Yin et al. *Advances in Difference Equations* (2019) 2019:364
https://doi.org/10.1186/s13662-019-2299-6

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Complete monotonicity related to the k-polygamma functions with applications

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Abstract

In this paper, we prove complete monotonicity of some functions involving k-polygamma functions. As an application of the main result, we also give new upper and lower bounds of the k-digamma function.

MSC: Primary 33B15; secondary 26A48

Keywords: Complete monotonicity; k-digamma function; k-polygamma functions

1 Introduction

The Euler gamma function is defined for all positive real numbers x by

$$\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}\,dt.$$

The logarithmic derivative of $\Gamma(x)$ is called the psi or digamma function. That is,

$$\psi(x)=\frac{d}{dx}\ln\Gamma(x)=\frac{\Gamma'(x)}{\Gamma(x)}=-\gamma-\frac{1}{x}+\sum_{n=1}^{\infty}\frac{x}{n(n+x)},$$

where $\gamma=0.5772\ldots$ is the Euler–Mascheroni constant. The polygamma functions $\psi^{(m)}(x)$ for $m\in\mathbb{N}$ are defined by

$$\psi^{(m)}(x)=\frac{d^m}{dx^m}\psi(x)=(-1)^m m!\sum_{n=0}^{\infty}\frac{1}{(n+x)^{m+1}},\quad x>0.$$

The gamma, digamma and polygamma functions play an important role in the theory of special functions, and are closely related to factorial, fractional differential equations, mathematical physics and crops up in many unexpected place in analysis [13–17, 22–28, 40–45]. For some of the work as regards origin, history, the complete monotonicity, and inequalities of these special functions one may refer to [1–12, 18–21, 29, 30, 33–39] and the references therein.

In 2007, Díaz and Pariguan [16] defined the k-analog of the gamma function for $k>0$ and $x>0$ as

$$\Gamma_k(x)=\int_0^\infty t^{x-1}e^{-\frac{t^k}{k}}\,dt=\lim_{n\to\infty}\frac{n!k^n(nk)^{\frac{x}{k}-1}}{x(x+k)\cdots(x+(n-1)k)},$$

where $\lim_{k\to 1} \Gamma_k(x) = \Gamma(x)$. Similarly, we may define the k-analog of the digamma and polygamma functions as

$$\psi_k(x) = \frac{d}{dx}\ln\Gamma_k(x) \quad \text{and} \quad \psi_k^{(m)}(x) = \frac{d^m}{dx^m}\psi_k(x).$$

Hence, the authors continued the study of this family of generalized functions, and suggested that many properties of classical gamma, digamma and polygamma functions have a counterpart in this more general setting. It would be natural to generalize the properties of classical functions to the k-gamma, digamma and polygamma functions.

It is well known that the k-analogues of the digamma and polygamma functions satisfy the following recursive formula and series identities (see [16, 31, 32]):

$$\Gamma_k(x+k) = x\Gamma_k(x), \quad x > 0, \tag{1.1}$$

$$\begin{aligned}\psi_k(x) &= \frac{\ln k - \gamma}{k} - \frac{1}{x} + \sum_{n=1}^{\infty}\frac{x}{nk(nk+x)}\\ &= -\int_0^{\infty}\frac{e^{-xt}}{1-e^{-kt}}\,dt,\end{aligned} \tag{1.2}$$

and

$$\begin{aligned}\psi_k^{(m)}(x) &= (-1)^{m+1}m!\sum_{n=0}^{\infty}\frac{1}{(nk+x)^{m+1}}\\ &= (-1)^{m+1}\int_0^{\infty}\frac{1}{1-e^{-kt}}t^m e^{-xt}\,dt.\end{aligned} \tag{1.3}$$

