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Shifts, rotations and distributional chaos

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Abstract

Let $R_{r_0}, R_{r_1} : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ be rotations on the unit circle $\mathbb{S}^1$ and define $f : \Sigma_2 \times \mathbb{S}^1 \longrightarrow \Sigma_2 \times \mathbb{S}^1$ as

$$f(x,t) = (\sigma(x), R_{r_{x_1}}(t)),$$

for $x = x_1x_2 \cdots \in \Sigma_2 := \{0,1\}^{\mathbb{N}}$, $t \in \mathbb{S}^1$, where $\sigma : \Sigma_2 \longrightarrow \Sigma_2$ is the shift, and r_0 and r_1 are rotational angles. It is first proved that the system $(\Sigma_2 \times \mathbb{S}^1, f)$ exhibits maximal distributional chaos for any $r_0, r_1 \in \mathbb{R}$ (no assumption of $r_0, r_1 \in \mathbb{R} \setminus \mathbb{Q}$), generalizing Theorem 1 in Wu and Chen (Topol. Appl. 162:91–99, 2014). It is also obtained that $(\Sigma_2 \times \mathbb{S}^1, f)$ is cofinitely sensitive and $(\hat{\mathscr{M}}^1, \hat{\mathscr{M}}^1)$-sensitive and that $(\Sigma_2 \times \mathbb{S}^1, f)$ is densely chaotic if and only if $r_1 - r_0 \in \mathbb{R} \setminus \mathbb{Q}$.

MSC: 54H20

Keywords: Distributional chaos; $\mathscr{F}$-sensitivity; $(\mathscr{F}_1, \mathscr{F}_2)$-sensitivity; Dense chaos

1 Introduction and preliminaries

A *discrete dynamical system* (briefly, *dynamical system*) is a pair (X, g), where X is a compact metric space and $g : X \longrightarrow X$ is a continuous map. Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $\mathbb{Z}^+ = \{0, 1, 2, \ldots\}$. Sharkovsky's amazing discovery [10], as well as Li and Yorke's famous work which introduced the concept of "chaos" known as Li–Yorke chaos today in a mathematically rigorous way [4], have activated sustained interest and provoked the recent rapid advancement of the frontier research on discrete chaos theory. In their study, Li and Yorke suggested considering "divergent pairs" (x, y), which are *proximal* but not *asymptotic* and called *Li–Yorke pairs*, i.e.,

$$\liminf_{n\to\infty} d\bigl(g^n(x), g^n(y)\bigr) = 0 \quad \text{and} \quad \limsup_{n\to\infty} d\bigl(g^n(x), g^n(y)\bigr) > 0.$$

A pair (x, y) is called a *Li–Yorke pair of modulus* δ if

$$\liminf_{n\to\infty} d\bigl(g^n(x), g^n(y)\bigr) = 0 \quad \text{and} \quad \limsup_{n\to\infty} d\bigl(g^n(x), g^n(y)\bigr) > \delta.$$

Clearly, (x, y) is a Li–Yorke pair if and only if it is a Li–Yorke pair of modulus δ for some $\delta > 0$. The set of Li–Yorke pairs of modulus δ is denoted by $\mathrm{LY}(g, \delta)$ and the set of Li–Yorke pairs by $\mathrm{LY}(g)$. According to the idea of Li and Yorke [4], a subset D of X is a δ-*scrambled set* of g, if $D \times D \setminus \Delta \subset \mathrm{LY}(g, \delta)$, where $\Delta = \{(x, x) \in X \times X : x \in X\}$. In particular, if $D \times D \setminus \Delta \subset \mathrm{LY}(g)$, then D is a *scrambled set* of g. If a scrambled set D of g is also uncountable, it is called a *Li–Yorke scrambled set* for g, and g is said to be *chaotic in the sense of Li–Yorke,* or *Li–Yorke chaotic*.

In 1985, Piórek [8] introduced the concept of *generic chaos*. Inspired by this, Snoha [11, 12] defined generic δ-chaos, dense chaos, and dense δ-chaos in 1990. The notion of *Li–Yorke sensitivity* was firstly introduced by Akin and Kolyada [1] in 2003. More recent results on chaos can be found in [2, 3, 7, 17, 18, 20–31].

Definition 1 ([1, 8, 11, 12]) A dynamical system (X, g) is

(1) *sensitive* if there exists $\varepsilon > 0$ such that for any $x \in X$ and any $\delta > 0$, there exist $y \in B(x, \delta) := \{z \in X : d(z, x) < \delta\}$ and $n \in \mathbb{N}$, such that $d(g^n(x), g^n(y)) \geq \varepsilon$;

(2) *Li–Yorke sensitive* if there exists $\varepsilon > 0$ such that for any $x \in X$ and any $\delta > 0$, there exists $y \in B(x, \delta)$ such that $(x, y) \in \mathrm{LY}(g, \varepsilon)$;

(3) *densely chaotic* if $\mathrm{LY}(g)$ is dense in $X \times X$;

(4) *densely δ-chaotic for some* $\delta > 0$ if $\mathrm{LY}(g, \delta)$ is dense in $X \times X$;

(5) *generically chaotic* if $\mathrm{LY}(g)$ is residual in $X \times X$;

(6) *generically δ-chaotic for some* $\delta > 0$ if $\mathrm{LY}(g, \delta)$ is residual in $X \times X$.

