Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Positive solutions for a boundary value problem of fractional differential equation with p-Laplacian operator

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Abstract
In this paper, the existence of a positive solution to a boundary value problem of fractional differential equations with the p-Laplacian operator is studied. By applying a monotone iterative method, some existence results of positive solutions are obtained. In addition, an example is included to illustrate the main results.

MSC: 34A08; 34B18; 35J05

Keywords: Fractional differential equation; Boundary value problem; p-Laplacian operator; Positive solution

1 Introduction

In this paper, we consider the following boundary value problem of fractional differential equations with a p-Laplacian operator:

$$D^{\gamma}\left(\phi_p\left(D^{\alpha}u(t)\right)\right)=f\left(t,u(t)\right),\quad 0<t<1, \tag{1.1}$$

$$u(0)=D^{\alpha}u(0)=0,\qquad D^{\beta}u(1)=aD^{\beta}u(\xi),\qquad D^{\alpha}u(1)=bD^{\alpha}u(\eta), \tag{1.2}$$

where $\alpha,\beta,\gamma\in\mathbb{R}$; $1<\alpha$, $\gamma\le 2$; $\beta>0$ and $1+\beta\le\alpha$; $\xi,\eta\in(0,1)$; $a,b\in[0,+\infty)$; $1-a\xi^{\alpha-\beta-1}>0$; $1-b^{p-1}\eta^{\gamma-1}>0$ and $\phi_p(s)=|s|^{p-2}s$, $p>1$, $\phi_q=(\phi_p)^{-1}$, $\frac{1}{p}+\frac{1}{q}=1$, D^{α} is the Riemann–Liouville differentiation and $f\in C([0,1]\times[0,+\infty),[0,+\infty))$.

Fractional differential equations arise in many engineering and scientific disciplines as the mathematical modeling of systems and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of a complex medium, polymer rheology, etc. Fractional differential equations also serve as an excellent tool for the description of hereditary properties of various materials and processes. In consequence, fractional differential equations have been of great interest. For details, see fractional two-point boundary value problems [29, 31, 32, 35, 57, 64], fractional boundary value problems at resonance [5, 8, 67, 69, 71], fractional multi-point problems with nonresonance [5, 8, 44, 48, 58, 61, 68], fractional initial value problems [6, 7, 34], fractional impulsive problems [48, 72], fractional integral boundary value problems [14, 40, 46, 62], fractional p-Laplace problems [15, 18, 20–22, 28, 36, 39, 45, 47, 49, 50, 52, 60, 65, 66, 70], fractional problems with lower and upper solution

[7, 39, 51, 59], fractional control problems, [41, 43, 53–56], fractional soliton problems [19, 24, 26, 42], fractional singular problems [17, 27, 30, 37, 38, 63].

On the other hand, the differential equations with p-Laplacian operator arise in the modeling of different physical and natural phenomena, non-Newtonian mechanics, nonlinear elasticity and glaciology, population biology, nonlinear flow laws, and system of Monge–Kantorovich partial differential equations. There are a very large number of papers devoted to the existence of solutions of the p-Laplacian operator [15, 18, 20–22, 28, 36, 39, 45, 47, 49, 50, 52, 60, 65, 66]. The approaches are mainly topological, fixed-point and continuation theorems, degree and fixed-point index theory.

In this paper, we study the existence of positive solutions for boundary value problem (1.1), by applying a monotone iterative method, some existence results of positive solutions are obtained. In the light of the above and to the best o four knowledge, this is the first study which discuss fractional p-Laplacian with lower and upper solution method. Taking into account that sometimes the corresponding research about the Riesz fractional derivative is interesting, see [16, 25], in the future work, we will focus our concentration on the Riesz derivative. Also, the reader can find some new methods for approximate solutions of fractional integro-differential equations involving the Caputo–Fabrizio derivative or extended fractional Caputo–Fabrizio derivative [1–4, 9–13, 23, 33]. The approximation solutions are interesting and need more concentration.

The organization of this paper is as follows. In Sect. 2, we present some necessary definitions from fractional calculus theory. In Sect. 3, we prove the main results about the existence of positive solution of the boundary value problem (1.1). In Sect. 4, we will give an example to illustrate our main results.

2 Preliminaries

In this section, we present some necessary definitions from fractional calculus theory.

Definition 2.1 ([32]) The Riemann–Liouville fractional integral of a function $x:(0,+\infty)\to\mathbb{R}$ of order $\alpha>0$ is given by

$$I^{\alpha}x(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}x(s)\,ds,$$

provided the right side is pointwise defined on $(0,\infty)$.

