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An oscillation criterion in 4*th*-order neutral differential equations with a continuously distributed delay

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Abstract

In this paper, a class of 4*th*-order neutral delay differential equations with continuously distributed delay is studied. We establish a new oscillation criterion using the Riccati transformation. An example illustrating the results is also given.

MSC: 34K10; 34K11

Keywords: 4*th*-order; Neutral differential equations; Oscillatory solutions

1 Introduction

In this work, we consider a 4*th*-order neutral differential equation with a continuously distributed delay of the form

$$\left[r(t)\left(\left[x(t)+p(t)x\left(\tau(t)\right)\right]'''\right)^{\alpha}\right]'+\int_a^b q(t,\xi)f\left(x\left(g(t,\xi)\right)\right)d\xi=0. \tag{1}$$

We assume that the following conditions hold:

(H_1) α is a quotient of odd positive integers;

(H_2) p, q, τ, $g\in C([t_0,\infty),R)$, $r(t)$, and $q(t,\xi)$ are positive, $0\le p(t)\le p<1$, $r(t)\in C^1([t_0,+\infty))$, $r'(t)\ge 0$, $\tau(t)\le t$, $g(t,\xi)\le t$, $\lim_{t\to\infty}\tau(t)=\infty$, $\lim_{t\to\infty}g(t,\xi)=\infty, q(t,\xi)$ is not zero on any half line $[t_\lambda,\infty)\times[a,b]$, $t_\lambda\ge t_0$, for $t\ge t_0$, $\xi\in[a,b]$, $g(t,\xi)$ is nondecreasing with respect to ξ.

(H_3) There exists a constant $k>0$ such that $f(u)/u^{\gamma}\ge k$ for $u\neq 0$.

We define the corresponding function $z(t)$ of a solution $x(t)$ of (1) by $z(t)=x(t)+p(t)x(\tau(t))$, we mean a non-trivial real function $x(t)\in C([t_x,\infty))$, $t_x\ge t_0$, satisfying (1) on $[t_x,\infty)$ and, moreover, having the properties: $z(t)$, $z'(t)$, $z''(t)$ and $r(t)[z'''(t)]^{\alpha}$ are continuously differentiable for all $t\in[t_x,\infty)$. We consider only those solutions $x(t)$ of (1) which satisfy $\sup\{|x(t)|:t\ge T\}>0$ for any $T\ge t_x$. A solution of (1) is called oscillatory if it has arbitrary large zeros, otherwise it is called nonoscillatory.

The oscillations of higher- and fourth-order differential equations have been studied by several authors, and several techniques have been proposed for obtaining oscillatory criteria for higher- and fourth-order differential equations. For treatments on this subject, we refer the reader to the texts [2, 5, 16–18, 21] and the articles [1, 3–15, 19–26]. In what

follows, we review some results that have provided the background and motivation for the present work.

Cesarano and Bazighifan [8], Moaaz et al. [21], and Zhang et al. [26] studied the oscillation of the fourth-order nonlinear differential equation with a continuously distributed delay

$$\left[r(t)\left(x'''(t)\right)^{\alpha}\right]' + \int_a^b q(t,\xi)f\left(x\left(g(t,\xi)\right)\right)d\xi = 0. \tag{2}$$

Li et al. [19] studied the oscillatory behavior of the fourth-order nonlinear differential equation

$$\left[r(t)z(t)\right]^{(4)} + q(t)x\left(\tau(t)\right) = 0. \tag{3}$$

Parhi and Tripathy [23] have considered the fourth-order neutral differential equations of the form

$$\left[r(t)\left(y(t) + p(t)y(t-\tau)\right)''\right]'' + q(t)G\left(y(t-\sigma)\right) = 0 \tag{4}$$

and

$$\left[r(t)\left(y(t) + p(t)y(t-\tau)\right)''\right]'' + q(t)G\left(y(t-\sigma)\right) = f(t) \tag{5}$$

and established the oscillation and asymptotic behavior of the above equations under the conditions

$$\int_{t_0}^{\infty} \frac{1}{r^{\frac{1}{\alpha}}(t)}\,d\dot{t} < \infty \tag{6}$$

and

$$\int_{t_0}^{\infty} \frac{1}{r^{\frac{1}{\alpha}}(t)}\,dt = \infty, \tag{7}$$

respectively.

Our aim in the present paper is to use the Riccati method to establish new conditions for the oscillation of all solutions of (1) under condition (7).

The proof of our main results is essentially based on the following lemmas.