A function f is said to be completely monotonic on an interval I if f has derivatives of all orders on I and satisfies $(-1)^n f^{(n)}(x) \geq 0$ for $x \in I$ and $n \geq 0$. A characterization of completely monotonic functions is given by the Bernstein–Widder theorem which reads that a function $f(x)$ on $x \in [0,\infty)$ is completely monotonic if and only if there exists a bounded and non-decreasing function $g(t)$ such that the integral

$$f(x) = \int_0^{\infty} e^{-xt}\,dg(t)$$

converges for $x \in [0,\infty)$. That is, a function $f(x)$ is completely monotonic on $x \in [0,\infty)$ if and only if it is a Laplace transform of a bounded and non-decreasing measure $g(t)$. From the above theorem it follows that completely monotonic functions on $[0,\infty)$ are always strictly completely monotonic unless they are constant (see [34]).

At present, these functions have been extensively studied. In [46], Yin *et al.* gave a concave theorem and some inequalities for the k-digamma function. Furthermore, Yin *et al.* [47] showed several monotonic and concave results related to the generalized digamma and polygamma functions. In [48], Zhao, Guo and Qi showed several complete monotonicity of two functions involving the tri- and tetra-gamma functions. Motivated by their work, we give a k-analog of their results. Furthermore, we also prove a new double inequality about k-polygamma functions. Finally, an application of the main result leads to new upper and lower bounds of the k-digamma function.

2 Main results

Lemma 2.1 *For* $k > 0$, *we have*

$$\psi_k(x) = \frac{\ln k}{k} + \frac{\psi(x/k)}{k} \tag{2.1}$$

and

$$\psi_k'(x) = \frac{1}{k^2}\psi'\left(\frac{x}{k}\right). \tag{2.2}$$

Proof Taking logarithms and differentiating on both sides of the formula

$$\Gamma_k(x) = k^{\frac{x}{k}-1}\Gamma\left(\frac{x}{k}\right), \tag{2.3}$$

we easily obtain Eq. (2.1). Differentiating on both sides of (2.1), we get (2.2). □

Lemma 2.2 *For* $k > 0$, *the following recursion formulas hold true*:

$$\begin{aligned} \psi_k'(x+k) &= \psi_k'(x) - \frac{1}{x^2}, \\ \psi_k''(x+k) &= \psi_k''(x) + \frac{2}{x^3}. \end{aligned} \tag{2.4}$$

Proof By using Eq. (1.1), we easily obtain the proof. □

Lemma 2.3 ([48, Eq. (12)]) *Let* $r > 0$. *Then*

$$\frac{1}{x^r} = \frac{1}{\Gamma(r)}\int_0^\infty t^{r-1}e^{-xt}\,dt. \tag{2.5}$$

Theorem 2.1 *Let* $k > 0$. *Then the function*

$$f_k(x) = (x+k)^2\left[\psi_k'(x) - \frac{1}{x^2} - \frac{1}{k(x+k)}\right]$$

is completely monotonic on $(0,\infty)$.

Proof By the integral representation (1.3) and integration by parts, we have

$$\begin{aligned} x\psi_k'(x) &= x\int_0^\infty \frac{te^{-xt}}{1-e^{-kt}}\,dt = \int_0^\infty \frac{-t}{1-e^{-kt}}\,de^{-xt} \\ &= \frac{-te^{-xt}}{1-e^{-kt}}\Big|_\infty^0 - \int_0^\infty \frac{d}{dt}\left(\frac{-t}{1-e^{-kt}}\right)e^{-xt}\,dt \\ &= \lim_{t\to 0}\frac{te^{(k-x)t}}{e^{kt}-1} + \int_0^\infty \frac{d}{dt}\left(\frac{te^{kt}}{e^{kt}-1}\right)e^{-xt}\,dt \\ &= \frac{1}{k} + \int_0^\infty \frac{e^{2kt}-e^{kt}-kte^{kt}}{(e^{kt}-1)^2}e^{-xt}\,dt. \end{aligned} \tag{2.6}$$

