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# On some results in intuitionistic fuzzy ideal convergence double sequence spaces

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## Abstract

Recently, spaces of ideal convergent sequences of bounded linear operators were studied by Khan et al. (Numer. Funct. Anal. Optim. 39:1278-1290, 2018). This has motivated us to propose the intuitionistic fuzzy *I*-convergent double sequence spaces determined by the bounded linear operator. In this paper, we investigate the algebraic and topological properties. We also study the concept of the ideal Cauchy and ideal convergence on the said spaces.

**Keywords:** Bounded linear operator; Intuitionistic fuzzy normed spaces; Ideal; Filter; *I*-convergence

## 1 Introduction

Zadeh [27] introduced the concept of fuzzy sets in 1965 and Goguen [6] extended it to $L$-fuzzy sets. As far as the theme of the concept of fuzzy sets is concerned, the idea has been utilized by the researchers around the globe heavily. Fuzzy sets have been put on to the metric spaces and emerged as fuzzy metric spaces studied by George et al. [5] and Amini et al. [1]. Furthermore, the idea of $I$ and $I^*$ convergent sequences in fuzzy normed spaces was due to Kumar et al. [14]. In 1986, Atanassov [2] started the study of intuitionistic fuzzy sets which is a generalization of fuzzy sets. Park [20] initiated the notion of intuitionistic fuzzy metric spaces moreover, Saadati et al. [21] extended this idea to intuitionistic fuzzy topological spaces. The study of the convergence of sequences in a fuzzy normed space is vital to fuzzy functional analysis, we feel that $I$-convergence in intuitionistic fuzzy normed space would yield a more general foundation. Later on, statistical convergence and ideal convergence of sequences concerning intuitionistic fuzzy normed space were studied by Mursaleen et al. [16, 19]. Moreover, the contributions to the study of intuitionistic fuzzy metric spaces and intuitionistic fuzzy normed spaces can be found in [7, 10, 11, 24].

The notion of ideal convergence was initiated by Kostyrko et al. [12] using the concept of the ideal $I$ as a subset of the set of positive integers which is a generalization of statistical convergence given by Fast [4] in 1951. Furthermore, it was examined from the sequence space viewpoint and connected with the summability theory by Šalát et al. [22, 23]; Tripathy et al. [25, 26] defined paranorm $I$-convergent sequence spaces; Khan et al. [9] studied the ideal convergent sequence of bounded linear operator. Later on, Das et al. [3] studied $I$ and $I^*$-convergence of double sequences. Mursaleen et al. [15, 17, 18] analyzed ideal convergence in random 2-normed spaces and probabilistic normed spaces.

Our aim for the present paper is to discuss the concept of intuitionistic fuzzy ideal convergence of double sequence spaces defined by the bounded linear operator which would yield a more convenient structure to deal with the inexactness of the sequence spaces in some situations.

## 2 Preliminaries

Now, we present some notations and basic definitions.

**Definition 2.1** ([12]) A family of sets $I \subseteq 2^Y$ is called an ideal in nonempty set $Y$, if

- $\emptyset \in I$;
- $I$ is additive; that is, $C, D \in I \Rightarrow C \cup D \in I$;
- $I$ is hereditary that is, $C \in I, D \subseteq C \Rightarrow D \in I$.

$I \subseteq 2^Y$ is said to be nontrivial if $I \neq 2^Y$. If $\{\{y\} : y \in Y\} \subseteq I$, then a nontrivial ideal $I \subseteq 2^Y$ is called admissible. $I$ is maximal if there cannot exist any nontrivial ideal $J$ containing $I$ as a subset.

**Definition 2.2** ([12]) Suppose $Y$ is a nonempty set. Then $\mathcal{F} \subset 2^Y$ is called a filter on $Y$ if and only if the following implications hold:

- $\emptyset \notin \mathcal{F}$;
- for $C, D \in \mathcal{F} \Rightarrow C \cap D \in \mathcal{F}$;
- for each $C \in \mathcal{F}$ and $D \supset C \Rightarrow D \in \mathcal{F}$.

For every ideal $I$ there corresponds a filter defined as

$$\mathcal{F}(I) = \left\{K \subset Y : K^c \in I, \text{ where } K^c = Y - K\right\}.$$

We take $I_2$ as a nontrivial ideal in $\mathbb{N} \times \mathbb{N}$ throughout the paper.

**Definition 2.3** ([26]) A double sequence $y = (y_{ij}) \in {}_2\omega$ is said to be $I_2$-convergent to $L$ if, for every $\epsilon > 0$, we have

$$\left\{(i,j) \in \mathbb{N} \times \mathbb{N} : |y_{ij} - L| \geq \epsilon\right\} \in I_2. \tag{2.1}$$

We write $I_2 - \lim y_{ij} = L$.