Definition 2.2 ([32]) The Riemann–Liouville fractional derivative of order $\alpha>0$ of a continuous function $x:(0,+\infty)\to\mathbb{R}$ is given by

$$D^{\alpha}x(t)=\frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n\int_0^t \frac{x(s)}{(t-s)^{\alpha-n+1}}\,ds,$$

where $n=[\alpha]+1$, $[\alpha]$ denotes the integer part of number α, provided the right side is pointwise defined on $(0,\infty)$.

Lemma 2.1 ([32])

(1) *If* $x\in L(0,1)$, $\rho>\sigma>0$, *then*

$$D^{\sigma}I^{\rho}x(t)=I^{\rho-\sigma}x(t),\qquad D^{\sigma}I^{\sigma}x(t)=x(t).$$

(2) *If* $\rho > 0$, $\lambda > 0$, *then*

$$D^\rho t^{\lambda-1} = \frac{\Gamma(\lambda)}{\Gamma(\lambda-\rho)} t^{\lambda-\rho-1}.$$

Lemma 2.2 ([32]) *Assume that* $x \in C(0,1) \cap L(0,1)$ *with a fractional derivative of order* $\alpha > 0$ *that belongs to* $C(0,1) \cap L(0,1)$. *Then*

$$I^\alpha D^\alpha x(t) = x(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_N t^{\alpha-N}, \quad c_i \in \mathbb{R}, i = 1,2,\ldots,N,$$

where N *is the smallest integer greater than or equal to* α.

3 Main results

In this section, we consider the existence of positive solution for problem (1.1).

Lemma 3.1 *If* $h \in C[0,1]$, $\alpha \in (1,2]$, $\beta > 0$ *and* $1 + \beta \le \alpha$, $\xi \in (0,1)$, $a \in [0,+\infty)$, $\mathscr{A} := a\xi^{\alpha-\beta-1}$, *then the boundary value problem*

$$D^\alpha u(t) + h(t) = 0, \quad 0 < t < 1, \tag{3.1}$$

$$u(0) = 0, \qquad D^\beta u(1) = aD^\beta u(\xi), \tag{3.2}$$

has an unique solution

$$u(t) = \int_0^1 G(t,s)h(s)\,ds, \tag{3.3}$$

where

$$G(t,s) = \begin{cases} \frac{t^{\alpha-1}(1-s)^{\alpha-\beta-1}-(1-\mathscr{A})(t-s)^{\alpha-1}-at^{\alpha-1}(\xi-s)^{\alpha-\beta-1}}{\Gamma(\alpha)(1-\mathscr{A})}, \\ 0 \le s \le t \le 1, s \le \xi, \\ \frac{t^{\alpha-1}(1-s)^{\alpha-\beta-1}-(1-\mathscr{A})(t-s)^{\alpha-1}}{\Gamma(\alpha)(1-\mathscr{A})}, \quad 0 < \xi \le s \le t \le 1, \\ \frac{t^{\alpha-1}(1-s)^{\alpha-\beta-1}-at^{\alpha-1}(\xi-s)^{\alpha-\beta-1}}{\Gamma(\alpha)(1-\mathscr{A})}, \quad 0 \le t \le s \le \xi < 1, \\ \frac{t^{\alpha-1}(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)(1-\mathscr{A})}, \quad 0 \le t \le s \le 1, \xi \le s. \end{cases} \tag{3.4}$$

Proof By applying Lemma 2.2, we can reduce Eq. (3.1) to an equivalent integral equation

$$u(t) = -I^\alpha h(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2}, \tag{3.5}$$

for some $c_1, c_2 \in \mathbb{R}$. Note that $u(0) = 0$, we have $c_2 = 0$. Consequently the general solution of Eq. (3.1) is

$$u(t) = -I^\alpha h(t) + c_1 t^{\alpha-1}. \tag{3.6}$$

By (3.6) and Lemma 2.1, we have

$$D^\beta u(t) = -D^\beta I^\alpha h(t) + c_1 D^\beta t^{\alpha-1} = -I^{\alpha-\beta} h(t) + c_1 \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)} t^{\alpha-\beta-1}.$$

So,

$$D^{\beta}u(1) = -\int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}h(s)\,ds + c_1\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}, \tag{3.7}$$

$$D^{\beta}u(\xi) = -\int_0^{\xi} \frac{(\xi-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}h(s)\,ds + c_1\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}\xi^{\alpha-\beta-1}. \tag{3.8}$$

By $D^{\beta}u(1) = aD^{\beta}u(\xi)$, combining with (3.7) and (3.8), we obtain

$$c_1 = \frac{1}{\Gamma(\alpha)(1-\mathscr{A})}\left\{\int_0^1 (1-s)^{\alpha-\beta-1}h(s)\,ds - a\int_0^{\xi}(\xi-s)^{\alpha-\beta-1}h(s)\,ds\right\}.$$

