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Variance-constrained resilient H_∞ state estimation for time-varying neural networks with randomly varying nonlinearities and missing measurements

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Abstract

This paper addresses the resilient H_∞ state estimation problem under variance constraint for discrete uncertain time-varying recurrent neural networks with randomly varying nonlinearities and missing measurements. The phenomena of missing measurements and randomly varying nonlinearities are described by introducing some Bernoulli distributed random variables, in which the occurrence probabilities are known a priori. Besides, the multiplicative noise is employed to characterize the estimator gain perturbation. Our main purpose is to design a time-varying state estimator such that, for all missing measurements, randomly varying nonlinearities and estimator gain perturbation, both the estimation error variance constraint and the prescribed H_∞ performance requirement are met simultaneously by providing some sufficient criteria. Finally, the feasibility of the proposed variance-constrained resilient H_∞ state estimation method is verified by some simulations.

Keywords: Time-varying neural networks; Resilient state estimation; Randomly varying nonlinearities; Missing measurements; H_∞ performance; Variance constraint

1 Introduction

In the past two decades, the popularization of the Internet has greatly changed our way of life through the rapid communication ways [1–3]. Accordingly, many scholars have witnessed successful applications of recurrent neural networks (RNNs) in wide fields including pattern recognition, image processing, associate memory and optimization domains [4–6]. Nevertheless, it should be noted that the nonlinearities are commonly inherent characteristics between neurons, which indeed affect the understanding and analysis of the neural networks (NNs). Thus, some efficient methods have been given to analyze different NNs. For example, an effective finite-time synchronization criterion has been proposed in [7] for coupled stochastic NNs, where both the Markovian switching parameters and saturation have been addressed. Moreover, some useful state estimation algorithms have been given in [8] for delayed NNs to guarantee the H_∞ as well as passivity and in [9] for bidirectional associative NNs subject to mixed time-delays. During the analysis and implementation of the methods related to RNNs, it should be noticed that the neuron

states may not always available in reality, so there is a need to estimate them by utilizing effective estimation methods [10–12]. Until now, many results have been published with respect to the state estimation problem of different types of dynamical networks [13–16]. Nevertheless, it is worth mentioning that much published literature is only applicable to time-invariant case, which might lead to certain application limitations. Hence, more and more researchers have paid attention on the analysis of the state estimation problem for time-varying system and proposed a variety of methods to analyze the behaviors of dynamics systems, see e.g. [17–20] and the references therein. To be specific, an event-based joint input and state estimation strategy have been presented in [19] to ensure that the covariance of the estimation error has an upper bound at any time for the addressed linear discrete time-varying systems. Accordingly, there exist the demands on the development of efficient analysis methods for time-varying RNNs.

During the network communications or transmissions, the perfect measurements can not be always available, thus increasing research effort has been made on the state estimation problems against the missing measurements [20–22]. The missing measurements are inevitable primarily due to the signal interference during the transmission including limited communication channels, noise interferences and so forth. For example, the study of missing measurements has been conducted in the areas of principal component analysis and partial least squares models [23]. Actually, it is well recognized that the existence of the missing measurements would lead to poor system performance for the addressed NNs [24, 25]. Accordingly, a large number of results have been given to deal with the state estimation problem for dynamics networks subject to missing measurements [26–29]. For example, in [27, 28], the H_∞ state estimation design strategies have been proposed for discrete delayed neural networks, where the impacts from the multiple missing measurements have been discussed. Nevertheless, few results have been reported to handle the problem of state estimation for time-varying dynamics networks with missing measurements [20, 30]. Recently, a new optimal state estimation algorithm has been developed in [20] for time-varying complex networks, where the impacts from both the stochastic coupling and the missing measurements within the framework of uncertain occurrence probabilities have been addressed. However, it is worthwhile to notice that few scholars have studied the state estimation problem of time-varying neural networks with missing measurements, which constitutes one of research motivations.

Recently, the phenomena of randomly varying nonlinearities have been modeled and discussed in various fields [31, 32]. The existence of randomly varying nonlinearities has brought more influences in physics, engineering, information science and other fields, and hence the dynamics system analysis problems with randomly varying nonlinearities have already received increasing interest [33–35]. For example, the state estimation algorithm in distributed way has been established in [33] for a class of discrete systems over sensor networks with incomplete measurements and randomly varying nonlinearities, and the probability-dependent H_∞ synchronization condition has been presented in [35] for dynamics networks by designing new control method. During the implementation process, the presented estimation method would be fragile/non-resilient due to various reasons, such as the numerical rounding error, finite-word length effects and analog-to-digital conversion accuracy [36, 37]. Accordingly, the non-fragile/resilient estimation schemes have wide application fields as in water quality processing [38] and synchronous generators [39]. In [40, 41], the non-fragile H_∞ state estimation algorithms have been given for dynamical

systems with Markov switching effects, where the possible estimator parameter variations have been modeled and discussed. The resilient H_∞ filtering methods have been given in [42] for switched delayed neural networks with ensured finite-time criterion and in [43] for time-varying dynamics systems through sensor networks with communication protocols. However, few resilient H_∞ estimation methods can be available for time-varying RNNs. On the other hand, from the engineering-oriented viewpoint on the state estimation problems, it is of great practical significance to ensure the estimation performance has a certain upper bound of related to the estimated error covariance [44]. Unlike to the minimal estimation of error covariance, the variance-constrained state estimation strategy could provide a looser evaluation by introducing a specific upper bound constraint, which can reflect the admissible accuracy of the proposed estimation methods. Very recently, a new variance-constrained filtering algorithm with distributed feature has been established in [45] for time-varying sensor networks subject to deception attacks and state estimation scheme; see [46] for time-varying complex networks to attenuate the impacts induced by randomly varying topologies. As such, we make the first attempt to handle the H_∞ resilient estimation problem under variance constraint for *time-varying* RNNs with randomly varying nonlinearities and missing measurements.

Inspired by the above discussions, we aim to design the resilient H_∞ state estimation approach for the addressed discrete time-varying RNNs subject to missing measurements and randomly varying nonlinearities under the variance constraint. In particular, the estimator gain perturbation is considered by employing the Gaussian white noise. By resorting to the matrix theory and stochastic analysis technique, a new nonlinear time-varying state estimation method is proposed, which can ensure the error variance boundedness and prescribed H_∞ performance requirements simultaneously. From an engineering viewpoint, the proposed recursive state estimation scheme has a time-varying feature applicable for handling the estimation problems of neural networks, which is suitable for the online estimation applications. Moreover, some sufficient conditions characterized by the matrix inequalities are given to ensure two mixed performance indices, which can better achieve satisfactory disturbance attenuation level and the estimation covariance performance, thus performing wide application domains. The major features of the paper can be summarized as follows: (1) the H_∞ state estimation problem under variance constraint is, for the first time, investigated for a class of discrete *time-varying RNNs* subject to missing measurements and randomly varying nonlinearities; (2) a new probability-dependent *time-varying* state estimation algorithm is proposed, which can be implemented in terms of the solutions to certain matrix inequalities; (3) the impacts caused by the missing measurements and randomly varying nonlinearities onto two estimation performance indices are discussed and examined simultaneously; and (4) the proposed estimation algorithm has time-varying characteristic applicable for online applications, which performs new advantages compared with the existing estimation results for time-invariant neural networks.

Notations. The symbols used throughout the paper are fairly standard. \mathbb{R}^r denotes the r dimensional Euclidean space. \mathbb{N}^+ stands for the sets of positive integers. For the matrix A and the vector x , A^T and x^T represents the transpose A and x , respectively. The identity matrix is denoted by I and the zero matrix is denoted by 0 . $\mathbb{E}\{x\}$ means the mathematical expectation of x . $X > 0$ means that X is a positive-definite symmetric matrix. In symmetric block matrices, we use an asterisk $*$ to represent a term that is induced by symmetry, and

diag{...} stands for a block-diagonal matrix. It is assumed that the matrices have compatible dimensions if it is not explicitly specified.

2 Problem formulation and preliminaries

In this paper, we consider the n -neurons time-varying neural network given by

$$\begin{aligned}
 x_{k+1} &= (A_k + \Delta A_k)x_k + \alpha_k B_{1k}f(x_k) + (1 - \alpha_k)B_{2k}g(x_k) + C_k v_{1k}, \\
 y_k &= \lambda_k D_k x_k + v_{2k}, \\
 z_k &= M_k x_k,
 \end{aligned}
 \tag{1}$$

where $x_k = [x_{1,k} \ x_{2,k} \ \dots \ x_{n,k}]^T \in \mathbb{R}^n$ represents the state vector of neural network, $y_k \in \mathbb{R}^m$ denotes the measurement output, $z_k \in \mathbb{R}^r$ stands for the controlled output, $A_k = \text{diag}\{a_{1,k}, a_{2,k}, \dots, a_{n,k}\}$ is the state coefficient matrix, C_k , D_k and M_k are known real matrices with appropriate dimensions, $v_{1k} \in \mathbb{R}^{n_w}$ and $v_{2k} \in \mathbb{R}^m$ are Gaussian white noises with zero mean values and covariances $V_1 > 0$ and $V_2 > 0$, respectively. $B_{1k} = [b_{1ij}(k)]_{n \times n}$ and $B_{2k} = [b_{2ij}(k)]_{n \times n}$ are the connection weight matrices. $f(x_k) = [f_1(x_{1,k}) \ \dots \ f_n(x_{n,k})]^T$ and $g(x_k) = [g_1(x_{1,k}) \ \dots \ g_n(x_{n,k})]^T$ are the neuron activation functions. ΔA_k describes the parameter uncertainty satisfying

$$\Delta A_k = H_k F_k N_k,
 \tag{2}$$

where H_k and N_k are appropriately dimensional known matrices, the unknown matrix F_k satisfies

$$F_k^T F_k \leq I, \quad \forall k \in \mathbb{N}^+.
 \tag{3}$$

The Bernoulli distributed random variables α_k and λ_k are utilized to describe the phenomena of randomly varying nonlinearities and missing measurements, respectively, and satisfy

$$\begin{aligned}
 \text{Prob}\{\alpha_k = 1\} &= \mathbb{E}\{\alpha_k\} = \bar{\alpha}, & \text{Prob}\{\alpha_k = 0\} &= 1 - \bar{\alpha}, \\
 \text{Prob}\{\lambda_k = 1\} &= \mathbb{E}\{\lambda_k\} = \bar{\lambda}, & \text{Prob}\{\lambda_k = 0\} &= 1 - \bar{\lambda},
 \end{aligned}$$

where $\bar{\alpha} \in [0, 1]$ and $\bar{\lambda} \in [0, 1]$ are known constants.

