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Pseudo almost automorphic solutions of quaternion-valued neural networks with infinitely distributed delays via a non-decomposing method

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Abstract

In this paper, we consider the existence and global exponential stability of pseudo almost automorphic solutions to quaternion-valued cellular neural networks with infinitely distributed delays. Unlike most previous studies of quaternion-valued cellular neural networks, we do not decompose the systems under consideration into real-valued or complex-valued systems, but rather directly study quaternion-valued systems. Our method and the results of this paper are new. An example is given to show the feasibility of our main results.

Keywords: Pseudo almost automorphic solution; Global exponential stability; Quaternion-valued neural networks

1 Introduction

The quaternion was introduced into mathematics in 1843 by Hamilton [1]. The skew field of quaternions is

$$\mathbb{H} := \{q | q = q^R + iq^I + jq^J + kq^K\},$$

where $q^R, q^I, q^J, q^K \in \mathbb{R}$ and i, j, k satisfy Hamilton's multiplication table formed by

$$i^2 = j^2 = k^2 = ijk = -1, \qquad ij = -ji = k, \qquad jk = -kj = i, \qquad ki = -ik = j,$$

and the norm of $q \in \mathbb{H}$ is

$$\|q\|_{\mathbb{H}} = \sqrt{\bar{q}q} = \sqrt{q\bar{q}} = \sqrt{(q^R)^2 + (q^I)^2 + (q^J)^2 + (q^K)^2},$$

where $\bar{q} = q^R - iq^I - jq^J - kq^K$. For $x = (x_1, x_2, \dots, x_n)^T \in \mathbb{H}^n$, we define $\|x\|_{\mathbb{H}^n} = \max_{1\le p\le n}\{\|x_p\|_{\mathbb{H}}\}$ and $|x|_{\mathbb{H}^n} = \sum_{p=1}^n \|x_p\|_{\mathbb{H}}$. Quaternion algebra is a non-commutative divisible algebra. It is because of its non-commutative nature that the study of quaternions is much more difficult than real and complex numbers. In recent years, with the rapid development of quaternion algebra and the wide application of quaternions in many fields, the study of quaternion algebra and quaternion analysis has attracted more and more scholars from various fields. Quaternion-valued differential equations, as special differential equations, are widely used in quantum mechanics, fluid mechanics, Frenet–Serret frame in differential geometry, dynamics model, robot operation, Kalman filter design, spatial rigid body dynamics, computer Graphics, and so on [2–10].

On the one hand, since quaternion-valued neural network models have more advantages than the real-value neural network models in dealing with affine transformation in three-dimensional space, color image compression, color night vision, satellite attitude control, and so on [11, 12], in recent years research on quaternion-valued neural networks has become a hot research topic. As we know, the design, implementation, and application of neural networks greatly depend on the dynamic behavior of neural networks. Therefore, there are some research results in this area. Since the quaternion multiplication does not satisfy the commutative law, most of the results are obtained by decomposing the considered quaternion-valued systems into real-valued systems or a complex-valued systems [13–19]. Only very few results on the stability and dissipation of quaternion-valued neural networks are obtained by direct method [20–22].

On the other hand, almost automorphicity is an extension of almost periodicity and pseudo automorphicity is a natural generalization of almost automorphicity. At the same time, for non-autonomous neural networks, periodicity, almost periodicity, and almost automorphicity are important dynamics [23–28]. At present, there are no results on the almost automorphicity of quaternion-valued neural networks obtained by direct method.

Inspired by the above discussion, in this paper, we are concerned with the following quaternion-valued neural network with infinitely distributed delays:

$$\begin{aligned}\dot{x}_p(t) &= -a_p(t)x_p(t) + \sum_{q=1}^{n} b_{pq}(t)f_q\bigl(x_q(t)\bigr)\\ &\quad + \sum_{q=1}^{n} c_{pq}(t)\int_{-\infty}^{t} k_{pq}(t-s)g_q\bigl(x_q(s)\bigr)\,ds + Q_p(t),\end{aligned} \tag{1}$$

where $p \in \mathbb{I}_n := \{1, 2, \dots, n\}$, $x_p(t) : \mathbb{R} \to \mathbb{H}$ denotes the activation of the pth neuron at time t; $a_p(t) : \mathbb{R} \to \mathbb{R}^+$ represents the rate at which the pth unit will reset its potential to the resting state in isolation when disconnected from the network, and external inputs at time t; $b_{pq}, c_{pq} : \mathbb{R} \to \mathbb{H}$ represent the connection weights and the distributively delayed connection weights between the qth neuron and the pth neuron at time t, respectively; $f_q, g_q : \mathbb{H} \to \mathbb{H}$ are the activation functions of signal transmission; $Q_p : \mathbb{R} \to \mathbb{H}$ is an external input on the pth unit at time t; the kernel function $k_{pq} : \mathbb{R} \to \mathbb{R}^+$ satisfies $\int_0^{+\infty} k_{pq}(s)\,ds = 1$.

The initial value of system (1) is given by

$$x_p(s) = \varphi_p(s), \quad s \in (-\infty, 0], p \in \mathbb{I}_n,$$

where $\varphi_p \in C((-\infty, 0], \mathbb{H})$.

The main purpose of this paper is to study the existence and global exponential stability of pseudo almost automorphic solutions to system (1). Our results and method are new, and our method can be used to study the existence and stability of almost periodic solutions, pseudo almost periodic solutions, almost automorphic solutions, and pseudo almost automorphic solutions for other types of quaternion-valued neural network models.

This paper is organized as follows. In Sect. 2, we introduce some basic definitions and lemmas. In Sect. 3, the existence of pseudo almost automorphic solutions of system (1) is discussed based on the contraction mapping principle. In Sect. 4, the global exponential stability of pseudo almost automorphic solutions is studied based on proof by contradiction. In Sect. 5, an example is given to illustrate the feasibility of our results of this paper.

2 Preliminaries

Let $BC(\mathbb{R},\mathbb{H}^n)$ be the set of all bounded continuous functions from $\mathbb{R}$ to $\mathbb{H}^n$.

Definition 1 Function $f\in BC(\mathbb{R},\mathbb{H}^n)$ is said to be almost automorphic if, for every sequence of real numbers $(s'_n)_{n\in\mathbb{N}}$, there exists a subsequence $(s_n)_{n\in\mathbb{N}}$ such that

$$g(t):=\lim_{n\to\infty}f(t+s_n)$$

is well defined for each $t\in\mathbb{R}$, and

$$\lim_{n\to\infty}g(t-s_n)=f(t)$$

for each $t\in\mathbb{R}$.

For convenience, we denote by $AA(\mathbb{R},\mathbb{H}^n)$ the set of all almost automorphic functions from $\mathbb{R}$ to $\mathbb{H}^n$.

Similar to the proofs of the corresponding results in Ref. [29], one can get the following.