So, the unique solution of the problem (3.1), (3.2) is

$$\begin{aligned} u(t) &= -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)\,ds + \frac{t^{\alpha-1}}{\Gamma(\alpha)(1-\mathscr{A})}\left\{\int_0^1 (1-s)^{\alpha-\beta-1}h(s)\,ds \right.\\ &\qquad \left. - a\int_0^{\xi}(\xi-s)^{\alpha-\beta-1}h(s)\,ds\right\} \\ &= \int_0^1 G(t,s)h(s)\,ds. \end{aligned}$$

The proof is completed. □

Lemma 3.2 *If $h \in C[0,1]$, $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $\phi_q = (\phi_p)^{-1}$, $\frac{1}{p}+\frac{1}{q} = 1$, $\alpha,\beta,\gamma \in \mathbb{R}$, $1 < \alpha$, $\gamma \le 2$, $\beta > 0$ and $1+\beta \le \alpha$, $0 < \xi,\eta < 1$, $a,b \in [0,+\infty)$, $\mathscr{B} =: b^{p-1}\eta^{\gamma-1}$, then the problem*

$$D^{\gamma}\bigl(\phi_p\bigl(D^{\alpha}u(t)\bigr)\bigr) = h(t), \quad 0 < t < 1, \tag{3.9}$$

$$u(0) = D^{\alpha}u(0) = 0, \qquad D^{\beta}u(1) = aD^{\alpha}u(\xi), \qquad D^{\alpha}u(1) = bD^{\alpha}u(\eta), \tag{3.10}$$

has an unique solution

$$u(t) = \int_0^1 G(t,s)\phi_q\left(\int_0^1 H(s,\tau)h(\tau)\,d\tau\right)ds,$$

where

$$H(t,s) = \begin{cases} \frac{[t(1-s)]^{\gamma-1} - b^{p-1}[t(\eta-s)]^{\gamma-1} - (1-\mathscr{B})(t-s)^{\gamma-1}}{(1-\mathscr{B})\Gamma(\gamma)}, & 0 \le s \le t \le 1, s \le \eta; \\ \frac{[t(1-s)]^{\gamma-1} - (1-\mathscr{B})(t-s)^{\gamma-1}}{(1-\mathscr{B})\Gamma(\gamma)}, & 0 < \eta \le s \le t \le 1; \\ \frac{[t(1-s)]^{\gamma-1} - b^{p-1}[t(\eta-s)]^{\gamma-1}}{(1-\mathscr{B})\Gamma(\gamma)}, & 0 \le t \le s \le \eta < 1; \\ \frac{[t(1-s)]^{\gamma-1}}{(1-\mathscr{B})\Gamma(\gamma)}, & 0 \le t \le s \le 1, \eta \le s. \end{cases} \tag{3.11}$$

$G(t, s)$ is defined by (3.4).

Proof From Eq. (3.9), and Lemma 2.2, we have

$$\phi_p\bigl(D^{\alpha}u(t)\bigr) = I^{\gamma}h(t) + d_1t^{\gamma-1} + d_2t^{\gamma-2}, \tag{3.12}$$

for some $d_1, d_2 \in \mathbb{R}$. Note that $D^\beta u(0) = 0$, we have $d_2 = 0$, then

$$\phi_p\big(D^\alpha u(t)\big) = \frac{1}{\Gamma(\gamma)} \int_0^t (t-\tau)^{\gamma-1} h(\tau)\, d\tau + d_1 t^{\gamma-1}. \tag{3.13}$$

So,

$$\phi_p\big(D^\alpha u(1)\big) = \frac{1}{\Gamma(\gamma)} \int_0^1 (1-\tau)^{\gamma-1} h(\tau)\, d\tau + d_1, \tag{3.14}$$

$$\phi_p\big(D^\alpha u(\eta)\big) = \frac{1}{\Gamma(\gamma)} \int_0^\eta (\eta-\tau)^{\gamma-1} h(\tau)\, d\tau + d_1 \eta^{\gamma-1}. \tag{3.15}$$

By $D^\alpha u(1) = bD^\alpha u(\eta)$, combining with (3.14) and (3.15), we have

$$d_1 = -\int_0^1 \frac{(1-\tau)^{\gamma-1}}{\Gamma(\gamma)(1-\mathscr{B})} h(\tau)\, d\tau + \int_0^\eta \frac{b^{p-1}(\eta-\tau)^{\gamma-1}}{\Gamma(\gamma)(1-\mathscr{B})} h(\tau)\, d\tau.$$