Lemma 1.1 *Let $\beta \geq 1$ be a ratio of two numbers, where U and V are constants. Then*

$$Uy - Vy^{\frac{\beta+1}{\beta}} \leq \frac{\beta^{\beta}}{(\beta+1)^{\beta+1}}\frac{U^{\beta+1}}{V^{\beta}}, \quad V > 0. \tag{8}$$

Lemma 1.2 *If the function z satisfies $z^{(i)} > 0$, $i = 0, 1, \ldots, n$, and $z^{(n+1)} < 0$, then*

$$\frac{z(t)}{t^n/n!} \geq \frac{z'(t)}{t^{n-1}/(n-1)!}. \tag{9}$$

Lemma 1.3 *Let $h \in C^n([t_0,\infty),(0,\infty))$. Assume that $h^{(n)}(t)$ is of a fixed sign and not identically zero on $[t_0,\infty)$ and that there exists $t_1 \geq t_0$ such that $h^{(n-1)}(t)h^{(n)}(t) \leq 0$ for all $t \geq t_1$. If $\lim_{t\to\infty} h(t) \neq 0$, then for every $\lambda \in (0,1)$ there exists $t_\lambda \geq t_0$ such that*

$$h(t) \geq \frac{\lambda}{(n-1)!} t^{n-1} \left|h^{(n-1)}(t)\right| \quad \text{for all } t \geq t_\lambda. \tag{10}$$

2 Main results

In this section, we shall establish some oscillation criteria for equation (1).

For convenience, we denote

$$\eta(t) = \int_{t_0}^{\infty} r^{-1/\alpha}(s)\,ds,$$

$$\rho'_+(t) := \max\{0, \rho'(t)\},$$

$$\vartheta'_+(t) := \max\{0, \vartheta'(t)\},$$

$$Q(t) = \int_a^b q(t,\xi)\,d\xi$$

and

$$Q^*(\upsilon) = \int_{\upsilon}^{\infty} kQ(s)(1-p)^{\alpha}\big(g(s,a)/s\big)^{3\alpha}\,ds. \tag{11}$$

Theorem 2.1 *Assume that (7) holds. If there exist positive functions $\rho, \vartheta \in C([t_0,\infty))$ such that*

$$\int_{t_0}^{\infty} \left(\Psi(s) - \frac{2^{\alpha}}{(\alpha+1)^{\alpha+1}} \frac{r(s)(\rho'(s))^{\alpha+1}}{\mu^{\alpha} s^{2\alpha} \rho^{\alpha}(s)}\right) ds = \infty \tag{12}$$

for some $\mu \in (0,1)$, where

$$\Psi(t) = k\rho(t)Q(t)(1-p)^{\alpha}\big(g(t,a)/t\big)^{3\alpha}, \tag{13}$$

and either

$$\int_{t_0}^{\infty} kQ(s)(1-p)^{\alpha}\big(g(s,a)/s\big)^{\alpha}\,ds = \infty \tag{14}$$

or

$$\int_{t_0}^{\infty} \big(Q^*(\upsilon)\big)^{\frac{1}{\alpha}} r^{\frac{-1}{\alpha}}(\upsilon)\,d\upsilon = \infty, \tag{15}$$

or

$$\int_{t_0}^{\infty} \left[\vartheta(t) \int_t^{\infty} \big(Q^*(\upsilon)\big)^{\frac{1}{\alpha}} r^{\frac{-1}{\alpha}}(\upsilon)\,d\upsilon - \frac{\vartheta'^2_+(t)}{4\vartheta(t)}\right] dt = \infty, \tag{16}$$

then all solutions of (1) are oscillatory.

Proof Let x be a nonoscillatory solution of equation (1) defined in the interval $[t_0, \infty)$. Without loss of generality, we can assume that $x(t)$ is eventually positive. It follows from (1) that there are two possible cases for $t \geq t_1$, where $t_1 \geq t_0$ is sufficiently large:

(C_1) $z'(t) > 0$, $z''(t) > 0$, $z'''(t) > 0$, $\big(r(t)\big(z'''(t)\big)^{\alpha}\big) < 0$,

(C_2) $z'(t) > 0$, $z''(t) < 0$, $z'''(t) > 0$, $\big(r(t)\big(z'''(t)\big)^{\alpha}\big) < 0$.