Lemma 1 *If $f,g\in AA(\mathbb{R},\mathbb{H})$ and if $\lambda\in\mathbb{R}$, then we have $f+g, fg, \lambda f\in AA(\mathbb{R},\mathbb{H})$.*

Lemma 2 *$x\in AA(\mathbb{R},\mathbb{H})$ and $\tau\in\mathbb{R}$, then $x(\cdot-\tau)\in AA(\mathbb{R},\mathbb{H})$.*

Lemma 3 *If $f\in C(\mathbb{R},\mathbb{H})$ satisfies the Lipschitz condition, $x\in AA(\mathbb{R},\mathbb{H})$, then $f(x(\cdot))$ belongs to $AA(\mathbb{R},\mathbb{H})$.*

Let

$$AA_0(\mathbb{R},\mathbb{H})=\left\{f\in BC(\mathbb{R},\mathbb{H})\,\middle|\,\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}\|f(t)\|_{\mathbb{H}}\,dt=0\right\}.$$

Definition 2 A function $f\in BC(\mathbb{R},\mathbb{H})$ is said to be pseudo almost automorphic if it can be expressed as $f=f_1+f_0$, where $f_1\in AA(\mathbb{R},\mathbb{H})$ and $f_0\in AA_0(\mathbb{R},\mathbb{H})$. The collection of such functions will be denoted by $PAA(\mathbb{R},\mathbb{H})$.

Lemma 4 *If $\varphi\in PAA(\mathbb{R},\mathbb{H})$, then $\varphi(\cdot-h)\in PAA(\mathbb{R},\mathbb{H})$.*

Proof Since $\varphi\in PAA(\mathbb{R},\mathbb{H})$, we can write $\varphi=\varphi_1+\varphi_0$, where $\varphi_1\in AA(\mathbb{R},\mathbb{H})$ and $\varphi_0\in AA_0(\mathbb{R},\mathbb{H})$. Then we have

$$\varphi(\cdot-h)=\varphi_1(\cdot-h)+\varphi_0(\cdot-h).$$

In view of Lemma 2, $\varphi_1(\cdot - h) \in AA(\mathbb{R}, \mathbb{H})$ and

$$\lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T} \|\varphi_0(s-h)\|_{\mathbb{H}}\,ds = \lim_{T\to+\infty} \frac{1}{2T}\int_{-T-h}^{T-h} \|\varphi_0(s)\|_{\mathbb{H}}\,ds = 0,$$

which implies that $\varphi_0(\cdot - h) \in AA_0(\mathbb{R}, \mathbb{H})$. So $\varphi(\cdot - h) \in PAA(\mathbb{R}, \mathbb{H})$. The proof is complete. □

Lemma 5 *If $\varphi, \psi \in PAA(\mathbb{R}, \mathbb{H})$, then $\varphi\psi \in PAA(\mathbb{R}, \mathbb{H})$.*

Proof We can write $\varphi(t) = \varphi_1(t) + \varphi_0(t)$, $\psi(t) = \psi_1(t) + \psi_0(t)$, where $\varphi_1, \psi_1 \in AA(\mathbb{R}, \mathbb{H})$ and $\varphi_0, \psi_0 \in AA_0(\mathbb{R}, \mathbb{H})$. Obviously,

$$\varphi(t)\psi(t) = \varphi_1(t)\psi_1(t) + \varphi_1(t)\psi_0(t) + \psi_1(t)\varphi_0(t) + \varphi_0(t)\psi_0(t).$$

By Lemma 1, $\varphi_1\psi_1 \in AA(\mathbb{R}, \mathbb{H})$. Since

$$\begin{aligned}
&\lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T} \|\varphi_1(t)\psi_0(t) + \psi_1(t)\varphi_0(t) + \varphi_0(t)\psi_0(t)\|_{\mathbb{H}}\,dt \\
&\quad\leq \lim_{T\to+\infty} \frac{\|\varphi_1\|_\infty}{2T}\int_{-T}^{T} \|\psi_0(t)\|_{\mathbb{H}}\,dt + \lim_{T\to+\infty} \frac{\|\psi_1\|_\infty}{2T}\int_{-T}^{T} \|\varphi_0(t)\|_{\mathbb{H}}\,dt \\
&\qquad+ \lim_{T\to+\infty} \frac{\|\varphi_0\|_\infty}{2T}\int_{-T}^{T} \|\psi_0(t)\|_{\mathbb{H}}\,dt = 0,
\end{aligned}$$

$\varphi_1\psi_0 + \psi_1\varphi_0 + \varphi_0\psi_0 \in AA_0(\mathbb{R}, \mathbb{H})$. Therefore, $\varphi\psi \in PAA(\mathbb{R}, \mathbb{H})$. The proof is complete. □

Lemma 6 *Let $g \in C(\mathbb{R}, \mathbb{H})$ and $\varphi \in PAA(\mathbb{R}, \mathbb{H})$. If there exists a positive constant L such that*

$$\|g(x) - g(y)\|_{\mathbb{H}} \leq L\|x - y\|_{\mathbb{H}}, \quad \forall x, y \in \mathbb{H},$$

then the function $g(\varphi(\cdot)) \in PAA(\mathbb{R}, \mathbb{H})$.

Proof Since $\varphi \in PAA(\mathbb{R}, \mathbb{H})$, we can write $\varphi(t) = \varphi_1(t) + \varphi_0(t)$. Hence,

$$g(\varphi(t)) = g(\varphi_1(t)) + g(\varphi(t)) - g(\varphi_1(t)).$$

By Lemma 3, we have $g(\varphi_1(\cdot)) \in AA(\mathbb{R}, \mathbb{H})$. Noticing that $\varphi_0 \in AA_0(\mathbb{R}, \mathbb{H})$, we have

$$\begin{aligned}
\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \|g(\varphi(t)) - g(\varphi_1(t))\|_{\mathbb{H}}\,dt &\leq \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} L\|\varphi(t) - \varphi_1(t)\|_{\mathbb{H}}\,dt \\
&= \lim_{T\to\infty} L\frac{1}{2T}\int_{-T}^{T} \|\varphi_0(t)\|_{\mathbb{H}}\,dt = 0,
\end{aligned}$$

which implies that $g(\varphi(\cdot)) - g(\varphi_1(\cdot)) \in AA_0(\mathbb{R}, \mathbb{H})$. Consequently, $g(\varphi(\cdot)) \in PAA(\mathbb{R}, \mathbb{H})$. The proof is complete. □

In the rest of this paper, we will adopt the following notation:

$$b_{pq}^+ = \sup_{t\in\mathbb{R}} \|b_{pq}(t)\|_{\mathbb{H}}, \qquad c_{pq}^+ = \sup_{t\in\mathbb{R}} \|c_{pq}(t)\|_{\mathbb{H}}$$

and make the following assumptions:

(H_1) For all $p, q \in \mathbb{I}_n$, $a_p \in AP(\mathbb{R}, \mathbb{R}^+)$, $b_{pq}, c_{pq}, Q_p \in PAA(\mathbb{R}, \mathbb{H})$, and $a_p^- = \inf_{t\in\mathbb{R}} a_p(t) > 0$.

(H_2) For all $q \in \mathbb{I}_n$, $f_q, g_q \in C(\mathbb{H}, \mathbb{H})$, and there exist constants L_q^f, L_q^g such that

$$\|f_q(x) - f_q(y)\|_{\mathbb{H}} \le L_q^f \|x - y\|_{\mathbb{H}}, \qquad \|g_q(x) - g_q(y)\|_{\mathbb{H}} \le L_q^g \|x - y\|_{\mathbb{H}}$$

for all $x, y \in \mathbb{H}$ and $f_q(0) = g_q(0) = 0$.

(H_3) For every pair of $p, q \in \mathbb{I}_n$, the kernel $k_{pq} \in C(\mathbb{R}, \mathbb{R}^+)$ and satisfies $\int_0^{+\infty} k_{pq}(s)\,ds = 1$.