**Definition 2.4** ([26]) A sequence $y = (y_{ij})$ is said to be $I_2$-Cauchy, if for each $\epsilon > 0$, there exist positive integers $m = m(\epsilon)$ and $n = n(\epsilon)$ such that the set

$$\left\{(i,j) \in \mathbb{N} \times \mathbb{N} : |y_{ij} - y_{mn}| \geq \epsilon\right\} \in I_2.$$

**Definition 2.5** ([19]) Let $(Y, \phi, \psi, *, \diamond)$ be an intuitionistic fuzzy normed space (IFNS). A sequence $y = (y_{ij})$ is termed convergent to $L \in Y$ under intuitionistic fuzzy norm $(\phi, \psi)$ if, for every $\epsilon, t > 0$, there exist $k_0 \in \mathbb{N}$ such that $\phi(y_{ij} - L, t) > 1 - \epsilon$ and $\psi(y_{ij} - L, t) < \epsilon$ for all $i, j \geq k_0$.

**Definition 2.6** ([19]) Let $(Y, \phi, \psi, *, \diamond)$ be an IFNS. A sequence $y = (y_{ij})$ is termed a Cauchy sequence with respect to the intuitionistic fuzzy norm $(\phi, \psi)$, if for every $\epsilon, t > 0$, $\exists\, k_0 \in \mathbb{N}$ such that $\phi(y_{ij} - y_{mn}, t) > 1 - \epsilon$ and $\psi(y_{ij} - y_{mn}, t) < \epsilon$, for all $i, j, m, n \geq k_0$.

*Remark* ([21]) If $*$ is a continuous $t$-norm, $\diamond$ is a continuous $t$-conorm and $p_i \in (0,1)$, $1 \leq i \leq 7$. Then:

- for any $p_1, p_2 \in (0,1)$ with $p_1 > p_2$, there exist $p_3, p_4 \in (0,1)$ such that

$$p_1 * p_3 \geq p_2 \quad \text{and} \quad p_1 \geq p_4 \diamond p_2;$$

- for any $p_5 \in (0,1)$, there exist $p_6, p_7 \in (0,1)$ such that $p_6 * p_6 \geq p_5$ and $p_7 \diamond p_7 \leq p_5$.

**Definition 2.7** ([19]) Let $I_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a nontrivial ideal and $(Y, \phi, \psi, *, \diamond)$ be an IFNS. A sequence $y = (y_{ij})$ in $Y$ is called $I_2$-convergent to $L \in Y$ with respect to the intuitionistic fuzzy norm $(\phi, \psi)$ if, for every $\epsilon$, $t > 0$, the set

$$\bigl\{(i,j) : \phi(y_{ij} - L, t) \leq 1 - \epsilon \text{ or } \psi(y_{ij} - L, t) \geq \epsilon\bigr\} \in I_2.$$

We write $I_2^{(\phi,\psi)} - \lim y_{ij} = L$.

**Definition 2.8** ([19]) Let $(Y, \phi, \psi, *, \diamond)$ be an IFNS. A sequence $y = (y_{ij})$ in $Y$ is called an $I_2$-Cauchy sequence with respect to the intuitionistic fuzzy norm $(\phi, \psi)$, if, for every $\epsilon$, $t > 0$, the set

$$\bigl\{(i,j) : \phi(y_{ij} - y_{mn}, t) \leq 1 - \epsilon \text{ or } \psi(y_{ij} - y_{mn}, t) \geq \epsilon\bigr\} \in I_2.$$

**Definition 2.9** ([13]) Let $U$ and $V$ be two normed linear spaces and $B : \mathcal{D}(B) \to V$ be a linear operator, where $\mathcal{D}(B) \subset U$. An operator $B$ is bounded, if there exists $k > 0$ such that

$$\|Bx\| \leq k\|x\|, \quad \text{for all } x \in \mathcal{D}(B).$$

We denote by $\mathcal{B}(U, V)$ the set of all bounded linear operators which is normed linear spaces normed by

$$\|B\| = \sup_{x \in U, \|x\|=1} \|Bx\|$$

and $\mathcal{B}(U, V)$ is a Banach space if $V$ is a Banach space.