Assume that Case (C_1) holds. Since $\tau(t) \leq t$ and $z'(t) > 0$, we get

$$\begin{aligned} z(t) &= x(t) + p(t)x\big(\tau(t)\big), \\ x(t) &= z(t) - p(t)x\big(\tau(t)\big) \\ &\geq z(t) - p(t)z\big(\tau(t)\big) \\ &= \big(1 - p(t)\big)z(t). \end{aligned}$$

From equation (1), we see that

$$\big(r(t)\big(z'''(t)\big)^{\alpha}\big)' = -\int_a^b q(t,\xi)f\big(x\big(g(t,\xi)\big)\big)\,d\xi \tag{17}$$

so that

$$\begin{aligned} \big(r(t)\big(z'''(t)\big)^{\alpha}\big)' &= -\int_a^b kq(t,\xi)x^{\alpha}\big(g(t,\xi)\big)\,d\xi \\ &\leq -\int_a^b kq(t,\xi)\big(1 - p\big(g(t,\xi)\big)\big)^{\alpha}z^{\alpha}\big(g(t,\xi)\big)\,d\xi. \end{aligned}$$

Since $g(t,\xi)$ is nondecreasing with respect to ξ and $z'(t) > 0$, we have

$$z\big(g(t,a)\big) \leq z\big(g(t,\xi)\big) \leq z\big(g(t,b)\big). \tag{18}$$

Thus

$$\big(r(t)\big(z'''(t)\big)^{\alpha}\big)' \leq -(1-p)^{\alpha}\int_a^b kq(t,\xi)z^{\alpha}\big(g(t,\xi)\big)\,d\xi \tag{19}$$

$$\leq -k(1-p)^{\alpha}z^{\alpha}\big(g(t,a)\big)\int_a^b q(t,\xi)\,d\xi \tag{20}$$

$$= -kQ(t)(1-p)^{\alpha}z^{\alpha}\big(g(t,a)\big). \tag{21}$$

Now, we define a generalized Riccati substitution by

$$\omega(t) := \rho(t)\frac{r(t)(z'''(t))^{\alpha}}{z^{\alpha}(t)}. \tag{22}$$

Then $\omega(t) > 0$. Differentiating and using (19), we obtain

$$\omega'(t) \leq \frac{\rho'(t)}{\rho(t)}\omega(t) - k\rho(t)Q(t)(1-p)^{\alpha}\frac{z^{\alpha}(g(t,a))}{z^{\alpha}(t)} \tag{23}$$

$$- \alpha\rho(t)\frac{r(t)(z'''(t))^{\alpha}}{z^{\alpha+1}(t)}z'(t). \tag{24}$$

From Lemma 1.2, we have that $z(t) \geq \frac{t}{3} z'(t)$, and hence

$$\frac{z(g(t,a))}{z(t)} \geq \frac{g^3(t,a)}{t^3}. \tag{25}$$

Since $r'(t) > 0$ and $(r(t)(z'''(t))^\alpha)' \leq 0$, we get $z^{(4)}(t) < 0$. It follows from Lemma 1.3 that

$$z'(t) \geq \frac{\mu}{2} t^2 z'''(t) \tag{26}$$

for all $\mu \in (0,1)$ and every sufficiently large t. Thus, by (24), (25), and (26), we get

$$\omega'(t) \leq \frac{\rho'(t)}{\rho(t)}\omega(t) - k\rho(t)Q(t)(1-p)^\alpha \left(\frac{g(t,a)}{t}\right)^{3\alpha} - \alpha\mu \frac{t^2}{2r^{1/\alpha}(t)\rho^{1/\alpha}(t)} \omega^{\frac{\alpha+1}{\alpha}}(t).$$

Using Lemma 1.1 with $U = \frac{\rho'(t)}{\rho(t)}$, $V = \frac{\alpha\mu t^2}{2r^{1/\alpha}(t)\rho^{1/\alpha}(t)}$ and $y = \omega$, we get

$$\omega'(t) \leq -\Psi(t) + \frac{2^\alpha}{(\alpha+1)^{\alpha+1}} \frac{r(t)(\rho'(t))^{\alpha+1}}{\mu^\alpha t^{2\alpha} \rho^\alpha(t)}. \tag{27}$$

This implies that

$$\int_{t_1}^{t} \left(\Psi(s) - r(s)\big(\rho'(s)/\alpha + 1\big)^{\alpha+1} \left(\frac{2}{\mu s^2 \rho(s)}\right)^\alpha \right) ds \leq \omega(t_1) \tag{28}$$

for some $\mu \in (0,1)$ which contradicts (12).