Distributional chaos. The notion of distributional chaos was first introduced in [9], where it was called "strong chaos", which is characterized by a distributional function of distances between trajectories of two points. It is described as follows.

Let (X, g) be a dynamical system. For any pair $(x, y) \in X \times X$, define the *lower and upper distributional functions* as

$$F_{x,y}(t, g) = \liminf_{n\to\infty} \frac{1}{n} \left|\left\{i : d\left(g^i(x), g^i(y)\right) < t, 0 \leq i < n\right\}\right|$$

and

$$F^*_{x,y}(t, g) = \limsup_{n\to\infty} \frac{1}{n} \left|\left\{i : d\left(g^i(x), g^i(y)\right) < t, 0 \leq i < n\right\}\right|,$$

respectively, where $|A|$ denotes the cardinality of set A. Both functions $F_{x,y}$ and $F^*_{x,y}$ are non-decreasing and $F_{x,y} \leq F^*_{x,y}$.

According to Schweizer and Smítal [9], a dynamical system (X, g) is *distributionally ε-chaotic* for some $\varepsilon > 0$ if there exists an uncountable subset $\mathcal{S} \subset X$ such that for any pair of distinct points $x, y \in \mathcal{S}$, one has that $F^*_{x,y}(t, g) = 1$ for all $t > 0$ and $F_{x,y}(\varepsilon, g) = 0$. The set $\mathcal{S}$ is called a *distributionally ε-scrambled set* and the pair (x, y) a *distributionally ε-chaotic pair*. If (X, g) is distributionally ε-chaotic for any $0 < \varepsilon < \mathrm{diam}(X)$, then (X, g) is said to exhibit *maximal distributional chaos*.

Let $\mathcal{P}$ be the collection of all subsets of $\mathbb{Z}^+$. A collection $\mathscr{F} \subset \mathcal{P}$ is called a *Furstenberg family* if it is hereditary upwards, i.e., $F_1 \subset F_2$ and $F_1 \in \mathscr{F}$ imply $F_2 \in \mathscr{F}$. A family $\mathscr{F}$ is *proper* if it is a proper subset of $\mathcal{P}$, i.e., neither empty nor the whole $\mathcal{P}$. It is easy to see that $\mathscr{F}$ is proper if and only if $\mathbb{Z}^+ \in \mathscr{F}$ and $\emptyset \notin \mathscr{F}$. Let $\mathscr{F}_{\mathrm{inf}}$ be the collection of all infinite subsets of $\mathbb{Z}^+$ and $\mathscr{F}_{cf}$ the family of cofinite subset, i.e., the collection of subsets of $\mathbb{Z}^+$ with finite complements.

For $A \subset \mathbb{Z}^+$, define

$$\overline{d}(A) = \limsup_{n\to\infty} \frac{1}{n} \left|A \cap [0, n-1]\right| \quad \text{and} \quad \underline{d}(A) = \liminf_{n\to\infty} \frac{1}{n} \left|A \cap [0, n-1]\right|.$$

Then, $\overline{d}(A)$ and $\underline{d}(A)$ are *the upper density* and *the lower density* of A, respectively. Fix any $\alpha \in [0,1]$ and denote by $\hat{\mathscr{M}}_\alpha$ (resp. $\hat{\mathscr{M}}^\alpha$) the family consisting of sets $A \subset \mathbb{Z}^+$ with $\underline{d}(A) \geq \alpha$ (resp. $\overline{d}(A) \geq \alpha$).

Using Furstenberg family, Wang et al. [6, 13, 14] introduced the notions of $\mathscr{F}$-sensitivity, $(\mathscr{F}_1, \mathscr{F}_2)$-sensitivity, and $(\mathscr{F}_1, \mathscr{F}_2)$-chaos for generalizing sensitivity, Li–Yorke sensitivity, and Li–Yorke chaos, respectively.

Let $\mathscr{F}$ be a Furstenberg family. According to Moothathu [6], a dynamical system (X, g) is

(1) *$\mathscr{F}$-sensitive* if there exists $\varepsilon > 0$ such that for any nonempty open subset U of X, $\{n \in \mathbb{Z}^+ : \operatorname{diam}(g^n(U)) \geq \delta\} \in \mathscr{F}$.
(2) *cofinitely sensitive* if there exists $\varepsilon > 0$ such that for any nonempty open subset U of X, $\{n \in \mathbb{Z}^+ : \operatorname{diam}(g^n(U)) \geq \delta\} \in \mathscr{F}_{cf}$.

