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Sixth-order finite difference scheme for the Helmholtz equation with inhomogeneous Robin boundary condition

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Abstract

In this paper, a class of sixth-order finite difference schemes for the Helmholtz equation with inhomogeneous Robin boundary condition is derived. This scheme is based on the sixth-order approximation for the Robin boundary condition by using the Helmholtz equation and the Taylor expansion, by which the ghost points in the scheme on the domain can be eliminated successfully. Some numerical examples are shown to verify its correctness and robustness with respect to the wave number.

Keywords: Helmholtz equation; Inhomogeneous Robin boundary condition; Sixth-order scheme; Finite difference scheme

1 Introduction

In this paper, we focus on the Helmholtz equation with inhomogeneous Robin boundary condition on four boundaries

$$-\Delta u - k^2 u = f, \quad \text{in } \Omega, \tag{1}$$

$$iku + \frac{\partial u}{\partial \mathbf{n}} = g, \quad \text{in } \Gamma_\Omega, \tag{2}$$

where u is the pressure field, k is the wave number, f is the body force, $\Omega := (0,1) \times (0,1)$, $i = \sqrt{-1}$, $\mathbf{n}$ is the unit out normal vector, g is a given function and $\Gamma_\Omega = \Gamma^b \cup \Gamma^r \cup \Gamma^t \cup \Gamma^l$ with $\Gamma^b := [0,1] \times \{0\}$, $\Gamma^r := \{1\} \times [0,1]$, $\Gamma^t := [1,0] \times \{1\}$, $\Gamma^l := \{0\} \times [1,0]$.

Equations (1)–(2) are a mathematical model describing the acoustic scattering that controls the wave propagation and scattering phenomena occurring in many fields. When the wave number k is large, the solution of the problem becomes highly oscillating and efficient numerical methods are required in order to get high performance simulation results. In this topic, various numerical methods were developed in the past decades, such as the finite difference method (see, e.g., [1–16]), the finite element method (see, e.g., [17–25]), the boundary element method (see, e.g., [26–30]), and other techniques (see, e.g., [17, 31–33]). For the finite difference method, two common methods are considered in the literature, namely the parameter method and the high-order method. The parameter method was studied in [9], the fourth-order finite difference schemes with the Dirichlet boundary condition were considered in [3, 10] and with the Neumann boundary condition in [1,

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4], the sixth-order finite difference schemes with the Dirichlet boundary condition were investigated in [3, 10] and with the Neumann boundary condition in [2]. For the problem with Robin boundary condition, Turkel et al. [12] considered a sixth-order scheme for the homogeneous case, i.e.,

$$\frac{\partial u}{\partial x} + iku = 0 \quad \text{at } x = x_0, \tag{3}$$

based on a radiation boundary condition, which is usually imposed in the far-field when the medium is constant. By differentiating (3) with respect to x several times, the high-order scheme for this kind of boundary condition was derived in [12]. Further research can be found in [14]. But in many cases, such as the Helmholtz equation after reduction from the large cavity electromagnetic scattering, the inhomogeneous Robin boundary condition (2) is necessary (see [6, 34–36]). Obviously, the high-order scheme in this case cannot be got following the process in [12] due to the existence of the function g. On the other hand, the Robin boundary condition is only imposed on a single boundary in [12, 14]. If two neighbor boundaries are imposed with this kind of boundary condition, there will be a corner point which satisfies two different boundary conditions and it is difficult to deal with when this two conditions are not compatible. Recently, the fourth-order scheme for the Helmholtz equation with inhomogeneous Robin boundary condition on four boundaries was derived in [7]. But to the best of our knowledge, no sixth-order difference scheme for this case was investigated.

In many cases, the body force f is equal to zero, then we simplify the Helmholtz equation with an inhomogeneous Robin boundary condition as follows:

$$-\Delta u - k^2 u = 0, \quad \text{in } \Omega, \tag{4}$$

$$\alpha u + \frac{\partial u}{\partial \mathbf{n}} = g, \quad \text{in } \Gamma_\Omega. \tag{5}$$

In this paper, by applying the one-sided approximation of the derivative and Taylor expansion carefully, we derive the sixth-order scheme for the inhomogeneous Robin boundary condition (5), by which the ghost points in the scheme on the domain can be eliminated successfully. The deduction here does not depend on the scheme on the domain. Then some numerical experiments are shown to verify the correctness and the robustness of the scheme with respect to the wave number, too.

2 Numerical scheme

Let $0 < h < 1$ denote an uniform mesh size with $h = \frac{1}{N-1}$ $(N \in Z^+, N > 1)$. Then, for any grid point (x_m, y_n), we have $x_m = mh, m = 1, \ldots, N$, $y_n = nh, n = 1, \ldots, N$. Write $u_{m,n} = u(x_m, y_n)$. Due to the sixth-order schemes for the Helmholtz equation with Dirichlet and Neumann boundary conditions having been investigated before, by applying the results in [2, 10], we can get the scheme on the domain:

$$\mathbf{A}^d \cdot \mathbf{U}^d_{m,n} = 0, \quad m = 1, \ldots, N, n = 1, \ldots, N, \tag{6}$$

where

$$\mathbf{A}^d = \left(A^d_1, A^d_2, A^d_3, A^d_4, A^d_5, A^d_6, A^d_7, A^d_8, A^d_9\right),$$

$$\mathbf{U}^d_{m,n} = (U_{m-1,n-1}, U_{m,n-1}, U_{m+1,n-1}, U_{m-1,n}, U_{m,n}, U_{m+1,n}, U_{m-1,n+1}, U_{m,n+1}, U_{m+1,n+1}),$$
$$A_0 := A^d_5, \qquad A_1 := A^d_2 = A^d_4 = A^d_6 = A^d_8, \qquad A_2 := A^d_1 = A^d_3 = A^d_7 = A^d_9.$$

$U_{m,n}$ is the numerical solution of $u(x_m, y_n)$, and A_0 is the coefficient of the central point $U_{m,n}$, A_1 is the coefficient of $U_{m,n-1}, U_{m-1,n}, U_{m+1,n}, U_{m,n+1}$, A_2 is the coefficient of four corner points $U_{m-1,n-1}, U_{m+1,n-1}, U_{m-1,n+1}, U_{m+1,n+1}$. They may take different values at different occurrences which will be specialized in Sect. 3.