Assume that Case (C_2) holds. Integrating (1) from t_1 to t, we obtain

$$-r(t_1)\big(z'''(t_1)\big)^\alpha \leq -\int_{t_1}^{t} kQ(t)x^\alpha\big(g(s,\xi)\big)\, ds. \tag{29}$$

From $z'(t) > 0$, $x(t) \geq (1-p(t))z(t)$ and $g(s,\xi) \leq t$, it follows that

$$\int_{t_1}^{t} kQ(s)(1-p)^\alpha z^\alpha\big(g(s,a)\big)\, ds \leq r(t_1)\big(z'''(t_1)\big)^\alpha. \tag{30}$$

From (25), we get

$$\int_{t_1}^{t} kQ(s)(1-p)^\alpha \big(g(s,a)/s\big)^{3\alpha}\, ds \leq r(t_1)\left(\frac{z'''(t_1)}{z(t_1)}\right)^\alpha, \tag{31}$$

which contradicts (12). Integrating (1) from t to ∞, we obtain

$$-r(t)\big(z'''(t)\big)^\alpha \leq -\int_{t}^{\infty} kQ(t)x^\alpha\big(g(s,\xi)\big)\, ds. \tag{32}$$

By virtue of $z'(t) > 0$, $x(t) \geq (1-p(t))z(t)$, $g(s,\xi) \leq t$, and (25), we obtain

$$-\big(z'''(t)\big) + \frac{z(t)}{r(t)^{1\backslash\alpha}} \left[\int_{t}^{\infty} kQ(s)(1-p)^\alpha \big(g(s,a)/s\big)^{3\alpha}\, ds\right]^{1/\alpha} \leq 0. \tag{33}$$

Integrating (33) from t_1 to t, we get

$$\int_{t_1}^{t} \frac{1}{r^{1/\alpha}(\upsilon)} \left(\int_{\upsilon}^{\infty} kQ(s)(1-p)^{\alpha} \big(g(s,a)/s\big)^{3\alpha} \, ds \right)^{1/\alpha} d\upsilon \leq -\frac{z''(t_1)}{z(t_1)}. \tag{34}$$

This yields

$$\int_{t_1}^{t} \big(Q^*(\upsilon)\big)^{\frac{1}{\alpha}} r^{\frac{-1}{\alpha}}(\upsilon) \, d\upsilon \leq -\frac{z''(t_1)}{z(t_1)}, \tag{35}$$

which contradicts (15). Integrating (33) from t to ∞, we get

$$\int_{t}^{\infty} \big(Q^*(\upsilon)\big)^{\frac{1}{\alpha}} r^{\frac{-1}{\alpha}}(\upsilon) \, d\upsilon \leq -\frac{z''(t)}{z(t)}, \tag{36}$$

so that

$$z''(t) + z(t) \int_{t}^{\infty} \big(Q^*(\upsilon)\big)^{\frac{1}{\alpha}} r^{\frac{-1}{\alpha}}(\upsilon) \, d\upsilon \leq 0. \tag{37}$$

Now, we define the Riccati substitution

$$\psi(t) := \vartheta(t) \frac{z'(t)}{z(t)}, \tag{38}$$

then $\psi(t) > 0$ for $t \geq t_1$ and

$$\psi'(t) := \vartheta'(t) \frac{z'(t)}{z(t)} + \vartheta(t) \frac{z''(t)z(t) - (z'(t))^2}{z^2(t)}. \tag{39}$$

From (37) and (38), it follows that

$$\psi'(t) \leq -\vartheta(t) \int_{t}^{\infty} \big(Q^*(\upsilon)\big)^{\frac{1}{\alpha}} r^{\frac{-1}{\alpha}}(\upsilon) \, d\upsilon + \frac{(\vartheta'(t))_+}{\vartheta(t)} \psi(t) - \frac{1}{\vartheta(t)} \psi^2(t). \tag{40}$$

Hence, we have

$$\psi'(t) \leq -\vartheta(t) \int_{t}^{\infty} \big(Q^*(\upsilon)\big)^{\frac{1}{\alpha}} r^{\frac{-1}{\alpha}}(\upsilon) \, d\upsilon + \frac{((\vartheta'(t))_+)^2}{4\vartheta(t)}. \tag{41}$$

Integrating (41) from t_1 to s, we get

$$\int_{t_1}^{s} \left(\vartheta(t) \int_{t}^{\infty} \big(Q^*(\upsilon)\big)^{\frac{1}{\alpha}} r^{\frac{-1}{\alpha}}(\upsilon) \, d\upsilon + \frac{((\vartheta'(t))_+)^2}{4\vartheta(t)} \right) dt \leq \psi(t_1) \tag{42}$$

for all large s, which contradicts (16).