## 3 Main results

*In this section, we introduce the following new sequence spaces:*

$${}_2S^{I_2}_{(\phi,\psi)}(B) = \bigl\{(x_{ij}) \in {}_2\ell_\infty : \bigl\{(i,j) : \phi\bigl(B(x_{ij}) - L, t\bigr) \leq 1 - \epsilon \text{ or } \psi\bigl(B(x_{ij}) - L, t\bigr) \geq \epsilon\bigr\} \in I_2\bigr\};$$
$${}_2S^{I_2}_{0(\phi,\psi)}(B) = \bigl\{(x_{ij}) \in {}_2\ell_\infty : \bigl\{(i,j) : \phi\bigl(B(x_{ij}), t\bigr) \leq 1 - \epsilon \text{ or } \psi\bigl(B(x_{ij}), t\bigr) \geq \epsilon\bigr\} \in I_2\bigr\}.$$

An open ball with center $x$ and radius $r$ with respect to $t$ is defined as

$${}_2\mathcal{B}_x(r,t)(B) = \bigl\{(y_{ij}) \in {}_2\ell_\infty : \phi\bigl(B(x_{ij}) - B(y_{ij}), t\bigr) > 1 - r \text{ and } \psi\bigl(B(x_{ij}) - B(y_{ij}), t\bigr) < r\bigr\}.$$

**Theorem 3.1** ${}_2S^{I_2}_{(\phi,\psi)}(B)$ *and* ${}_2S^{I_2}_{0(\phi,\psi)}(B)$ *are linear spaces.*

*Proof* Let $x = (x_{ij}), y = (y_{ij}) \in {}_2S^{I_2}_{(\phi,\psi)}(B)$ and $\alpha$, $\beta$ be scalars. For a given $\epsilon > 0$, we obtain

$$A_1 = \left\{(i,j) : \phi\left(B(x_{ij}) - L_1, \frac{t}{2|\alpha|}\right) \leq 1-\epsilon \text{ and } \psi\left(B(x_{ij}) - L_1, \frac{t}{2|\alpha|}\right) \geq \epsilon\right\} \in I_2;$$
$$A_2 = \left\{(i,j) : \phi\left(B(y_{ij}) - L_2, \frac{t}{2|\beta|}\right) \leq 1-\epsilon \text{ and } \psi\left(B(y_{ij}) - L_2, \frac{t}{2|\beta|}\right) \geq \epsilon\right\} \in I_2.$$
$$A_1^c = \left\{(i,j) : \phi\left(B(x_{ij}) - L_1, \frac{t}{2|\alpha|}\right) > 1-\epsilon \text{ and } \psi\left(B(x_{ij}) - L_1, \frac{t}{2|\alpha|}\right) < \epsilon\right\} \in \mathcal{F}(I_2);$$
$$A_2^c = \left\{(i,j) : \phi\left(B(y_{ij}) - L_2, \frac{t}{2|\beta|}\right) > 1-\epsilon \text{ and } \psi\left(B(y_{ij}) - L_2, \frac{t}{2|\beta|}\right) < \epsilon\right\} \in \mathcal{F}(I_2).$$

Define $A_3 = A_1 \cup A_2$, so that $A_3 \in I_2$. It implies that $A_3^c$ is a nonempty set in $\mathcal{F}(I_2)$. Now, we have to show that, for each $(x_{ij}), (y_{ij}) \in {}_2S^{I_2}_{(\phi,\psi)}(B)$, $A_3^c \subset \{(i,j) : \phi((\alpha B(x_{ij}) + \beta B(y_{ij})) - (\alpha L_1 + \beta L_2), t) > 1-\epsilon$ and $\psi((\alpha B(x_{ij}) + \beta B(y_{ij})) - (\alpha L_1 + \beta L_2), t) < \epsilon\}$. Let $m, n \in A_3^c$. In this case

$$\phi\left(B(x_{mn}) - L_1, \frac{t}{2|\alpha|}\right) > 1-\epsilon \quad \text{and} \quad \psi\left(B(x_{mn}) - L_1, \frac{t}{2|\alpha|}\right) < \epsilon,$$
$$\phi\left(B(y_{mn}) - L_2, \frac{t}{2|\beta|}\right) > 1-\epsilon \quad \text{and} \quad \psi\left(B(y_{mn}) - L_2, \frac{t}{2|\beta|}\right) < \epsilon.$$

We have

$$\begin{aligned}
&\phi\big(\big(\alpha B(x_{mn}) + \beta B(y_{mn})\big) - (\alpha L_1 + \beta L_2), t\big) \\
&\quad \geq \phi\left(\alpha B(x_{mn}) - \alpha L_1, \frac{t}{2}\right) \\
&\qquad * \phi\left(\beta B(y_{mn}) - \beta L_2, \frac{t}{2}\right) \\
&\quad \geq \phi\left(B(x_{mn}) - L_1, \frac{t}{2|\alpha|}\right) * \phi\left(B(y_{mn}) - L_2, \frac{t}{2|\beta|}\right) \\
&\quad \geq (1-\epsilon) * (1-\epsilon) \\
&\quad > 1-\epsilon.
\end{aligned}$$