Next, we derive the scheme at the interior point of the boundary. Without loss of generality, we only present the deduction in detail for the top boundary $\varGamma^t$, which satisfies

$$\alpha u + \frac{\partial u}{\partial y} = g^t, \tag{7}$$

where g^t is a given function on the top boundary. By applying the Taylor formula, we have

$$\frac{u_{m,n+1} - u_{m,n-1}}{2h} = \left(\frac{\partial u}{\partial y}\right)_{m,n} + \frac{h^2}{6}\left(\frac{\partial^3 u}{\partial y^3}\right)_{m,n} + \frac{h^4}{120}\left(\frac{\partial^5 u}{\partial y^5}\right)_{m,n} + \mathcal{O}(h^6). \tag{8}$$

It is valid that

$$\frac{\partial^3 u}{\partial y^3} = -k^2\frac{\partial u}{\partial y} - \frac{\partial^3 u}{\partial x^2 \partial y}, \qquad \frac{\partial^5 u}{\partial y^5} = -k^2\frac{\partial^3 u}{\partial y^3} - \frac{\partial^5 u}{\partial x^2 \partial y^3}. \tag{9}$$

On the one hand, $\frac{\partial^3 u}{\partial x^2 \partial y}$ can be written as

$$\left(\frac{\partial^3 u}{\partial x^2 \partial y}\right)_{m,n} = \delta_x^2\delta_y u_{m,n} - \frac{h^2}{12}\left(\frac{\partial^5 u}{\partial x^4 \partial y} + 2\frac{\partial^5 u}{\partial x^2 \partial y^3}\right)_{m,n} + \mathcal{O}(h^4), \tag{10}$$

where $\delta_x^2 u_{m,n} = \frac{u_{m-1,n} - 2u_{m,n} + u_{m+1,n}}{h^2}$, $\delta_y u_{m,n} = \frac{u_{m,n+1} - u_{m,n-1}}{2h}$. Using (4), we get

$$\frac{\partial^5 u}{\partial x^4 \partial y} + \frac{\partial^5 u}{\partial x^2 \partial y^3} = -k^2\frac{\partial^3 u}{\partial x^2 \partial y}. \tag{11}$$

Setting $n = N$, then combining (8) with (6), (9), (10) and (11), we can eliminate $U_{m-1,N+1}$ and $U_{m+1,N+1}$ as follows:

$$\begin{aligned} &2\left(\frac{1}{6} + \frac{k^2h^2}{180}\right)A_2(U_{m-1,N-1} + U_{m+1,N-1}) + \left[\left(\frac{2}{3} - \frac{k^2h^2}{90}\right)A_2 + \left(\frac{1}{6} + \frac{k^2h^2}{180}\right)A_1\right]U_{m,N-1} \\ &\quad + \left(\frac{1}{6} + \frac{k^2h^2}{180}\right)A_1(U_{m-1,N} + U_{m+1,N}) \\ &\quad + \left[\left(-\frac{2}{3} + \frac{k^2h^2}{90}\right)A_2 + \left(\frac{1}{6} + \frac{k^2h^2}{180}\right)A_1\right]U_{m,N+1} \\ &\quad + \left(\frac{1}{6} + \frac{k^2h^2}{180}\right)A_0U_{m,N} + 2h\left(1 - \frac{k^2h^2}{6} + \frac{k^4h^4}{120}\right)A_2\left(\frac{\partial u}{\partial y}\right)_{m,N} \\ &\quad + \frac{h^5A_2}{90}\left(\frac{\partial^5 u}{\partial x^2 \partial y^3}\right)_{m,N} = 0. \end{aligned} \tag{12}$$

But the ghost point $U_{m,N+1}$ and $\frac{\partial^5 u}{\partial x^2 \partial y^3}$ are left in the scheme which need to be dealt with. By using the Taylor formula, (4) and (7) repeatedly, we obtain, after setting $n = N$,

$$\left(\frac{\partial^5 u}{\partial x^2 \partial y^3}\right)_{m,N} = -\alpha\left(\frac{\partial^4 u}{\partial x^2 \partial y^2}\right)_{m,N} - k^2\left(\frac{\partial^2 g^t}{\partial x^2}\right)_{m,N} - \left(\frac{\partial^4 g^t}{\partial x^4}\right)_{m,N}, \tag{13}$$

$$\left(\frac{\partial^3 u}{\partial y^3}\right)_{m,N} = -k^2\left(\frac{\partial u}{\partial y}\right)_{m,N} + \alpha\left(\frac{\partial^2 u}{\partial x^2}\right)_{m,N} - \left(\frac{\partial^2 g^t}{\partial x^2}\right)_{m,N}, \tag{14}$$

$$\begin{aligned}\left(\frac{\partial^5 u}{\partial y^5}\right)_{m,N} &= k^4\left(\frac{\partial u}{\partial y}\right)_{m,N} - \alpha k^2\left(\frac{\partial^2 u}{\partial x^2}\right)_{m,N} + \alpha\left(\frac{\partial^4 u}{\partial x^2 \partial y^2}\right)_{m,N} \\ &\quad + 2k^2\left(\frac{\partial^2 g^t}{\partial x^2}\right)_{m,N} + \left(\frac{\partial^4 g^t}{\partial x^4}\right)_{m,N}.\end{aligned} \tag{15}$$

According to (8), (14) and (15), we have

$$\begin{aligned}u_{m+1,N} = u_{m-1,N} &+ 2h\left(1 - \frac{k^2h^2}{6} + \frac{k^4h^4}{120}\right)\left(\frac{\partial u}{\partial y}\right)_{m,N} + 2h\alpha\left(\frac{h^2}{6} - \frac{k^2h^4}{120}\right)\left(\frac{\partial^2 u}{\partial x^2}\right)_{m,N} \\ &+ \frac{h^5\alpha}{60}\left(\frac{\partial^4 u}{\partial x^2 y^2}\right)_{m,N} - 2h\left(\frac{h^2}{6} - \frac{2k^2h^4}{120}\right)\left(\frac{\partial^2 g^t}{\partial x^2}\right)_{m,N}.\end{aligned} \tag{16}$$

Since we need a sixth-order finite difference scheme, we need a fourth-order approximation to $\frac{\partial^2 u}{\partial x^2}$ in (16). Obviously, it is valid that

$$u_{m+1,N} + u_{m-1,N} = 2u_{m,N} + h^2\left(\frac{\partial^2 u}{\partial x^2}\right)_{m,N} + \frac{h^4}{12}\left(\frac{\partial^4 u}{\partial x^4}\right)_{m,N} + \mathcal{O}(h^6). \tag{17}$$

By using (4), we have

$$\left(\frac{\partial^4 u}{\partial x^4}\right)_{m,N} = -\left(\frac{\partial^4 u}{\partial x^2 \partial y^2}\right)_{m,N} - k^2\left(\frac{\partial^2 u}{\partial x^2}\right)_{m,N}. \tag{18}$$