(H_4) $K = \max_{1\le p\le n}\{\frac{1}{a_p^-}\sum_{q=1}^n [b_{pq}^+ L_q^f + c_{pq}^+ L_q^g]\} < 1$.

3 The existence of pseudo almost automorphic solutions

Before stating and proving our existence theorem, we first prove two lemmas.

Lemma 7 *Assume that assumptions (H_1)–(H_3) hold and $x_q \in PAA(\mathbb{R}, \mathbb{H})$ for all $q \in \mathbb{I}_n$, then for every pair of $p, q \in \mathbb{I}_n$, the function $\varphi_p : t \to \int_{-\infty}^t k_{pq}(t-s) g_q(x_q(s))\,ds$ belongs to $PAA(\mathbb{R}, \mathbb{H})$.*

Proof Because $x_q \in PAA(\mathbb{R}, \mathbb{H})$, so $x_q \in BC(\mathbb{R}, \mathbb{H})$. Since

$$\|\varphi_p(t)\|_{\mathbb{H}} \le \int_{-\infty}^t \|k_{pq}(t-s) g_q(x_q(s))\,ds\|_{\mathbb{H}} \le L_q^g \|x_q\| \int_0^{+\infty} k_{pq}(s)\,ds = L_q^g \|x_q\|,$$

we see that the integral $\int_{-\infty}^t k_{pq}(t-s) g_q(x_q(s))\,ds$ is absolutely convergent and the function φ_p is bounded. In addition, it is easy to show that φ_p is continuous. Hence, $\varphi_p \in BC(\mathbb{R}, \mathbb{H})$.

Now, we prove that $\varphi_p \in PAA(\mathbb{R}, \mathbb{H})$.

By Lemma 6, we have $g_q(x_q(\cdot)) \in PAA(\mathbb{R}, \mathbb{H})$. Hence, we can write $g_q(x_q(t)) = u_q(t) + v_q(t)$, where $u_q \in AA(\mathbb{R}, \mathbb{H})$ and $v_q \in AA_0(\mathbb{R}, \mathbb{H})$. Consequently,

$$\begin{aligned}\varphi_p(t) &= \int_{-\infty}^t k_{pq}(t-s) u_q(s)\,ds + \int_{-\infty}^t k_{pq}(t-s) v_q(s)\,ds \\ &:= \varphi_p^1(t) + \varphi_p^0(t).\end{aligned}$$

Step 1. We will prove that $\varphi_p^1 \in AA(\mathbb{R}, \mathbb{H})$. Let (s_n') be a sequence of real numbers, we can extract a subsequence (s_n) of (s_n') such that

$$\lim_{n\to+\infty} u_q(t + s_n) = \bar{u}_q(t) \quad \text{and} \quad \lim_{n\to+\infty} \bar{u}_q(t - s_n) = u_q(t)$$

for each $t \in \mathbb{R}$. Denote

$$\bar{\varphi}_p^1(t) = \int_{-\infty}^t k_{pq}(t-s) \bar{u}_q(s)\,ds.$$

Then we have

$$\begin{aligned}&\|\varphi_p^1(t + s_n) - \bar{\varphi}_p^1(t)\|_{\mathbb{H}} \\ &\quad = \left\| \int_{-\infty}^{t+s_n} k_{pq}(t + s_n - s) u_q(s)\,ds - \int_{-\infty}^t k_{pq}(t-s) \bar{u}_q(s)\,ds \right\|_{\mathbb{H}} \\ &\quad \le \int_{-\infty}^t k_{pq}(t-s) \|u_q(s + s_n) - \bar{u}_q(s)\|_{\mathbb{H}}\,ds.\end{aligned}$$

By the Lebesgue dominated convergence theorem, we obtain

$$\lim_{n\to+\infty}\varphi_p^1(t+s_n)=\bar{\varphi}_p^1(t)$$

for each $t\in\mathbb{R}$. Similarly, we can obtain

$$\lim_{n\to+\infty}\bar{\varphi}_p^1(t-s_n)=\varphi_p^1(t)$$

for each $t\in\mathbb{R}$, which implies that $\varphi_p^1\in AA(\mathbb{R},\mathbb{H})$.

Step 2. We will prove that $\varphi_p^0\in AA_0(\mathbb{R},\mathbb{H})$. Since

$$\begin{aligned}
&\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\|\varphi_p^0(t)\|_{\mathbb{H}}\,dt\\
&\quad\leq\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\int_0^{+\infty}k_{pq}(\delta)\|v_q(t-\delta)\|_{\mathbb{H}}\,d\delta\,dt\\
&\quad\leq\int_0^{+\infty}k_{pq}(\delta)\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\|v_q(t-\delta)\|_{\mathbb{H}}\,dt\,d\delta=0,
\end{aligned}$$

$\varphi_p^0\in AA_0(\mathbb{R},\mathbb{H})$. Consequently, $\varphi_p(\cdot)\in PAA(\mathbb{R},\mathbb{H})$. The proof is complete. □

Let $\mathbb{X}=PAA(\mathbb{R},\mathbb{H}^n)$, then $(\mathbb{X},\|\cdot\|_0)$ is a Banach space, where $\|x\|_0=\sup_{t\in\mathbb{R}}\|x(t)\|_{\mathbb{H}^n}$ for $x\in\mathbb{X}$. Let

$$\begin{aligned}
\varphi^0(t)={}&\biggl(\int_{-\infty}^{t}e^{\int_s^t-a_1(u)\,du}Q_1(s)\,ds,\int_{-\infty}^{t}e^{\int_s^t-a_2(u)\,du}Q_2(s)\,ds,\\
&\ldots,\int_{-\infty}^{t}e^{\int_s^t-a_n(u)\,du}Q_n(s)\,ds\biggr)^T
\end{aligned}$$

and take a constant $\bar{\omega}>\|\varphi^0\|_0$.

Lemma 8 *Let* (H_1)–(H_3) *hold. For every* $\varphi=(\varphi_1,\varphi_2,\ldots,\varphi_n)^T\in PAA(\mathbb{R},\mathbb{H}^n)$ *and* $p\in\mathbb{I}_n$, *we have*

$$\begin{aligned}
(\Lambda_p\varphi)(t)={}&\int_{-\infty}^{t}e^{\int_s^t-a_p(u)\,du}\Biggl[\sum_{q=1}^{n}b_{pq}(s)f_q\bigl(\varphi_q(s)\bigr)\\
&+\sum_{q=1}^{n}c_{pq}(s)\int_{-\infty}^{s}k_{pq}(s-u)g_q\bigl(x_q(u)\bigr)\,du+Q_p(s)\Biggr]\,ds
\end{aligned}$$

is pseudo almost automorphic.