So, the unique solution of problem (3.1), (3.2) is

$$\begin{aligned} \phi_p\big(D^\alpha u(t)\big) &= \int_0^t \frac{(t-\tau)^{\gamma-1}}{\Gamma(\gamma)} h(\tau)\, d\tau - \int_0^1 \frac{t^{\gamma-1}(1-\tau)^{\gamma-1}}{\Gamma(\gamma)(1-\mathscr{B})} h(\tau)\, d\tau \\ &\quad + \int_0^\eta \frac{b^{p-1} t^{\gamma-1}(\eta-\tau)^{\gamma-1}}{\Gamma(\gamma)(1-\mathscr{B})} h(\tau)\, d\tau \\ &= -\int_0^1 H(t,\tau) h(\tau)\, d\tau. \end{aligned}$$

Therefore,

$$D^\alpha u(t) + \phi_q\left(\int_0^1 H(t,\tau)h(\tau)\, d\tau\right) = 0.$$

Combining with the boundary conditions $u(0) = 0$, $D^\beta u(1) = aD^\alpha u(\xi)$, by Lemma 3.1, we obtain the unique solution of problem (3.9), (3.10)

$$u(t) = \int_0^1 G(t,s)\phi_q\left(\int_0^1 H(s,\tau)h(\tau)\, d\tau\right) ds.$$

The proof is completed. □

Lemma 3.3 *Suppose* $1 - \mathscr{A} > 0$, $1 - \mathscr{B} > 0$. *The functions* $G(t,s)$ *and* $H(t,s)$ *have the following properties*:

(1) $G(t,s), H(t,s) \in C([0,1] \times [0,1])$ *and* $G(t,s) > 0$, $H(t,s) > 0$ *for* $t, s \in (0,1)$;

(2) *there exist functions* $\omega, \omega_1 \in C((0,1),(0,+\infty))$ *such that*

$$\omega(s) \geq \max_{0\leq t\leq 1} G(t,s), \qquad \omega_1(s) \geq \max_{0\leq t\leq 1} H(t,s),$$

where

$$g(t,s) = \frac{t^{\alpha-1}(1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)},$$
$$g_1(t,s) = \frac{[t(1-s)]^{\gamma-1} - (t-s)^{\gamma-1}}{\Gamma(\gamma)},$$
$$\omega(s) = g(s,s) + \frac{\mathscr{A}(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)(1-\mathscr{A})}, \qquad \omega_1(s) = g_1(s,s) + \frac{\mathscr{B}(1-s)^{\gamma-1}}{\Gamma(\gamma)(1-\mathscr{B})}, \quad s \in (0,1).$$

Proof It is obvious that $g(t,s) > 0$, $g_1(t,s) > 0$ for $s,t \in (0,1)$. We note that $g(t,s)$, $g_1(t,s)$ are decreasing with respect to t for $s \leq t$ and increasing with respect to t for $t \leq s$. Hence,

$$\max_{0\leq t\leq 1} g(t,s) = g(s,s) = \frac{s^{\alpha-1}(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)}, \quad s \in (0,1);$$
$$\max_{0\leq t\leq 1} g_1(t,s) = g_1(s,s) = \frac{s^{\gamma-1}(1-s)^{\gamma-1}}{\Gamma(\gamma)}, \quad s \in (0,1).$$

We first prove the statement (1). From the definitions of $G(t,s)$ and $H(t,s)$, it is clear that $G(t,s), H(t,s) \in C([0,1] \times [0,1])$.

For $0 < s \leq t < 1$, $s \leq \xi$, we have

$$\begin{aligned}
G(t,s) &= \frac{t^{\alpha-1}(1-s)^{\alpha-\beta-1} - (1-\mathscr{A})(t-s)^{\alpha-1} - at^{\alpha-1}(\xi-s)^{\alpha-\beta-1}}{\Gamma(\alpha)(1-\mathscr{A})} \\
&= \frac{1}{\Gamma(\alpha)}\left(1 + \frac{\mathscr{A}}{1-\mathscr{A}}\right)t^{\alpha-1}(1-s)^{\alpha-\beta-1} - \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \\
&\quad - \frac{at^{\alpha-1}(\xi-s)^{\alpha-\beta-1}}{\Gamma(\alpha)(1-\mathscr{A})} \\
&= \frac{t^{\alpha-1}(1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)} \\
&\quad + \frac{at^{\alpha-1}[\xi^{\alpha-\beta-1}(1-s)^{\alpha-\beta-1} - (\xi-s)^{\alpha-1}]}{\Gamma(\alpha)(1-\mathscr{A})} \\
&\geq g(t,s) + \frac{at^{\alpha-1}}{\Gamma(\alpha)(1-\mathscr{A})}g(\xi,s) \\
&> 0.
\end{aligned}$$

By using an analogous argument, we have $G(t,s) > 0$ for $0 < \xi \leq s \leq t < 1$ or $0 < t \leq s \leq \xi < 1$ or $0 < t \leq s < 1$, $\xi \leq s$.