Combining (17) and (18), we can obtain the fourth-order approximation to $\frac{\partial^2 u}{\partial x^2}$ and approximating $\frac{\partial^2 u}{\partial x^2}$ in (18) with the second-order central difference formula and $\frac{\partial^4 u}{\partial x^2 \partial y^2}$ in (13) and (18) with the second-order central difference formula in the x-direction and the one-sided difference formula in the y-direction, namely

$$\begin{aligned}\left(\frac{\partial^4 u}{\partial x^2 \partial y^2}\right)_{m,N} = \frac{1}{2h^4}\Bigg[& 8(u_{m+1,N-1} - 2u_{m,N-1} + u_{m-1,N-1}) - (u_{m+1,N-2} - 2u_{m,N-2} \\ & + u_{m-1,N-2}) - 7(u_{m+1,N} - 2u_{m,N} + u_{m-1,N}) + 6h\left(\left(\frac{\partial u}{\partial y}\right)_{m+1,N}\right. \\ & \left. - 2\left(\frac{\partial u}{\partial y}\right)_{m,N} + \left(\frac{\partial u}{\partial y}\right)_{m-1,N}\right)\Bigg] + \mathcal{O}(h^2),\end{aligned} \tag{19}$$

then combining (8) with (6) and (9)–(19) yields the sixth-order scheme for the interior point on the top boundary as follows:

$$\mathbf{A}^t \cdot \mathbf{U}^t_{m,N} = \mathbf{C}^t \cdot \mathbf{G}^t_{m,N}, \quad m = 2, \ldots, N-1, \tag{20}$$

where

$$
\begin{aligned}
&\mathbf{A}^t = \left(A_1^t, A_2^t, A_3^t, A_4^t, A_5^t, A_6^t, A_7^t, A_8^t, A_9^t\right),\\
&\mathbf{U}^t_{m,N} = (U_{m-1,N-1}, U_{m,N-1}, U_{m+1,N-1}, U_{m-1,N}, U_{m,N}, U_{m+1,N}, U_{m-1,N-2},\\
&\qquad U_{m,N-2}, U_{m+1,N-2}),\\
&\mathbf{C}^t = \left(C_1^t, C_2^t, C_3^t, C_4^t\right),\\
&\mathbf{G}^t_{m,N} = \left(g^t_{m,N}, \left(\frac{\partial^2 g^t}{\partial x^2}\right)_{m,N}, \left(\frac{\partial^4 g^t}{\partial x^4}\right)_{m,N}, g^t_{m-1,N} + g^t_{m+1,N}\right),\\
&A_1^t = 2\left(\frac{1}{6} + \frac{k^2h^2}{180}\right)A_2 + \frac{4\tilde{B}}{h^4}, \qquad A_3^t = A_1^t, \qquad A_2^t = 2\left(\frac{1}{6} + \frac{k^2h^2}{180}\right)A_1 - \frac{8\tilde{B}}{h^4},\\
&A_4^t = \left(\frac{1}{6} + \frac{k^2h^2}{180}\right)A_1 + 2hB\alpha\left(1 + \frac{k^2h^2}{12}\right)\left(\frac{1}{6} - \frac{k^2h^2}{120}\right) - (7 + 6h\alpha)\frac{\tilde{B}}{2h^4}, \qquad A_6^t = A_4^t,\\
&A_5^t = \left(\frac{1}{6} + \frac{k^2h^2}{180}\right)A_0 - 2h\alpha(A_2 + B)\left(1 - \frac{k^2h^2}{6} + \frac{k^4h^4}{120}\right)\\
&\qquad - 4hB\alpha\left(1 + \frac{k^2h^2}{12}\right)\left(\frac{1}{6} - \frac{k^2h^2}{120}\right) + (14 + 12h\alpha)\frac{\tilde{B}}{2h^4},\\
&A_7^t = -\frac{\tilde{B}}{2h^4}, \qquad A_9^t = A_7^t, \qquad A_8^t = \frac{\tilde{B}}{h^4},\\
&C_1^t = \frac{6\tilde{B}}{h^3} - 2h(A_2 + B)\left(1 - \frac{k^2h^2}{6} + \frac{k^4h^4}{120}\right),\\
&C_2^t = 2hB\left(\frac{h^2}{6} - \frac{k^2h^4}{120}\right) + k^2h^5\left(\frac{A_2}{90} - \frac{B}{60}\right),\\
&C_3^t = h^5\left(\frac{A_2}{90} - \frac{B}{60}\right), \qquad C_4^t = -\frac{3\tilde{B}}{h^3},\\
&B = \left(-\frac{2}{3} + \frac{k^2h^2}{90}\right)A_2 + \left(\frac{1}{6} + \frac{k^2h^2}{180}\right)A_1,\\
&\tilde{B} = \frac{Bh^5\alpha}{6}\left(\frac{1}{6} - \frac{k^2h^2}{120}\right) - h^5\alpha\left(\frac{A_2}{90} - \frac{B}{60}\right).
\end{aligned}
$$

Setting $m = 1$ in (6) and following a similar process to that deriving (20), we can obtain the sixth-order scheme for the interior point on the left boundary,

$$
\mathbf{A}^l \cdot \mathbf{U}^l_{1,n} = \mathbf{C}^l \cdot \mathbf{G}^l_{1,n}, \quad n = 2, \ldots, N-1, \tag{21}
$$

where

$$
\begin{aligned}
&\mathbf{A}^l = \mathbf{A}^t, \qquad \mathbf{C}^l = \mathbf{C}^t,\\
&\mathbf{U}^l_{1,n} = (U_{2,n-1}, U_{2,n}, U_{2,n+1}, U_{1,n-1}, U_{1,n}, U_{1,n+1}, U_{3,n-1}, U_{3,n}, U_{3,n+1}),\\
&\mathbf{G}^l_{1,n} = \left(g^l_{1,n}, \left(\frac{\partial^2 g^l}{\partial x^2}\right)_{1,n}, \left(\frac{\partial^4 g^l}{\partial x^4}\right)_{1,n}, g^l_{1,n-1} + g^l_{1,n+1}\right).
\end{aligned}
$$

By setting $n = 1$ and $m = N$ in (6), respectively, and using the analogous deduction in (20) and (21), we can deduce the following schemes on the bottom and right boundaries, respectively:

$$\mathbf{A}^b \cdot \mathbf{U}^b_{m,1} = \mathbf{C}^b \cdot \mathbf{G}^b_{m,1}, \quad m = 2, \dots, N-1, \tag{22}$$

where

$$\begin{aligned}
&\mathbf{A}^b = \mathbf{A}^t, \qquad \mathbf{C}^b = \mathbf{C}^t, \\
&\mathbf{U}^b_{m,1} = (U_{m-1,2}, U_{m,2}, U_{m+1,2}, U_{m-1,1}, U_{m,1}, U_{m+1,1}, U_{m-1,3}, U_{m,3}, U_{m+1,3}), \\
&\mathbf{G}^b_{m,1} = \left(g^b_{m,1}, \left(\frac{\partial^2 g^b}{\partial y^2}\right)_{m,1}, \left(\frac{\partial^4 g^b}{\partial y^4}\right)_{m,1}, g^b_{m-1,1} + g^b_{m+1,1} \right),
\end{aligned}$$

and

$$\mathbf{A}^r \cdot \mathbf{U}^r_{N,n} = \mathbf{C}^r \cdot \mathbf{G}^r_{N,n}, \quad n = 2, \dots, N-1, \tag{23}$$

where

$$\begin{aligned}
&\mathbf{A}^r = \mathbf{A}^t, \qquad \mathbf{C}^l = \mathbf{C}^t, \\
&\mathbf{U}^r_{N,n} = (U_{N-1,n-1}, U_{N-1,n}, U_{N-1,n+1}, U_{N,n-1}, U_{N,n}, U_{N,n+1}, U_{N-2,n-1}, U_{N-2,n}, U_{N-2,n+1}), \\
&\mathbf{G}^r_{N,n} = \left(g^r_{N,n}, \left(\frac{\partial^2 g^r}{\partial y^2}\right)_{N,n}, \left(\frac{\partial^4 g^r}{\partial y^4}\right)_{N,n}, g^r_{N,n-1} + g^r_{N,n+1} \right).
\end{aligned}$$