Definition 2 ([13, 14]) Let $\mathscr{F}_1$, $\mathscr{F}_2$ be Furstenberg families. A dynamical system (X, g) is

(1) *$(\mathscr{F}_1, \mathscr{F}_2)$-sensitive* if there exists some $\varepsilon > 0$ such that for any $x \in X$ and any $\delta > 0$, there exists $y \in B(x, \delta)$ such that $\{n \in \mathbb{Z}^+ : d(g^n(x), g^n(y)) < t\} \in \mathscr{F}_1$ for any $t > 0$ and $\{n \in \mathbb{Z}^+ : d(g^n(x), g^n(y)) > \varepsilon\} \in \mathscr{F}_2$.
(2) *$(\mathscr{F}_1, \mathscr{F}_2)$-chaotic* if there exists an uncountable subset $D \subset X$ such that for any $(x, y) \in D \times D \setminus \Delta$, there exists $\delta > 0$ such that $\{n \in \mathbb{Z}^+ : d(g^n(x), g^n(y)) < t\} \in \mathscr{F}_1$ for any $t > 0$ and $\{n \in \mathbb{Z}^+ : d(g^n(x), g^n(y)) > \delta\} \in \mathscr{F}_2$.

From Definition 2, it can be verified that a dynamical system is Li–Yorke chaotic (resp., distributionally chaotic, Li–Yorke sensitive) if and only if it is $(\mathscr{F}_{\text{inf}}, \mathscr{F}_{\text{inf}})$-chaotic (resp., $(\hat{\mathscr{M}}^1, \hat{\mathscr{M}}^1)$-chaotic, $(\mathscr{F}_{\text{inf}}, \mathscr{F}_{\text{inf}})$-sensitive).

Shift and rotation. Let $\Sigma = \{0, 1\}$ and consider a product space $\Sigma_2 = \Sigma^{\mathbb{N}}$ with the product topology which is compact and metrizable. Let Σ_2 endow with the following metric:

$$d_1(x, y) = \begin{cases} 0, & x = y, \\ \frac{1}{\min\{m \geq 1 : x_m \neq y_m\}}, & x \neq y, \end{cases}$$

for any $x = x_1x_2\cdots$, $y = y_1y_2\cdots \in \Sigma_2$.

Define the *shift* $\sigma : \Sigma_2 \longrightarrow \Sigma_2$ by $\sigma(x) = x_2x_3\cdots$ for any $x = x_1x_2\cdots \in \Sigma_2$. Clearly, σ is continuous. If X is a closed and invariant subset of Σ_2, then $(X, \sigma|_X)$ is called a *shift space* or *subshift*.

Any element A of the set Σ^n is called an *n-word* over Σ and the *length* of A is n, denoted by $|A|$. A *word* over Σ is an element of the set $\bigcup_{n \in \mathbb{N}} \Sigma^n$. Let $A = a_1 \cdots a_n \in \Sigma^n$ and $B = b_1 \cdots b_m \in \Sigma^m$. Denote $AB = a_1 \cdots a_n b_1 \cdots b_m$ and $\overline{A} = \overline{a_1} \cdots \overline{a_n}$, where

$$\overline{a_i} = \begin{cases} 0, & a_i = 1, \\ 1, & a_i = 0. \end{cases}$$

Clearly, $AB \in \Sigma^{n+m}$ and $\overline{A} \in \Sigma^n$. For any $a \in \Sigma$, denote a^n as an n-length concatenation of a (for example, $0^3 = 000$), and $a^\infty = aa\cdots$ as an infinite concatenation. If $x = x_1x_2\cdots \in \Sigma_2$ and $i \leq j \in \mathbb{N}$, then let $x_{[i,j]} = x_ix_{i+1}\cdots x_j$ and $x_{(i,j]} = x_{[i+1,j]}$. For any $B = b_1 \cdots b_n \in \bigcup_{n \in \mathbb{N}} \Sigma^n$, the set $[B] = \{x_1x_2\cdots \in \Sigma_2 : x_i = b_i, 1 \leq i \leq n\}$ is called the *cylinder* generated by B. For any $n \in \mathbb{N}$, let $\mathcal{B}_n = \{[b_1 \cdots b_n] : b_i \in \Sigma, 1 \leq i \leq n\}$.

The function $\wp : \bigcup_{n\in\mathbb{N}} \Sigma^n \longrightarrow \mathbb{Z}^+$ is defined by

$$\wp(A) = |A| - \sum_{j=1}^{|A|} a_j = \left|\left\{j : 1 \leq j \leq |A|, a_j = 0\right\}\right|,$$

for any $A = a_1 \cdots a_{|A|} \in \bigcup_{n\in\mathbb{N}} \Sigma^n$.