Proof By (H_1)–(H_3), according to Lemmas 5–7, for every $\varphi=(\varphi_1,\varphi_2,\ldots,\varphi_n)^T\in PAA(\mathbb{R},\mathbb{H}^n)$, we have that, for each $p\in\mathbb{I}_n$,

$$\begin{aligned}
\Upsilon_p(t):={}&\sum_{q=1}^{n}b_{pq}(t)f_q\bigl(\varphi_q(t)\bigr)+\sum_{q=1}^{n}c_{pq}(t)\int_{-\infty}^{t}k_{pq}(t-s)g_q\bigl(x_q(s)\bigr)\,ds\\
&+Q_p(t)\in AA(\mathbb{R},\mathbb{H})
\end{aligned}$$

is pseudo almost automorphic. Consequently, for every $p \in \mathbb{I}_n$, Υ_p can be expressed as $\Upsilon_p = \Upsilon_p^1 + \Upsilon_p^0$, where $\Upsilon_p^1 \in AA(\mathbb{R},\mathbb{H})$, $\Upsilon_p^0 \in AA_0(\mathbb{R},\mathbb{H})$. So

$$\begin{aligned}(\Lambda_p\varphi)(t) &= \int_{-\infty}^{t} e^{\int_s^t -a_p(u)\,du}\Upsilon_p^1(s)\,ds + \int_{-\infty}^{t} e^{\int_s^t -a_p(u)\,du}\Upsilon_p^0(s)\,ds \\ &= \big(\Lambda_p^1\varphi\big)(t) + \big(\Lambda_p^0\varphi\big)(t).\end{aligned}$$

Step 1. We will prove $\Lambda_p^1\varphi \in AA(\mathbb{R},\mathbb{H})$. Let $(s_n')_{n\in\mathbb{N}}$ be a sequence of real numbers, we can extract a subsequence $(s_n)_{n\in\mathbb{N}}$ of $(s_n')_{n\in\mathbb{N}}$ such that, for every $t \in \mathbb{R}$ and $p \in \mathbb{I}_p$,

$$\lim_{n\to+\infty} a_p(t+s_n) = \bar{a}_p(t), \qquad \lim_{n\to+\infty} \bar{a}_p(t-s_n) = a_p(t)$$

and

$$\lim_{n\to+\infty} \Upsilon_p^1(t+s_n) = \bar{\Upsilon}_p^1(t), \qquad \lim_{n\to+\infty} \bar{\Upsilon}_p^1(t-s_n) = \Upsilon_p^1(t).$$

Set

$$\big(\bar{\Lambda}_p^1\varphi\big)(t) = \int_{-\infty}^{t} e^{-\int_s^t \bar{a}_p(u)\,du}\bar{\Upsilon}_p^1(s)\,ds,$$

then we have

$$\begin{aligned}&\lim_{n\to+\infty}\big\|\big(\Lambda_p^1\varphi\big)(t+s_n) - \big(\bar{\Lambda}_p^1\varphi\big)(t)\big\|_{\mathbb{H}} \\ &\quad= \lim_{n\to+\infty}\bigg\|\int_{-\infty}^{t+s_n} e^{-\int_s^{t+s_n} a_p(u)\,du}\Upsilon_p^1(s)\,ds - \int_{-\infty}^{t} e^{-\int_s^t \bar{a}_p(u)\,du}\bar{\Upsilon}_p^1(s)\,ds\bigg\|_{\mathbb{H}} \\ &\quad= \lim_{n\to+\infty}\bigg\|\int_{-\infty}^{t} e^{-\int_u^t a_p(\delta+s_n)\,d\delta}\Upsilon_p^1(u+s_n)\,du - \int_{-\infty}^{t} e^{-\int_s^t \bar{a}_p(\delta)\,d\delta}\bar{\Upsilon}_p^1(s)\,ds\bigg\|_{\mathbb{H}} \\ &\quad\le \lim_{n\to+\infty}\int_{-\infty}^{t} e^{-\int_u^t a_p(\delta+s_n)\,d\delta}\big\|\big(\Upsilon_p^1(s+s_n) - \bar{\Upsilon}_p^1(s)\big)\big\|_{\mathbb{H}}\,ds \\ &\qquad+ \lim_{n\to+\infty}\bigg\|\int_{-\infty}^{t}\big(e^{-\int_u^t a_p(\delta+s_n)\,d\delta} - e^{-\int_u^t \bar{a}_p(\delta)\,d\delta}\big)\bar{\Upsilon}_p^1(s)\,ds\bigg\|_{\mathbb{H}}.\end{aligned}$$

By the Lebesgue dominated convergence theorem, we obtain that $\lim_{n\to+\infty}(\Lambda_p^1\varphi)(t+s_n) = (\bar{\Lambda}_p^1\varphi)(t)$ for each $t \in \mathbb{R}$ and $p \in \mathbb{I}_p$. Similarly, we can prove that $\lim_{n\to+\infty}(\bar{\Lambda}_p^1\varphi)(t-s_n) = (\Lambda_p^1\varphi)(t)$ for each $t \in \mathbb{R}$ and $p \in \mathbb{I}_p$. Hence, the function $\Lambda_p^1\varphi \in AA(\mathbb{R},\mathbb{H})$.

Step 2. We will prove that $\Lambda_p^0\varphi \in AA_0(\mathbb{R},\mathbb{H})$. For all $p \in \mathbb{I}_p$, we have

$$\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}\big\|\big(\Lambda_p^0\varphi\big)(s)\,ds\big\|_{\mathbb{H}}\,dt \le \Omega_1 + \Omega_2,$$

where

$$\begin{aligned}\Omega_1 &= \lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}\int_{-T}^{t}\big\|e^{\int_s^t -a_p(u)\,du}\Upsilon_p^0(s)\big\|_{\mathbb{H}}\,ds\,dt, \\ \Omega_2 &= \lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}\int_{-\infty}^{-T}\big\|e^{\int_s^t -a_p(u)\,du}\Upsilon_p^0(s)\big\|_{\mathbb{H}}\,ds\,dt.\end{aligned}$$

Let $\zeta = t - s$, by Fubini's theorem one has

$$\begin{aligned}
&\frac{1}{2T}\int_{-T}^{T}\int_{-T}^{t}\big\|e^{\int_s^t -a_p(u)\,du}\Upsilon_p^0(s)\big\|_{\mathbb{H}}\,ds\,dt \\
&\quad\le \frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} e^{-(t-s)a_p^-}\big\|\Upsilon_p^0(s)\big\|_{\mathbb{H}}\,ds\,dt \\
&\quad= \frac{1}{2T}\bigg(\int_{-T}^{T}\int_{0}^{t+T} e^{-\zeta a_p^-}\big\|\Upsilon_p^0(t-\zeta)\big\|_{\mathbb{H}}\,d\zeta\bigg)\,dt \\
&\quad\le \frac{1}{2T}\bigg(\int_{-T}^{T}\int_{0}^{+\infty} e^{-\zeta a_p^-}\big\|\Upsilon_p^0(t-\zeta)\big\|_{\mathbb{H}}\,d\zeta\bigg)\,dt \\
&\quad= \int_{0}^{+\infty} e^{-\zeta a_p^-}\bigg(\frac{1}{2T}\int_{-T}^{T}\big\|\Upsilon_p^0(t-\zeta)\big\|_{\mathbb{H}}\,dt\bigg)\,d\zeta \\
&\quad\le \int_{0}^{+\infty} e^{-\zeta a_p^-}\bigg(\frac{T+\zeta}{T}\frac{1}{2(T+\zeta)}\int_{-T-\zeta}^{T-\zeta}\big\|\Upsilon_p^0(u)\big\|_{\mathbb{H}}\,du\bigg)\,d\zeta.
\end{aligned}$$

Since the function $\Upsilon_p^0 \in AA_0(\mathbb{R},\mathbb{H})$,

$$\lim_{T\to+\infty}\frac{1}{2T}\int_{-T-\zeta}^{T-\zeta}\big\|\Upsilon_p^0(u)\big\|_{\mathbb{H}}\,du = 0.$$

Consequently, by the Lebesgue dominated convergence theorem, we obtain

$$\varOmega_1 = \lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}\int_{-T}^{t}\big\|e^{\int_s^t -a_p(u)\,du}\Upsilon_p^0(s)\big\|_{\mathbb{H}}\,ds\,dt = 0.$$

