Ablowitz, M.J., Clarkson, P.A.: Soliton, Nonlinear Evolution Equations and Inverse Scattering. Cambridge University Press, Cambridge (1991)
2. Karasu, A., Sakovich, S.Y.: Bäcklund transformation and special solutions for the Drinfeld–Sokolov–Satsuma–Hirota system of coupled equations. *J. Phys. A, Math. Gen.* **34**, 7355–7358 (2001)
3. Wang, D.S., Liu, J., Zhang, Z.F.: Integrability and equivalence relationships of six integrable coupled Korteweg–de Vries equations. *Math. Methods Appl. Sci.* **39**, 3516–3530 (2016)
4. Wang, D.S., Liu, J.: Integrability aspects of some two-component KdV systems. *Appl. Math. Lett.* **79**, 211–219 (2018)
5. Balakhnev, M.Y., Demskoi, D.K.: Auto-Bäcklund transformations and superposition formulas for solutions of Drinfeld–Sokolov systems. *Appl. Math. Comput.* **219**, 3625–3637 (2012)
6. Guo, B.L., Ling, L.M.: Rogue wave, breathers and bright-dark-rogue solutions for the coupled Schrödinger equations. *Chin. Phys. Lett.* **28**, 110202 (2011)
7. Xu, X.X.: A deformed reduced semi-discrete Kaup–Newell equation, the related integrable family and Darboux transformation. *Appl. Math. Comput.* **251**, 275–283 (2015)
8. Xu, X.X., Sun, Y.P.: An integrable coupling hierarchy of Dirac integrable hierarchy, its Liouville integrability and Darboux transformation. *J. Nonlinear Sci. Appl.* **10**, 3328–3343 (2017)
9. Liu, L., Wen, X.Y., Wang, D.S.: A new lattice hierarchy: Hamiltonian structures, symplectic map and N-fold Darboux transformation. *Appl. Math. Model.* **67**, 201–218 (2019)
10. Chen, J.C., Ma, Z.Y., Hu, Y.H.: Nonlocal symmetry, Darboux transformation and soliton-cnoidal wave interaction solution for the shallow water wave equation. *J. Math. Anal. Appl.* **460**, 987–1003 (2018)
11. Liu, N., Wen, X.Y., Xu, L.: Dynamics of bright and dark multi-soliton solutions for two higher-order Toda lattice equations for nonlinear waves. *Adv. Differ. Equ.* **2018**, 289 (2018)
12. Wen, X.Y.: Modulational instability and higher-order rogue wave solutions for an integrable generalization of the nonlinear Schrödinger equation in monomode optical fibers. *Adv. Differ. Equ.* **2016**, 311 (2016)
13. Wen, X.Y., Yan, Z.Y.: Modulational instability and dynamics of multi-rogue wave solutions for the discrete Ablowitz–Ladik equation. *J. Math. Phys.* **59**, 073511 (2018)
14. Wen, X.Y., Wang, D.S.: Modulational instability and higher order-rogue wave solutions for the generalized discrete Hirota equation. *Wave Motion* **79**, 84–97 (2018)
15. Zhang, N., Xia, T.C., Hu, B.B.: A Riemann–Hilbert approach to complex Sharma–Tasso–Olver equation on half line. *Commun. Theor. Phys.* **68**, 580–594 (2017)
16. Wang, D.S., Guo, B.L., Wang, X.L.: Long-time asymptotics of the focusing Kundu–Eckhaus equation with nonzero boundary conditions. *J. Differ. Equ.* **266**, 5209–5253 (2019)
17. Wang, D.S., Wang, X.L.: Long-time asymptotics and the bright N -soliton solutions of the Kundu–Eckhaus equation via the Riemann–Hilbert approach. *Nonlinear Anal., Real World Appl.* **41**, 334–361 (2018)