Assumption 1 For the activation functions $f(\cdot)$ and $g(\cdot)$ with $f(0) = g(0) = 0$, there exist four scalars $\lambda_i^-, \lambda_i^+, \sigma_i^-$ and σ_i^+ satisfying the following conditions:

$$\begin{aligned}
 \lambda_i^- &\leq \frac{f_i(s_1) - f_i(s_2)}{s_1 - s_2} \leq \lambda_i^+, \\
 \sigma_i^- &\leq \frac{g_i(s_1) - g_i(s_2)}{s_1 - s_2} \leq \sigma_i^+, \quad \forall s_1, s_2 \in \mathbb{R}, s_1 \neq s_2.
 \end{aligned}
 \tag{4}$$

In order to estimate the states of neurons, the following time-varying nonlinear state estimator is constructed:

$$\begin{aligned} \hat{x}_{k+1} &= A_k \hat{x}_k + \bar{\alpha} B_{1k} f(\hat{x}_k) + (1 - \bar{\alpha}) B_{2k} g(\hat{x}_k) + (K_k + \delta_k \bar{K}_k)(y_k - \bar{\lambda} D_k \hat{x}_k), \\ \hat{z}_k &= M_k \hat{x}_k, \end{aligned} \tag{5}$$

where \hat{x}_k is the estimation of neuron state x_k , \bar{K}_k is a known real matrix with appropriate dimension, δ_k is zero mean Gaussian white noise with unity covariance. K_k is the estimator gain matrix to be determined later. In the sequel, suppose that α_k , λ_k , v_{1k} , v_{2k} and δ_k are mutually independent.

Remark 1 During the implementation process, the state estimation performance of the neural networks is usually affected by the numerical rounding error, finite-word length effects and analog-to-digital conversion accuracy, which inevitably lead to the possible deviations of the estimation values provided by the state estimator. Thus, the presented estimation method might be a fragile/non-resilient one. Therefore, we aim to take the estimator gain perturbations into account during the estimator design with hope to provide a resilient time-varying state estimator with admissible adjustment ability. Accordingly, the estimator gain perturbations are modeled by a zero mean Gaussian white noise δ_k and the nominal matrix \bar{K}_k , where the changes of the white noise δ_k are utilized to characterize the admissible errors of the estimator gain. As such, a resilient state estimation scheme under prescribed performance indices is expected to be given later for the addressed *time-varying* RNNs.

Let the estimation error be $e_k = x_k - \hat{x}_k$ and the controlled output estimation error be $\tilde{z}_k = z_k - \hat{z}_k$. Then the estimation error dynamics can be obtained from (1) and (5) as follows:

$$\begin{aligned} e_{k+1} &= [A_k - (K_k + \delta_k \bar{K}_k) \bar{\lambda} D_k] e_k + \Delta A_k x_k + \tilde{\alpha}_k B_{1k} f(x_k) + \bar{\alpha} B_{1k} f(e_k) \\ &\quad - \tilde{\alpha}_k B_{2k} g(x_k) + (1 - \bar{\alpha}) B_{2k} g(e_k) - (K_k + \delta_k \bar{K}_k)(\bar{\lambda}_k D_k x_k + v_{2k}) + C_k v_{1k}, \\ \tilde{z}_k &= M_k e_k, \end{aligned} \tag{6}$$

with $f(e_k) = f(x_k) - f(\hat{x}_k)$, $g(e_k) = g(x_k) - g(\hat{x}_k)$, $\tilde{\alpha}_k = \alpha_k - \bar{\alpha}$ and $\tilde{\lambda}_k = \lambda_k - \bar{\lambda}$.

To simplify the notation, we can define

$$\begin{aligned} \eta_k &= \begin{bmatrix} x_k^T & e_k^T \end{bmatrix}^T, \quad v_k = \begin{bmatrix} v_{1k}^T & v_{2k}^T \end{bmatrix}^T, \\ f(\eta_k) &= \begin{bmatrix} f^T(x_k) & f^T(e_k) \end{bmatrix}^T, \quad g(\eta_k) = \begin{bmatrix} g^T(x_k) & g^T(e_k) \end{bmatrix}^T. \end{aligned}$$

Considering (1), (6) and the above notations, we can easily derive the following augmented system:

$$\begin{aligned} \eta_{k+1} &= (\mathcal{A}_k + \tilde{\lambda}_k \mathcal{D}_{1k} + \tilde{\lambda}_k \delta_k \mathcal{D}_{2k} + \delta_k \mathcal{A}_{2k}) \eta_k + (\mathcal{B}_{1k} + \tilde{\alpha}_k \mathcal{B}_k) f(\eta_k) \\ &\quad + (\mathcal{B}_{2k} - \tilde{\alpha}_k \check{\mathcal{B}}_k) g(\eta_k) + (\mathcal{C}_{1k} + \delta_k \mathcal{C}_{2k}) v_k, \\ \tilde{z}_k &= \mathcal{M}_k \eta_k, \end{aligned} \tag{7}$$

where

$$\begin{aligned}
 A_k &= \begin{bmatrix} A_k + \Delta A_k & 0 \\ \Delta A_k & A_k - \bar{\lambda} K_k D_k \end{bmatrix}, & D_{1k} &= \begin{bmatrix} 0 & 0 \\ -K_k D_k & 0 \end{bmatrix}, \\
 D_{2k} &= \begin{bmatrix} 0 & 0 \\ -\bar{K}_k D_k & 0 \end{bmatrix}, & A_{2k} &= \begin{bmatrix} 0 & 0 \\ 0 & -\bar{\lambda} \bar{K}_k D_k \end{bmatrix}, & B_{1k} &= \begin{bmatrix} \bar{\alpha} B_{1k} & 0 \\ 0 & \bar{\alpha} B_{1k} \end{bmatrix}, \\
 B_k &= \begin{bmatrix} B_{1k} & 0 \\ B_{1k} & 0 \end{bmatrix}, & B_{2k} &= \begin{bmatrix} (1 - \bar{\alpha}) B_{2k} & 0 \\ 0 & (1 - \bar{\alpha}) B_{2k} \end{bmatrix}, & \check{B}_k &= \begin{bmatrix} B_{2k} & 0 \\ B_{2k} & 0 \end{bmatrix}, \\
 C_{1k} &= \begin{bmatrix} C_k & 0 \\ C_k & -K_k \end{bmatrix}, & C_{2k} &= \begin{bmatrix} 0 & 0 \\ 0 & -\bar{K}_k \end{bmatrix}, & M_k &= \begin{bmatrix} 0 & M_k \end{bmatrix}.
 \end{aligned}$$

Next, we introduce the covariance matrix X_k described by

$$X_k = \mathbb{E}\{\eta_k \eta_k^T\} = \mathbb{E}\left\{ \begin{bmatrix} x_k \\ e_k \end{bmatrix} \begin{bmatrix} x_k \\ e_k \end{bmatrix}^T \right\}. \tag{8}$$

The main purpose of this paper is to design a time-varying nonlinear state estimator (5) such that the following two requirements are achieved simultaneously.

(R1) Let the scalar $\gamma > 0$, the positive-definite matrices W_φ and W_ϕ be given. For the initial state η_0 , the estimation error \check{z}_k satisfies the following constraint:

$$J_1 := \mathbb{E}\left\{ \sum_{k=0}^{N-1} (\|\check{z}_k\|^2 - \gamma^2 \|v_k\|_{W_\varphi}^2) \right\} - \gamma^2 \mathbb{E}\{\eta_0^T W_\phi \eta_0\} < 0, \tag{9}$$

where $\|v_k\|_{W_\varphi}^2 = v_k^T W_\varphi v_k$.

(R2) The estimation error covariance satisfies the following bounded constraint:

$$J_2 := \mathbb{E}\{e_k e_k^T\} \leq \Psi_k, \tag{10}$$

where Ψ_k ($0 \leq k < N$) is a set of pre-defined upper matrix, which reflects the admissible estimation precision demand with respect to the specific situation.

Remark 2 In this paper, our main purpose is to design a time-varying state estimator such that, for all missing measurements, randomly varying nonlinearities and estimator gain perturbations, both the estimation error variance constraint and the prescribed H_∞ performance requirement are met simultaneously by providing some sufficient criteria. On the one hand, the H_∞ performance requirement within finite horizon is used to reflect the attenuation capacity of the presented estimation algorithm against the effects from the exogenous disturbances and initial value. On the other hand, the upper bound constraint on the estimation error covariance is also introduced to evaluate the estimation accuracy. Compared with the traditional estimation method in the minimum variance framework, a prescribed upper bound constraint regarding the estimation error covariance is needed to be ensured, which could provide another evaluation way to the characterize the estimation performance.

Remark 3 In the practical circumstances, both the nonlinear effects and random coupling are inevitable [7]. Moreover, there is a common case that the NNs consist of a large number of highly interconnected neurons. Thus, it is generally difficult to obtain all states of the neurons due to the inherent interaction effects. Consequently, it is necessary to propose a proper way estimating the state information of all neurons, which has wide theoretical importance and practical significance. For example, a battery state-of-charge estimator has been designed in [47] to testify the usefulness of the merging fuzzy neural networks with a novel learning structure. As such, the desirable state estimation strategy with resilience ability for *time-varying RNNs* subject to missing measurements and randomly varying nonlinearities can provide important solvable method, thereby enriching the state estimation techniques for NNs.

At the end of this section, the following lemmas are introduced to facilitate further derivations.

Lemma 1 ([26]) *Suppose that $\Gamma = \text{diag}\{\mu_1, \mu_2, \dots, \mu_n\}$ is a diagonal matrix. For $x = [x_1 \ x_2 \ \dots \ x_n]^T \in \mathbb{R}^n$, let $\mathcal{H}(x) = [h_1(x_1) \ h_2(x_2) \ \dots \ h_n(x_n)]^T$ be a continuous nonlinear function satisfying*

$$l_i^- \leq \frac{h_i(s)}{s} \leq l_i^+, \quad s \in \mathbb{R}, s \neq 0$$

with l_i^+ and l_i^- being constant scalars. Then

$$\begin{bmatrix} x \\ \mathcal{H}(x) \end{bmatrix}^T \begin{bmatrix} \Gamma L_1 & -\Gamma L_2 \\ -\Gamma L_2 & \Gamma \end{bmatrix} \begin{bmatrix} x \\ \mathcal{H}(x) \end{bmatrix} \leq 0,$$

where $L_1 = \text{diag}\{l_1^+ l_1^-, l_2^+ l_2^-, \dots, l_n^+ l_n^-\}$ and $L_2 = \text{diag}\{\frac{l_1^+ + l_1^-}{2}, \frac{l_2^+ + l_2^-}{2}, \dots, \frac{l_n^+ + l_n^-}{2}\}$.