On the other hand, since Υ_p^0 is bounded, we have

$$\begin{aligned}
\varOmega_2 &\le \lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}\int_{-\infty}^{-T} e^{-(t-s)a_p^-}\big\|\Upsilon_p^0(s)\big\|_{\mathbb{H}}\,ds\,dt \\
&= \lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}\int_{T+t}^{+\infty} e^{-\zeta a_p^-}\big\|\Upsilon_p^0(t-\zeta)\big\|_{\mathbb{H}}\,d\zeta\,dt \\
&\le \lim_{T\to+\infty}\frac{\sup_{t\in\mathbb{R}}\|\Upsilon_p^0(t)\|_{\mathbb{H}}}{2T}\int_{-T}^{T}\int_{T+t}^{+\infty} e^{-\zeta a_p^-}\,d\zeta\,dt \\
&= \lim_{T\to+\infty}\frac{\sup_{t\in\mathbb{R}}\|\Upsilon_p^0(t)\|_{\mathbb{H}}}{2T}\frac{1}{a_p^-}\int_{-T}^{T} e^{-(t+T)a_p^-}\,dt \\
&= \lim_{T\to+\infty}\frac{\sup_{t\in\mathbb{R}}\|\Upsilon_p^0(t)\|_{\mathbb{H}}}{2T}\frac{1}{(a_p^-)^2}\big[1-e^{-2a_p^- T}\big] = 0.
\end{aligned}$$

Hence, $\varLambda_p^0\varphi \in AA_0(\mathbb{R},\mathbb{H})$ for all $p \in \mathbb{I}_p$. Therefore, $\varLambda_p\varphi \in PAA(\mathbb{R},\mathbb{H})$. The proof is complete. □

Theorem 1 *Suppose* (H_1)–(H_4) *hold. Then system* (1) *has a pseudo almost automorphic solution that is contained in* $\mathbb{X}_0 = \{\varphi \mid \varphi \in \mathbb{X}, \|\varphi - \varphi^0\|_0 \le \frac{K\bar{\omega}}{1-K}\}$.

Proof Let $x = (x_1, x_2, \dots, x_n)^T \in C(\mathbb{R}, \mathbb{H}^n)$ satisfy

$$\begin{aligned} x_p(t) &= \int_{-\infty}^{t} e^{\int_s^t -a_p(u)\,du} \Bigg[\sum_{q=1}^{n} b_{pq}(s) f_q\big(x_q(s)\big) \\ &\quad + \sum_{q=1}^{n} c_{pq}(s) \int_{-\infty}^{s} k_{pq}(s-u) g_q\big(x_q(u)\big)\,du + Q_p(s) \Bigg]\,ds, \quad p \in \mathbb{I}_n, \end{aligned} \tag{2}$$

then we can deduce that

$$\begin{aligned} \dot{x}_p(t) &= \int_{-\infty}^{t} -a_p(t) e^{\int_s^t -a_p(u)\,du} \Bigg[\sum_{q=1}^{n} b_{pq}(s) f_q\big(x_q(s)\big) \\ &\quad + \sum_{q=1}^{n} c_{pq}(s) \int_{-\infty}^{s} k_{pq}(s-u) g_q\big(x_q(u)\big)\,du + Q_p(s) \Bigg]\,ds \\ &\quad + e^{-\int_t^t a_p(u)\,du} \Bigg[\sum_{q=1}^{n} b_{pq}(s) f_q\big(x_q(s)\big) \\ &\quad + \sum_{q=1}^{n} c_{pq}(s) \int_{-\infty}^{s} k_{pq}(s-u) g_q\big(x_q(u)\big)\,du + Q_p(s) \Bigg] \\ &= -a_p(t)x_p(t) + \sum_{q=1}^{n} b_{pq}(t) f_q\big(x_q(t)\big) + \sum_{q=1}^{n} c_{pq}(t) \int_{-\infty}^{t} k_{pq}(t-s) g_q\big(x_q(s)\big)\,ds + Q_p(t), \end{aligned}$$

that is, x satisfies system (1).

Define an operator $T : \mathbb{X}_0 \to AA(\mathbb{R}, \mathbb{H}^n)$ by

$$T = (T_1, T_2, \dots, T_n)^T,$$

where, for any $\varphi \in AA(\mathbb{R}, \mathbb{H}^n)$ and $p \in \mathbb{I}_n$,

$$\begin{aligned} (T_p\varphi)(t) &= \int_{-\infty}^{t} e^{\int_s^t -a_p(u)\,du} \Bigg[\sum_{q=1}^{n} b_{pq}(s) f_q\big(\varphi_q(s)\big) \\ &\quad + \sum_{q=1}^{n} c_{pq}(s) \int_{-\infty}^{s} k_{pq}(s-u) g_q\big(x_q(u)\big)\,du + Q_p(s) \Bigg]\,ds. \end{aligned}$$

Obviously, for any $\varphi \in \mathbb{X}_0$, we have

$$\|\varphi\|_0 \le \|\varphi - \varphi_0\|_0 + \|\varphi_0\|_0 \le \frac{K\bar{\omega}}{1-K} + \bar{\omega} = \frac{\bar{\omega}}{1-K}.$$

Step 1. We prove that for every $\varphi \in \mathbb{X}_0$, $T\varphi \in \mathbb{X}_0$. Since

$$\begin{aligned} &\big\| T\varphi(t) - \varphi^0(t) \big\|_{\mathbb{H}^n} \\ &\quad \le \max_{1 \le p \le n} \Bigg\{ \int_{-\infty}^{t} \Bigg\| e^{\int_s^t -a_p(u)\,du} \sum_{q=1}^{n} b_{pq}(s) f_q\big(\varphi_q(s)\big) \Bigg\|_{\mathbb{H}} \,ds \end{aligned}$$

$$
\begin{aligned}
&\quad + \int_{-\infty}^{t} \left\| e^{\int_s^t -a_p(u)\,du} \sum_{q=1}^{n} c_{pq}(s) \int_{-\infty}^{s} k_{pq}(s-v) g_q\bigl(x_q(v)\bigr)\,dv \right\|_{\mathbb{H}} ds \\
&\leq \max_{1\leq p\leq n} \left\{ \sum_{q=1}^{n} \left[\int_{-\infty}^{t} e^{-a_p^-(t-s)} b_{pq}^+ L_q^f \|\varphi\|_0\,ds + \int_{-\infty}^{t} e^{-a_p^-(t-s)} c_{pq}^+ L_q^g \|\varphi\|_0\,ds \right] \right\} \\
&\leq \max_{1\leq p\leq n} \frac{1}{a_p^-} \left\{ \sum_{q=1}^{n} \bigl[b_{pq}^+ L_q^f \|\varphi\|_0 + c_{pq}^+ L_q^g \|\varphi\|_0 \bigr] \right\} \\
&\leq \frac{\bar{\omega}}{1-K} \max_{1\leq p\leq n} \left\{ \frac{1}{a_p^-} \sum_{q=1}^{n} \bigl[b_{pq}^+ L_q^f + c_{pq}^+ L_q^g \bigr] \right\} \\
&\leq \frac{K\bar{\omega}}{1-K},
\end{aligned}
$$

which implies that $T\varphi \in \mathbb{X}_0$.