For the sixth-order scheme at four vertices of the domain, we take the upper right vertex as an example for illustration. Using (4), (14), (15), (17), the right boundary condition $\alpha u + \frac{\partial u}{\partial x} = g^r$ at (x_N, y_N), (x_N, y_{N-1}), (x_N, y_{N-2}) and the top boundary condition $\alpha u + \frac{\partial u}{\partial y} = g^t$ at (x_N, y_N), (x_{N-1}, y_N), (x_{N-2}, y_N), we have

$$\begin{aligned}
&u_{N,N+1} + u_{N+1,N} \\
&\quad = u_{N-1,N} + u_{N,N-1} + 2h\left(1 - \frac{k^2h^2}{6} + \frac{k^4h^4}{120}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\right)_{N,N} \\
&\qquad - 2k^2h\alpha\left(\frac{h^2}{6} - \frac{k^2h^4}{120}\right)u_{N,N} + \frac{h^5\alpha}{30}\left(\frac{\partial^4 u}{\partial x^2 \partial y^2}\right)_{N,N} \\
&\qquad + \left(\frac{k^2h^5}{30} - \frac{h^3}{3}\right)\left(\frac{\partial^2 g^t}{\partial x^2} + \frac{\partial^2 g^r}{\partial y^2}\right)_{N,N} + \frac{h^5}{60}\left(\frac{\partial^4 g^t}{\partial x^4} + \frac{\partial^4 g^r}{\partial y^4}\right)_{N,N} + \mathcal{O}(h^6),
\end{aligned} \tag{24}$$

$$\begin{aligned}
&u_{N-1,N+1} + u_{N+1,N-1} \\
&\quad = 2u_{N-1,N-1} + 2h\left(1 - \frac{k^2h^2}{6} + \frac{k^4h^4}{120}\right)\left[\left(\frac{\partial u}{\partial x}\right)_{N,N-1} + \left(\frac{\partial u}{\partial y}\right)_{N-1,N}\right] \\
&\qquad + 2h\alpha\left(\frac{h^2}{6} - \frac{k^2h^4}{120}\right)\left(1 + \frac{k^2h^2}{12}\right)\left[\left(\frac{\partial^2 u}{\partial x^2}\right)_{N-1,N} + \left(\frac{\partial^2 u}{\partial y^2}\right)_{N,N-1}\right] \\
&\qquad + \left[\frac{h^3\alpha}{6}\left(\frac{h^2}{6} - \frac{k^2h^4}{120}\right) + \frac{h^5\alpha}{60}\right]\left[\left(\frac{\partial^4 u}{\partial x^2 \partial y^2}\right)_{N-1,N} + \left(\frac{\partial^4 u}{\partial x^2 \partial y^2}\right)_{N,N-1}\right]
\end{aligned}$$

$$
\begin{aligned}
&+\left(\frac{k^2h^5}{30}-\frac{h^3}{3}\right)\left[\left(\frac{\partial^2 g^t}{\partial x^2}\right)_{N-1,N}+\left(\frac{\partial^2 g^r}{\partial y^2}\right)_{N,N-1}\right]\\
&+\frac{h^5}{60}\left[\left(\frac{\partial^4 g^t}{\partial x^4}\right)_{N-1,N}+\left(\frac{\partial^4 g^r}{\partial y^4}\right)_{N,N-1}\right]+\mathcal{O}\left(h^6\right),
\end{aligned}\tag{25}
$$

$$
\begin{aligned}
&u_{N-2,N+1}+u_{N+1,N-2}\\
&\quad=u_{N-2,N-1}+u_{N-1,N-2}+2h\left(1-\frac{k^2h^2}{6}+\frac{k^4h^4}{120}\right)\left[\left(\frac{\partial u}{\partial x}\right)_{N,N-2}+\left(\frac{\partial u}{\partial y}\right)_{N-2,N}\right]\\
&\qquad+2h\alpha\left(\frac{h^2}{6}-\frac{k^2h^4}{120}\right)\left(1+\frac{k^2h^2}{12}\right)\left[\left(\frac{\partial^2 u}{\partial x^2}\right)_{N-2,N}+\left(\frac{\partial^2 u}{\partial y^2}\right)_{N,N-2}\right]\\
&\qquad+\left[\frac{h^3\alpha}{6}\left(\frac{h^2}{6}-\frac{k^2h^4}{120}\right)+\frac{h^5\alpha}{60}\right]\left[\left(\frac{\partial^4 u}{\partial x^2\partial y^2}\right)_{N-2,N}+\left(\frac{\partial^4 u}{\partial x^2\partial y^2}\right)_{N,N-2}\right]\\
&\qquad+\left(\frac{k^2h^5}{30}-\frac{h^3}{3}\right)\left[\left(\frac{\partial^2 g^t}{\partial x^2}\right)_{N-2,N}+\left(\frac{\partial^2 g^r}{\partial y^2}\right)_{N,N-2}\right]\\
&\qquad+\frac{h^5}{60}\left[\left(\frac{\partial^4 g^t}{\partial x^4}\right)_{N-2,N}+\left(\frac{\partial^4 g^r}{\partial y^4}\right)_{N,N-2}\right]+\mathcal{O}\left(h^6\right).
\end{aligned}\tag{26}
$$

Setting both m in the top boundary scheme and n in the right boundary scheme to N, adding the resulting formula, using (24)–(26) and following the process in (20), we can get the scheme for the top right vertex as follows:

$$
\mathbf{A}^{tr}\cdot\mathbf{U}^{tr}_{N,N}=\mathbf{C}^{tr}\cdot\mathbf{G}^{tr}_{N,N},\tag{27}
$$