Lemma 2 ([48]) *Let R, F, H and Q be real matrices of appropriate dimensions with Q and F satisfying $Q = Q^T$ and $FF^T \leq I$. Then $Q + RFH + H^T F^T R^T < 0$ holds if and only if there exists $\epsilon > 0$ such that*

$$Q + \epsilon RR^T + \epsilon^{-1} H^T H < 0.$$

Lemma 3 *If the activation functions $f(s)$ and $g(s)$ satisfy the conditions in Assumption 1, then the following inequalities hold:*

$$\begin{aligned} f(s)f^T(s) &\leq \left\{ \frac{\rho + \frac{1}{\rho}}{2(1-\rho)} (\Lambda_2 + \Lambda_0)^2 + \frac{1}{\rho(1-\rho)} (\Lambda_2 - \Lambda_0)^2 \right\} \|s\|^2, \\ g(s)g^T(s) &\leq \left\{ \frac{\rho + \frac{1}{\rho}}{2(1-\rho)} (\Sigma_2 + \Sigma_0)^2 + \frac{1}{\rho(1-\rho)} (\Sigma_2 - \Sigma_0)^2 \right\} \|s\|^2, \quad \rho \in (0, 1), \end{aligned} \tag{11}$$

where

$$\begin{aligned}
 \Lambda_2 &= \text{diag} \left\{ \frac{\lambda_1^+ + \lambda_1^-}{2}, \frac{\lambda_2^+ + \lambda_2^-}{2}, \dots, \frac{\lambda_n^+ + \lambda_n^-}{2} \right\} = \text{diag} \{ \lambda_i^1 \}, \\
 \Lambda_0 &= \text{diag} \left\{ \frac{\lambda_1^+ - \lambda_1^-}{2}, \frac{\lambda_2^+ - \lambda_2^-}{2}, \dots, \frac{\lambda_n^+ - \lambda_n^-}{2} \right\} = \text{diag} \{ \lambda_i^0 \}, \\
 \Sigma_2 &= \text{diag} \left\{ \frac{\sigma_1^+ + \sigma_1^-}{2}, \frac{\sigma_2^+ + \sigma_2^-}{2}, \dots, \frac{\sigma_n^+ + \sigma_n^-}{2} \right\} = \text{diag} \{ \sigma_i^1 \}, \\
 \Sigma_0 &= \text{diag} \left\{ \frac{\sigma_1^+ - \sigma_1^-}{2}, \frac{\sigma_2^+ - \sigma_2^-}{2}, \dots, \frac{\sigma_n^+ - \sigma_n^-}{2} \right\} = \text{diag} \{ \sigma_i^0 \}.
 \end{aligned} \tag{12}$$

Proof The proof this lemma can be easily obtained, hence it is omitted for brevity. □

3 Discussions of H_∞ performances and covariance constraint

In this section, both the H_∞ performance requirement and the upper bounded constraint of the estimation error covariance matrix are considered. The sufficient criteria are proposed accordingly based on the recursive matrix inequality technique.

3.1 H_∞ performance requirement

Firstly, a sufficient condition is obtained to ensure that the output estimation error satisfies the specified H_∞ performance index over the finite horizon.

Theorem 1 *Consider the time-varying RNNs (1) subject to randomly varying nonlinearities and missing measurements. Suppose that the scalars $\gamma > 0$, $\bar{\alpha} \in [0, 1]$ and $\bar{\lambda} \in [0, 1]$, matrices $W_\phi > 0$ and $W_\psi > 0$, state estimator gain matrix K_k in (5) are given. If $Q_0 \leq \gamma^2 W_\phi$ and there exists a series of positive-definite real-value matrices $\{Q_k\}_{1 \leq k \leq N+1}$ satisfying the following recursive matrix inequality:*

$$\Phi = \begin{bmatrix} \Phi_{11} & \Lambda_{21} + A_k^T Q_{k+1} B_{1k} & \Sigma_{22} + A_k^T Q_{k+1} B_{2k} & 0 \\ * & \Phi_{22} & \Phi_{23} & 0 \\ * & * & \Phi_{33} & 0 \\ * & * & * & \Phi_{44} \end{bmatrix} < 0, \tag{13}$$

where

$$\begin{aligned}
 \Phi_{11} &= A_k^T Q_{k+1} A_k + \bar{\lambda}(1 - \bar{\lambda}) D_{1k}^T Q_{k+1} D_{1k} + \bar{\lambda}(1 - \bar{\lambda}) D_{2k}^T Q_{k+1} D_{2k} \\
 &\quad + A_{2k}^T Q_{k+1} A_{2k} + M_k^T M_k - Q_k - \Lambda_{11} - \Sigma_{12}, \\
 \Phi_{22} &= B_{1k}^T Q_{k+1} B_{1k} + \bar{\alpha}(1 - \bar{\alpha}) B_k^T Q_{k+1} B_k - \bar{F}, \\
 \Phi_{23} &= B_{1k}^T Q_{k+1} B_{2k} - \bar{\alpha}(1 - \bar{\alpha}) B_k^T Q_{k+1} \check{B}_k, \\
 \Phi_{33} &= B_{2k}^T Q_{k+1} B_{2k} + \bar{\alpha}(1 - \bar{\alpha}) \check{B}_k^T Q_{k+1} \check{B}_k - \bar{H}, \\
 \Phi_{44} &= C_{1k}^T Q_{k+1} C_{1k} + C_{2k}^T Q_{k+1} C_{2k} - \gamma^2 W_\phi, \\
 \Lambda_{11} &= \begin{bmatrix} \Gamma_1 \Lambda_1 & 0 \\ 0 & \Gamma_2 \Lambda_1 \end{bmatrix}, \quad \Lambda_{21} = \begin{bmatrix} \Gamma_1 \Lambda_2 & 0 \\ 0 & \Gamma_2 \Lambda_2 \end{bmatrix}, \quad \Sigma_{12} = \begin{bmatrix} \Gamma_3 \Sigma_1 & 0 \\ 0 & \Gamma_4 \Sigma_1 \end{bmatrix},
 \end{aligned}$$

$$\Sigma_{22} = \begin{bmatrix} \Gamma_3 \Sigma_2 & 0 \\ 0 & \Gamma_4 \Sigma_2 \end{bmatrix}, \quad \bar{F} = \begin{bmatrix} \Gamma_1 & 0 \\ 0 & \Gamma_2 \end{bmatrix}, \quad \bar{H} = \begin{bmatrix} \Gamma_3 & 0 \\ 0 & \Gamma_4 \end{bmatrix},$$

$$\Gamma_1 = \text{diag}\{\mu_{11}, \mu_{12}, \dots, \mu_{1n}\}, \quad \Gamma_2 = \text{diag}\{\mu_{21}, \mu_{22}, \dots, \mu_{2n}\},$$

$$\Gamma_3 = \text{diag}\{\mu_{31}, \mu_{32}, \dots, \mu_{3n}\}, \quad \Gamma_4 = \text{diag}\{\mu_{41}, \mu_{42}, \dots, \mu_{4n}\},$$

$$\Lambda_1 = \text{diag}\{\lambda_1^+ \lambda_1^-, \lambda_2^+ \lambda_2^-, \dots, \lambda_n^+ \lambda_n^-\}, \quad \Sigma_1 = \text{diag}\{\sigma_1^+ \sigma_1^-, \sigma_2^+ \sigma_2^-, \dots, \sigma_n^+ \sigma_n^-\},$$

then the H_∞ performance constraint defined in (9) can be achieved for all nonzero v_k .

Proof Define

$$V_k = \eta_{k+1}^T Q_{k+1} \eta_{k+1} - \eta_k^T Q_k \eta_k. \tag{14}$$

Next, it follows from the augmented system (7) that

$$\begin{aligned} \mathbb{E}\{V_k\} &= \mathbb{E}\left\{\eta_k^T \mathcal{A}_k^T Q_{k+1} \mathcal{A}_k \eta_k + \bar{\lambda}(1 - \bar{\lambda}) \eta_k^T \mathcal{D}_{1k}^T Q_{k+1} \mathcal{D}_{1k} \eta_k + \bar{\lambda}(1 - \bar{\lambda}) \eta_k^T \mathcal{D}_{2k}^T \right. \\ &\quad \times Q_{k+1} \mathcal{D}_{2k} \eta_k + \eta_k^T A_{2k}^T Q_{k+1} A_{2k} \eta_k + f^T(\eta_k) \mathcal{B}_{1k}^T Q_{k+1} \mathcal{B}_{1k} f(\eta_k) \\ &\quad + \bar{\alpha}(1 - \bar{\alpha}) f^T(\eta_k) B_k^T Q_{k+1} B_k f(\eta_k) + g^T(\eta_k) \mathcal{B}_{2k}^T Q_{k+1} \mathcal{B}_{2k} g(\eta_k) \\ &\quad + \bar{\alpha}(1 - \bar{\alpha}) g^T(\eta_k) \check{B}_k^T Q_{k+1} \check{B}_k g(\eta_k) + 2\eta_k^T \mathcal{A}_k^T Q_{k+1} \mathcal{B}_{1k} f(\eta_k) \\ &\quad + 2\eta_k^T A_{2k}^T Q_{k+1} \mathcal{B}_{2k} g(\eta_k) + 2f^T(\eta_k) \mathcal{B}_{1k}^T Q_{k+1} \mathcal{B}_{2k} g(\eta_k) \\ &\quad \left. - 2\bar{\alpha}(1 - \bar{\alpha}) f^T(\eta_k) B_k^T Q_{k+1} \check{B}_k g(\eta_k) + v_k^T C_{1k}^T Q_{k+1} C_{1k} v_k + v_k^T C_{2k}^T Q_{k+1} C_{2k} v_k - \eta_k^T Q_k \eta_k\right\}. \end{aligned} \tag{15}$$

Adding the zero term $\tilde{z}_k^T \tilde{z}_k - \gamma^2 v_k^T W_\varphi v_k - \tilde{z}_k^T \tilde{z}_k + \gamma^2 v_k^T W_\varphi v_k$ to $\mathbb{E}\{V_k\}$ leads to

$$\mathbb{E}\{V_k\} = \mathbb{E}\left\{ \begin{bmatrix} \bar{\eta}_k^T & v_k^T \end{bmatrix} \tilde{\Phi} \begin{bmatrix} \bar{\eta}_k \\ v_k \end{bmatrix} - \tilde{z}_k^T \tilde{z}_k + \gamma^2 v_k^T W_\varphi v_k \right\},$$

where

$$\begin{aligned} \bar{\eta}_k^T &= \begin{bmatrix} \eta_k^T & f^T(\eta_k) & g^T(\eta_k) \end{bmatrix}, \\ \tilde{\Phi} &= \begin{bmatrix} \tilde{\Phi}_{11} & \mathcal{A}_k^T Q_{k+1} \mathcal{B}_{1k} & \mathcal{A}_k^T Q_{k+1} \mathcal{B}_{2k} & 0 \\ * & \tilde{\Phi}_{22} & \Phi_{23} & 0 \\ * & * & \tilde{\Phi}_{33} & 0 \\ * & * & * & \Phi_{44} \end{bmatrix} < 0, \end{aligned} \tag{16}$$

$$\begin{aligned} \tilde{\Phi}_{11} &= \mathcal{A}_k^T Q_{k+1} \mathcal{A}_k + \bar{\lambda}(1 - \bar{\lambda}) \mathcal{D}_{1k}^T Q_{k+1} \mathcal{D}_{1k} + \bar{\lambda}(1 - \bar{\lambda}) \mathcal{D}_{2k}^T Q_{k+1} \mathcal{D}_{2k} \\ &\quad + A_{2k}^T Q_{k+1} A_{2k} + \mathcal{M}_k^T \mathcal{M}_k - Q_k, \\ \tilde{\Phi}_{22} &= \mathcal{B}_{1k}^T Q_{k+1} \mathcal{B}_{1k} + \bar{\alpha}(1 - \bar{\alpha}) B_k^T Q_{k+1} B_k, \\ \tilde{\Phi}_{33} &= \mathcal{B}_{2k}^T Q_{k+1} \mathcal{B}_{2k} + \bar{\alpha}(1 - \bar{\alpha}) \check{B}_k^T Q_{k+1} \check{B}_k, \end{aligned}$$

and Φ_{23} as well as Φ_{44} being defined below (13).