Also

$$\begin{aligned}
\psi\big(\big(\alpha B(x_{mn}) + \beta B(y_{mn})\big) - (\alpha L_1 + \beta L_2), t\big) &\leq \psi\left(\alpha B(x_{mn}) - \alpha L_1, \frac{t}{2}\right) \\
&\quad \diamond \psi\left(\beta B(y_{mn}) - \beta L_2, \frac{t}{2}\right) \\
&\leq \psi\left(B(x_{mn}) - L_1, \frac{t}{2|\alpha|}\right) \\
&\quad \diamond \psi\left(B(y_{mn}) - L_2, \frac{t}{2|\beta|}\right) \\
&\leq \epsilon \diamond \epsilon \\
&< \epsilon.
\end{aligned}$$

This implies

$$A_3^c \subset \big\{(i,j): \phi\big(\big(\alpha B(x_{ij}) + \beta B(y_{ij})\big) - (\alpha L_1 + \beta L_2), t\big) > 1 - \epsilon$$
$$\text{and } \psi\big(\big(\alpha B(x_{ij}) + \beta B(y_{ij})\big) - (\alpha L_1 + \beta L_2), t\big) < \epsilon\big\}.$$

Hence ${}_2S^{I_2}_{(\phi,\psi)}(B)$ is a linear space.

In a similar way, we can prove that ${}_2S^{I_2}_{0(\phi,\psi)}(B)$ is linear space. □

**Theorem 3.2** *Every open ball ${}_2\mathcal{B}_x(r,t)(B)$ is an open set in ${}_2S^{I_2}_{(\phi,\psi)}(B)$.*

*Proof* Suppose $y \in {}_2\mathcal{B}_x(r,t)(B)$. Then, from the definition of ${}_2\mathcal{B}_x(r,t)(B)$, we have

$$\phi\big(B(x_{ij}) - B(y_{ij}), t\big) > 1 - r \quad \text{and} \quad \psi\big(B(x_{ij}) - B(y_{ij}), t\big) < r. \tag{3.1}$$

From (3.1), there exists $t_0 \in (0,t)$ such that $\phi(B(x_{ij}) - B(y_{ij}), t_0) > 1 - r$ and $\psi(B(x_{ij}) - B(y_{ij}), t_0) < r$. Setting $p_0 = \phi(B(x_{ij}) - B(y_{ij}), t_0)$. This implies that $p_0 > 1 - r$. Thus, there exists $s \in (0,1)$ with $p_0 > 1 - s > 1 - r$. For given $p_0$, $s$ with $p_0 > 1 - s$, there exist $p_1, p_2 \in (0,1)$ such that $p_0 * p_1 > 1 - s$ and $(1 - p_0) \diamond (1 - p_2) \le s$. Select $p_3 = \max\{p_1, p_2\}$ and consider the ball ${}_2\mathcal{B}_y(1 - p_3, t - t_0)(B)$. Now, we show

$${}_2\mathcal{B}_y(1 - p_3, t - t_0)(B) \subset {}_2\mathcal{B}_x(r,t)(B).$$

Let $z = (z_{ij}) \in {}_2\mathcal{B}_y(1 - p_3, t - t_0)(B)$. Then $\phi(B(y_{ij}) - B(z_{ij}), t - t_0) > p_3$ and $\psi(B(y_{ij}) - B(z_{ij}), t - t_0) < 1 - p_3$.

Therefore

$$\begin{aligned}
\phi\big(B(x_{ij}) - B(z_{ij}), t\big) &\ge \phi\big(B(x_{ij}) - B(y_{ij}), t_0\big) * \phi\big(B(y_{ij}) - B(z_{ij}), t - t_0\big) \\
&\ge (p_0 * p_3) \\
&\ge (p_0 * p_1) \\
&\ge (1 - s) > 1 - r
\end{aligned}$$

and

$$\begin{aligned}
\psi\big(B(x_{ij}) - B(z_{ij}), t\big) &\le \psi\big(B(x_{ij}) - B(y_{ij}), t_0\big) \diamond \psi\big(B(y_{ij}) - B(z_{ij}), t - t_0\big) \\
&\le (1 - p_0) \diamond (1 - p_3) \\
&\le (1 - p_0) \diamond (1 - p_2) \\
&\le s < r.
\end{aligned}$$

The above inequalities imply that $z \in {}_2\mathcal{B}_x(r,t)(B)$. Thus ${}_2\mathcal{B}_y(1 - p_3, t - t_0)(B) \subset {}_2\mathcal{B}_x(r,t)(B)$. □

*Remark* Let ${}_2S^{I_2}_{(\phi,\psi)}(B)$ be an IFNS. Define ${}_2\tau^{I_2}_{(\phi,\psi)}(B) = \{K \subset {}_2S^{I_2}_{(\phi,\psi)}(B) :$ for each $x \in K, \exists\ t > 0$ and $r \in (0,1)$ such that ${}_2\mathcal{B}_x(r,t)(B) \subset K\}$. Then ${}_2\tau^{I}_{(\phi,\psi)}(B)$ is a topology on ${}_2S^{I_2}_{(\phi,\psi)}(B)$.