Step 2. We will prove that the mapping T is a contraction mapping of $\mathbb{X}_0$. For any $\varphi, \phi \in \mathbb{X}_0$, we have

$$
\begin{aligned}
&\bigl\| T\varphi(t) - T\psi(t) \bigr\|_{\mathbb{H}^n} \\
&\leq \max_{1\leq p\leq n} \left\{ \int_{-\infty}^{t} \left\| e^{\int_s^t -a_p(u)\,du} \sum_{q=1}^{n} b_{pq}(s) \bigl[f_q\bigl(\varphi_q(s)\bigr) - f_q\bigl(\psi_q(s)\bigr) \bigr] \right\|_{\mathbb{H}} ds \right. \\
&\quad \left. + \int_{-\infty}^{t} \left\| e^{\int_s^t -a_p(u)\,du} \sum_{q=1}^{n} c_{pq}(s) \int_{-\infty}^{s} k_{pq}(s-v) \bigl[g_q(\varphi_q(v) - g_q(\psi_q(v)) \bigr]\,dv \right\|_{\mathbb{H}} ds \right\} \\
&\leq \max_{1\leq p\leq n} \left\{ \int_{-\infty}^{t} e^{-a_p^-(t-s)} \sum_{q=1}^{n} b_{pq}^+ L_q^f \|\varphi - \psi\|_0\,ds \right. \\
&\quad \left. + \int_{-\infty}^{t} e^{-a_p^-(t-s)} \sum_{q=1}^{n} c_{pq}^+ L_q^g \|\varphi - \psi\|_0 \right\} \\
&\leq \max_{1\leq p\leq n} \left\{ \frac{1}{a_p^-} \sum_{q=1}^{n} \bigl[b_{pq}^+ L_q^f + c_{pq}^+ L_q^g \bigr] \right\} \|\varphi - \psi\|_0 \\
&= K\|\varphi - \psi\|_0,
\end{aligned}
$$

which means that the mapping T is a contracting mapping. Therefore, there exists a unique fixed point $\varphi^* \in \mathbb{X}_0$ such that $T\varphi^* = \varphi^*$, that is, system (1) has a pseudo almost automorphic solution. The proof is complete. □

4 Global exponential stability

In this section, for $x = (x_1, x_2, \ldots, x_n)^T \in C((-\infty, 0], \mathbb{H}^n)$, we denote

$$
\|x\|_\tau = \sum_{p=1}^{n} \sup_{t\in(-\infty,0]} \bigl\| x_p(t) \bigr\|_{\mathbb{H}}.
$$

Definition 3 Let $x = (x_1, x_2, \ldots, x_n)^T$ be a pseudo almost automorphic solution of system (1) with the initial value $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_n)^T \in C((-\infty, 0], \mathbb{H}^n)$ and $y = (y_1, y_2, \ldots, y_n)^T$

be an arbitrary solution of system (1) with the initial value $\psi = (\psi_1, \psi_2, \ldots, \psi_n)^T \in C((-\infty, 0], \mathbb{H}^n)$, respectively. If there exist positive constants η and M such that

$$\left|x(t) - y(t)\right|_{\mathbb{H}^n} \leq M\|\varphi - \psi\|_\tau e^{-\eta t}, \quad t \geq 0,$$

then the pseudo almost automorphic solution x of system (1) is said to be globally exponentially stable.

Theorem 2 *Assume that* (H_1)*–*(H_4) *hold, and suppose further that there exists a positive constant* λ_0 *such that*

$$\int_0^{+\infty} k_{pq}(s)e^{\lambda_0 s}\,ds < +\infty.$$

Then system (1) *has a pseudo almost automorphic solution that is globally exponentially stable.*

Proof By Theorem 1, system (1) has a pseudo almost automorphic solution, let $x(t)$ be the pseudo almost automorphic solution with the initial value $\varphi(t)$, and $y(t)$ be an arbitrary solution with the initial value $\psi(t)$. Set $z_p(t) = y_p(t) - x_p(t)$, $\phi_p(t) = \psi_p(t) - \varphi_p(t)$, we have

$$\begin{aligned} &\dot{z}_p(t) + a_p(t)z_p(t) \\ &\quad = \sum_{q=1}^n b_{pq}(t)\big[f_q\big(z_q(t)\big) + x_q(t)) - f_q\big(x_q(t)\big)\big] \\ &\qquad + \sum_{q=1}^n c_{pq}(t)\int_{-\infty}^t k_{pq}(t-s)\big[g_q\big(z_p(s) + x_q(s)\big) - g_q\big(x_q(s)\big)\big]\,ds, \end{aligned} \tag{3}$$

where $p \in \mathbb{I}_n$. Let Θ_p be defined by

$$\Theta_p(\omega) = a_p^- - \omega - \sum_{q=1}^n \left[b_{pq}^+ L_q^f + c_{pq}^+ L_q^g \int_0^{+\infty} k_{pq}(s)e^{\omega s}\,ds\right],$$

where $p \in \mathbb{I}_n$, $\omega \in [0, +\infty)$ and $\Theta_p(\omega) \to -\infty; \omega \to +\infty$, there exist $\varepsilon_p^* > 0$ such that $\Theta_p(\varepsilon_p) > 0$ for $\varepsilon_p \in (0, \varepsilon_p^*)$. Let $\eta = \min\{\varepsilon_1^*, \varepsilon_2^*, \ldots, \varepsilon_n^*\}$, we obtain

$$\Theta_p(\eta) \geq 0, \quad p = 1, 2, \ldots, n.$$

So we can take a positive constant λ satisfying $0 < \lambda < \min\{\eta, a_1^-, a_2^-, \ldots, a_n^-, \lambda_0\}$ such that $\Theta_p(\lambda) > 0$, which implies that, for $p = 1, 2, \ldots, n$,

$$\frac{1}{a_p^- - \lambda}\sum_{q=1}^n \left[b_{pq}^+ L_q^f + c_{pq}^+ L_q^g \int_0^{+\infty} k_{pq}(s)e^{\lambda s}\,ds\right] < 1. \tag{4}$$

Multiplying both sides of (3) by $e^{\int_0^s a_p(u)\,du}$ and integrating on $[0,t]$, we have

$$\begin{aligned} z_p(t) = \phi_p(0)e^{-\int_0^t a_p(u)\,du} + \int_0^t e^{-\int_s^t a_p(u)\,du} \\ \times \Bigg[\sum_{q=1}^n b_{pq}(t)\big[f_q\big(z_q(t)\big) + x_q(t)\big) - f_q\big(x_q(t)\big)\big] \\ + \sum_{q=1}^n c_{pq}(t)\int_{-\infty}^s k_{pq}(s-\mu)\big[g_q\big(z_p(\mu) + x_q(\mu)\big) - g_q\big(x_q(\mu)\big)\big]\,d\mu\Bigg]\,ds. \end{aligned} \tag{5}$$

Let

$$M = \max_{1\le p\le n}\left(\frac{a_p^-}{\sum_{q=1}^n [b_{pq}^+ L_q^f + c_{pq}^+ L_q^g]}\right).$$

In view of (H_4), $M > 1$, and we can deduce that

$$\left(\frac{1}{M} - \frac{1}{a_p^- - \lambda}\sum_{q=1}^n\left[b_{pq}^+ L_q^f + c_{pq}^+ L_q^g\int_0^{+\infty} k_{pq}(s)e^{\lambda s}\,ds\right]\right) < 0. \tag{6}$$

It is easy to see that

$$\|z(t)\|_0 = \|\phi(t)\|_\tau \le \|\phi\|_\tau \le M\|\phi\|_\tau e^{-\lambda t}, \quad t\in(-\infty,0].$$