By using (2.6) and integration by parts, we also easily obtain

$$\begin{aligned} x^2\psi_k'(x) &= \frac{x}{k} + x\int_0^\infty \frac{e^{2kt} - e^{kt} - kte^{kt}}{(e^{kt}-1)^2} e^{-xt}\,dt \\ &= \frac{x}{k} + \frac{1}{2} + x\int_0^\infty \frac{e^{kt}((k^2t - 2k)e^{kt} + 2k + k^2t)}{(e^{kt}-1)^3} e^{-xt}\,dt. \end{aligned} \tag{2.7}$$

Furthermore, direct computation results in

$$f_k(x) = x^2\psi_k'(x) + 2kx\psi_k'(x) + k^2\psi_k'(x) - 2 - \frac{x}{k} - \frac{2k}{x} - \frac{k^2}{x^2}. \tag{2.8}$$

Considering (2.6)–(2.8) and Lemma 2.3, we easily get

$$f_k(x) = \frac{1}{2} + \int_0^\infty \frac{W_k(t)}{(e^{kt}-1)^3} e^{-xt}\,dt,$$

where

$$W_k(t) = \bigl(k^2t - 2k\bigr)e^{kt} + 2k + k^2t.$$

Next, we shall prove $W_k(t) > 0$ for $t \in (0,\infty)$. Simple calculation gives

$$W_k'(t) = k^3te^{kt} - k^2e^{kt} + k^2$$

and

$$W_k''(t) = k^4te^{kt} > 0.$$

From the facts that $W_k'(0^+) = \lim_{t\to 0^+} W_k'(t) = 0$ and $W_k(0^+) = \lim_{t\to 0^+} W_k(t) = 0$, it follows that the functions $W_k'(t)$ and $W_k(t)$ are increasing and positive on $(0,\infty)$. By computation, we get

$$-f_k'(x) = \int_0^\infty \frac{tW_k(t)}{(e^{kt}-1)^3} e^{-xt}\,dt.$$

In consequence, the function $-f_k'(x)$ is completely monotonic on $(0,\infty)$. This means that

$$(-1)^n\bigl(-f_k'(x)\bigr)^{(n)} = (-1)^{n+1}\bigl(f_k(x)\bigr)^{(n+1)} > 0.$$

It is easy to check that $f_k(x) \ge \frac{1}{2} > 0$. Consequently, the function $f_k(x)$ is completely monotonic on $(0,\infty)$. □

Corollary 2.1 *For $x > 0$ and $k > 0$, we have*

$$\frac{1}{k(x+k)} + \frac{1}{x^2} + \frac{a}{(x+k)^2} < \psi_k'(x) < \frac{1}{k(x+k)} + \frac{1}{x^2} + \frac{b}{(x+k)^2}$$

with the best possible constants $a = \frac{1}{2}$ and $b = \frac{\pi^2}{6} - 1$.

Proof Complete monotonicity of the function $f_k(x)$ implies that the function $f_k(x)$ is decreasing on $(0,\infty)$. Therefore, we have

$$\lim_{x\to\infty} f_k(x) = f_k(\infty) < f_k(x) < f_k(0^+) = \lim_{x\to 0^+} f_k(x).$$

Applying Lemma 2.2, we get

$$f_k(x) = (x+k)^2\left[\psi_k'(x+k) - \frac{1}{(x+k)^2}\right].$$

It is easily seen that $f_k(0^+) = k^2\psi_k'(k) - 1 = \frac{\pi^2}{6} - 1$. On the other hand, using the asymptotic formula (see [1])

$$\psi'(x) \sim \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \cdots, \quad x\to\infty,$$

and (2.2), we can conclude

$$f_k(x) = \frac{1}{2} + o\left(\frac{1}{(x+k)}\right) \to \frac{1}{2}, \quad x\to\infty.$$

This completes the proof. □

Remark 2.1 In [47, Lemma 2.4], Yin *et al.* gave an estimation of $\psi_k'(x)$ as follows:

$$\frac{1}{kx} < \psi_k'(x) < \frac{1}{kx} + \frac{1}{x^2}.$$

Here, we give another inequality of $\psi_k'(x)$.