**Theorem 3.3** *The spaces* ${}_2S^{I_2}_{(\phi,\psi)}(B)$ *and* ${}_2S^{I_2}_{0(\phi,\psi)}(B)$ *are Hausdorff.*

*Proof* Let $x$, $y \in {}_2S^{I_2}_{(\phi,\psi)}(B)$ be two distinct points. Then $0 < \phi(B(x) - B(y), t) < 1$ and $0 < \psi(B(x) - B(y), t) < 1$. Putting $p_1 = \phi(B(x) - B(y), t)$, $p_2 = \psi(B(x) - B(y), t)$ and $p = \max\{p_1, 1 - p_2\}$.

For each $p_0 \in (p, 1)$ there exist $p_3$ and $p_4$ such that $p_3 * p_3 \geq p_0$ and $(1 - p_4) \diamond (1 - p_4) \leq (1 - p_0)$.

Put $p_5 = \max\{p_3, p_4\}$ and consider the open balls ${}_2\mathcal{B}_x(1 - r_p, \frac{t}{2})$ and ${}_2\mathcal{B}_y(1 - p_5, \frac{t}{2})$.

Clearly ${}_2\mathcal{B}_x(1 - p_5, \frac{t}{2}) \cap {}_2\mathcal{B}_y(1 - p_5, \frac{t}{2}) = \emptyset$.

If there exists $z \in {}_2\mathcal{B}_x(1 - p_5, \frac{t}{2}) \cap {}_2\mathcal{B}_y(1 - p_5, \frac{t}{2})$, then

$$\begin{aligned} p_1 &= \phi\bigl(B(x) - B(y), t\bigr) \\ &\geq \phi\left(B(x) - B(z), \frac{t}{2}\right) * \phi\left(B(z) - B(y), \frac{t}{2}\right) \\ &\geq p_5 * p_5 \\ &\geq p_3 * p_3 \geq p_0 > p_1 \end{aligned}$$

and

$$\begin{aligned} p_2 &= \psi\bigl(B(x) - B(y), t\bigr) \\ &\leq \psi\left(B(x) - B(z), \frac{t}{2}\right) \diamond \psi\left(B(z) - B(y), \frac{t}{2}\right) \\ &\leq (1 - p_5) \diamond (1 - p_5) \\ &\leq (1 - p_4) \diamond (1 - p_4) \\ &\leq (1 - p_0) < p_2, \end{aligned}$$

which is a contradiction. Hence ${}_2S^{I_2}_{(\phi,\psi)}(B)$ is Hausdorff space. Similarly, one can show that ${}_2S^{I_2}_{0(\phi,\psi)}(B)$ is Hausdorff space. □

**Theorem 3.4** ${}_2S^{I_2}_{(\phi,\psi)}(B)$ *is an IFNS and* ${}_2\tau^{I_2}_{(\phi,\psi)}(B)$ *is a topology on* ${}_2S^{I_2}_{(\phi,\psi)}(B)$. *Then a sequence* $(x_{ij}) \in {}_2S^{I_2}_{(\phi,\psi)}(B)$, $x_{ij} \to x$ *if and only if* $\phi(B(x_{ij}) - B(x), t) \to 1$ *and* $\psi(B(x_{ij}) - B(x), t) \to 0$ *as* $i, j \to \infty$.

*Proof* Choose $t_0 > 0$. Suppose $x_{ij} \to x$. Then for $0 < r < 1$, $\exists\, N_0 \in \mathbb{N}$ such that $(x_{ij}) \in {}_2S^{I_2}_{(\phi,\psi)}(B)$ for all $i, j \geq N_0$,

$$\begin{aligned} {}_2S^{I_2}_{(\phi,\psi)}(B) = \Bigl\{(x_{ij}) \in {}_2\ell_\infty : \bigl\{(i,j) : \phi\bigl(B(x_{ij}) - B(x), t\bigr) \leq 1 - \epsilon \text{ or } \psi\bigl(B(x_{ij}) - B(x), t\bigr) \geq \epsilon\bigr\} \\ \in I_2\Bigr\} \end{aligned}$$

such that ${}_2S^{I_2}_{(\phi,\psi)}(B) \in \mathcal{F}(I_2)$. Then $1 - \phi(B(x_{ij}) - B(x), t) < r$ and $\psi(B(x_{ij}) - B(x), t) < r$.