We claim that

$$\|z(t)\|_0 \le M\|\phi\|_\tau e^{-\lambda t}, \quad t > 0. \tag{7}$$

To prove (7), we show for any $\xi > 1$ that the following inequality holds:

$$\|z(t)\|_0 \le \xi M\|\phi\|_\tau e^{-\lambda t}, \quad t > 0. \tag{8}$$

If (8) is false, then there must be some $t_1 > 0$ and some $p \in \{1,2,\dots,n\}$ such that

$$\|z(t_1)\|_0 = \|z_p(t_1)\|_0 = \xi M\|\phi\|_\tau e^{-\lambda t_1}, \quad t > 0 \tag{9}$$

and

$$\|z(t)\|_0 < \xi M\|\phi\|_\tau e^{-\lambda t}, \quad t\in(-\infty,t_1). \tag{10}$$

By (4), (5), (6), (10), and (H_3), we have

$$\begin{aligned} \|z_p(t_1)\|_0 \le \|\phi_p\|_\tau e^{-t_1 a_p^-} + \int_0^{t_1} e^{-(t_1-s)a_p^-}\Bigg[\sum_{q=1}^n b_{pq}^+ L_q^f\|z_q(s)\|_{\mathbb{H}} \\ + \sum_{q=1}^n c_{pq}^+\int_{-\infty}^s k_{pq}(s-\mu)L_q^g\|z_q(s)\|_{\mathbb{H}}\,d\mu\Bigg]\,ds \end{aligned}$$

$$
\begin{aligned}
&\le \|\phi_p\|_\tau e^{-t_1 a_p^-} + \int_0^{t_1} e^{-(t_1-s)a_p^-}\Bigg[\sum_{q=1}^n b_{pq}^+ L_q^f \xi M\|\phi\|_\tau e^{-\lambda s} \\
&\quad + \sum_{q=1}^n c_{pq}^+ \int_{-\infty}^s k_{pq}(\mu) L_q^g \xi M\|\phi\|_\tau e^{-\lambda(s-\mu)}\,d\mu\Bigg]\,ds \\
&\le \|\phi_p\|_\tau e^{-t_1 a_p^-} + \int_0^{t_1} e^{-(t_1-s)a_p^-}\xi M\|\phi\|_\tau e^{-\lambda s}\Bigg[\sum_{q=1}^n b_{pq}^+ L_q^f \\
&\quad + \sum_{q=1}^n c_{pq}^+ \int_{-\infty}^s k_{pq}(\mu) L_q^g e^{\lambda\mu}\,d\mu\Bigg]\,ds \\
&\le \xi M\|\phi\|_\tau e^{-\lambda t_1}\Bigg[\frac{e^{(\lambda-a_p^-)t_1}}{\xi M} + \frac{1}{a_p^- - \lambda}\Bigg(\sum_{q=1}^n b_{pq}^+ L_q^f \\
&\quad + \sum_{q=1}^n d_{pq}^+ \int_{-\infty}^s k_{pq}(\mu) L_q^g e^{\lambda\mu}\,d\mu\big(1-e^{(\lambda-a_p^-)t_1}\big)\Bigg)\Bigg] \\
&\le \xi M\|\phi\|_\tau e^{-\lambda t_1}\Bigg[e^{(\lambda-a_p^-)t_1}\Bigg(\frac{1}{M} - \frac{1}{a_p^- - \lambda}\sum_{q=1}^n\Bigg[b_{pq}^+ L_q^f \\
&\quad + c_{pq}^+ L_q^g \int_0^{+\infty} k_{pq}(s) e^{\lambda s}\,ds\Bigg]\Bigg) \\
&\quad + \Bigg(\frac{1}{a_p^- - \lambda}\Bigg[b_{pq} L_q^f + c_{pq} L_q^g \int_0^{+\infty} k_{pq}(s) e^{\lambda s}\,ds\Bigg]\Bigg)\Bigg] \\
&\le \xi M\|\phi\|_\tau e^{-\lambda t_1}\Bigg[\frac{1}{a_p^- - \lambda}\sum_{q=1}^n\Bigg[b_{pq}^+ L_q^f + c_{pq}^+ L_q^g \int_0^{+\infty} k_{pq}(s) e^{\lambda s}\,ds\Bigg]\Bigg] \\
&< \xi M\|\phi\|_\tau e^{-\lambda t_1},
\end{aligned}
$$

which contradicts equality (9), and so (8) holds. Let $\xi \to 1$, then (7) holds. Hence, the pseudo almost automorphic solution of (1) is globally exponentially stable. The proof is completed. □

5 Example

In this section, we give an example to show the feasibility of our obtained results in this paper.

Example 1 In system (1), let $n = 2$, $x_p(t) = x_p^R(t) + i x_p^I(t) + j x_p^J(t) + k x_p^K(t) \in \mathbb{H}$, $k_{pq}(t) = e^{-t}$, and take

$$
\begin{aligned}
f_q(x_q) &= \frac{1}{40}\sin\big(x_q^R + x_q^I\big) + i\frac{1}{50}\sin\big(x_q^I - x_q^K\big) \\
&\quad + j\frac{1}{64}\arctan\big(x_q^R - 2x_q^J\big) + k\frac{1}{48}\sin\big(x_q^J - x_q^K\big), \\
g_q(x_q) &= \frac{1}{125}\sin\big(x_q^R - 3x_q^J\big) + i\frac{1}{200}\sin\big(x_q^I - x_q^J\big) \\
&\quad + j\frac{1}{175}\arctan\big(2x_q^R - x_q^J\big) + k\frac{1}{130}\sin\big(x_q^R + x_q^K\big),
\end{aligned}
$$

$$\begin{pmatrix} a_1(t) \\ a_2(t) \end{pmatrix} = \begin{pmatrix} 2.3 + 0.1 \sin\sqrt{2}t \\ 2.5 + 0.2\cos\sqrt{5}t \end{pmatrix},$$

$$\begin{aligned} &\begin{pmatrix} b_{11}(t) & b_{12}(t) \\ b_{21}(t) & b_{22}(t) \end{pmatrix} \\ &\quad = \begin{pmatrix} \frac{1}{100+t^2} + i0.06\sin t & \frac{1}{25+t^2} + k0.1\sin t \\ 0.15 - i0.11\cos t + k0.2\cos\sqrt{3}t + \frac{1}{100+t^2} & \frac{1}{16+t^2} + j0.2\sin\sqrt{3}t \end{pmatrix}, \\ &\begin{pmatrix} c_{11}(t) & c_{12}(t) \\ c_{21}(t) & c_{22}(t) \end{pmatrix} \\ &\quad = \begin{pmatrix} \frac{1}{16+t^2} + 0.1\sin\sqrt{7}t + i0.02\sin\sqrt{3}t & 0.13 + k0.1\sin\sqrt{11}t \\ \frac{1}{100+t^2} + 0.15 - i0.01\cos\sqrt{6}t + j0.02\cos\sqrt{5}t & 0.11 + k0.2\sin t \end{pmatrix}, \\ &\begin{pmatrix} Q_1(t) \\ Q_2(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\sin\sqrt{5}t + i\frac{1}{12}\sin\sqrt{8}t + j\frac{1}{8}\cos\sqrt{5}t + k\frac{1}{15}\sin t + \frac{3}{1+t^2} \\ \frac{1}{5}\sin\sqrt{2}t + i\frac{1}{12}\sin\sqrt{7}t + j\frac{1}{7}\cos t + k\frac{1}{10}\sin\sqrt{7}t - \frac{\sin t}{1+t^2} \end{pmatrix}. \end{aligned}$$