Hence, $G(t,s) > 0$ for $t,s \in (0,1)$.

For $0 < s \leq t < 1$, $s \leq \eta$, we have

$$\begin{aligned}
H(t,s) &= \frac{[t(1-s)]^{\gamma-1} - b^{p-1}[t(\eta-s)]^{\gamma-1} - (1-\mathscr{B})(t-s)^{\gamma-1}}{\Gamma(\gamma)(1-\mathscr{B})} \\
&= \left(1 + \frac{\mathscr{B}}{1-\mathscr{B}}\right)\frac{[t(1-s)]^{\gamma-1}}{\Gamma(\gamma)} - \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)} - \frac{b^{p-1}[t(\eta-s)]^{\gamma-1}}{\Gamma(\gamma)(1-\mathscr{B})}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{[t(1-s)]^{\gamma-1} - (t-s)^{\gamma-1}}{\Gamma(\gamma)} + \frac{b^{p-1}t^{\gamma-1}[\eta^{\gamma-1}(1-s)^{\gamma-1} - (\eta-s)^{\gamma-1}]}{\Gamma(\gamma)(1-\mathscr{B})} \\
&= g_1(t,s) + \frac{b^{p-1}t^{\gamma-1}}{1-\mathscr{B}} g_1(\eta,s) \\
&> 0.
\end{aligned}
$$

Similarly, $H(t,s) > 0$ for $0 < \eta \le s \le t < 1$ or $0 < t \le s \le \eta < 1$ or $0 < t \le s < 1$, $\eta \le s$.

Hence, $H(t,s) > 0$ for $t,s \in (0,1)$.

Now follows the proof of the statement (2).

For $0 \le s \le t \le 1$, $s \le \xi$, one has

$$
\begin{aligned}
&\max_{0\le t\le 1} G(t,s) \\
&\quad = \max_{0\le t\le 1}\left(g(t,s) + \frac{at^{\alpha-1}[\xi^{\alpha-\beta-1}(1-s)^{\alpha-\beta-1} - (\xi-s)^{\alpha-1}]}{\Gamma(\alpha)(1-\mathscr{A})}\right) \\
&\quad \le g(s,s) + \frac{\mathscr{A}(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)(1-\mathscr{A})} = \omega(s).
\end{aligned}
$$

For $0 < \xi \le s \le t \le 1$, one has

$$
\begin{aligned}
&\max_{0\le t\le 1} G(t,s) \\
&\quad = \max_{0\le t\le 1} \frac{t^{\alpha-1}(1-s)^{\alpha-\beta-1} - (1-\mathscr{A})(t-s)^{\alpha-1}}{\Gamma(\alpha)(1-\mathscr{A})} \\
&\quad = \max_{0\le t\le 1}\left(\frac{1}{\Gamma(\alpha)}\left(1 + \frac{\mathscr{A}}{1-\mathscr{A}}\right)t^{\alpha-1}(1-s)^{\alpha-\beta-1} - \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\right) \\
&\quad \le g(s,s) + \frac{\mathscr{A}(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)(1-\mathscr{A})} = \omega(s).
\end{aligned}
$$

For $0 \le t \le s \le \xi < 1$,

$$
\begin{aligned}
&\max_{0\le t\le 1} G(t,s) \\
&\quad = \max_{0\le t\le 1} \frac{t^{\alpha-1}(1-s)^{\alpha-\beta-1} - at^{\alpha-1}(\xi-s)^{\alpha-\beta-1}}{\Gamma(\alpha)(1-\mathscr{A})} \\
&\quad = \max_{0\le t\le 1}\left(\frac{t^{\alpha-1}(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)} + \frac{at^{\alpha-1}[(\xi-\xi s)^{\alpha-\beta-1} - (\xi-s)^{\alpha-1}]}{\Gamma(\alpha)(1-\mathscr{A})}\right) \\
&\quad \le g(s,s) + \frac{\mathscr{A}(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)(1-\mathscr{A})} = \omega(s).
\end{aligned}
$$