The proof of the theorem is complete. □

Let $\rho(t) = t^3$ and $\vartheta(t) = t$. As a consequence of Theorem 2.1, we obtain the following oscillation criterion.

Corollary *Assume that* (7) *holds and for some constant* $\lambda_0 \in (0,1)$

$$\int_{t_0}^{\infty}\left(\Psi^*(t)-\left(\frac{3}{\alpha+1}\right)^{\alpha+1}\left(\frac{2}{\lambda_0}\right)^{\alpha}t^{2-3\alpha}r(t)\right)ds=\infty, \tag{43}$$

where

$$\Psi^*(t)=kt^3Q(t)(1-p)^{\alpha}\big(g(t,a)/t\big)^{3\alpha}. \tag{44}$$

If either (14) or (15) is satisfied, or

$$\int_{t_0}^{\infty}\left[t\int_t^{\infty}\big(Q^*(\upsilon)\big)^{\frac{1}{\alpha}}r^{\frac{-1}{\alpha}}(\upsilon)\,d\upsilon-\frac{1}{4t}\right]dt=\infty, \tag{45}$$

then all solutions of (1) are oscillatory.

3 Example

In this section, we give the following example to illustrate our main results.

Example Consider the differential equation

$$\left(\left[x(t)+\frac{1}{2}x\left(\frac{t}{3}\right)\right]'''\right)'+\int_0^1\big(\nu/t^4\big)\xi x\left(\frac{t-\xi}{2}\right)d\xi=0, \tag{46}$$

where $\nu>0$ is a constant. Let

$$\begin{aligned}&\alpha=1,\qquad r(t)=1,\qquad p(t)=\frac{1}{2},\qquad \tau(t)=\frac{t}{3},\\&g(t,a)=\frac{t}{2},\qquad q(t,\xi)=\big(\nu/t^4\big)\xi,\qquad f(x)=x,\end{aligned} \tag{47}$$

we get

$$\begin{aligned}\eta(s)&=\int_{t_0}^{\infty}ds=\infty,\\Q(t)&=\int_0^1 q(t,\xi)\,d\xi=\frac{\nu}{2t^4}\\Q^*(t)&=\int_{t_0}^{\infty}kQ(s)(1-p)^{\alpha}\big(g(s,a)/s\big)^{3\alpha}\,ds\\&=\int_{t_0}^{\infty}\frac{\nu}{32s^4}\,ds=\frac{\nu}{96t^3}.\end{aligned}$$

If we now set $k=1$, then

$$\int_{t_0}^{\infty}\left(kt^3Q(t)(1-p)^{\alpha}\big(g(t,a)/t\big)^{3\alpha}-\left(\frac{3}{\alpha+1}\right)^{\alpha+1}\left(\frac{2}{\lambda_0}\right)^{\alpha}t^{2-3\alpha}r(t)\right)ds \tag{48}$$

$$=\left(\frac{\nu}{32}-\frac{9}{2\lambda_0}\right)\int_{t_0}^{\infty}\frac{1}{t}\,dt=\infty,\quad \text{if } \nu>\frac{144}{\lambda_0}\text{ for some constant }\lambda_0\in(0,1), \tag{49}$$

and

$$\begin{aligned}
&\int_{t_0}^{\infty}\left[t\int_{t}^{\infty}\left(Q^{*}(\upsilon)\right)^{\frac{1}{\alpha}}r^{\frac{-1}{\alpha}}(\upsilon)\,d\upsilon-\frac{1}{4t}\right]dt\\
&=\int_{t_0}^{\infty}\left[t\int_{t}^{\infty}\frac{\nu}{96t^{3}}\,dt-\frac{1}{4t}\right]dt\\
&=\int_{t_0}^{\infty}\left[t\frac{\nu}{192t^{2}}-\frac{1}{4t}\right]dt\\
&=\left(\frac{\nu}{192}-\frac{1}{4}\right)\int_{t_0}^{\infty}\frac{1}{t}\,dt\\
&=\infty,\quad \text{if } \nu>48.
\end{aligned}$$

Thus, by Corollary, every solution of equation (46) is oscillatory.

4 Conclusion

The results of this paper are presented in a form which is essentially new and of high degree of generality. In this paper, using a Riccati transformation technique, we offer some new sufficient conditions which ensure that any solution of Eq. (1) oscillates under the condition $\int_{t_0}^{\infty}\frac{1}{r^{\frac{1}{\alpha}}(t)}\,dt=\infty$. Further, we can consider the case of $z(t)=x(t)-p(t)x(\tau(t))$, and we can try to get some oscillation criteria of Eq. (1) in the future work.

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