Hence $\phi(B(x_{ij}) - B(x), t) \to 1$ and $\psi(B(x_{ij}) - B(x), t) \to 0$ as $i, j \to \infty$.

On the other way around, if for each $t > 0$, $\phi(B(x_{ij}) - B(x), t) \to 1$ and $\psi(B(x_{ij}) - B(x), t) \to 0$ as $i, j \to \infty$. Then for $0 < r < 1$, $\exists\, N_0 \in \mathbb{N}$ such that $1 - \phi(B(x_{ij}) - B(x), t) < r$ and $\psi(B(x_{ij}) - B(x), t) < r$, for all $i, j \geq N_0$. This implies that $\phi(B(x_{ij}) - B(x), t) > 1 - r$ and $\psi(B(x_{ij}) - B(x), t) < r$ for all $i, j \geq N_0$. Thus $(x_{ij}) \in {}_2S^{I_2}_{(\phi,\psi)}(B)$ for all $i, j \geq N_0$ and hence $x_{ij} \to x$. □

**Theorem 3.5** *Let ${}_2S^{I_2}_{(\phi,\psi)}(B)$ be an IFNS. If a double sequence $x = (x_{ij})$ is $I_2$-convergent with respect to the intuitionistic fuzzy norms $(\phi,\psi)$, then the $I_2^{(\phi,\psi)}$-limit is unique.*

*Proof* Let $I_2^{(\phi,\psi)} - \lim x = L_1$ and $I_2^{(\phi,\psi)} - \lim x = L_2$. For a given $\epsilon > 0$, select $s > 0$ in such a way that $(1-s) * (1-s) > 1-\epsilon$ and $s \diamond s < \epsilon$. Then, for any $t > 0$, we define the following sets:

$$P_{\phi,1}(s,t) = \left\{(i,j) \in \mathbb{N}\times\mathbb{N} : \phi\left(B(x_{ij}) - L_1, \frac{t}{2}\right) \leq 1-s\right\},$$

$$P_{\phi,2}(s,t) = \left\{(i,j) \in \mathbb{N}\times\mathbb{N} : \phi\left(B(x_{ij}) - L_2, \frac{t}{2}\right) \leq 1-s\right\},$$

$$P_{\psi,1}(s,t) = \left\{(i,j) \in \mathbb{N}\times\mathbb{N} : \psi\left(B(x_{ij}) - L_1, \frac{t}{2}\right) \geq s\right\},$$

$$P_{\psi,2}(s,t) = \left\{(i,j) \in \mathbb{N}\times\mathbb{N} : \psi\left(B(x_{ij}) - L_2, \frac{t}{2}\right) \geq s\right\}.$$

Since $I_2^{(\phi,\psi)} - \lim x = L_1$, we obtain $P_{\phi,1}(s,t)$ and $P_{\psi,1}(s,t) \in I_2$.

Moreover, using $I_2^{(\phi,\psi)} - \lim x = L_2$, we have $P_{\phi,2}(s,t)$ and $P_{\psi,2}(s,t) \in I_2$. Now, suppose that

$$P_{\phi,\psi}(s,t) = \big(P_{\phi,1}(s,t) \cup P_{\phi,2}(s,t)\big) \cap \big({}_{\psi,1}(s,t) \cup P_{\psi,2}(s,t)\big) \in I_2.$$

Thus, $P_{\phi,\psi}(s,t) \in I_2$, implies that $P^c_{\phi,\psi}(s,t)$ is a nonempty set in $\mathcal{F}(I_2)$.

If $(i,j) \in P^c_{\phi,\psi}(s,t)$, then two possibilities arise:

$$(i,j) \in P^c_{\phi,1}(s,t) \cap P_{\phi,2}(s,t) \quad \text{or} \quad (i,j) \in P^c_{\psi,1}(s,t) \cap P_{\psi,2}(s,t).$$

Firstly, we consider that $(i,j) \in P^c_{\phi,1}(s,t) \cap P^c_{\phi,2}(s,t)$. Then we get

$$\phi(L_1 - L_2, t) \geq \phi\left(B(x_{ij}) - L_1, \frac{t}{2}\right) * \phi\left(B(x_{ij}) - L_2, \frac{t}{2}\right) > (1-s) * (1-s) > 1-\epsilon.$$

Since $\epsilon > 0$ was arbitrary, we obtain $\phi(L_1 - L_2, t) = 1$ for every $t > 0$, which yields $L_1 = L_2$. Under other conditions, if $(i,j) \in P^c_{\psi,1}(s,t) \cap P^c_{\psi,2}(s,t)$, then we may write

$$\psi(L_1 - L_2, t) \leq \psi\left(B(x_{ij}) - L_1, \frac{t}{2}\right) \diamond \psi\left(B(x_{ij}) - L_2, \frac{t}{2}\right) < s \diamond s < \epsilon.$$