18. Wazwaz, A.M., El-Tantawy, S.A.: Solving the $(3 + 1)$ -dimensional KP-Boussinesq and BKP-Boussinesq equations by the simplified Hirota’s method. *Nonlinear Dyn.* **88**, 3017–3021 (2017)
19. Yue, Y.F., Huang, L.L., Chen, Y.: Localized waves and interaction solutions to an extended $(3 + 1)$ -dimensional Jimbo–Miwa equation. *Appl. Math. Lett.* **89**, 70–77 (2019)
20. Liu, Y.Q., Wen, X.Y., Wang, D.S.: The N -soliton solution and localized wave interaction solutions of the $(2 + 1)$ -dimensional generalized Hirota–Satsuma–Ito equation. *Comput. Math. Appl.* **77**, 947–966 (2019)
21. Liu, Y.Q., Wen, X.Y., Wang, D.S.: Novel interaction phenomena of localized waves in the generalized $(3 + 1)$ -dimensional KP equation. *Comput. Math. Appl.* **78**, 1–19 (2019)
22. Ma, W.X., Zhou, Y.: Lump solutions to nonlinear partial differential equations via Hirota bilinear forms. *J. Differ. Equ.* **264**, 2633–2659 (2018)
23. Yong, X.L., Ma, W.X., Huang, Y.H., Liu, Y.: Lump solutions to the Kadomtsev–Petviashvili I equation with a self-consistent source. *Comput. Math. Appl.* **75**, 3414–3419 (2018)
24. He, C.H., Tang, Y.N., Ma, W.X., Ma, J.L.: Interaction phenomena between a lump and other multi-solitons for the $(2 + 1)$ -dimensional BLMP and Ito equations. *Nonlinear Dyn.* **95**, 29–42 (2019)
25. Ren, B., Ma, W.X., Yu, J.: Rational solutions and their interaction solutions of the $(2 + 1)$ -dimensional modified dispersive water wave equation. *Comput. Math. Appl.* **77**, 2086–2095 (2019)
26. Yu, J.P., Jian, J., Sun, Y.L., Wu, S.P.: $(n + 1)$ -dimensional reduced differential transform method for solving partial differential equations. *Appl. Math. Comput.* **273**, 697–705 (2016)
27. Yu, J.P., Sun, Y.L.: Lump solutions to dimensionally reduced Kadomtsev–Petviashvili-like equations. *Nonlinear Dyn.* **87**, 1405–1412 (2017)
28. Yu, J.P., Sun, Y.L.: Study of lump solutions to dimensionally reduced generalized KP equations. *Nonlinear Dyn.* **87**, 2755–2763 (2017)
29. Ding, D.J., Jin, D.Q., Dai, C.Q.: Analytical solutions of differential-difference sine-Gordon equation. *Therm. Sci.* **21**, 1701–1705 (2017)
30. Kong, L.Q., Liu, J., Jin, D.Q., Ding, D.J., Dai, C.Q.: Soliton dynamics in the three-spine α -helical protein with inhomogeneous effect. *Nonlinear Dyn.* **87**, 83–92 (2017)
31. Zhang, B., Zhang, X.L., Dai, C.Q.: Discussions on localized structures based on equivalent solution with different forms of breaking soliton model. *Nonlinear Dyn.* **87**, 2385–2393 (2017)
32. Wang, Y.Y., Zhang, Y.P., Dai, C.Q.: Re-study on localized structures based on variable separation solutions from the modified tanh-function method. *Nonlinear Dyn.* **83**, 1331–1339 (2016)
33. Wang, Y.Y., Chen, L., Dai, C.Q., Zheng, J., Fan, Y.: Exact vector multipole and vortex solitons in the media with spatially modulated cubic-quintic nonlinearity. *Nonlinear Dyn.* **90**, 1269–1275 (2017)
34. Ma, W.X., Li, J., Khalique, C.M.: A study on lump solutions to a generalized Hirota–Satsuma–Ito equation in $(2 + 1)$ -dimensions. *Complexity* **2018**, Article ID 9059858 (2018)