Consider the unit circle $\mathbb{S}^1$ defined by

$$\mathbb{S}^1 = \left\{(x,y) : x^2 + y^2 = 1\right\} \subset \mathbb{R}^2.$$

Identifying $\mathbb{R}^2$ with the complex plane $\mathbb{C}$, one can write

$$\mathbb{S}^1 = \left\{e^{2\pi i\theta} : 0 \leq \theta < 1\right\} \subset \mathbb{C}.$$

Consider a rotation R_r of angle $2\pi r$ on the circle, given by

$$R_r\left(e^{2\pi i\theta}\right) = e^{2\pi i(\theta+r)}.$$

There is a natural distance $d_2(z_1, z_2)$ between points z_1 and z_2 on $\mathbb{S}^1$, given by the *arc length distance*. It is normalized by dividing with 2π.

This paper considers the following dynamical system.

Let $R_{r_0}, R_{r_1} : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ be rotations and define $f : \Sigma_2 \times \mathbb{S}^1 \longrightarrow \Sigma_2 \times \mathbb{S}^1$ by

$$f(x,t) = \left(\sigma(x), R_{r_{x_1}}(t)\right),$$

for any $x = x_1x_2 \cdots \in \Sigma_2$, $t \in \mathbb{S}^1$ (so, $r_{x_1} = r_0$ or r_1). Note that the nth iteration of f at the point $(x,t) \in \Sigma_2 \times \mathbb{S}^1$ is given by

$$f^n(x,t) = \left(\sigma^n(x), R_{r_{x_n}} \circ \cdots \circ R_{r_{x_2}} \circ R_{r_{x_1}}(t)\right).$$

Let d be the product metric on the product space $\Sigma_2 \times \mathbb{S}^1$, i.e.,

$$d\left((x,t_1),(y,t_2)\right) = \max\left\{d_1(x,y), d_2(t_1,t_2)\right\},$$

for any (x,t_1), $(y,t_2) \in \Sigma_2 \times \mathbb{S}^1$.

Wang et al. [15] proved that a dynamical system having a regular shift-invariant set is distributionally chaotic and posed the following question:

Question 3 ([15]) Is the system $(\Sigma_2 \times \mathbb{S}^1, f)$ distributionally chaotic when $r_0, r_1 \in \mathbb{R} \setminus \mathbb{Q}$ with $r_0 \neq \pm r_1$?

Recently, Wu and Chen [19] gave a positive answer to this question and proved that $(\Sigma_2 \times \mathbb{S}^1, f)$ is Li–Yorke sensitive. By further discussing $(\Sigma_2 \times \mathbb{S}^1, f)$, we in this paper investigate the chaos of $(\Sigma_2 \times \mathbb{S}^1, f)$ for any $r_0, r_1 \in \mathbb{R}$ (not assuming $r_0, r_1 \in \mathbb{R} \setminus \mathbb{Q}$). More precisely, we first prove that $(\Sigma_2 \times \mathbb{S}^1, f)$ is distributionally chaotic for any $r_0, r_1 \in \mathbb{R}$. Meanwhile, we obtain that (1) it is cofinitely sensitive and $(\hat{\mathscr{M}}^1, \hat{\mathscr{M}}^1)$-sensitive, and (2) it is densely chaotic if and only if $r_1 - r_0 \in \mathbb{R} \setminus \mathbb{Q}$.

2 Distributional chaos for $(\Sigma_2 \times \mathbb{S}^1, f)$

The following lemma will be the key to prove the distributional chaoticity for $(\Sigma_2 \times \mathbb{S}^1, f)$.

Lemma 4 ([5, 16]) *There exists an uncountable subset E of Σ_2 such that for any distinct points $x = x_1x_2\cdots$, $y = y_1y_2\cdots$ in E, they satisfy $x_n = y_n$ for infinitely many n and $x_m \neq y_m$ for infinitely many m.*

Theorem 5 *There exists an uncountable subset $\mathcal{T} \subset \Sigma_2 \times \mathbb{S}^1$ which is a distributionally β-scrambled set of f for any $0 < \beta \leq \operatorname{diam}(\Sigma_2 \times \mathbb{S}^1) = 1$, i.e., $(\Sigma_2 \times \mathbb{S}^1, f)$ exhibits maximal distributional chaos.*