where

$$
\begin{aligned}
\mathbf{A}^{tr}&=\left(A^{tr}_1,A^{tr}_2,A^{tr}_3,A^{tr}_4,A^{tr}_5,A^{tr}_6,A^{tr}_7,A^{tr}_8,A^{tr}_9\right),\\
\mathbf{U}^{tr}_{N,N}&=(U_{N-1,N-1},U_{N-1,N}+U_{N,N-1},U_{N,N},U_{N-2,N-2},U_{N-1,N-2}+U_{N-2,N-1},\\
&\qquad U_{N-2,N}+U_{N,N-2},U_{N-3,N-2}+U_{N-2,N-3},U_{N-3,N-1}+U_{N-1,N-3},U_{N-3,N}+U_{N,N-3}),\\
\mathbf{C}^{tr}&=\left(C^{tr}_1,C^{tr}_2,C^{tr}_3,C^{tr}_4,C^{tr}_5,C^{tr}_6,C^{tr}_7,C^{tr}_8,C^{tr}_9,C^{tr}_{10},C^{tr}_{11},C^{tr}_{12}\right),\\
\mathbf{G}^{tr}_{N,N}&=\Bigg(g^t_{N,N}+g^r_{N,N},g^t_{N-1,N}+g^r_{N,N-1},g^t_{N-2,N}+g^r_{N,N-2},g^t_{N-3,N}+g^r_{N,N-3},\\
&\qquad\left(\frac{\partial^2 g^t}{\partial x^2}\right)_{N-2,N}+\left(\frac{\partial^2 g^r}{\partial y^2}\right)_{N,N-2},\left(\frac{\partial^2 g^t}{\partial x^2}\right)_{N-1,N}+\left(\frac{\partial^2 g^r}{\partial y^2}\right)_{N,N-1},\\
&\qquad\left(\frac{\partial^2 g^t}{\partial x^2}+\frac{\partial^2 g^r}{\partial y^2}\right)_{N,N},\left(\frac{\partial^4 g^t}{\partial x^4}\right)_{N-2,N}+\left(\frac{\partial^4 g^r}{\partial y^4}\right)_{N,N-2},\\
&\qquad\left(\frac{\partial^4 g^t}{\partial x^4}\right)_{N-1,N}+\left(\frac{\partial^4 g^r}{\partial y^4}\right)_{N,N-1},\left(\frac{\partial^4 g^t}{\partial x^4}+\frac{\partial^4 g^r}{\partial y^4}\right)_{N,N},\\
&\qquad\left(\frac{\partial g^t}{\partial x}+\frac{\partial g^r}{\partial y}\right)_{N,N},g^t_{N+1,N}+g^r_{N,N+1}\Bigg),\\
A^{tr}_1&=\left(4-\frac{16}{h^4}s\right)A^t_1+\frac{8h\alpha}{15}A^t_4-\frac{4s\tilde{B}}{h^8},
\end{aligned}
$$

$$
\begin{aligned}
A_2^{tr} &= \left[-2hp\alpha - \frac{4\alpha}{h}\left(\frac{h^2}{6} - \frac{k^2h^4}{120}\right)\left(1 + \frac{k^2h^2}{12}\right) + \frac{s}{2h^4}(22 + 12h\alpha)\right]A_1^t + A_2^t \\
&\quad + \left[2 - \frac{h\alpha}{120}(56 + 48h\alpha)\right]A_4^t - \frac{\tilde{B}}{2h^4}\left[\frac{2\alpha}{h}\left(\frac{h^2}{6} - \frac{k^2h^4}{120}\right)\left(1 + \frac{k^2h^2}{12}\right)\right. \\
&\quad \left. - \frac{s}{2h^4}(7 + 6h\alpha)\right], \\
A_3^{tr} &= \left[\frac{4\alpha}{h}\left(\frac{h^2}{6} - \frac{k^2h^4}{120}\right)\left(1 + \frac{k^2h^2}{12}\right) - \frac{s}{2h^4}(14 + 12h\alpha)\right]A_1^t \\
&\quad + \left[-4hp\alpha - 2k^2h\alpha\left(\frac{h^2}{6} - \frac{k^2h^4}{120}\right) + \frac{h\alpha}{120}(49 + 84h\alpha + 36h^2\alpha^2)\right]A_4^t + 2A_5^t, \\
A_4^{tr} &= -\frac{sA_1^t}{h^4} + \frac{h\alpha A_4^t}{120} - \frac{\tilde{B}s}{h^8}, \qquad A_5^{tr} = \frac{5sA_1^t}{h^4} - \frac{h\alpha A_4^t}{15} - \frac{\tilde{B}}{2h^4}\left(1 - \frac{17s}{2h^4}\right), \\
A_6^{tr} &= \left[\frac{2\alpha}{h}\left(\frac{h^2}{6} - \frac{k^2h^4}{120}\right)\left(1 + \frac{k^2h^2}{12}\right) - \frac{s}{2h^4}(8 + 6h\alpha)\right]A_1^t + \frac{h\alpha A_4^t}{120}(7 + 6h\alpha) \\
&\quad - \frac{\tilde{B}}{2h^4}\left[-2hp\alpha - \frac{4\alpha}{h}\left(\frac{h^2}{6} - \frac{k^2h^4}{120}\right)\left(1 + \frac{k^2h^2}{12}\right) + \frac{s}{2h^4}(14 + 12h\alpha)\right] + \frac{\tilde{B}}{h^4}, \\
A_7^{tr} &= \frac{\tilde{B}s}{4h^8}, \qquad A_8^{tr} = -\frac{2\tilde{B}s}{h^8}, \\
A_9^{tr} &= -\frac{\tilde{B}}{2h^4}\left[\frac{2\alpha}{h}\left(\frac{h^2}{6} - \frac{k^2h^4}{120}\right)\left(1 + \frac{k^2h^2}{12}\right) - \frac{s}{2h^4}(7 + 6h\alpha)\right], \\
C_1^{tr} &= -\frac{3sA_1^t}{h^3} - \left[2hp - \frac{h\alpha}{120}(42h + 18h^2\alpha)\right]A_4^t - 2hp(A_2 + B) + \frac{6\tilde{B}}{h^3}, \\
C_2^{tr} &= \left(-2hp + \frac{6s}{h^3}\right)A_1^t - \frac{2h^2\alpha}{5}A_4^t + \frac{3\tilde{B}s}{2h^7} - \frac{3\tilde{B}}{h^3}, \\
C_3^{tr} &= -\frac{3sA_1^t}{h^3} + \frac{h^2\alpha A_4^t}{20} + \frac{\tilde{B}}{2h^4}\left(2hp - \frac{6s}{h^3}\right), \qquad C_4^{tr} = \frac{3\tilde{B}s}{2h^7}, \\
C_5^{tr} &= \frac{\tilde{B}}{2h^4}\left(-\frac{h^3}{3} + \frac{k^2h^5}{30}\right), \qquad C_6^{tr} = \left(\frac{h^3}{3} - \frac{k^2h^5}{30}\right)A_1^t, \\
C_7^{tr} &= \left(\frac{h^3}{3} - \frac{k^2h^5}{30}\right)A_4^t + 2hB\left(\frac{h^2}{6} - \frac{k^2h^4}{120}\right) + k^2\left(\frac{h^5A_2}{90} - \frac{h^5B}{60}\right), \\
C_8^{tr} &= \frac{h\tilde{B}}{120}, \qquad C_9^{tr} = -\frac{h^5A_1^t}{60}, \qquad C_{10}^{tr} = -\frac{h^5A_4^t}{60} + \frac{h^5A_2}{90} - \frac{h^5B}{60}, \\
C_{11}^{tr} &= -\frac{3h^3\alpha A_4^t}{20}, \qquad C_{12}^{tr} = -\frac{3\tilde{B}}{h^3}, \\
s &= \frac{h^3\alpha}{6}\left(\frac{h^2}{6} - \frac{k^2h^4}{120}\right) + \frac{h^5\alpha}{60}, \qquad p = 1 - \frac{k^2h^2}{6} + \frac{k^4h^4}{120},
\end{aligned}
$$

and $g_{N+1,N}^t, g_{N,N+1}^r$ only need to Taylor expand at (x_N, y_N).