Moreover, based on Assumption 1 and Lemma 1, we have

$$\begin{aligned} & \begin{bmatrix} x_k \\ f(x_k) \end{bmatrix}^T \begin{bmatrix} \Gamma_1 \Lambda_1 & -\Gamma_1 \Lambda_2 \\ -\Gamma_1 \Lambda_2 & \Gamma_1 \end{bmatrix} \begin{bmatrix} x_k \\ f(x_k) \end{bmatrix} \leq 0, \\ & \begin{bmatrix} e_k \\ f(e_k) \end{bmatrix}^T \begin{bmatrix} \Gamma_2 \Lambda_1 & -\Gamma_2 \Lambda_2 \\ -\Gamma_2 \Lambda_2 & \Gamma_2 \end{bmatrix} \begin{bmatrix} e_k \\ f(e_k) \end{bmatrix} \leq 0, \\ & \begin{bmatrix} x_k \\ g(x_k) \end{bmatrix}^T \begin{bmatrix} \Gamma_3 \Sigma_1 & -\Gamma_3 \Sigma_2 \\ -\Gamma_3 \Sigma_2 & \Gamma_3 \end{bmatrix} \begin{bmatrix} x_k \\ g(x_k) \end{bmatrix} \leq 0, \\ & \begin{bmatrix} e_k \\ g(e_k) \end{bmatrix}^T \begin{bmatrix} \Gamma_4 \Sigma_1 & -\Gamma_4 \Sigma_2 \\ -\Gamma_4 \Sigma_2 & \Gamma_4 \end{bmatrix} \begin{bmatrix} e_k \\ g(e_k) \end{bmatrix} \leq 0, \quad k = 1, 2, \dots, n, \end{aligned}$$

where Λ_2 and Σ_2 are defined in (12), $\Lambda_1, \Sigma_1, \Gamma_1, \Gamma_2, \Gamma_3$ and Γ_4 are defined in (13). From the above inequalities, we can deduce that

$$\begin{aligned} & \begin{bmatrix} \eta_k \\ f(\eta_k) \end{bmatrix}^T \begin{bmatrix} \Lambda_{11} & -\Lambda_{21} \\ -\Lambda_{21} & \bar{F} \end{bmatrix} \begin{bmatrix} \eta_k \\ f(\eta_k) \end{bmatrix} \leq 0, \\ & \begin{bmatrix} \eta_k \\ g(\eta_k) \end{bmatrix}^T \begin{bmatrix} \Sigma_{12} & -\Sigma_{22} \\ -\Sigma_{22} & \bar{H} \end{bmatrix} \begin{bmatrix} \eta_k \\ g(\eta_k) \end{bmatrix} \leq 0, \end{aligned} \tag{17}$$

where $\Lambda_{11}, \Lambda_{21}, \Sigma_{12}, \Sigma_{22}, \bar{F}$ and \bar{H} are mentioned below (13). Then, together with (16)–(17), this yields

$$\begin{aligned} \mathbb{E}\{V_k\} & \leq \mathbb{E} \left\{ \begin{bmatrix} \bar{\eta}_k^T & v_k^T \end{bmatrix} \tilde{\Phi} \begin{bmatrix} \bar{\eta}_k \\ v_k \end{bmatrix} - \tilde{z}_k^T \tilde{z}_k + \gamma^2 v_k^T W_\varphi v_k - [\eta_k^T \Lambda_{11} \eta_k - 2\eta_k^T \Lambda_{21} f(\eta_k) \right. \\ & \quad \left. + f^T(\eta_k) \bar{F} f(\eta_k)] - [\eta_k^T \Sigma_{12} \eta_k - 2\eta_k^T \Sigma_{22} g(\eta_k) + g^T(\eta_k) \bar{H} g(\eta_k)] \right\} \\ & = \mathbb{E} \left\{ \begin{bmatrix} \bar{\eta}_k^T & v_k^T \end{bmatrix} \Phi \begin{bmatrix} \bar{\eta}_k \\ v_k \end{bmatrix} - \tilde{z}_k^T \tilde{z}_k + \gamma^2 v_k^T W_\varphi v_k \right\}. \end{aligned} \tag{18}$$

Summarizing (18) from 0 to $N - 1$ with respect to k , it is not difficult to see that

$$\begin{aligned} \sum_{k=0}^{N-1} \mathbb{E}\{V_k\} & = \mathbb{E}\{\eta_N^T Q_N \eta_N - \eta_0^T Q_0 \eta_0\} \\ & \leq \mathbb{E} \left\{ \sum_{k=0}^{N-1} \begin{bmatrix} \bar{\eta}_k^T & v_k^T \end{bmatrix} \Phi \begin{bmatrix} \bar{\eta}_k \\ v_k \end{bmatrix} \right\} - \mathbb{E} \left\{ \sum_{k=0}^{N-1} (\tilde{z}_k^T \tilde{z}_k - \gamma^2 v_k^T W_\varphi v_k) \right\}, \end{aligned} \tag{19}$$

where Φ is defined in (13). Therefore, we can derive the following inequality:

$$J_1 \leq \mathbb{E} \left\{ \sum_{k=0}^{N-1} \begin{bmatrix} \bar{\eta}_k^T & v_k^T \end{bmatrix} \Phi \begin{bmatrix} \bar{\eta}_k \\ v_k \end{bmatrix} + \eta_0^T (Q_0 - \gamma^2 W_\varphi) \eta_0 \right\} - \mathbb{E}\{\eta_N^T Q_N \eta_N\}. \tag{20}$$

According to $\Phi < 0, Q_N > 0$ and the initial condition $Q_0 \leq \gamma^2 W_\varphi$, it follows that $J_1 < 0$. The proof in Theorem 1 is now complete. \square

So far, we have presented the criterion to guarantee the H_∞ performance within the finite horizon. Next, we are ready to propose the sufficient condition to ensure the upper bound constraint with respect to the covariance matrix X_k .

3.2 Variance constraint analysis

In this subsection, the upper bound constraint of the covariance matrix X_k is ensured by providing a sufficient criterion.

Theorem 2 Consider the time-varying RNNs (1) subject to randomly varying nonlinearities and missing measurements. Let the scalars $\bar{\alpha} \in [0, 1]$, $\bar{\lambda} \in [0, 1]$ and the estimator gain matrix K_k in (5) be given. Under the initial condition $G_0 = X_0$, if there exists a set of positive-definite matrices $\{G_k\}_{1 \leq k \leq N+1}$ satisfying

$$G_{k+1} \geq \Psi(G_k), \tag{21}$$

where

$$\begin{aligned} \Psi(G_k) = & (1 + \varepsilon_1 + \varepsilon_2) \mathcal{A}_k G_k \mathcal{A}_k^T + \bar{\lambda}(1 - \bar{\lambda}) \mathcal{D}_{1k} G_k \mathcal{D}_{1k}^T + \bar{\lambda}(1 - \bar{\lambda}) \mathcal{D}_{2k} G_k \mathcal{D}_{2k}^T \\ & + A_{2k} G_k A_{2k}^T + (1 + \varepsilon_1^{-1} + \varepsilon_3) \text{tr}(G_k) \mathcal{B}_{1k} Y_1 \mathcal{B}_{1k}^T + \bar{\alpha}(1 - \bar{\alpha})(1 + \varepsilon_4^{-1}) \\ & \times \text{tr}(G_k) \check{B}_k Y_2 \check{B}_k^T + \bar{\alpha}(1 - \bar{\alpha})(1 + \varepsilon_4) \text{tr}(G_k) B_k Y_1 B_k^T + (1 + \varepsilon_2^{-1} \\ & + \varepsilon_3^{-1}) \text{tr}(G_k) \mathcal{B}_{2k} Y_2 \mathcal{B}_{2k}^T + C_{1k} V C_{1k}^T + C_{2k} V C_{2k}^T, \end{aligned} \tag{22}$$

$$V = \text{diag}\{V_1, V_2\},$$

$$Y_1 = \frac{\rho + \frac{1}{\rho}}{2(1 - \rho)} (\Lambda_2 + \Lambda_0)^2 + \frac{1}{\rho(1 - \rho)} (\Lambda_2 - \Lambda_0)^2,$$

$$Y_2 = \frac{\rho + \frac{1}{\rho}}{2(1 - \rho)} (\Sigma_2 + \Sigma_0)^2 + \frac{1}{\rho(1 - \rho)} (\Sigma_2 - \Sigma_0)^2,$$

then we have $G_k \geq X_k$ ($\forall k \in 1, 2, \dots, N + 1$).

Proof As we know from (8), the equation of state covariance X_k can be calculated by

$$\begin{aligned} X_{k+1} = & \mathbb{E}\{\eta_{k+1} \eta_{k+1}^T\} \\ = & \mathbb{E}\{\mathcal{A}_k \eta_k \eta_k^T \mathcal{A}_k^T + \bar{\lambda}(1 - \bar{\lambda}) \mathcal{D}_{1k} \eta_k \eta_k^T \mathcal{D}_{1k}^T + \bar{\lambda}(1 - \bar{\lambda}) \mathcal{D}_{2k} \eta_k \eta_k^T \mathcal{D}_{2k}^T + A_{2k} \eta_k \\ & \times \eta_k^T A_{2k}^T + \bar{\alpha}(1 - \bar{\alpha}) B_k f(\eta_k) f^T(\eta_k) B_k^T + \mathcal{B}_{1k} f(\eta_k) \eta_k^T \mathcal{A}_k^T + \mathcal{A}_k \eta_k f^T(\eta_k) \\ & \times \mathcal{B}_{1k}^T + \mathcal{B}_{2k} g(\eta_k) g^T(\eta_k) \mathcal{B}_{2k}^T + \bar{\alpha}(1 - \bar{\alpha}) \check{B}_k g(\eta_k) g^T(\eta_k) \check{B}_k^T + \mathcal{B}_{2k} g(\eta_k) \eta_k^T \\ & \times \mathcal{A}_k^T + \mathcal{A}_k \eta_k g^T(\eta_k) \mathcal{B}_{2k}^T + \mathcal{B}_{2k} g(\eta_k) f^T(\eta_k) \mathcal{B}_{1k}^T + \mathcal{B}_{1k} f(\eta_k) g^T(\eta_k) \mathcal{B}_{2k}^T \\ & + \mathcal{B}_{1k} f(\eta_k) f^T(\eta_k) \mathcal{B}_{1k}^T - \bar{\alpha}(1 - \bar{\alpha}) B_k f(\eta_k) g^T(\eta_k) \check{B}_k^T - \bar{\alpha}(1 - \bar{\alpha}) \check{B}_k g(\eta_k) \\ & \times f^T(\eta_k) B_k^T + C_{1k} V C_{1k}^T + C_{2k} V C_{2k}^T\}. \end{aligned}$$