Proof Let $\mathscr{L}_1 = L_1 = 2$ and $L_{k+1} = 2^{\mathscr{L}_k} + 2\mathscr{L}_k$, $\mathscr{L}_{k+1} = \mathscr{L}_k + L_{k+1}$ for any $k \in \mathbb{N}$. From Lemma 4, it follows that there exists an uncountable subset $E \subset \Sigma_2$ such that for any two distinct points $x = x_1x_2\cdots$, $y = y_1y_2\cdots \in E$, $x_n = y_n$ holds for infinitely many n and $x_m \neq y_m$ holds for infinitely many m. For any $x = x_1x_2\cdots \in E$, take $\widetilde{x} = \widetilde{x}_1\widetilde{x}_2\widetilde{x}_3\cdots$ as

$$\widetilde{x} = x_1^{\mathscr{L}_1}\overline{\widetilde{x}_{[1,\mathscr{L}_1]}}x_2^{2^{\mathscr{L}_1}+\mathscr{L}_1}\cdots\overline{\widetilde{x}_{[1,\mathscr{L}_k]}}x_{k+1}^{2^{\mathscr{L}_k}+\mathscr{L}_k}\cdots.$$

Set $\mathcal{T} = \{\widetilde{x} : x \in E\} \times \{1\}$. Clearly, $\mathcal{T}$ is uncountable.

Now, we claim that $\mathcal{T}$ is a distributionally β-scrambled set of f for any $0 < \beta \leq 1$.

Fixing any pair of distinct points $(x, 1), (y, 1) \in \mathcal{T}$, it follows from the construction of of $\mathcal{T}$ that there exist two different points $u = u_1u_2\cdots$, $v = v_1v_2\cdots \in E$ and two increasing sequences $\{p_k\}_{k=1}^{\infty}$, $\{q_k\}_{k=1}^{\infty}$ of $\mathbb{N}$ such that for any $k \in \mathbb{N}$,

(a) $\widetilde{u} = x$, $\widetilde{v} = y$,

(b) $u_{p_k} = v_{p_k}$, $u_{q_k} = \overline{v_{q_k}}$.

(1) Note that $\wp(\widetilde{u}_{[1,2\mathscr{L}_{p_k}]}) = \wp(\widetilde{v}_{[1,2\mathscr{L}_{p_k}]})$ for any $k \in \mathbb{N}$. This implies that

$$R_{r_{x_{2\mathscr{L}_{p_k}}}} \circ \cdots \circ R_{r_{x_2}} \circ R_{r_{x_1}} = R_{r_{y_{2\mathscr{L}_{p_k}}}} \circ \cdots \circ R_{r_{y_2}} \circ R_{r_{y_1}}. \tag{2.1}$$

For any $2\mathscr{L}_{p_k} \leq j \leq \mathscr{L}_{p_k+1} - \mathscr{L}_{p_k}$, the first $\mathscr{L}_{p_k}$ respective symbols of $\sigma^j(x)$ and $\sigma^j(y)$ coincide, implying that

$$d_1\big(\sigma^j(x), \sigma^j(y)\big) \leq \frac{1}{\mathscr{L}_{p_k}} \to 0 \quad (k \to \infty).$$

Meanwhile, applying (2.1) yields that for any $2\mathscr{L}_{p_k} \leq j \leq \mathscr{L}_{p_k+1} - \mathscr{L}_{p_k}$,

$$R_{r_{x_j}} \circ \cdots \circ R_{r_{x_2}} \circ R_{r_{x_1}}(1) = R_{r_{y_j}} \circ \cdots \circ R_{r_{y_2}} \circ R_{r_{y_1}}(1).$$

Then, for any $t > 0$, there exists some $K \in \mathbb{N}$ such that for any $k \geq K$ and any $2\mathscr{L}_{p_k} \leq j \leq \mathscr{L}_{p_k+1} - \mathscr{L}_{p_k}$,

$$d\big(f^j(x, 1), f^j(y, 1)\big) < t.$$

Consequently,

$$F^*_{(x,1),(y,1)}(t, f)$$

$$
\begin{aligned}
&= \limsup_{n\to\infty} \frac{1}{n}\left|\left\{k : d\left(f^k(x,1),f^k(y,1)\right) < t, 0 \le k < n\right\}\right| \\
&\ge \limsup_{k\to\infty} \frac{1}{\mathscr{L}_{p_k+1} - \mathscr{L}_{p_k}}\left|\left\{n : d\left(f^j(x,1),f^j(y,1)\right) < t, 0 \le j < \mathscr{L}_{p_k+1} - \mathscr{L}_{p_k}\right\}\right| \\
&\ge \limsup_{k\to\infty} \frac{\mathscr{L}_{p_k+1} - 3\mathscr{L}_{p_k}}{\mathscr{L}_{p_k+1} - \mathscr{L}_{p_k}} \\
&= \limsup_{k\to\infty} \frac{2^{\mathscr{L}_{p_k}}}{2^{\mathscr{L}_{p_k}} + 2\mathscr{L}_{p_k}} \\
&= 1.
\end{aligned}
$$