Similarly, we can derive the sixth-order scheme for the other vertices by the symmetry. Setting both m in the top boundary scheme to 1 and n in the left boundary scheme to N, following the process in (27), we can get the scheme for the top left vertex as follows:

$$
\mathbf{A}^{tl} \cdot \mathbf{U}_{1,N}^{tl} = \mathbf{C}^{tl} \cdot \mathbf{G}_{1,N}^{tl}, \tag{28}
$$

where

$$\begin{aligned}
&\mathbf{A}^{tl} = \mathbf{A}^{tr}, \qquad \mathbf{C}^{tl} = \mathbf{C}^{tr},\\
&\mathbf{U}^{tl}_{1,N} = (U_{2,N-1}, U_{1,N-1} + U_{2,N}, U_{1,N}, U_{3,N-2}, U_{2,N-2} + U_{3,N-1}, U_{3,N} + U_{1,N-2},\\
&\qquad U_{4,N-2} + U_{3,N-3}, U_{4,N-1} + U_{2,N-3}, U_{4,N} + U_{1,N-3}),\\
&\mathbf{G}^{tl}_{1,N} = \biggl(g^t_{1,N} + g^l_{1,N}, g^t_{2,N} + g^l_{1,N-1}, g^t_{3,N} + g^l_{1,N-2}, g^t_{4,N} + g^l_{1,N-3},\\
&\qquad \left(\frac{\partial^2 g^t}{\partial x^2}\right)_{3,N} + \left(\frac{\partial^2 g^l}{\partial y^2}\right)_{1,N-2}, \left(\frac{\partial^2 g^t}{\partial x^2}\right)_{2,N} + \left(\frac{\partial^2 g^l}{\partial y^2}\right)_{1,N-1}, \left(\frac{\partial^2 g^t}{\partial x^2} + \frac{\partial^2 g^l}{\partial y^2}\right)_{1,N},\\
&\qquad \left(\frac{\partial^4 g^t}{\partial x^4}\right)_{3,N} + \left(\frac{\partial^4 g^l}{\partial y^4}\right)_{1,N-2}, \left(\frac{\partial^4 g^t}{\partial x^4}\right)_{2,N} + \left(\frac{\partial^4 g^l}{\partial y^4}\right)_{1,N-1}, \left(\frac{\partial^4 g^t}{\partial x^4} + \frac{\partial^4 g^l}{\partial y^4}\right)_{1,N},\\
&\qquad \left(-\frac{\partial g^t}{\partial x} + \frac{\partial g^l}{\partial y}\right)_{1,N}, g^t_{0,N} + g^l_{1,N+1}\biggr),
\end{aligned}$$

and $g^t_{0,N}, g^l_{1,N+1}$ only need to Taylor expand at (x_1, y_N).

Setting both m in the bottom boundary scheme to 1 and n in the left boundary scheme to 1, following the process in (27), we can get the scheme for the bottom left vertex as follows:

$$\mathbf{A}^{bl} \cdot \mathbf{U}^{bl}_{1,1} = \mathbf{C}^{bl} \cdot \mathbf{G}^{bl}_{1,1}, \tag{29}$$

where

$$\begin{aligned}
&\mathbf{A}^{bl} = \mathbf{A}^{tr}, \qquad \mathbf{C}^{bl} = \mathbf{C}^{tr},\\
&\mathbf{U}^{bl}_{1,1} = (U_{2,2}, U_{1,2} + U_{2,1}, U_{1,1}, U_{3,3}, U_{2,3} + U_{3,2}, U_{3,1} + U_{1,3},\\
&\qquad U_{4,3} + U_{3,4}, U_{4,2} + U_{2,4}, U_{4,1} + U_{1,4}),\\
&\mathbf{G}^{bl}_{1,1} = \biggl(g^b_{1,1} + g^l_{1,1}, g^b_{2,1} + g^l_{1,2}, g^b_{3,1} + g^l_{1,3}, g^b_{4,1} + g^l_{1,4},\\
&\qquad \left(\frac{\partial^2 g^b}{\partial x^2}\right)_{3,1} + \left(\frac{\partial^2 g^l}{\partial y^2}\right)_{1,3}, \left(\frac{\partial^2 g^b}{\partial x^2}\right)_{2,1} + \left(\frac{\partial^2 g^l}{\partial y^2}\right)_{1,2}, \left(\frac{\partial^2 g^b}{\partial x^2} + \frac{\partial^2 g^l}{\partial y^2}\right)_{1,1},\\
&\qquad \left(\frac{\partial^4 g^b}{\partial x^4}\right)_{3,1} + \left(\frac{\partial^4 g^l}{\partial y^4}\right)_{1,3}, \left(\frac{\partial^4 g^b}{\partial x^4}\right)_{2,1} + \left(\frac{\partial^4 g^l}{\partial y^4}\right)_{1,2}, \left(\frac{\partial^4 g^b}{\partial x^4} + \frac{\partial^4 g^l}{\partial y^4}\right)_{1,1},\\
&\qquad \left(-\frac{\partial g^b}{\partial x} - \frac{\partial g^l}{\partial y}\right)_{1,1}, g^b_{0,1} + g^l_{1,0}\biggr),
\end{aligned}$$

and $g^b_{0,1}, g^l_{1,0}$ only need to be Taylor expanded at (x_1, y_1).

Setting both m in the bottom boundary scheme to N and n in the left boundary scheme to 1, following the process in (27), we can get the scheme for the bottom left vertex as follows:

$$\mathbf{A}^{br} \cdot \mathbf{U}^{br}_{N,1} = \mathbf{C}^{br} \cdot \mathbf{G}^{bl}_{N,1}, \tag{30}$$