Therefore, we obtain $\psi(L_1 - L_2, t) = 0$, for all $t > 0$, which yields $L_1 = L_2$. Hence, in all cases, we find that the $I_2^{(\phi,\psi)}$-limit is unique. □

**Theorem 3.6** *A sequence $x = (x_{ij}) \in {}_2S^{I_2}_{(\phi,\psi)}(B)$ is $I_2$-convergent with respect to the intuitionistic fuzzy norm $(\phi,\psi)$ if and only if it is $I_2$-Cauchy with respect to same norm.*

*Proof* Suppose that the sequence $x = (x_{ij}) \in {}_2S^{I_2}_{(\phi,\psi)}(B)$ is $I_2$-convergent, i.e., $I_2^{(\phi,\psi)} - \lim x = L$. Take $s > 0$, in such a way that $(1-s) * (1-s) > 1-\epsilon$ and $s \diamond s < \epsilon$. For all $t > 0$, we get

$$P = \big\{(i,j) \in \mathbb{N}\times\mathbb{N} : \phi\big(B(x_{ij}) - L, t\big) \leq 1-s \text{ or } \psi\big(B(x_{ij}) - L, t\big) \geq s\big\} \in I_2.$$

This implies $P^c = \{(i,j) \in \mathbb{N} \times \mathbb{N} : \phi(B(x_{ij}) - L, t) > 1 - s \text{ and } \psi(B(x_{ij}) - L, t) < s\} \in \mathcal{F}(I_2)$. Suppose $(m,n) \in P^c$. Then we obtain

$$\phi\bigl(B(x_{mn}) - L, t\bigr) > 1 - s \quad \text{and} \quad \psi\bigl(B(x_{mn}) - L, t\bigr) < s.$$

Let $Q = \{(i,j) \in \mathbb{N} \times \mathbb{N} : \phi(B(x_{ij}) - B(x_{mn}), t) \leq 1 - \epsilon \text{ or } \psi(B(x_{ij}) - B(x_{mn}), t) \geq \epsilon\}$.

Furthermore, we prove the inclusion $Q \subset P$. Let $(i,j) \in Q$, we have

$$\phi\left(B(x_{ij}) - B(x_{mn}), \frac{t}{2}\right) \leq 1 - \epsilon \quad \text{and} \quad \psi\left(B(x_{ij}) - B(x_{mn}), \frac{t}{2}\right) \geq \epsilon.$$

There are two possible cases, firstly we consider $\phi(B(x_{ij}) - B(x_{mn}), t) \leq 1 - \epsilon$. Then we have $\phi(B(x_{ij}) - L, \frac{t}{2}) \leq 1 - s$, therefore $(i,j) \in P$. On the other hand, if $\phi(B(x_{ij}) - L, \frac{t}{2}) > 1 - s$ then

$$\begin{aligned} 1 - \epsilon &\geq \phi\bigl(B(x_{ij}) - B(x_{mn}), t\bigr) \\ &\geq \phi\left(B(x_{ij}) - L, \frac{t}{2}\right) * \phi\left(B(x_{mn}) - L, \frac{t}{2}\right) > (1 - s) * (1 - s) > 1 - \epsilon, \end{aligned}$$

which is impossible. Hence $Q \subset P$.

Similarly, consider $\psi(B(x_{ij}) - B(x_{mn}), t) \geq \epsilon$. Then we have $\psi(B(x_{ij}) - L, \frac{t}{2}) \geq s$, hence $(i,j) \in P$. Otherwise, if $\psi(B(x_{ij}) - L, \frac{t}{2}) < s$, then

$$\epsilon \leq \psi\bigl(B(x_{ij}) - B(x_{mn}), t\bigr) \leq \psi\left(B(x_{ij}) - L, \frac{t}{2}\right) \diamond \psi\left(B(x_{mn}) - L, \frac{t}{2}\right) < s \diamond s < \epsilon,$$

which is impossible. Hence $Q \subset P$. Thus in both cases we conclude that $Q \subset P$. Therefore $Q \in I$. Hence $x$ is $I_2$ Cauchy with respect to the intuitionistic fuzzy norm $(\phi, \psi)$.