(2) It is easy to see that for any $2\mathscr{L}_{q_k} \le j \le \mathscr{L}_{q_k+1} - 1$, the first respective symbols of $\sigma^j(x)$ and $\sigma^j(y)$ are distinct, implying that $d_1(\sigma^j(x),\sigma^j(y)) = 1$. Then, for any $2\mathscr{L}_{q_k} \le j \le \mathscr{L}_{q_k+1} - 1$ and any $0 < \beta \le 1$,

$$
d\left(f^j(x,1),f^j(y,1)\right) \ge \beta.
$$

Therefore,

$$
\begin{aligned}
&F_{(x,1),(y,1)}(\beta,f) \\
&\quad= \liminf_{n\to\infty} \frac{1}{n}\left|\left\{k : d\left(f^k(x,1),f^k(y,1)\right) < \beta, 0 \le k < n\right\}\right| \\
&\quad\le \liminf_{k\to\infty} \frac{1}{\mathscr{L}_{q_k+1} - 1}\left|\left\{j : d\left(f^{kj}(x,1),f^j(y,1)\right) < \beta, 0 \le j \le \mathscr{L}_{q_k+1} - 1\right\}\right| \\
&\quad\le \liminf_{k\to\infty} \frac{2\mathscr{L}_{q_k}}{\mathscr{L}_{q_k+1} - 1} \\
&\quad= \liminf_{k\to\infty} \frac{2\mathscr{L}_{q_k}}{3\mathscr{L}_{q_k} + 2^{\mathscr{L}_{q_k}}} \\
&\quad= 0.
\end{aligned}
$$

Hence, $\mathcal{T}$ is a distributionally β-scrambled set for any $0 < \beta \le 1$, i.e., $(\Sigma_2 \times \mathbb{S}^1, f)$ exhibits maximal distributional chaos. □

3 Other chaos for $(\Sigma_2 \times \mathbb{S}^1, f)$

This section shall show that $(\Sigma_2 \times \mathbb{S}^1, f)$ is cofinitely sensitive and $(\hat{\mathscr{M}}^1, \hat{\mathscr{M}}^1)$-sensitive and obtain a sufficient and necessary condition for a dense chaos.

Theorem 6 *The system $(\Sigma_2 \times \mathbb{S}^1, f)$ is cofinitely sensitive.*

Proof Given any point $(x,t) \in \Sigma_2 \times \mathbb{S}^1$ and any $\delta > 0$, take $K = [\frac{1}{\delta}] + 2$ and assume that $x = x_1x_2x_3\cdots$. Choose $y = y_1y_2y_3\cdots \in \Sigma_2$ by the following formula:

$$
y_n = \begin{cases} x_n, & 1 \le n \le K, \\ \overline{x_n}, & n > K. \end{cases}
$$

Clearly, $d((x,t),(y,t)) < \delta$. For any $n > K$, it can be verified that

$$d\big(f^n(x,t),f^n(y,t)\big) \geq d_1\big(\sigma^n(x),\sigma^n(y)\big) = 1,$$

implying that $(\Sigma_2 \times \mathbb{S}^1, f)$ is cofinitely sensitive, due to the arbitrariness of δ and (x,t) □

Theorem 7 *The system $(\Sigma_2 \times \mathbb{S}^1, f)$ is $(\hat{\mathscr{M}}^1, \hat{\mathscr{M}}^1)$-sensitive. In particular, it is Li–Yorke sensitive.*

Proof Given any $(x,t) \in \Sigma_2 \times \mathbb{S}^1$ and any $\delta > 0$, choose $L_1 = \mathscr{L}_1 = [\frac{1}{\delta}] + 2$ and $L_{k+1} = 2^{\mathscr{L}_k} + 2\mathscr{L}_k$, $\mathscr{L}_{k+1} = \mathscr{L}_k + L_{k+1}$ for any $k \in \mathbb{N}$. Take $y = y_1y_2y_3\cdots \in \Sigma_2$ as the following inductive method:

(1) $y_{[1,\mathscr{L}_2]} = x_{[1,\mathscr{L}_1]}\overline{x_{[1,\mathscr{L}_1]}}x_{[2\mathscr{L}_1+1,\mathscr{L}_2]}$;
(2) $y_{[\mathscr{L}_{2n}+1,\mathscr{L}_{2n+1}]} = \overline{y_{[1,\mathscr{L}_{2n}]}x_{[2\mathscr{L}_{2n}+1,\mathscr{L}_{2n+1}]}}$ for any $n \in \mathbb{N}$;
(3) $y_{[\mathscr{L}_{2n+1}+1,\mathscr{L}_{2n+2}]} = \overline{y_{[1,\mathscr{L}_{2n+1}]}}x_{[2\mathscr{L}_{2n+1}+1,\mathscr{L}_{2n+2}]}$ for any $n \in \mathbb{N}$.