where

$$
\begin{aligned}
&\mathbf{A}^{br} = \mathbf{A}^{tr}, \qquad \mathbf{C}^{br} = \mathbf{C}^{tr},\\
&\mathbf{U}^{br}_{N,1} = (U_{N-1,2}, U_{N,2} + U_{N-1,1}, U_{N,1}, U_{N-2,3}, U_{N-1,3} + U_{N-2,2}, U_{N-2,1} + U_{N,3},\\
&\qquad U_{N-3,3} + U_{N-2,4}, U_{N-3,2} + U_{N-1,4}, U_{N-3,1} + U_{N,4}),\\
&\mathbf{G}^{br}_{N,1} = \biggl(g^b_{N,1} + g^r_{N,1}, g^b_{N-1,1} + g^r_{N,2}, g^b_{N-2,1} + g^r_{N,3}, g^b_{N-3,1} + g^r_{N,4},\\
&\qquad \left(\frac{\partial^2 g^b}{\partial x^2}\right)_{N-2,1} + \left(\frac{\partial^2 g^r}{\partial y^2}\right)_{N,3}, \left(\frac{\partial^2 g^b}{\partial x^2}\right)_{N-1,1} + \left(\frac{\partial^2 g^r}{\partial y^2}\right)_{N,2}, \left(\frac{\partial^2 g^b}{\partial x^2} + \frac{\partial^2 g^r}{\partial y^2}\right)_{N,1},\\
&\qquad \left(\frac{\partial^4 g^b}{\partial x^4}\right)_{N-2,1} + \left(\frac{\partial^4 g^r}{\partial y^4}\right)_{N,3}, \left(\frac{\partial^4 g^b}{\partial x^4}\right)_{N-1,1} + \left(\frac{\partial^4 g^r}{\partial y^4}\right)_{N,2}, \left(\frac{\partial^4 g^b}{\partial x^4} + \frac{\partial^4 g^r}{\partial y^4}\right)_{N,1},\\
&\qquad \left(\frac{\partial g^b}{\partial x} - \frac{\partial g^r}{\partial y}\right)_{N,1}, g^b_{N+1,1} + g^r_{N,0}\biggr),
\end{aligned}
$$

and $g^b_{N+1,1}, g^r_{N,0}$ only need to be Taylor expanded at (x_N, y_1).

3 Numerical results

In this section, we present some numerical experiments to verify the correctness and robustness of the scheme derived above. In all our results, the errors are measured in l^∞-norm.

Setting $\alpha = ik$ in (4) and the exact solution $u(x,y) = e^{i(k_1x+k_2y)}$, the function $g(x,y)$ on the boundary can be easily determined by (5) as follows:

$$
g(x,y) = \begin{cases} i(k-k_2)e^{ik_1x}, & (x,y) \in \Gamma^b,\\ i(k+k_1)e^{i(k_1+k_2y)}, & (x,y) \in \Gamma^r,\\ i(k+k_2)e^{i(k_1x+k_2)}, & (x,y) \in \Gamma^t,\\ i(k-k_1)e^{ik_2y}, & (x,y) \in \Gamma^l, \end{cases} \tag{31}
$$

where $k_1 = k\cos\theta$ and $k_2 = k\sin\theta$ are the wave numbers in the x and y directions, respectively, and θ is the propagation direction. Firstly, taking the sixth-order scheme EB (scheme (111) in [3]) on the domain for example, combining with the sixth-order scheme derived above for the inhomogeneous Robin boundary condition, we show the convergence order in Fig. 1, which is consistent with the theoretical prediction. Next, to illustrate the correctness and robustness of the high-order scheme derived above, we compare it with some well-known ones in the literature. Let $k = 10$, $\theta = \frac{\pi}{4}$, $N = 20, 40, 80, 160$, respectively, we show the error in Tables 1–3, which is good agreement with the theoretical precondition. Here, we use SFD as a standard for the standard second-order scheme (5) in [10], 5PT as a standard for the classical 5-point finite difference scheme (102) in [3], RD5 as a standard for the second-order reduced dispersion 5-point scheme (108) in [3], GFEM as a standard for the Galerkin finite element method (99) in [3], GLSFEM as a standard for the stabilized finite element method (105) in [3], ACFS and CFS as a standard for two compact fourth-order finite difference schemes (2.5) and (2.10) in [4], EBm ($m = 4, 6$) as a standard for the schemes (10) and (14) in [10], HO as a standard for the high-order scheme (24) in [1], QSFEM as a standard for the quasi-stabilized finite element method in Sect. 4.3.2 of

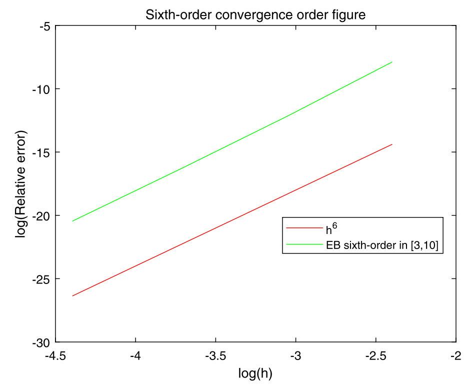


Figure 1 Convergence orders

Table 1 Errors of different second-order schemes ($k = 10$)

N	20	40	80	160
SFD	7.03e–002	1.72e–002	4.28e–003	1.07e–003
5PT	5.22e–002	1.60e–002	4.20e–003	1.06e–003
GFEM	8.93e–002	2.07e–002	5.07e–003	1.26e–003
GLSFEM	3.38e–002	1.03e–002	2.71e–003	6.86e–004
RD5	6.26e–002	1.35e–002	3.24e–003	8.02e–004
QOFDT2	3.35e–001	8.83e–002	2.23e–002	5.58e–003

Table 2 Errors of different fourth-order schemes ($k = 10$)

N	20	40	80	160
ACFS	1.18e–003	7.23e–005	4.50e–006	2.81e–007
CFS	1.19e–003	7.36e–005	4.60e–006	2.87e–007
EB	2.02e–003	1.22e–004	7.58e–006	4.73e–007
EB4	2.46e–003	1.29e–004	7.69e–006	4.75e–007
HO	1.19e–003	7.36e–005	4.60e–006	2.87e–007
QOFDT4	1.65e–002	1.08e–003	6.89e–005	4.31e–006
T2QOFD	1.60e–002	1.00e–003	6.30e–005	3.94e–006

Table 3 Errors of different sixth-order schemes ($k = 10$)

N	20	40	80	160
EB	5.47e–006	8.36e–008	1.30e–009	2.00e–011
EB6	4.44e–005	2.03e–007	1.31e–009	1.94e–011
FLAME	4.64e–005	2.33e–007	1.67e–009	1.97e–011
HO ($\Gamma = 0$)	1.85e–005	2.83e–007	4.38e–009	6.85e–011
QSFEM	4.62e–005	2.29e–007	1.62e–009	1.93e–011
QOFD	4.62e–005	2.29e–007	1.62e–009	1.93e–011
QOFDT6	8.52e–004	1.41e–005	2.24e–007	3.50e–009
T4QOFD	7.78e–004	1.28e–005	2.03e–007	3.17e–009
T6QOFD	4.23e–005	2.13e–007	1.57e–009	1.93e–011

[3], FLAME as a standard for the flexible approximation scheme in Sect. 4.3.3 of [3], and QOFD as a standard for the quasi-optimal finite difference scheme in Sect. 4.3.4 of [3], and TmQOFD and QOFDTm ($m = 2, 4, 6$) as a standard for alternative schemes in Sect. 4.4 of [3].