On the basis of the inequality $ab^T + ba^T \leq \zeta aa^T + \zeta^{-1} bb^T$ with $\zeta > 0$, we can obtain

$$\begin{aligned} & \mathbb{E}\{\mathcal{A}_k \eta_k f^T(\eta_k) \mathcal{B}_{1k}^T + \mathcal{B}_{1k} f(\eta_k) \eta_k^T \mathcal{A}_k^T\} \\ & \leq \mathbb{E}\{\varepsilon_1 \mathcal{A}_k \eta_k \eta_k^T \mathcal{A}_k^T + \varepsilon_1^{-1} \mathcal{B}_{1k} f(\eta_k) f^T(\eta_k) \mathcal{B}_{1k}^T\}, \end{aligned}$$

$$\begin{aligned} & \mathbb{E}\{\mathcal{A}_k \eta_k g^T(\eta_k) \mathcal{B}_{2k}^T + \mathcal{B}_{2k} g(\eta_k) \eta_k^T \mathcal{A}_k^T\} \\ & \leq \mathbb{E}\{\varepsilon_2 \mathcal{A}_k \eta_k \eta_k^T \mathcal{A}_k^T + \varepsilon_2^{-1} \mathcal{B}_{2k} g(\eta_k) g^T(\eta_k) \mathcal{B}_{2k}^T\}, \\ & \mathbb{E}\{\mathcal{B}_{1k} f(\eta_k) g^T(\eta_k) \mathcal{B}_{2k}^T + \mathcal{B}_{2k} g(\eta_k) f^T(\eta_k) \mathcal{B}_{1k}^T\} \\ & \leq \mathbb{E}\{\varepsilon_3 \mathcal{B}_{1k} f(\eta_k) f^T(\eta_k) \mathcal{B}_{1k}^T + \varepsilon_3^{-1} \mathcal{B}_{2k} g(\eta_k) g^T(\eta_k) \mathcal{B}_{2k}^T\}, \\ & \mathbb{E}\{-\bar{\alpha}(1-\bar{\alpha}) \mathcal{B}_k f(\eta_k) g^T(\eta_k) \check{\mathcal{B}}_k^T - \bar{\alpha}(1-\bar{\alpha}) \check{\mathcal{B}}_k g(\eta_k) f^T(\eta_k) \mathcal{B}_k^T\} \\ & \leq \mathbb{E}\{\varepsilon_4 \bar{\alpha}(1-\bar{\alpha}) \mathcal{B}_k f(\eta_k) f^T(\eta_k) \mathcal{B}_k^T + \varepsilon_4^{-1} \bar{\alpha}(1-\bar{\alpha}) \check{\mathcal{B}}_k g(\eta_k) g^T(\eta_k) \check{\mathcal{B}}_k^T\}, \end{aligned}$$

where ε_i ($i = 1, 2, 3, 4$) are positive scalars. Then it is straightforward to see that

$$\begin{aligned} X_{k+1} & \leq \mathbb{E}\{(1 + \varepsilon_1 + \varepsilon_2) \mathcal{A}_k \eta_k \eta_k^T \mathcal{A}_k^T + \bar{\lambda}(1-\bar{\lambda}) \mathcal{D}_{1k} \eta_k \eta_k^T \mathcal{D}_{1k}^T + \bar{\lambda}(1-\bar{\lambda}) \mathcal{D}_{2k} \eta_k \eta_k^T \mathcal{D}_{2k}^T \\ & \quad + \eta_k^T \mathcal{D}_{2k}^T \mathcal{D}_{2k} + \mathcal{A}_{2k} \eta_k \eta_k^T \mathcal{A}_{2k}^T + (1 + \varepsilon_1^{-1} + \varepsilon_3) \mathcal{B}_{1k} f(\eta_k) f^T(\eta_k) \mathcal{B}_{1k}^T \\ & \quad + \bar{\alpha}(1-\bar{\alpha})(1 + \varepsilon_4^{-1}) \check{\mathcal{B}}_k g(\eta_k) g^T(\eta_k) \check{\mathcal{B}}_k^T + \bar{\alpha}(1-\bar{\alpha})(1 + \varepsilon_4) \mathcal{B}_k f(\eta_k) f^T(\eta_k) \mathcal{B}_k^T \\ & \quad + (1 + \varepsilon_2^{-1} + \varepsilon_3^{-1}) \mathcal{B}_{2k} g(\eta_k) g^T(\eta_k) \mathcal{B}_{2k}^T + \mathcal{C}_{1k} \mathcal{V} \mathcal{C}_{1k}^T + \mathcal{C}_{2k} \mathcal{V} \mathcal{C}_{2k}^T\}. \end{aligned}$$

Furthermore, it follows from Lemma 3 that

$$\begin{aligned} \mathbb{E}\{f(\eta_k) f^T(\eta_k)\} & \leq \mathbb{E}\{Y_1 \|\eta_k\|^2\} = \mathbb{E}\{Y_1 \eta_k^T \eta_k\}, \\ \mathbb{E}\{g(\eta_k) g^T(\eta_k)\} & \leq \mathbb{E}\{Y_2 \|\eta_k\|^2\} = \mathbb{E}\{Y_2 \eta_k^T \eta_k\}, \end{aligned}$$

where Y_1 and Y_2 are defined in (22). Thus, one has

$$\begin{aligned} X_{k+1} & \leq \mathbb{E}\{(1 + \varepsilon_1 + \varepsilon_2) \mathcal{A}_k \eta_k \eta_k^T \mathcal{A}_k^T + \bar{\lambda}(1-\bar{\lambda}) \mathcal{D}_{1k} \eta_k \eta_k^T \mathcal{D}_{1k}^T \\ & \quad + \bar{\lambda}(1-\bar{\lambda}) \mathcal{D}_{2k} \eta_k \eta_k^T \mathcal{D}_{2k}^T + \mathcal{A}_{2k} \eta_k \eta_k^T \mathcal{A}_{2k}^T + (1 + \varepsilon_1^{-1} + \varepsilon_3) \mathcal{B}_{1k} Y_1 \eta_k^T \eta_k \mathcal{B}_{1k}^T \\ & \quad + \bar{\alpha}(1-\bar{\alpha})(1 + \varepsilon_4^{-1}) \check{\mathcal{B}}_k Y_2 \eta_k^T \eta_k \check{\mathcal{B}}_k^T + \bar{\alpha}(1-\bar{\alpha})(1 + \varepsilon_4) \mathcal{B}_k Y_1 \eta_k^T \eta_k \\ & \quad \times \mathcal{B}_k^T + (1 + \varepsilon_2^{-1} + \varepsilon_3^{-1}) \mathcal{B}_{2k} Y_2 \eta_k^T \eta_k \mathcal{B}_{2k}^T + \mathcal{C}_{1k} \mathcal{V} \mathcal{C}_{1k}^T + \mathcal{C}_{2k} \mathcal{V} \mathcal{C}_{2k}^T\}. \end{aligned} \tag{23}$$

According to the feature of the trace, we can obtain

$$\mathbb{E}\{\eta_k^T \eta_k\} = \mathbb{E}\{\text{tr}(\eta_k \eta_k^T)\} = \text{tr}(X_k). \tag{24}$$

Combining (23) with (24) results in

$$\begin{aligned} X_{k+1} & \leq (1 + \varepsilon_1 + \varepsilon_2) \mathcal{A}_k X_k \mathcal{A}_k^T + \bar{\lambda}(1-\bar{\lambda}) \mathcal{D}_{1k} X_k \mathcal{D}_{1k}^T + \bar{\lambda}(1-\bar{\lambda}) \mathcal{D}_{2k} X_k \mathcal{D}_{2k}^T \\ & \quad + \mathcal{A}_{2k} X_k \mathcal{A}_{2k}^T + (1 + \varepsilon_1^{-1} + \varepsilon_3) \text{tr}(X_k) \mathcal{B}_{1k} Y_1 \mathcal{B}_{1k}^T + \bar{\alpha}(1-\bar{\alpha})(1 + \varepsilon_4^{-1}) \\ & \quad \times \text{tr}(X_k) \check{\mathcal{B}}_k Y_2 \check{\mathcal{B}}_k^T + \bar{\alpha}(1-\bar{\alpha})(1 + \varepsilon_4) \text{tr}(X_k) \mathcal{B}_k Y_1 \mathcal{B}_k^T + (1 + \varepsilon_2^{-1} \\ & \quad + \varepsilon_3^{-1}) \text{tr}(X_k) \mathcal{B}_{2k} Y_2 \mathcal{B}_{2k}^T + \mathcal{C}_{1k} \mathcal{V} \mathcal{C}_{1k}^T + \mathcal{C}_{2k} \mathcal{V} \mathcal{C}_{2k}^T \\ & = \Psi(X_k). \end{aligned}$$

Noting that $G_0 \geq X_0$ and letting $G_k \geq X_k$, we can derive the following inequality:

$$\Psi(G_k) \geq \Psi(X_k) \geq X_{k+1}. \tag{25}$$

Then, from (21) and (25), we arrive at

$$G_{k+1} \geq \Psi(G_k) \geq \Psi(X_k) \geq X_{k+1}. \tag{26}$$

Therefore, the proof is complete. □

Based on the above theorems, a sufficient condition can be presented to guarantee the specified H_∞ performance and estimation error variance constraint by solving the recursive matrix inequalities.

Theorem 3 Consider the time-varying RNNs (1) and suppose that the estimator gain matrix K_k is given. For given scalars $\gamma > 0$, $\bar{\alpha} \in [0, 1]$ and $\bar{\lambda} \in [0, 1]$, positive-definite matrices $W_\varphi > 0$ and $W_\phi > 0$, under the initial conditions $Q_0 \leq \gamma^2 W_\phi$ and $G_0 = X_0$, if there are two series of positive-definite real-valued matrices $\{Q_k\}_{1 \leq k \leq N+1}$ and $\{G_k\}_{1 \leq k \leq N+1}$ satisfying the following matrix inequalities:

$$\begin{bmatrix} \mathcal{E}_{11} & \mathcal{E}_{12} & \mathcal{E}_{13} & \mathcal{E}_{14} & 0 \\ * & \mathcal{E}_{22} & 0 & 0 & 0 \\ * & * & \mathcal{E}_{33} & 0 & \mathcal{E}_{35} \\ * & * & * & \mathcal{E}_{44} & 0 \\ * & * & * & * & \mathcal{E}_{55} \end{bmatrix} < 0, \tag{27}$$

$$\begin{bmatrix} \Upsilon_{11} & \Upsilon_{12} & \Upsilon_{13} & \Upsilon_{14} \\ * & -G_k & 0 & 0 \\ * & * & \Upsilon_{33} & 0 \\ * & * & * & \Upsilon_{44} \end{bmatrix} < 0, \tag{28}$$

where

$$\begin{aligned} \mathcal{E}_{11} &= -\Lambda_{11} - \Sigma_{12} - Q_k, \\ \mathcal{E}_{12} &= \begin{bmatrix} \Lambda_{21} + A_k^T Q_{k+1} B_{1k} & \Sigma_{22} + A_k^T Q_{k+1} B_{2k} \end{bmatrix}, \\ \mathcal{E}_{13} &= \begin{bmatrix} 0 & A_k^T Q_{k+1} & \varrho_2 D_{1k}^T Q_{k+1} \end{bmatrix}, \\ \mathcal{E}_{14} &= \begin{bmatrix} \varrho_2 D_{2k}^T Q_{k+1} & A_{2k}^T Q_{k+1} & M_k^T \end{bmatrix}, \\ \mathcal{E}_{22} &= \begin{bmatrix} -\bar{F} + B_{1k}^T Q_{k+1} B_{1k} + \bar{\alpha}(1 - \bar{\alpha})B_k^T Q_{k+1} B_k & \Omega_1 \\ * & \Omega_2 \end{bmatrix}, \\ \mathcal{E}_{33} &= \text{diag}\{-\gamma^2 W_\varphi, -Q_{k+1}, -Q_{k+1}\}, \\ \mathcal{E}_{35} &= \begin{bmatrix} C_{1k}^T Q_{k+1} & C_{2k}^T Q_{k+1} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \\ \mathcal{E}_{44} &= \text{diag}\{-Q_{k+1}, -Q_{k+1}, -I\}, \quad \mathcal{E}_{55} = \text{diag}\{-Q_{k+1}, -Q_{k+1}\}, \\ \Upsilon_{11} &= -G_{k+1} + \varrho_5 \text{tr}(G_k) B_{1k} Y_1 B_{1k}^T + \varrho_1 \text{tr}(G_k) B_k Y_1 B_k^T \\ &\quad + \varrho_3 \text{tr}(G_k) B_{2k} Y_2 B_{2k}^T + \varrho_6 \text{tr}(G_k) \check{B}_k Y_2 \check{B}_k^T, \end{aligned}$$