Clearly, $d((x,t),(y,t)) = d_1(x,y) = \frac{1}{\mathscr{L}_1+1} < \delta$. Similarly as in the proof of Theorem 5, it can be verified that

(1) for any $\varepsilon > 0$, $\{n \in \mathbb{Z}^+ : d(f^n(x,t),f^n(y,t)) < t\} \in \hat{\mathscr{M}}^1$;
(2) for any $0 < \beta \leq 1$, $\{n \in \mathbb{Z}^+ : d(f^n(x,t),f^n(y,t)) \geq \beta\} \in \hat{\mathscr{M}}^1$,

implying that $(\Sigma_2 \times \mathbb{S}^1, f)$ is $(\hat{\mathscr{M}}^1, \hat{\mathscr{M}}^1)$-sensitive, due to the arbitrariness of δ and (x,t). □

Theorem 8 *The system $(\Sigma_2 \times \mathbb{S}^1, f)$ is densely chaotic if and only if $r_1 - r_0 \in \mathbb{R} \setminus \mathbb{Q}$.*

Proof ($\Longrightarrow$) Suppose on the contrary that $r_1 - r_0 \in \mathbb{Q}$. Then there exist $t_0 \in \mathbb{S}^1$ and $\eta > 0$ such that

$$\xi := \inf\left\{d_2(z_1,z_2) : z_1 \in B(t_0,\eta), z_2 \in \bigcup_{n\in\mathbb{Z}} R^n_{r_1-r_0}\big(B(1,\eta)\big)\right\} > 0.$$

Fix any $x \in \Sigma_2$ and set $U = B(x,\eta) \times B(1,\eta)$ and $V = B(x,\eta) \times B(t_0,\eta)$. Clearly, both U and V are nonempty open subsets of $\Sigma_2 \times \mathbb{S}^1$.

Now, we claim that $(U \times V) \cap \mathrm{LY}(f) = \emptyset$.

In fact, for any $((y,t_1),(z,t_2)) \in U \times V$ and any $n \in \mathbb{Z}^+$, it can be verified that for any $n \in \mathbb{Z}^+$,

$$\begin{aligned} &R_{r_{y_n}} \circ \cdots \circ R_{r_{y_2}} \circ R_{r_{y_1}}(t_1) \\ &\quad = e^{2\pi i(\sum_{k=1}^n r_{y_k})}t_1 = e^{2\pi i[r_0\wp(y_{[1,n]})+r_1(n-\wp(y_{[1,n]}))]}t_1 \\ &\quad = e^{2\pi i[r_0\wp(z_{[1,n]})+r_1(n-\wp(z_{[1,n]}))]} \cdot e^{2\pi i[(r_1-r_0)(\wp(z_{[1,n]})-\wp(y_{[1,n]}))]}t_1 \end{aligned}$$

and

$$R_{r_{z_n}} \circ \cdots \circ R_{r_{z_2}} \circ R_{r_{z_1}}(t_2) = e^{2\pi i(\sum_{k=1}^n r_{z_k})}t_2 = e^{2\pi i[r_0\wp(z_{[1,n]})+r_1(n-\wp(z_{[1,n]}))]}t_2,$$

implying that

$$d_2\big(R_{r_{y_n}} \circ \cdots \circ R_{r_{y_2}} \circ R_{r_{y_1}}(t_1), R_{r_{z_n}} \circ \cdots \circ R_{r_{z_2}} \circ R_{r_{z_1}}(t_2)\big)$$

$$= d_2\big(e^{2\pi i[(r_1-r_0)(\wp(z_{[1,n]})-\wp(y_{[1,n]}))]}t_1, t_2\big)$$
$$= d_2\big(R_{r_1-r_0}^{\wp(z_{[1,n]})-\wp(y_{[1,n]})}(t_1), t_2\big) \geq \xi.$$

Therefore,

$$\liminf_{n\to\infty} d\big(f^n(y,t_1), f^n(z,t_2)\big)$$
$$\geq \liminf_{k\to\infty} d_2\big(R_{r_{y_n}} \circ \cdots \circ R_{r_{y_2}} \circ R_{r_{y_1}}(t_1), R_{r_{z_n}} \circ \cdots \circ R_{r_{z_2}} \circ R_{r_{z_1}}(t_2)\big) \geq \xi > 0.$$

This means that $(U \times V) \cap \mathrm{LY}(f) = \emptyset$. Hence, f is not densely chaotic, which is a contradiction.