Then we use the SFD scheme in [10] in the second-order scheme, the CFS scheme in [4] in the fourth-order scheme, the EB scheme in [10] in the sixth-order scheme and the parameter scheme in [9]. Setting $\theta = \frac{\pi}{4}$, and $k = 100, 200, 500$ and $N = 100, 200, 400, 800$,

Table 4 Errors of different schemes ($k = 100$)

N	100	200	400	800
Second-order scheme in [10]	3.03e–000	9.52e–001	2.38e–001	5.91e–002
Fourth-order scheme in [4]	6.33e–002	4.22e–003	2.65e–004	1.65e–005
Sixth-order scheme in [10]	2.21e–003	3.13e–005	4.74e–007	7.37e–009
Parameter scheme in [9]	1.40e–001	3.39e–002	8.60e–003	2.16e–003

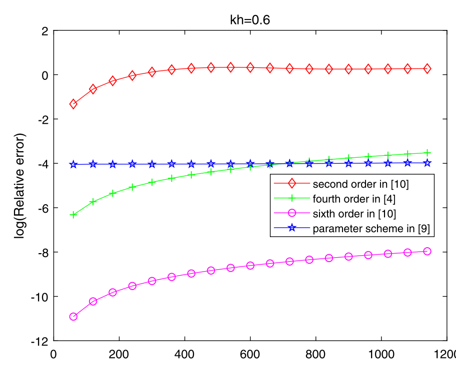
Table 5 Errors of different schemes ($k = 200$)

N	100	200	400	800
Second-order scheme in [10]	3.51e–000	2.88e–000	1.84e–000	4.87e–001
Fourth-order scheme in [4]	1.32e–000	1.23e–001	8.15e–003	5.12e–004
Sixth-order scheme in [10]	6.19e–001	4.63e–003	6.29e–005	9.50e–007
Parameter scheme in [9]	3.79e–000	1.44e–001	3.42e–002	8.99e–003

Table 6 Errors of different schemes ($k = 500$)

N	100	200	400	800
Second-order scheme in [10]	1.00e–000	3.66e–000	3.13e–000	2.79e–000
Fourth-order scheme in [4]	1.00e–000	3.00e–000	7.24e–001	4.91e–002
Sixth-order scheme in [10]	1.50e–000	3.07e–000	5.25e–002	6.27e–004
Parameter scheme in [9]	8.13e–000	4.00e–000	6.86e–000	5.53e–002

Figure 2 Development of the relative error with respect to k



respectively, we show the error generated by different schemes in Tables 4–6. And letting $kh = 0.6$, we collect the relationship between the relative error and the wave number k in Fig. 2. As is well known, the relative error increases as the wave number increases. But compared with the second- and fourth-order schemes and the parameter one, the sixth-order method investigated here can achieve the best computational accuracy in all tested cases.

Finally, we consider a practical model which is reduced from the large cavity electromagnetic scattering and has been investigated in [6, 34–36]. In this problem, $\Omega := (0,1) \times (0, \frac{1}{4})$, $\Gamma_\Omega = \Gamma^b \cup \Gamma^r \cup \Gamma^t \cup \Gamma^l$ with $\Gamma^b := [0,1] \times \{0\}$, $\Gamma^r := \{1\} \times [0, \frac{1}{4}]$, $\Gamma^t := [1,0] \times \{\frac{1}{4}\}$, $\Gamma^l := \{0\} \times [\frac{1}{4}, 0]$, $f = 0$, $u = 0$ on $\Gamma^b \cup \Gamma^r \cup \Gamma^l$, $\frac{\partial u}{\partial y} + iku = g^t$ on Γ^t, which is the lowest-order approximation of the radiation boundary condition (see [6, 18]). Setting $g^t = -2ik \cos\theta e^{ik\sin\theta x}$ and $\theta = \frac{\pi}{4}$ (see [6, 36]), we show that the real part, the image part and magnitude of the solution with $k = 128\pi, N = 512$ in Figs. 3–4, which is consistent with that illustrated in [6, 36]. The results confirm the correctness of the scheme deduced above again.

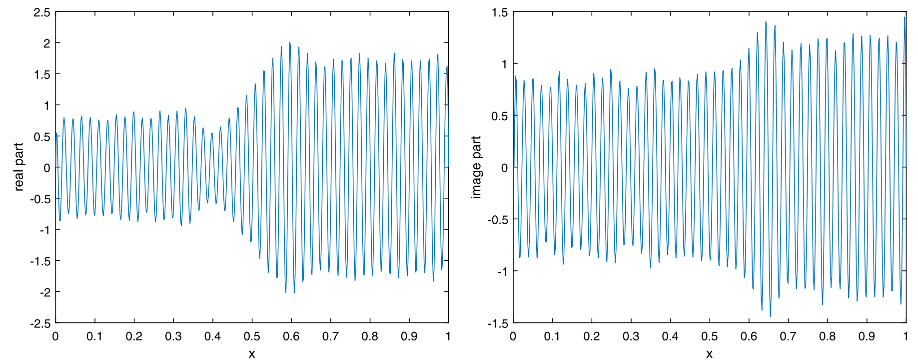


Figure 3 Real and image parts of the solution at the line $y = 1/4$ with $k = 128\pi$, $N = 512$

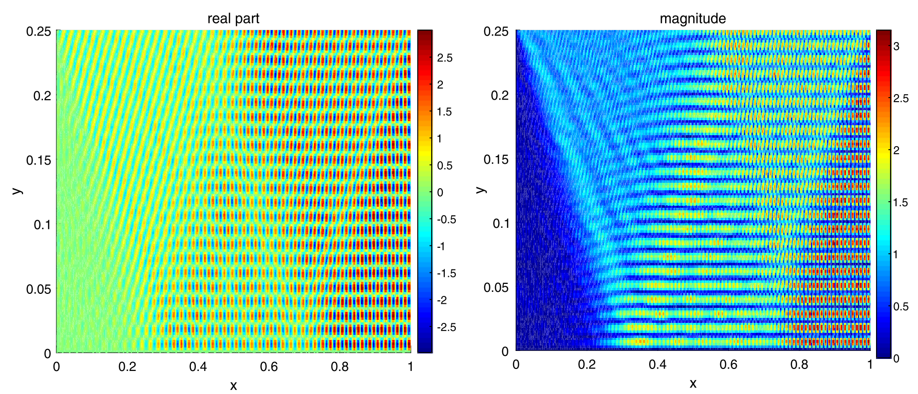


Figure 4 Real part (left) and magnitude (right) of the solution with $k = 128\pi$, $N = 512$

4 Conclusions

In this work, we derived a class of sixth-order finite difference scheme with inhomogeneous Robin boundary condition for solving the Helmholtz equation. We show some numerical examples to illustrate the efficiency and the correctness of the scheme. In all tests, compared with the second-order, fourth-order and parameter schemes, the sixth-order scheme has higher accuracy.

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Availability of data and materials

The datasets used or analyzed during the current study are available from the corresponding author on request.

Competing interests

The authors declare that no competing interests exist.

Authors' contributions

YZ derived the scheme and implements the numerical examples; KW proposed the problem and supervised the deduction of the scheme and simulation of the numerical examples; RG suggested some details. All authors read and approved the final manuscript.

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