$$\begin{aligned}
 \Upsilon_{12} &= \varrho_4 A_k G_k, \\
 \Upsilon_{13} &= \begin{bmatrix} \varrho_2 \mathcal{D}_{1k} G_k & \varrho_2 \mathcal{D}_{2k} G_k \end{bmatrix}, \\
 \Upsilon_{14} &= \begin{bmatrix} A_{2k} G_k & C_{1k} V & C_{2k} V \end{bmatrix}, \\
 \Upsilon_{33} &= \text{diag}\{-G_k, -G_k\}, \\
 \Upsilon_{44} &= \text{diag}\{-G_k, -V, -V\}, \\
 \Omega_1 &= B_{1k}^T Q_{k+1} B_{2k} - \bar{\alpha}(1 - \bar{\alpha}) B_k^T Q_{k+1} \check{B}_k, \\
 \Omega_2 &= -\bar{H} + B_{2k}^T Q_{k+1} B_{2k} + \bar{\alpha}(1 - \bar{\alpha}) \check{B}_k^T Q_{k+1} \check{B}_k, \\
 \varrho_1 &= \sqrt{\bar{\alpha}(1 - \bar{\alpha})(1 + \varepsilon_4)}, & \varrho_2 &= \sqrt{\bar{\lambda}(1 - \bar{\lambda})}, \\
 \varrho_3 &= \sqrt{1 + \varepsilon_2^{-1} + \varepsilon_3^{-1}}, & \varrho_4 &= \sqrt{1 + \varepsilon_1 + \varepsilon_2}, \\
 \varrho_5 &= \sqrt{1 + \varepsilon_1^{-1} + \varepsilon_3}, & \varrho_6 &= \sqrt{\bar{\alpha}(1 - \bar{\alpha})(1 + \varepsilon_4^{-1})},
 \end{aligned}$$

then both the H_∞ performance requirement and the upper constraints of estimation error covariance can be satisfied simultaneously.

Proof Under the initial conditions, according to the above analysis of H_∞ performance and estimation error covariance in Theorems 1–2, the inequality (27) implies (13) and (28) yields (22). As such, both the H_∞ performance requirement and variance constraint are guaranteed, which ends the proof. \square

Remark 4 So far, some sufficient conditions are given to ensure the desirable performance requirements. To be specific, a sufficient condition is firstly established in Theorem 1 to ensure that the output estimation error satisfies the specified H_∞ performance index over the finite horizon provided that the estimator gain is given. Secondly, the upper bound constraint of the estimation error covariance matrix is guaranteed in Theorem 2. Subsequently, based on the above two theorems, a sufficient condition is presented in Theorem 3 to ensure the specified H_∞ performance and estimation error variance constraint by solving some recursive matrix inequalities. Finally, the design problem of a discrete time-varying state estimator is discussed in the next section, where the estimator gains can be obtained at each sampling step by solving several recursive matrix inequalities.

4 Design of the estimator gain matrix

In this section, a sufficient criterion is proposed to deal with the design problem of discrete time-varying state estimator, which can be solved by several recursive matrix inequalities.

Theorem 4 *Given the disturbance attenuation level $\gamma > 0$, the scalars $\bar{\alpha} \in [0, 1]$ and $\bar{\lambda} \in [0, 1]$, the positive-definite matrices $W_\varphi > 0$ and $W_\phi = \begin{bmatrix} W_{\phi 1} & W_{\phi 2} \\ W_{\phi 2}^T & W_{\phi 4} \end{bmatrix} > 0$ and a set of pre-defined variance upper bound matrices $\{\Psi_k\}_{0 \leq k \leq N+1}$, under the initial conditions*

$$\begin{cases} \begin{bmatrix} L_0 - \gamma^2 W_{\phi 1} & W_{\phi 2} \\ W_{\phi 2}^T & Z_0 - \gamma^2 W_{\phi 4} \end{bmatrix} \leq 0, \\ \mathbb{E}\{e_0 e_0^T\} = G_{2,0} \leq \Psi_0, \end{cases} \tag{29}$$

if there exist series of positive-definite matrices $\{L_k\}_{1 \leq k \leq N+1}$, $\{Z_k\}_{1 \leq k \leq N+1}$, $\{G_{1k}\}_{1 \leq k \leq N+1}$ and $\{G_{2k}\}_{1 \leq k \leq N+1}$, positive scalars $\{\epsilon_{1,k}\}_{0 \leq k \leq N+1}$ and $\{\epsilon_{2,k}\}_{0 \leq k \leq N+1}$, matrices $\{K_k\}_{0 \leq k \leq N+1}$ and $\{G_{3k}\}_{1 \leq k \leq N+1}$ with appropriate dimensions satisfying the following recursive matrix inequalities:

$$\begin{bmatrix} \Theta_{11} & \Theta_{12} & \Theta_{13} & \Theta_{14} & 0 & 0 \\ * & \Theta_{22} & 0 & 0 & 0 & \mathcal{W}_k^T \\ * & * & \Theta_{33} & 0 & \Theta_{35} & \mathcal{Y}_k^T \\ * & * & * & \Theta_{44} & 0 & 0 \\ * & * & * & * & \Theta_{55} & 0 \\ * & * & * & * & * & -\epsilon_{1,k}I \end{bmatrix} < 0, \tag{30}$$

$$\begin{bmatrix} \Pi_{11} & \Pi_{12} & \Pi_{13} & \Pi_{14} & 0 \\ * & \Pi_{22} & 0 & 0 & \mathcal{X}_k^T \\ * & * & \Pi_{33} & 0 & 0 \\ * & * & * & \Pi_{44} & 0 \\ * & * & * & * & -\epsilon_{2,k}I \end{bmatrix} < 0, \tag{31}$$

$$G_{2,k+1} - \Psi_{k+1} \leq 0, \tag{32}$$

where

$$\Theta_{11} = \begin{bmatrix} -\Gamma_1 \Lambda_1 - \Gamma_3 \Sigma_1 + \epsilon_{1,k} N_k N_k^T - L_k & 0 \\ 0 & -\Gamma_2 \Lambda_1 - \Gamma_4 \Sigma_1 - Z_k \end{bmatrix},$$

$$\Theta_{12} = \begin{bmatrix} \bar{\alpha} A_k^T L_{k+1} B_{1k} + \Gamma_1 \Lambda_2 & 0 & (1 - \bar{\alpha}) A_k^T L_{k+1} B_{2k} + \Gamma_3 \Sigma_2 & 0 \\ 0 & \Omega_5 & 0 & \Omega_6 \end{bmatrix},$$

$$\Theta_{13} = \begin{bmatrix} 0 & A_k^T L_{k+1} & 0 & 0 & -Q_2 D_k^T K_k^T Z_{k+1} \\ 0 & 0 & \Omega_7 & 0 & 0 \end{bmatrix},$$

$$\Theta_{14} = \begin{bmatrix} 0 & -Q_2 D_k^T \bar{K}_k^T Z_{k+1} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\bar{\lambda} D_k^T \bar{K}_k^T Z_{k+1} & M_k^T \end{bmatrix},$$

$$\Theta_{22} = \begin{bmatrix} \Omega_8 & 0 & -\bar{\alpha}(1 - \bar{\alpha}) B_{1k}^T Z_{k+1} B_{2k} & 0 \\ * & -\Gamma_2 + \bar{\alpha}^2 B_{1k}^T Z_{k+1} B_{1k} & 0 & \Omega_3 \\ * & * & \Omega_9 & 0 \\ * & * & * & \Omega_4 \end{bmatrix},$$

$$\Theta_{33} = \text{diag}\{-\gamma^2 W_\varphi, -L_{k+1}, -Z_{k+1}, -L_{k+1}, -Z_{k+1}\},$$

$$\Theta_{35} = \begin{bmatrix} C_k^T L_{k+1} & C_k^T Z_{k+1} & 0 & 0 \\ 0 & -K_k^T Z_{k+1} & 0 & -\bar{K}_k^T Z_{k+1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\Theta_{44} = \text{diag}\{-L_{k+1}, -Z_{k+1}, -L_{k+1}, -Z_{k+1}, -I\},$$

$$\Theta_{55} = \text{diag}\{-L_{k+1}, -Z_{k+1}, -L_{k+1}, -Z_{k+1}\},$$

$$\begin{aligned}
 \Pi_{11} &= \begin{bmatrix} \Omega_{10} & \Omega_{11} \\ * & \Omega_{12} \end{bmatrix}, \\
 \Pi_{12} &= \begin{bmatrix} \varrho_4 A_k G_{1k} & \varrho_4 A_k G_{3k}^T \\ \varrho_4 (A_k - \bar{\lambda} K_k D_k) G_{3k} & \varrho_4 (A_k - \bar{\lambda} K_k D_k) G_{2k} \end{bmatrix}, \\
 \Pi_{13} &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ -\varrho_2 K_k D_k G_{1k} & -\varrho_2 K_k D_k G_{3k}^T & -\varrho_2 \bar{K}_k D_k G_{1k} & -\varrho_2 \bar{K}_k D_k G_{3k}^T \end{bmatrix}, \\
 \Pi_{14} &= \begin{bmatrix} 0 & 0 & C_k V_1 & 0 & 0 & 0 \\ -\bar{\lambda} \bar{K}_k D_k G_{3k} & -\bar{\lambda} \bar{K}_k D_k G_{2k} & C_k V_1 & -K_k V_2 & 0 & -\bar{K}_k V_2 \end{bmatrix}, \\
 \Pi_{22} &= \begin{bmatrix} -G_{1k} & -G_{3k}^T \\ * & -G_{2k} \end{bmatrix}, \\
 \Pi_{33} &= \text{diag}\{\Pi_{22}, \Pi_{22}\}, \\
 \Pi_{44} &= \text{diag}\{\Pi_{22}, -V_1, -V_2, -V_1, -V_2\}, \\
 \Omega_3 &= \bar{\alpha}(1 - \bar{\alpha})B_{1k}^T Z_{k+1} B_{2k}, \\
 \Omega_4 &= -\Gamma_4 + (1 - \bar{\alpha})^2 B_{2k}^T Z_{k+1} B_{2k}, \\
 \Omega_5 &= \bar{\alpha}(A_k^T - \bar{\lambda} D_k^T K_k^T) Z_{k+1} B_{1k} + \Gamma_2 \Lambda_2, \\
 \Omega_6 &= (1 - \bar{\alpha})(A_k^T - \bar{\lambda} D_k^T K_k^T) Z_{k+1} B_{2k} + \Gamma_4 \Sigma_2, \\
 \Omega_7 &= A_k^T Z_{k+1} - \bar{\lambda} D_k^T K_k^T Z_{k+1}, \\
 \Omega_8 &= -\Gamma_1 + \bar{\alpha} B_{1k}^T L_{k+1} B_{1k} + \bar{\alpha}(1 - \bar{\alpha}) B_{1k}^T Z_{k+1} B_{1k}, \\
 \Omega_9 &= -\Gamma_3 + (1 - \bar{\alpha}) B_{2k}^T L_{k+1} B_{2k} + \bar{\alpha}(1 - \bar{\alpha}) B_{2k}^T Z_{k+1} B_{2k}, \\
 \Omega_{10} &= -G_{1,k+1} + \bar{\alpha}^2 \varrho_5 \text{tr}(G_k) B_{1k} Y_1 B_{1k}^T + \varrho_1 \text{tr}(G_k) B_{1k} Y_1 B_{1k}^T + (1 - \bar{\alpha})^2 \varrho_3 \\
 &\quad \times \text{tr}(G_k) B_{2k} Y_2 B_{2k}^T + \varrho_6 \text{tr}(G_k) B_{2k} Y_2 B_{2k}^T + \epsilon_{2,k} H_k^T H_k, \\
 \Omega_{11} &= -G_{3,k+1}^T + \varrho_1 \text{tr}(G_k) B_{1k} Y_1 B_{1k}^T + \varrho_6 \text{tr}(G_k) B_{2k} Y_2 B_{2k}^T + \epsilon_{2,k} H_k^T H_k, \\
 \Omega_{12} &= -G_{2,k+1} + \bar{\alpha}^2 \varrho_5 \text{tr}(G_k) B_{1k} Y_1 B_{1k}^T + \varrho_1 \text{tr}(G_k) B_{1k} Y_1 B_{1k}^T + (1 - \bar{\alpha})^2 \varrho_3 \\
 &\quad \times \text{tr}(G_k) B_{2k} Y_2 B_{2k}^T + \varrho_6 \text{tr}(G_k) B_{2k} Y_2 B_{2k}^T + \epsilon_{2,k} H_k^T H_k, \\
 \mathcal{W}_k &= \begin{bmatrix} \bar{\alpha} H_k^T L_{k+1} B_{1k} & \bar{\alpha} H_k^T Z_{k+1} B_{1k} & (1 - \bar{\alpha}) H_k^T L_{k+1} B_{2k} & (1 - \bar{\alpha}) H_k^T Z_{k+1} B_{2k} \end{bmatrix}, \\
 \mathcal{Y}_k &= \begin{bmatrix} 0 & H_k^T L_{k+1} & H_k^T Z_{k+1} & 0 & 0 \end{bmatrix}, \\
 \mathcal{N}_k^T &= \begin{bmatrix} N_k & 0 \end{bmatrix}, \\
 \mathcal{X}_k &= \begin{bmatrix} \varrho_4 N_k G_{1k} & \varrho_4 N_k G_{3k}^T \end{bmatrix}, \\
 \mathcal{F}_k^T &= \begin{bmatrix} H_k^T & H_k^T \end{bmatrix},
 \end{aligned}$$