By computing, we have $L_q^f = \frac{1}{20}$, $L_q^g = \frac{1}{25}$, $a_1^- = 2.2$, $a_2^- = 2.3$, $b_{11}^+ = 0.061$, $b_{12}^+ = 0.108$, $b_{21}^+ = 0.274$, $b_{22}^+ = 0.210$, $c_{11}^+ = 0.120$, $c_{12}^+ = 0.130$, $c_{21}^+ = 0.153$, $c_{22}^+ = 0.228$. So (H_1) and (H_2) are satisfied. Besides, it is easy to obtain that

$$\max_{1\le p\le 2}\left\{\frac{1}{a_p^-}\left[\sum_{q=1}^{2} b_{pq}^+ L_q^f + \sum_{q=1}^{2} .c_{pq}^+ L_q^g\right]\right\} \approx 0.024 < 1.$$

Therefore, all of the conditions of Theorem 2 are satisfied. Thus, according to Theorem 2, system (1) has a pseudo almost automorphic solution that is globally exponentially stable (see Figs. 1–3).

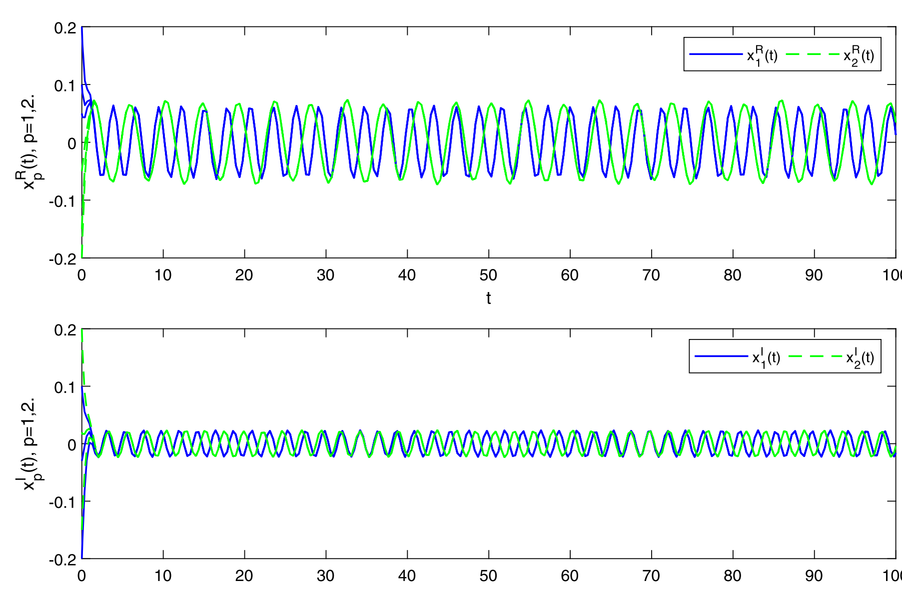

Figure 1 Curves of $x_p^R(t)$ and $x_p^I(t)$, $p = 1, 2$

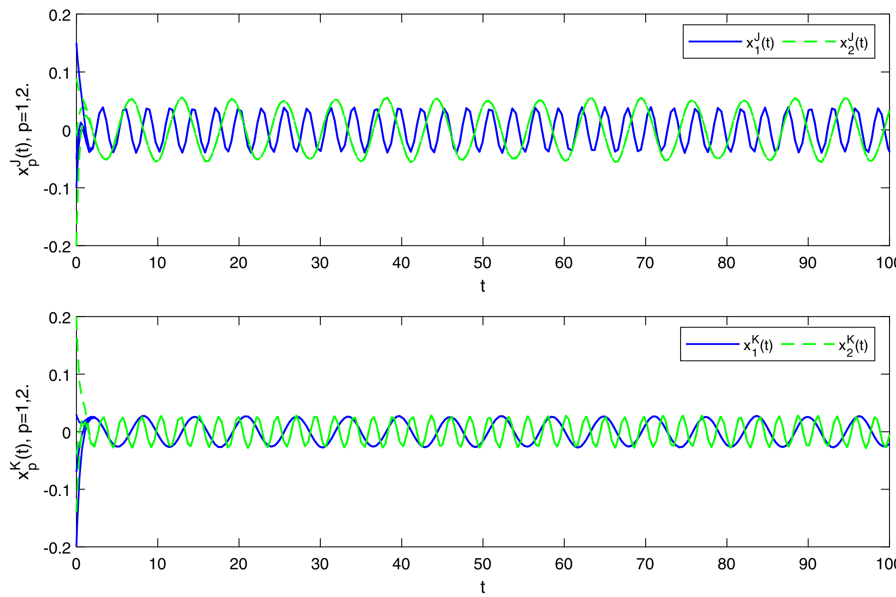

Figure 2 Curves of $x_p^J(t)$ and $x_p^K(t)$, $p = 1, 2$

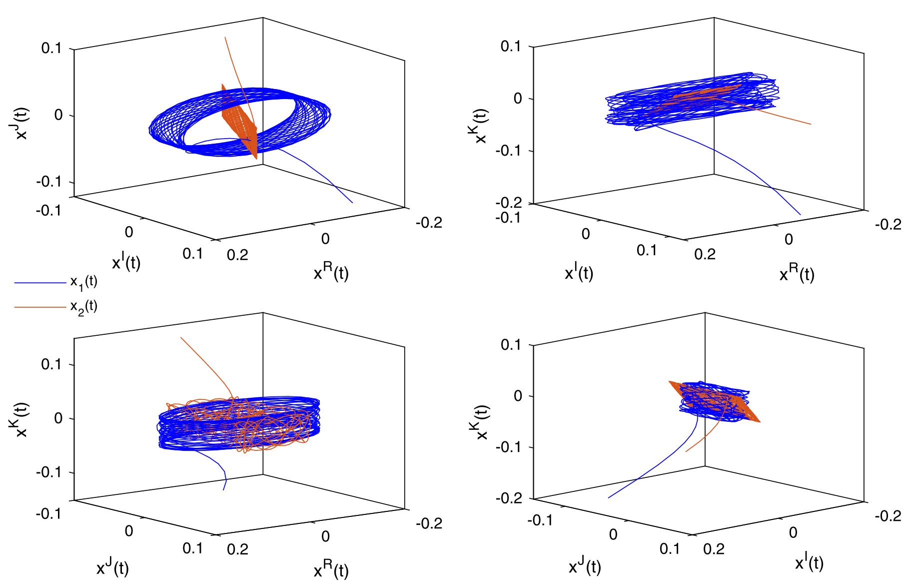

Figure 3 Curves of $x_p^R(t)$, $x_p^I(t)$, $x_p^J(t)$, and $x_p^K(t)$ in 3-dimensional space for stable case, $p = 1, 2$

6 Conclusion

In this paper, we have obtained the existence and global exponential stability of pseudo almost automorphic solutions to quaternion-valued cellular neural networks with infinitely distributed delays via direct method. Our method and the results of this paper are new, and our method can be used to study the existence and stability of almost periodic solutions, pseudo almost periodic solutions, almost automorphic solutions, and pseudo almost automorphic solutions for other types of quaternion-valued neural network models.

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Authors' contributions
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