Theorem 2.2 *Let* $0<k\le 1$. *Then the functions*

$$\alpha_k(x) = k\big(\psi_k'(x)\big)^2 + \psi_k''(x) - \frac{k(x^2+12k^2)}{12x^4(x+k)^2}$$

and

$$\beta_k(x) = \frac{k(x+12k)}{12x^4(x+k)} - k\big(\psi_k'(x)\big)^2 - \psi_k''(x)$$

are completely monotonic on $(0,\infty)$. *As a direct result, for* $0<k\le 1$ *and* $x\in(0,\infty)$, *we have the following double inequality*:

$$\frac{k(x^2+12k^2)}{12x^4(x+k)^2} < k\big(\psi_k'(x)\big)^2 + \psi_k''(x) < \frac{k(x+12k)}{12x^4(x+k)}. \tag{2.9}$$

Proof By the recursion formula (2.4), we get

$$\begin{aligned}\alpha_k(x) - \alpha_k(x+k) &= k\big[\psi_k'(x) + \psi_k'(x+k)\big]\big[\psi_k'(x) - \psi_k'(x+k)\big] \\ &\quad + \psi_k''(x) - \psi_k''(x+k) - \left[\frac{k(x^2+12k^2)}{12x^4(x+k)^2} - \frac{k((x+k)^2+12k^2)}{12(x+k)^4(x+2k)^2}\right]\end{aligned}$$

$$
\begin{aligned}
&= \frac{2k}{x^2}\left[\psi_k'(x) - \frac{1}{2x^2} - \frac{1}{kx} - \frac{(x^2+12k^2)}{24x^2(x+k)^2} + \frac{x^2((x+k)^2+12k^2)}{24(x+k)^4(x+2k)^2}\right] \\
&= \frac{2k}{x^2} g_k(x),
\end{aligned}
$$

where

$$
\begin{aligned}
g_k(x) = \psi_k'(x) &- \frac{1}{x^2} - \frac{k^2}{2(x+k)^4} - \frac{2k}{(x+k)^3} \\
&+ \frac{7}{2(x+k)^2} - \frac{43}{6k(x+k)} + \frac{37}{6k(x+2k)} + \frac{13}{6(x+2k)^2}.
\end{aligned}
$$

Applying (1.3) and (2.5), we have

$$
g_k(x) = \frac{1}{12k}\int_0^\infty \frac{q_k(x)}{e^{kt}-1} e^{-(x+2k)t}\,dt,
$$

where

$$
\begin{aligned}
q_k(x) = {} & e^{2kt}\bigl(k^3t^3 - 12k^2t^2 + 12kt + 42t - 86\bigr) \\
& + e^{kt}\bigl(-k^3t^3 + 12k^2t^2 - 16t + 160\bigr) - 26t - 74.
\end{aligned}
$$

Direct calculation yields

$$
\begin{aligned}
q_k'(x) = {} & e^{2kt}\bigl(2k^4t^3 - 21k^3t^2 + 84kt + 42 - 160k\bigr) \\
& + e^{kt}\bigl(-k^4t^3 + 9k^3t^2 + \bigl(24k^2 - 16k\bigr)t + 160k - 16\bigr) - 26
\end{aligned}
$$

and

$$
q_k''(x) = e^{kt}\lambda_k(x),
$$

where

$$
\begin{aligned}
\lambda_k(x) = {} & e^{kt}\bigl(4k^5t^3 - 36k^4t^2 + 168k^2t - 42k^3t + 168k - 320k^2\bigr) \\
& - k^5t^3 + 6k^4t^2 + \bigl(42k^3 - 16k^2\bigr)t + 184k^2 - 32k.
\end{aligned}
$$