then we can conclude that the estimator design problem is solvable.

Proof Firstly, the matrices Q_k and G_k are decomposed as follows:

$$Q_k = \begin{bmatrix} L_k & 0 \\ * & Z_k \end{bmatrix}, \quad G_k = \begin{bmatrix} G_{1k} & G_{3k}^T \\ * & G_{2k} \end{bmatrix}.$$

Secondly, in order to deal with parameter uncertainty, we rewrite (27) as follows:

$$\begin{bmatrix} \mathcal{E}_{11} & \mathcal{E}_{12}^0 & \mathcal{E}_{13}^0 & \mathcal{E}_{14} & 0 \\ * & \mathcal{E}_{22} & 0 & 0 & 0 \\ * & * & \mathcal{E}_{33} & 0 & \mathcal{E}_{35} \\ * & * & * & \mathcal{E}_{44} & 0 \\ * & * & * & * & \mathcal{E}_{55} \end{bmatrix} + \bar{N}_k F_k \bar{H}_k + \bar{H}_k^T F_k^T \bar{N}_k^T < 0,$$

where

$$\begin{aligned} \mathcal{E}_{12}^0 &= \begin{bmatrix} \bar{\alpha} A_k^T L_{k+1} B_{1k} + \Gamma_1 A_2 & 0 & (1 - \bar{\alpha}) A_k^T L_{k+1} B_{2k} + \Gamma_3 \Sigma_2 & 0 \\ & \Omega_5 & & \Omega_6 \end{bmatrix}, \\ \mathcal{E}_{13}^0 &= \begin{bmatrix} 0 & A_k^T L_{k+1} & 0 & 0 & -\varrho_2 D_k^T K_k^T Z_{k+1} \\ 0 & 0 & \Omega_7 & 0 & 0 \end{bmatrix}, \\ \bar{N}_k^T &= [\mathcal{N}_k^T \quad 0 \quad 0 \quad 0 \quad 0], \\ \bar{H}_k &= \begin{bmatrix} 0 & \mathcal{W}_k & \mathcal{Y}_k & 0 & 0 \end{bmatrix}. \end{aligned}$$

Subsequently, it follows from Lemma 2 that

$$\begin{bmatrix} \mathcal{E}_{11} & \mathcal{E}_{12}^0 & \mathcal{E}_{13}^0 & \mathcal{E}_{14} & 0 \\ * & \mathcal{E}_{22} & 0 & 0 & 0 \\ * & * & \mathcal{E}_{33} & 0 & \mathcal{E}_{35} \\ * & * & * & \mathcal{E}_{44} & 0 \\ * & * & * & * & \mathcal{E}_{55} \end{bmatrix} + \epsilon_{1,k} \bar{N}_k \bar{N}_k^T + \epsilon_{1,k}^{-1} \bar{H}_k^T \bar{H}_k < 0.$$

Similarly, (28) can be rewritten as

$$\begin{bmatrix} -G_{k+1} & \Upsilon_{12}^0 & \Upsilon_{13} & \Upsilon_{14} \\ * & -G_k & 0 & 0 \\ * & * & \Upsilon_{33} & 0 \\ * & * & * & \Upsilon_{44} \end{bmatrix} + \tilde{N}_k F_k \tilde{H}_k + \tilde{H}_k^T F_k^T \tilde{N}_k^T < 0,$$

where

$$\begin{aligned} \Upsilon_{12}^0 &= \begin{bmatrix} \varrho_4 A_k G_{1k} & \varrho_4 A_k G_{3k}^T \\ \varrho_4 (A_k - \bar{\lambda} K_k D_k) G_{3k} & \varrho_4 (A_k - \bar{\lambda} K_k D_k) G_{2k} \end{bmatrix}, \\ \tilde{N}_k^T &= [\mathcal{F}_k^T \quad 0 \quad 0 \quad 0], \\ \tilde{H}_k &= \begin{bmatrix} 0 & \mathcal{X}_k & 0 & 0 \end{bmatrix}. \end{aligned}$$

Then it follows from Lemma 2 that

$$\begin{bmatrix} -G_{k+1} & \Upsilon_{12}^0 & \Upsilon_{13} & \Upsilon_{14} \\ * & -G_k & 0 & 0 \\ * & * & \Upsilon_{33} & 0 \\ * & * & * & \Upsilon_{44} \end{bmatrix} + \epsilon_{2,k} \tilde{N}_k \tilde{N}_k^T + \epsilon_{2,k}^{-1} \tilde{H}_k^T \tilde{H}_k < 0.$$

Now, it should be noted that (30) implies (27). Similarly, we can see that (31) leads to (28). As such, both the estimation error covariance constraint and H_∞ performance requirement of system (7) are ensured. The proof of Theorem 4 is complete. \square

Remark 5 Up to now, we have discussed the variance-constrained resilient H_∞ state estimation problem for a class of *time-varying* RNNs with randomly varying nonlinearities and missing measurements. By applying the recursive matrix inequality technique, some criteria have been established to guarantee the prescribed H_∞ performance and the estimation error covariance constraints for the addressed estimation problem of time-varying neural networks within the finite-horizon framework. It should be noticed that the proposed estimation approach has the following three advantages: (i) the disturbance effects can be effectively attenuated by the H_∞ performance index over finite horizon; (ii) the prescribed upper bound of the estimation error covariance can be guaranteed by verifying certain matrix inequalities; and (iii) the newly designed state estimation approach can be applied to the online calculations and implementations for solving the estimation problems of time-varying RNNs, which constitutes another appealing feature.

Remark 6 In fact, almost all existing estimation schemes can be applied to *time-invariant NNs* only, but we have made one of the first attempts to discuss the characteristics of the *time-varying RNNs* and address two combined performance indices to meet the practical requirements, which are the essential superiority of the proposed result. For example, compared with the non-fragile/resilient state estimation method in [10], our estimation scheme has the advantage to reveal the whole impacts from missing measurements and randomly varying nonlinearities onto the estimation algorithm performance, which can present a new treatment way. In contrast to the results in [11, 12], the superiority dealing with the time-varying characteristics can be observed from our new state estimation scheme.

5 An illustrative example

In this section, we give a simulation to illustrate the feasibility of proposed estimation approach under variance constraint. The parameters of time-varying RNNs (1) are given as follows:

$$\begin{aligned} A_k &= \begin{bmatrix} -0.5 & 0 \\ 0 & -0.1 \sin(2k) \end{bmatrix}, & B_{1k} &= \begin{bmatrix} -\sin(2k) & 0.5 \\ -0.2 & 0.5 \end{bmatrix}, & \Gamma_1 &= \begin{bmatrix} 0.9 & 0 \\ 0 & 0.9 \end{bmatrix}, \\ B_{2k} &= \begin{bmatrix} -0.27 \sin(k) & 0.2 \\ -0.1 & -0.14 \end{bmatrix}, & C_k &= \begin{bmatrix} -0.1 & -0.3 \sin(2k) \end{bmatrix}^T, & \Gamma_2 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \\ D_k &= \begin{bmatrix} -0.55 \sin(k) & 1.5 \end{bmatrix}, & \bar{K}_k &= \begin{bmatrix} -0.27 \sin(2k) & 0.15 \end{bmatrix}^T, & \Gamma_3 &= \begin{bmatrix} 1.1 & 0 \\ 0 & 1.1 \end{bmatrix}, \end{aligned}$$

Table 1 The values of estimator gain K_k

k	K_k
1	$K_1 = [-0.1701 \ 0.1102]^T$
2	$K_2 = [-0.1481 \ 0.2838]^T$
3	$K_3 = [-0.2438 \ 0.1024]^T$
\vdots	\vdots

$$\Gamma_4 = \begin{bmatrix} 1.3 & 0 \\ 0 & 1.3 \end{bmatrix}, \quad H_k = \begin{bmatrix} 0.1 & 0.15 \end{bmatrix}^T, \quad N_k = \begin{bmatrix} 0.2 & 0.1 \end{bmatrix}, \quad F_k = \sin(0.6k),$$

$$M_k = \begin{bmatrix} -0.01 & -0.12 \sin(k) \end{bmatrix}, \quad \bar{\lambda} = 0.34, \quad \bar{\alpha} = 0.1, \quad \rho = 0.5,$$

$$\varepsilon_1 = 0.5, \quad \varepsilon_2 = 0.3, \quad \varepsilon_3 = 0.2, \quad \varepsilon_4 = 0.1.$$