($\Longleftarrow$) For any nonempty open subsets U, V of $\Sigma_2 \times \mathbb{S}^1$, choose $(u,t_1) \in U$, $(v,t_2) \in V$ and $K \in \mathbb{N}$ such that $B((u,t_1), \frac{1}{K}) \subset U$ and $B((v,t_2), \frac{1}{K}) \subset V$. From $r_1 - r_0 \in \mathbb{R} \setminus \mathbb{Q}$, it follows that there exists an increasing sequence $\{p_n\}_{n=1}^{\infty} \subset \mathbb{N}$ such that $d_2(R_{r_1-r_0}^{p_n}(t_1), t_2) < \frac{1}{n}$ for any $n \in \mathbb{N}$. Let $L_1 = \mathscr{L}_1 = K$ and $L_{n+1} = 2\mathscr{L}_n + p_n$, $\mathscr{L}_{n+1} = \mathscr{L}_n + L_n$ for any $n \in \mathbb{N}$. Take $\hat{u} = \hat{u}_1\hat{u}_2\hat{u}_3\cdots$ and $\hat{v} = \hat{v}_1\hat{v}_2\hat{v}_3\cdots \in \Sigma_2$ as

$$\hat{u} = u_{[1,\mathscr{L}_1]}\overline{u_{[1,\mathscr{L}_1]}}1^{p_1}0^{\mathscr{L}_1}\cdots\overline{u_{[1,\mathscr{L}_n]}}1^{p_n}0^{\mathscr{L}_n}\cdots$$

and

$$\hat{v} = v_{[1,\mathscr{L}_1]}\overline{v_{[1,\mathscr{L}_1]}}0^{p_1+\mathscr{L}_1}\cdots\overline{v_{[1,\mathscr{L}_n]}}0^{p_n+\mathscr{L}_n}\cdots,$$

respectively. Clearly, $(\hat{u}, t_1) \in U$ and $(\hat{v}, t_2) \in V$.

(1) Noting that $\hat{u}_{2\mathscr{L}_n+1} = 1$ and $\hat{v}_{2\mathscr{L}_n+1} = 0$, it follows that

$$d\big(f^{2\mathscr{L}_n}(\hat{u},t_1), f^{2\mathscr{L}_n}(\hat{v},t_2)\big) \geq d_1\big(\sigma^{2\mathscr{L}_n}(\hat{u}), \sigma^{2\mathscr{L}_n}(\hat{v})\big) = 1,$$

implying that

$$\limsup_{n\to\infty} d\big(f^n(\hat{u},t_1), f^n(\hat{v},t_2)\big) \geq \limsup_{n\to\infty} d\big(f^{2\mathscr{L}_n}(\hat{u},t_1), f^{2\mathscr{L}_n}(\hat{v},t_2)\big) \geq 1.$$

(2) From the fact that the first $\mathscr{L}_n$ respective symbols of $\sigma^{2\mathscr{L}_n+p_n}(\hat{u})$ and $\sigma^{2\mathscr{L}_n+p_n}(\hat{v})$ coincide, it follows that

$$d_1\big(\sigma^{2\mathscr{L}_n+p_n}(\hat{u}), \sigma^{2\mathscr{L}_n+p_n}(\hat{v})\big) \leq \frac{1}{\mathscr{L}_n} < \frac{1}{n}. \tag{3.1}$$

Meanwhile, it can verified that

$$d_2\big(R_{\hat{u}_{2\mathscr{L}_n+p_n}} \circ \cdots \circ R_{\hat{u}_2} \circ R_{\hat{u}_1}(t_1), R_{\hat{v}_{2\mathscr{L}_n+p_n}} \circ \cdots \circ R_{\hat{v}_2} \circ R_{\hat{v}_1}(t_2)\big) = d_2\big(R_{r_1-r_0}^{p_n}(t_1), t_2\big) < \frac{1}{n}.$$

This, together with (3.1), implies that

$$d\big(f^{2\mathscr{L}_n+p_n}(\hat{u},t_1), f^{2\mathscr{L}_n+p_n}(\hat{v},t_2)\big) < \frac{1}{n}.$$

Then,

$$\liminf_{n\to\infty} d\big(f^n(\hat{u},t_1),f^n(\hat{v},t_2)\big) \le \liminf_{n\to\infty} d\big(f^{2\mathscr{L}_n+p_n}(\hat{u},t_1),f^{2\mathscr{L}_n+p_n}(\hat{v},t_2)\big) = 0.$$

Therefore, $((\hat{u},t_1),(\hat{v},t_2)) \in (U \times V) \cap \mathrm{LY}(f)$. Hence, $(\Sigma_2 \times \mathbb{S}^1, f)$ is densely chaotic. □

Applying Theorem 8, the following can be verified.

Corollary 9 *The following statements are equivalent*:

(1) $r_1 - r_0 \in \mathbb{R} \setminus \mathbb{Q}$;
(2) $(\Sigma_2 \times \mathbb{S}^1, f)$ *is densely chaotic*;
(3) $(\Sigma_2 \times \mathbb{S}^1, f)$ *is densely* δ-*chaotic for any* $0 < \delta < 1$;
(4) $(\Sigma_2 \times \mathbb{S}^1, f)$ *is generically chaotic*;
(5) $(\Sigma_2 \times \mathbb{S}^1, f)$ *is generically* δ-*chaotic for any* $0 < \delta < 1$.

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