Further computation gives

$$
\begin{aligned}
\lambda_k'(x) = {} & k^2\bigl(-16 + 42k + 12k^2t - 3k^3t^2\bigr) \\
& + 2k^2e^{kt}\bigl(168 - 57k^2t - 12k^3t^2 + 2k^4t^3 - 181k + 84kt\bigr), \\
\lambda_k''(x) = {} & 2k^3\bigl[-3k(-2+kt) + e^{kt}\bigl(252 - 81k^2t - 6k^3t^2 + 2k^4t^3 + 14k(-17+6t)\bigr)\bigr], \\
\lambda_k'''(x) = {} & -6k^5 + 2k^4e^{kt}\bigl[336 - 319k + \bigl(84k - 93k^2\bigr)t + 2k^4t^3\bigr].
\end{aligned}
$$

Since $0 < k \le 1$, the function $336 - 319k + (84k - 93k^2)t + 2k^4t^3$ attains minimum value $14 - 3\sqrt{6}$ as $t \to \sqrt{3/2}$ and $k \to 1$. This implies $\lambda_k'''(x) > 0$. From the facts $\lambda_k'(0) = k^2(320 - 320k) > 0$ and $\lambda_k''(0) = 2k^3(252 - 232k) > 0$, it follows that the functions $\lambda_k'(x), \lambda_k''(x), \lambda_k'''(x)$

are increasing and positive on $(0,\infty)$. Thus, the derivative $q_k''(x)$ is positive, and so the function $q_k'(x)$ is increasing on $(0,\infty)$. Since $q_k'(0)=0$, the function $q_k'(t)$ is positive and $q_k(t)$ is increasing on $(0,\infty)$. Since $q_k(0)=0$, $q_k(t)$ is positive on $(0,\infty)$.

Positivity of $q_k(t)$ leads to the complete monotonicity of $g_k(x)$ on $(0,\infty)$. Since $\frac{2k}{x^2}$ is completely monotonic on $(0,\infty)$ and the product of finite completely monotonic functions is also completely monotonic, the difference $\alpha_k(x)-\alpha_k(x+k)$ is completely monotonic on $(0,\infty)$. That is,

$$(-1)^n\big(\alpha_k(x)-\alpha_k(x+k)\big)^{(n)}=(-1)^n\big(\alpha_k(x)\big)^{(n)}-(-1)^n\big(\alpha_k(x+k)\big)^{(n)}>0.$$

By mathematical induction, we get

$$(-1)^n\big(\alpha_k(x)\big)^{(n)}>(-1)^n\big(\alpha_k(x+k)\big)^{(n)}>\cdots>(-1)^n\big(\alpha_k(x+ik)\big)^{(n)}\to 0.$$

So, we prove that the function $\alpha_k(x)$ is completely monotonic on $(0,\infty)$. A completely similar method may apply to the function $\beta_k(x)$. Here, we omit the details for the sake of simplicity. □

Remark 2.2 Taking $k=1$ in inequality (2.9), we obtain [48, Theorem 1(8)].

3 An application

In this section, we shall give an application to obtain the bounds of the k-digamma function by using Theorem 2.2.

Lemma 3.1 *For $x>0$ and $0<k\le 1$, we have $\psi_k'(x)e^{k\psi_k(x)}<\frac{1}{k}$.*

Proof By using inequality (2.9), we have

$$\frac{d}{dx}\big(k\psi_k(x)+\ln\psi_k'(x)\big)>0,\quad x>0.$$

This means that $k\psi_k(x)+\ln\psi_k'(x)$ is strictly increasing on $(0,\infty)$. By [31] for $x>0$ and $0<k\le 1$, we have

$$\frac{1}{k}\ln x-\frac{1}{x}<\psi_k(x)<\frac{1}{k}\ln x.$$

This gives

$$x\psi_k'(x)e^{-\frac{k}{x}}<\psi_k'(x)e^{k\psi_k(x)}<x\psi_k'(x). \tag{3.1}$$

Using Eq. (2.2) and an asymptotic representation of $\psi(x)$, we can get

$$\psi_k'(x)\sim\frac{1}{kx}+\frac{1}{2x^2}+\frac{k}{6x^3}-\cdots,\quad x\to\infty.$$