Moreover, the activation functions can be taken as

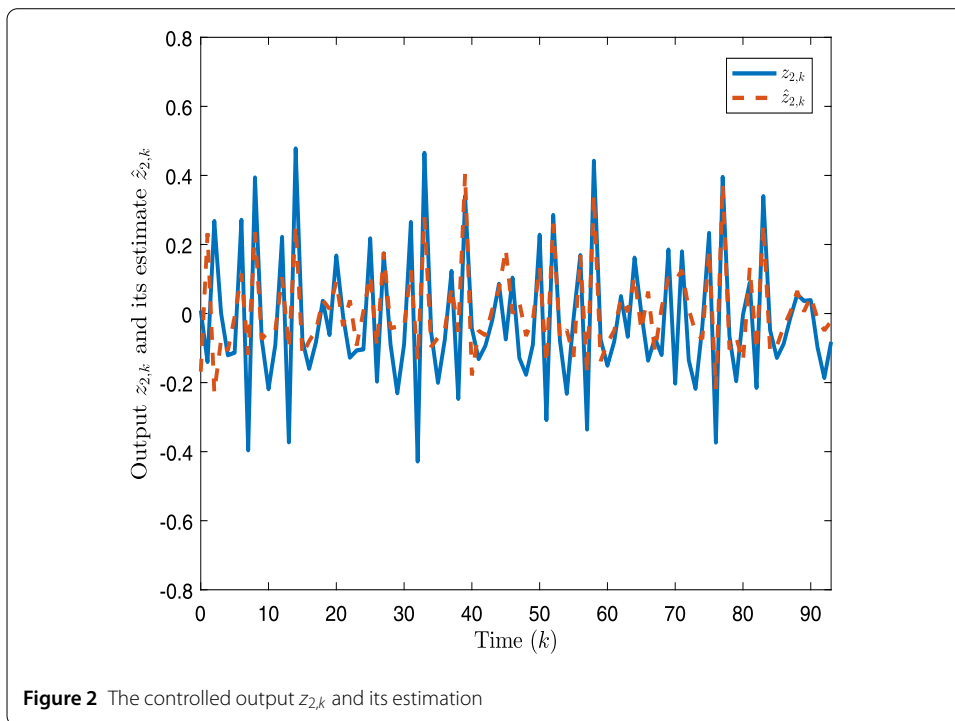
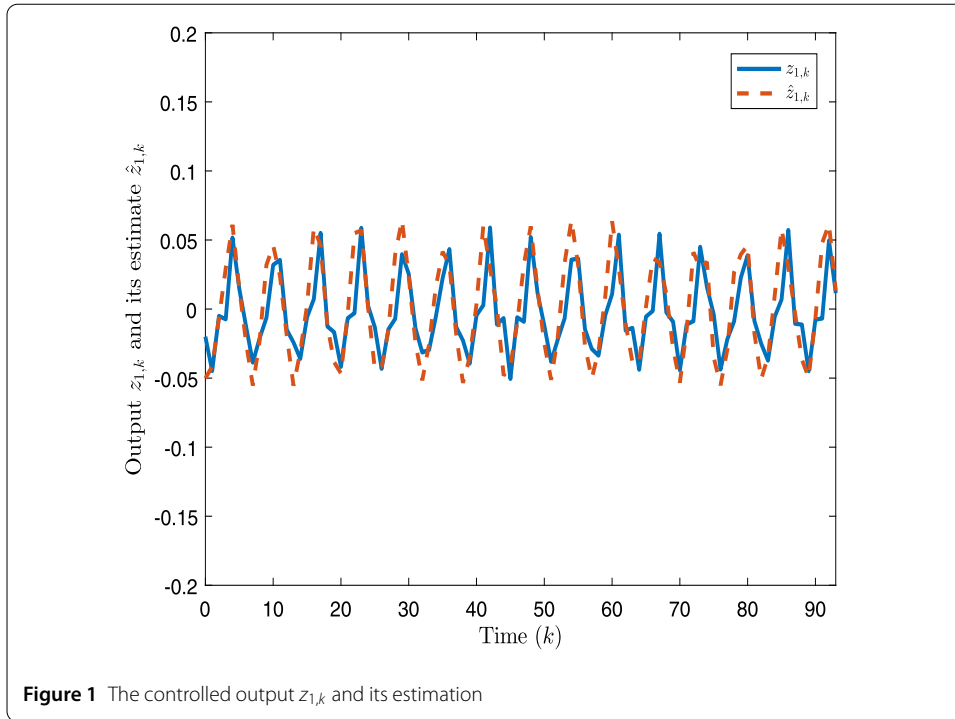
$$f(x_k) = g(x_k) = \begin{bmatrix} \tanh(x_{1,k}) \\ \tanh(0.8x_{2,k}) \end{bmatrix}$$

with $x_k = [x_{1,k} \ x_{2,k}]^T$ being the neuron state vector of neural network. It is easy to obtain $\Lambda_0 = \text{diag}\{0.1, 0.1\}$, $\Lambda_1 = \text{diag}\{0.2, 0.2\}$, $\Lambda_2 = \text{diag}\{0.9, 0.9\}$, $\Sigma_0 = \text{diag}\{0.2, 0.2\}$, $\Sigma_1 = \text{diag}\{0.3, 0.3\}$ and $\Sigma_2 = \text{diag}\{0.5, 0.5\}$. Let the disturbance attenuation level be $\gamma = 0.9$ and $N = 94$, weighted matrices as $W_\varphi(1) = W_\varphi(2) = 1$, upper bound matrices as $\{\Psi_k\}_{1 \leq k \leq N} = \text{diag}\{0.3, 0.3\}$, and covariances as $V_1 = V_2 = 1$. Choose the parameters' initial matrices satisfying (29). Then the matrix inequalities (30)–(32) in Theorem 4 can be solved, and K_k is designed as in Table 1.

Suppose the initial states as $x_0 = [-1.5 \ 0.3]^T$ and $\hat{x}_0 = [-1.2 \ 0.3]^T$. Based on the state estimation method in Theorem 4, the simulation results can be shown in Figs. 1–4. Figures 1–2 plot the output z_k and its estimation \hat{z}_k , respectively. Figure 3 depicts the output estimation error \tilde{z}_k . The error variance upper bound and actual error variance are plotted in Fig. 4, which indeed illustrates that the actual error variance below the error variance upper bound. From the simulations, we can conclude that the newly presented variance-constrained resilient H_∞ estimation algorithm is efficient.

6 Conclusions

In this paper, we have discussed the variance-constrained resilient H_∞ state estimation problem for a class of time-varying neural networks with randomly varying nonlinearities and missing measurements. Two random variables that obey Bernoulli distribution have been adopted to describe the phenomena of randomly varying nonlinearities and missing measurements. A new variance-constrained H_∞ state estimation method has been designed based on the available information. By applying the recursive matrix inequality technique, some criteria have been established to guarantee the prescribed H_∞ performance and the estimation error covariance constraints for the addressed estimation problem of the time-varying neural networks. In addition, the gain matrix of state estimator has been obtained by testing the feasibility of the concerned recursive matrix inequalities. Finally, the validity of the proposed estimation method has been verified by a simulation example. Our future research topics include the state estimation problems for



time-varying RNNs with the finite-time criterion as in [36] and the inaccuracy occurrence probability as mentioned in [49].

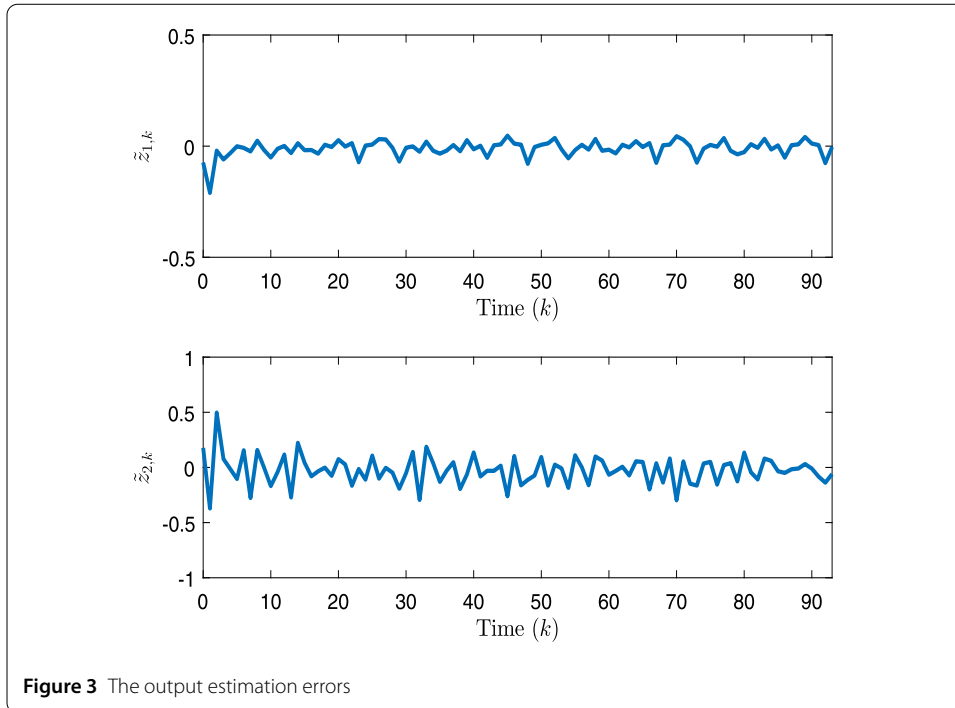


Figure 3 The output estimation errors

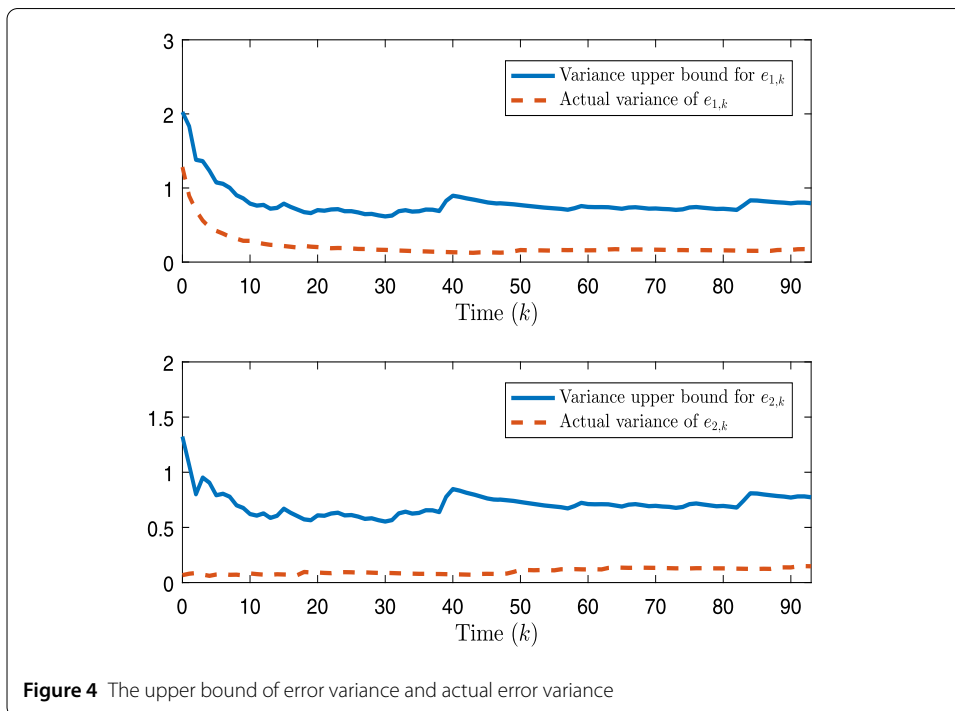


Figure 4 The upper bound of error variance and actual error variance

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