For $0 \le t \le s \le 1$, $\xi \le s$,

$$
\begin{aligned}
\max_{0\le t\le 1} G(t,s) &= \max_{0\le t\le 1} \frac{t^{\alpha-1}(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)(1-\mathscr{A})} \\
&= \max_{0\le t\le 1}\left(\frac{1}{\Gamma(\alpha)}\left(1 + \frac{\mathscr{A}}{1-\mathscr{A}}\right)t^{\alpha-1}(1-s)^{\alpha-\beta-1}\right) \\
&\le g(s,s) + \frac{\mathscr{A}(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)(1-\mathscr{A})} = \omega(s).
\end{aligned}
$$

So,

$$\max_{0\le t\le 1} G(t,s) \le \omega(s), \quad s \in (0,1).$$

For $0 \le s \le t \le 1$, $s \le \eta$, one has

$$\begin{aligned}\max_{0\le t\le 1} H(t,s) &\le \max_{0\le t\le 1}\left(g_1(t,s) + \frac{b^{p-1}t^{\gamma-1}}{1-\mathscr{B}} g_1(\eta,s)\right)\\ &\le g_1(s,s) + \frac{\mathscr{B}(1-s)^{\gamma-1}}{\Gamma(\gamma)(1-\mathscr{B})} = \omega_1(s).\end{aligned}$$

For $0 < \eta \le s \le t \le 1$, one has

$$\begin{aligned}&\max_{0\le t\le 1} H(t,s)\\ &\quad = \max_{0\le t\le 1} \frac{[t(1-s)]^{\gamma-1} - (1-\mathscr{B})(t-s)^{\gamma-1}}{\Gamma(\gamma)(1-\mathscr{B})}\\ &\quad = \max_{0\le t\le 1}\left(g_1(t,s) + \frac{\mathscr{B}t^{\gamma-1}(1-s)^{\gamma-1}}{\Gamma(\gamma)(1-\mathscr{B})}\right)\\ &\quad \le g_1(s,s) + \frac{\mathscr{B}(1-s)^{\gamma-1}}{\Gamma(\gamma)(1-\mathscr{B})} = \omega_1(s).\end{aligned}$$

For $0 \le t \le s \le \eta < 1$,

$$\begin{aligned}\max_{0\le t\le 1} H(t,s) &= \max_{0\le t\le 1} \frac{[t(1-s)]^{\gamma-1} - b^{p-1}[t(\eta-s)]^{\gamma-1}}{\Gamma(\gamma)(1-\mathscr{B})}\\ &= \max_{0\le t\le 1}\left(\frac{[t(1-s)]^{\gamma-1}}{\Gamma(\gamma)} + \frac{b^{\gamma-1}t^{\gamma-1}}{1-\mathscr{B}} g_1(\eta,s)\right)\\ &\le g_1(s,s) + \frac{\mathscr{B}(1-s)^{\gamma-1}}{\Gamma(\gamma)(1-\mathscr{B})} = \omega_1(s).\end{aligned}$$

For $0 \le t \le s \le 1$, $\eta \le s$,

$$\begin{aligned}\max_{0\le t\le 1} H(t,s) &= \max_{0\le t\le 1} \frac{[t(1-s)]^{\gamma-1}}{\Gamma(\gamma)(1-\mathscr{B})}\\ &= \max_{0\le t\le 1}\left(\frac{[t(1-s)]^{\gamma-1}}{\Gamma(\gamma)} + \frac{\mathscr{B}[t(1-s)]^{\gamma-1}}{\Gamma(\gamma)(1-\mathscr{B})}\right)\\ &\le g_1(s,s) + \frac{\mathscr{B}(1-s)^{\gamma-1}}{\Gamma(\gamma)(1-\mathscr{B})} = \omega_1(s).\end{aligned}$$

So,

$$\max_{0\le t\le 1} H(t,s) \le \omega_1(s), \quad s \in (0,1).$$

It is obvious that $\omega(s), \omega_1(s) \in C((0,1),(0,+\infty))$. The proof is completed. □

Let $E = C[0,1]$ be a Banach space with the maximum norm $\|u\| = \max_{0\le t\le 1}|u(t)|$. Define the cone $P \subset E$ by $P = \{u \in E \mid u(t) \ge 0, 0 \le t \le 1\}$.

Lemma 3.4 *Let $T : P \to E$ be the operator defined by*

$$(Tu)(t) = \int_0^1 G(t,s)\phi_q\left(\int_0^s H(s,\tau)f\big(\tau,u(\tau)\big)\,d\tau\right)ds.$$

Then $T : P \to P$ is completely continuous.