35. Ma, W.X.: Interaction solutions to the Hirota–Satsuma–Ito equation in $(2 + 1)$ -dimensions. *Front. Math. China* **14**, 619–629 (2019). <https://doi.org/10.1007/s11464-019-0771-y>

36. Ma, W.X.: A search for lump solutions to a combined fourth-order nonlinear PDE in $(2 + 1)$ -dimensions. *J. Appl. Anal. Comput.* **9**, 1319–1332 (2019)
37. Ablowitz, M.J., Kaup, D.J., Newell, A.C., Segur, H.: The inverse scattering transform-Fourier analysis for nonlinear problems. *Stud. Appl. Math.* **53**, 249–315 (1974)
38. He, J.S., Zhang, H.R., Wang, L.H., Fokas, A.S.: Generating mechanism for higher-order rogue waves. *Phys. Rev. E* **87**, 052914 (2013)
39. Zhuang, J.H., Liu, Y.Q., Chen, X., Wu, J.J., Wen, X.Y.: Diverse solitons and interaction solutions for the $(2 + 1)$ -dimensional CDGKS equation. *Mod. Phys. Lett. B* **33**(16), 1950174 (2019)
40. Liu, C., Yang, Z.Y., Zhao, L.C., Yang, W.L.: Vector breathers and the inelastic interaction in a three-mode nonlinear optical fiber. *Phys. Rev. A* **89**, 055803 (2014)
41. Ma, W.X.: Lump solutions to the Kadomtsev–Petviashvili equation. *Phys. Lett. A* **379**, 1975–1978 (2015)
42. Wang, D.S., Shi, Y.R., Feng, W.X., Wen, L.: Dynamical and energetic instabilities of $F = 2$ spinor Bose–Einstein condensates in an optical lattice. *Physica D* **351–352**, 30–41 (2017)
43. Boiti, M., Leon, J.J.-P., Manna, M., Pempinelli, F.: On the spectral transform of a Korteweg–de Vries equation in two spatial dimensions. *Inverse Probl.* **2**, 271–279 (1986)
44. Dai, C.Q., Wu, S.S., Cen, X.: New exact solutions of the $(2 + 1)$ -dimensional asymmetric Nizhnik–Novikov–Veselov system. *Int. J. Theor. Phys.* **47**, 1286–1293 (2008)
45. Wazwaz, A.M.: Structures of multiple soliton solutions of the generalized, asymmetric and modified Nizhnik–Novikov–Veselov equations. *Appl. Math. Comput.* **218**, 11344–11349 (2012)
46. Fan, E.G.: Quasi-periodic waves and an asymptotic property for the asymmetrical Nizhnik–Novikov–Veselov equation. *J. Phys. A, Math. Theor.* **42**, 095206 (2009)
47. Zhao, Z.L., Chen, Y., Han, B.: Lump soliton, mixed lump stripe and periodic lump solutions of a $(2 + 1)$ -dimensional asymmetrical Nizhnik–Novikov–Veselov equation. *Mod. Phys. Lett. B* **31**, 1750157 (2017)
48. Hirota, R.: *The Direct Method in Soliton Theory*. Cambridge University Press, New York (2004)
49. Satsuma, J., Ablowitz, M.J.: Two-dimensional lumps in nonlinear dispersive systems. *J. Math. Phys.* **20**, 1496–1503 (1979)
50. Zhang, Y., Liu, Y.P., Tang, X.Y.: M-lump solutions to a $(3 + 1)$ -dimensional nonlinear evolution equation. *Comput. Math. Appl.* **76**, 592–601 (2018)

Submit your manuscript to a SpringerOpen[®] journal and benefit from:

- Convenient online submission
- Rigorous peer review
- Open access: articles freely available online
- High visibility within the field
- Retaining the copyright to your article

Submit your next manuscript at ► [springeropen.com](https://www.springeropen.com)