Contrarily, suppose that $x = (x_{ij})$ is $I_2$-Cauchy but not $I_2$-convergent with respect to the intuitionistic fuzzy norm $(\phi, \psi)$. Then there exist $p$ and $q$ such that

$$P(\epsilon, t) = \bigl\{(i,j) \in \mathbb{N} \times \mathbb{N} : \phi\bigl(B(x_{ij}) - B(x_{pq}), t\bigr) \leq 1 - \epsilon \text{ or } \psi\bigl(B(x_{ij}) - B(x_{pq}), t\bigr) \geq s\bigr\} \in I_2,$$

and

$$Q(\epsilon, t) = \left\{(i,j) \in \mathbb{N} \times \mathbb{N} : \phi\left(B(x_{ij}) - L, \frac{t}{2}\right) > 1 - \epsilon \text{ or } \psi\left(B(x_{ij}) - L, \frac{t}{2}\right) < \epsilon\right\} \in I_2.$$

Equivalently, $Q^c(\epsilon, t) \in \mathcal{F}(I_2)$. Since

$$\phi\bigl(B(x_{ij}) - B(x_{mn}), t\bigr) \geq 2\phi\left(B(x_{ij}) - L, \frac{t}{2}\right) > 1 - \epsilon$$

and

$$\psi\bigl(B(x_{jk}) - B(x_{pq}), t\bigr) \leq 2\phi\left(B(x_{ij}) - L, \frac{t}{2}\right) < \epsilon,$$

if $\phi(B(x_{ij}) - L, \frac{t}{2}) > \frac{1-\epsilon}{2}$ and $\psi(B(x_{ij}) - L, \frac{t}{2}) < \frac{\epsilon}{2}$, respectively, we obtain $P^c(\epsilon, t) \in I_2$ and so $P(\epsilon, t) \in \mathcal{F}(I_2)$, which contradicts our assumption. $\square$

**Theorem 3.7** *Suppose* ${}_2S^{I_2}_{(\phi,\psi)}(B)$ *be an intuitionistic fuzzy normed space such that* ${}_2S^{I_2}_{(\phi,\psi)}(B)$ *has a convergent subsequence. Then* ${}_2S^{I_2}_{(\phi,\psi)}(B)$ *is complete.*

*Proof* Let $(x_{i_m j_n})$ be a subsequence of Cauchy sequence $(x_{ij})$ that converges to $x$. We show that $(x_{ij}) \to x$ as $(i,j) \to \infty$. Let $t > 0$ and $\epsilon \in (0,1)$. Choose $s \in (0,1)$ in a such way that $s \diamond s \leq \epsilon$ and $(1-s) * (1-s) \geq 1-\epsilon$. Since $(x_{ij})$ is a Cauchy sequence, $\exists\ N_0 \in \mathbb{N}$ such that $\phi(B(x_{ij}) - B(x_{pq}), \frac{t}{2}) > 1-s$ and $\psi(B(x_{ij}) - B(x_{pq}), \frac{t}{2}) < s$, for all $i, j, p$ and $q \geq N_0$. Since $(x_{m_i n_j}) \to x$, there is positive integer $i_k, j_l > N_0$ such that $\phi(B(x_{i_k j_l}) - B(x), \frac{t}{2}) > 1-s$ and $\psi(B(x_{i_k j_l}) - B(x), \frac{t}{2}) < s$. Then, if $i, j \geq N_0$,

$$\begin{aligned}
\phi\big(B(x_{ij}) - B(x), t\big) &\geq \phi\left(B(x_{ij}) - B(x_{i_k j_l}), \frac{t}{2}\right) * \phi\left(B(x_{i_k j_l}) - B(x), \frac{t}{2}\right) \\
&\geq (1-s) * (1-s) \\
&\geq 1-\epsilon
\end{aligned}$$

or

$$\begin{aligned}
\psi\big(B(x_{ij}) - B(x), t\big) &\leq \psi\left(B(x_{ij}) - B(x_{i_k j_l}), \frac{t}{2}\right) \diamond \psi\left(B(x_{i_k j_l}) - B(x), \frac{t}{2}\right) \\
&\leq s \diamond s \\
&\leq \epsilon.
\end{aligned}$$

Since $B$ is a bounded linear operator, $x_{ij} \to x$ as $(i,j) \to \infty$. Hence ${}_2S^{I_2}_{(\phi,\psi)}(B)$ is complete. □

## 4 Conclusion

The concept of intuitionistic fuzzy convergence of sequences has been studied by numerous researchers. In the present work, we introduced a more general type of convergence of sequence spaces, namely the intuitionistic fuzzy ideal convergence of double sequence spaces defined by a bounded linear operator. We investigated some algebraic and topological properties on these spaces. Furthermore, we contributed new tools to work with the convergence problems of sequences in the intuitionistic fuzzy settings, occurring in various fields of science and engineering.

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**Authors' contributions**
Each of the authors contributed to each part of this study equally, all authors read and approved the final manuscript.

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