Proof The operator $T : P \to P$ is continuous in view of nonnegative and continuity of $G(t,s)$, $H(t,s)$ and $f(t,u)$. Furthermore it is easy see by the Arzela–Ascoli theorem and Lebesgue dominated convergence theorem that $T : P \to P$ is completely continuous. □

For convenience, we introduce the following notation:

$$L = \left[\int_0^1 \omega(s)\phi_q\left(\int_0^s \omega_1(\tau)\,d\tau\right)ds\right]^{-1}.$$

Theorem 3.1 *Assume there exists a constant $\lambda > 0$ such that*

(C_1) $f(t,x_1) \le f(t,x_2)$ *for any* $0 \le t \le 1$, $0 \le x_1 \le x_2 \le \lambda$;

(C_2) $\max_{0\le t\le 1} f(t,\lambda) \le \phi_p(\lambda L)$;

(C_3) $f(t,0) \not\equiv 0$ *for* $0 \le t \le 1$.

Then problem (1.1), (1.2) *has two positive solution u^* and v^*, such that*

$0 < \|u^*\| \le \lambda$ *and* $\lim_{n\to\infty} T^n u_0 = u^*$, *where* $u_0(t) = \lambda$,

$0 < \|v^*\| \le \lambda$ *and* $\lim_{n\to\infty} T^n v_0 = v^*$, *where* $v_0(t) = 0$.

Proof Define $P_\lambda = \{u \in P \mid \|u\| \le \lambda\}$. In what follows, we first prove $TP_\lambda \subseteq P_\lambda$.

Let $u \in P_\lambda$, then $0 \le u(t) \le \|u\| \le \lambda$. By assumption (C_1) and (C_2), we have

$$0 \le f\big(t,u(t)\big) \le f(t,\lambda) \le \phi_p(\lambda L).$$

For any $u \in P_\lambda$, by Lemma 3.4, we know that $Tu \in P$, and as a result

$$\begin{aligned}
\|Tu\| &= \max_{0\le t\le 1}\int_0^1 G(t,s)\phi_q\left(\int_0^s H(s,\tau)f\big(\tau,u(\tau)\big)\,d\tau\right)ds \\
&\le \int_0^1 \omega(s)\phi_q\left(\int_0^s \phi_p(\lambda L)\omega_1(\tau)\,d\tau\right)ds \\
&= \lambda L\int_0^1 \omega(s)\phi_q\left(\int_0^s \omega_1(\tau)\,d\tau\right)ds \\
&= \lambda.
\end{aligned}$$

Hence $Tu \in P_\lambda$. Thus, we get $TP_\lambda \subseteq P_\lambda$.

Let $u_0(t) = \lambda$, $0 \le t \le 1$, then $\|u_0\| = \lambda$ and $u_0 \in P_\lambda$. Let $u_1(t) = Tu_0(t)$, then $u_1 \in P_\lambda$. Define

$$u_{n+1} = Tu_n = T^{n+1}u_0, \quad n = 0,1,2,\ldots.$$

Since $TP_\lambda \subseteq P_\lambda$, one has $u_n \in P_\lambda$ $(n = 0,1,2,\ldots)$. From Lemma 3.3, T is compact; we assert that $\{u_n\}_{n=1}^\infty$ has a convergent subsequence $\{u_{n_k}\}_{k=1}^\infty$ and there exists $u^* \in P_\lambda$ such that

$u_{n_k} \to u^*$. From the definition of T and (C_1), for any $t \in [0,1]$, we have

$$\begin{aligned}
u_1(t) &= (Tu_0)(t) \\
&= \int_0^1 G(t,s)\phi_q\left(\int_0^s H(s,\tau)f\bigl(\tau,u_0(\tau)\bigr)\,d\tau\right)ds \\
&\le \int_0^1 \omega(s)\phi_q\left(\int_0^s \phi_p(\lambda L)\omega_1(\tau)\,d\tau\right)ds \\
&= \lambda L\int_0^1 \omega(s)\phi_q\left(\int_0^s \omega_1(\tau)\,d\tau\right)ds \\
&= \lambda \\
&= u_0(t).
\end{aligned}$$

So,

$$u_2(t) = Tu_1(t) \le Tu_0(t) = u_1(t), \quad 0 \le t \le 1.$$

Hence, by the induction we have

$$u_{n+1} \le u_n, \quad 0 \le t \le 1, n = 0,1,2,\ldots.$$

Thus, there exists $u^* \in P_\lambda$ such that $u_n \to u^*$. Applying the continuity of T and $u_{n+1} = Tu_n$, we get $Tu^* = u^*$.

Let $v_0 = 0$, $0 \le t \le 1$, then $v_0 \in P_\lambda$. Let $v_1 = Tv_0$, then $v_1 \in P_\lambda$. Define

$$v_{n+1} = Tv_n = T^{n+1}v_0, \quad n = 0,1,2,\ldots.$$

Since $T: P_\lambda \to P_\lambda$, we have $v_n \subseteq P_\lambda$, $n = 0,1,2,\ldots$. Since T is completely continuous, we assert that $\{v_n\}_{n=1}^\infty$ is a sequentially compact set.