Furthermore, we get $\lim_{x\to\infty}x\psi_k'(x)=\frac{1}{k}$. Hence, by inequality (3.1), we find that $\lim_{x\to\infty}k\psi_k(x)+\ln\psi_k'(x)=\ln(\frac{1}{k})$. So the proof follows from the monotonicity of the function $k\psi_k(x)+\ln\psi_k'(x)$. □

Lemma 3.2 *Let $0 < k \leq 1$. Then the function $A_k(x) = \frac{1}{k}e^{k\psi_k(x+k)} - \frac{x}{k}$ is strictly decreasing and strictly convex on $(-k, \infty)$.*

Proof Simple computation yields

$$A_k'(x) = \psi_k'(x+k)e^{k\psi_k(x+k)} - \frac{1}{k}$$

and

$$A_k''(x) = \left[k\left(\psi_k'(x+k)\right)^2 + \psi_k''(x+k)\right]e^{k\psi_k(x+k)}.$$

By applying Lemma (3.1) and inequality (2.9), we easily obtain $A_k'(x) < 0$ and $A_k''(x) > 0$. The proof is complete. □

Theorem 3.1 *For $0 < k \leq 1$ and $x > 0$, we have*

$$\frac{\ln k}{k} + \frac{1}{k}\ln\left(\frac{x}{k} + \frac{1}{2}\right) - \frac{1}{x} < \psi_k(x) < \frac{\ln k}{k} + \frac{1}{k}\ln\left(\frac{x}{k} + e^{-\gamma}\right) - \frac{1}{x}. \tag{3.2}$$

The constants $\frac{1}{2}$ and $e^{-\gamma}$ in (3.2) are the best possible as $x \to \infty$.

Proof Direct calculation results in $\lim_{x\to 0^+} A_k(x) = e^{k[\psi_k(k) - \frac{\ln k}{k}]} = e^{-\gamma}$ and $\lim_{x\to\infty} A_k(x) = \frac{1}{2}$. Noting that the function $A_k(x)$ is strictly increasing on $(0, \infty)$, we easily complete the proof. □

4 Conclusion

In this paper, we mainly proved the following theorems: Let $k > 0$. Then the function

$$f_k(x) = (x+k)^2\left[\psi_k'(x) - \frac{1}{x^2} - \frac{1}{k(x+k)}\right]$$

is completely monotonic on $(0, \infty)$.

Let $0 < k \leq 1$. Then the functions

$$\alpha_k(x) = k\left(\psi_k'(x)\right)^2 + \psi_k''(x) - \frac{k(x^2 + 12k^2)}{12x^4(x+k)^2}$$

and

$$\beta_k(x) = \frac{k(x + 12k)}{12x^4(x+k)} - k\left(\psi_k'(x)\right)^2 - \psi_k''(x)$$

are completely monotonic on $(0, \infty)$. As an application of Theorem 2.2, we also give new upper and lower bounds of the k-digamma function.

Acknowledgements

The authors would like to thank the editor and the anonymous referee for their valuable suggestions and comments, which helped us to improve this paper greatly.

Funding
This work was supported by National Natural Science Foundation of China (Grant No. 11701320 and 11601036), the Science and Technology Foundations of Shandong Province (Grant No. J17KA161 and J18KB105) and Science Foundation of Binzhou University (Grant No. BZXYL1704).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript and read and approved the final manuscript.

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Received: 8 May 2019 Accepted: 16 August 2019 Published online: 28 August 2019

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