Since $v_1(t) = Tv_0(t) = (T0)(t) \ge 0$, $0 \le t \le 1$, one has

$$v_2(t) = Tv_1(t) \ge (T0)(t) = v_1(t), \quad 0 \le t \le 1.$$

Hence, by the induction we have

$$v_{n+1} \ge v_n, \quad 0 \le t \le 1, n = 0,1,2,\ldots.$$

Thus, there exists $v^* \in P_\lambda$ such that $v_n \to v^*$. Applying the continuity of T and $v_{n+1} = Tv_n$, we get $Tv^* = v^*$.

It is well known that each fixed point of operator T in P is a solution of problem (1.1). Furthermore, if $f(t,0) \not\equiv 0$, $0 \le t \le 1$, then the zero function is not the solution of problem (1.1). Hence, we have $\|u^*\| > 0$, $\|v^*\| > 0$. Then u^* and v^* are two positive solutions of problem (1.1). The proof is completed. □

By applying Theorem 3.1, we can get the following corollary.

Corollary 3.1 *Assume* (C_3) *holds, and the following conditions hold*:

(C_4) $f(t,x_1) \le f(t,x_2)$ *for any* $0 \le t \le 1$, $0 \le x_1 \le x_2$;

(C_5) $\lim_{x\to\infty} \max_{0\le t\le 1} \frac{f(t,x)}{x^{p-1}} \le \phi_p(L)$ (*particularly* $\lim_{x\to\infty} \max_{0\le t\le 1} \frac{f(t,x)}{x^{p-1}} = 0$).

Then problem (1.1) *have two positive solutions* u^* *and* v^*.

4 Example

Let $p = \frac{3}{2}$, $\alpha = \frac{4}{3}$, $\gamma = \frac{3}{2}$, $\beta = \frac{1}{4}$, $\xi = \frac{1}{3}$, $\eta = \frac{1}{4}$, $a = b = \frac{1}{2}$. We consider the following boundary value problem:

$$\begin{cases} D^{\frac{3}{2}}(\phi_{\frac{3}{2}}(D^{\frac{4}{3}}x(t))) = f(t,x(t)), \quad 0 < t < 1, \\ x(0) = D^{\frac{4}{3}}x(0) = 0, \qquad D^{\frac{1}{4}}x(1) = \frac{1}{2}D^{\frac{1}{4}}x(\frac{1}{3}), \qquad D^{\frac{4}{3}}x(1) = \frac{1}{2}D^{\frac{4}{3}}x(\frac{1}{4}), \end{cases} \tag{4.1}$$

where

$$f(t,x) = \frac{1}{24}\bigl(1 + xe^t + x^{\frac{3}{2}}\bigr).$$

It is obvious that

$$1 - a\xi^{\alpha-\beta-1} = 1 - \frac{1}{2}\left(\frac{1}{3}\right)^{\frac{1}{12}} > 0, \qquad 1 - b^{p-1}\eta^{\gamma-1} = 1 - \frac{1}{2^{\frac{3}{2}}} > 0.$$

By computation, we can obtain $L \approx 0.2318$. We choose $\lambda = 5$. So, $f(t,x)$ satisfies

(1) $f(t,x_1) \le f(t,x_2)$ for any $0 \le t \le 1$, $0 \le x_1 \le x_2 \le 5$;

(2) $\max_{0\le t\le 1} f(t,\lambda) = f(1,5) \approx \frac{25.7703}{24} \approx 1.0738 < \phi_{\frac{3}{2}}(\lambda L) \approx 1.0766$;

(3) $f(t,0) = \frac{1}{24} \ne 0$ for $0 \le t \le 1$.

Then applying Theorem 3.1, problem (4.1) has two positive solutions u^* and v^* such that

$0 < \|u^*\| \le 5$ and $\lim_{n\to\infty} T^n u_0 = u^*$, where $u_0(t) = 5$,

$0 < \|v^*\| \le 5$ and $\lim_{n\to\infty} T^n v_0 = v^*$, where $v_0(t) = 0$.

Acknowledgements
Not applicable.

Funding
This work is supported by the Scientific Research Foundation of Hunan Provincial Education Department (16A198, 18C1019), National Natural Science Foundation of China (11571207), and the Taishan Scholar project.

Availability of data and materials
Data sharing not applicable to this article as no data sets were generated or analysed during the current study.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 11 April 2019 Accepted: 31 July 2019 Published online: 20 August 2019

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