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Uniqueness of meromorphic solutions sharing values with a meromorphic function to $w(z+1)w(z-1) = h(z)w^m(z)$

BaoQin Chen[1] and Sheng Li[1*]

*Correspondence: lish_ls@sina.com
[1]Faculty of Mathematics and Computer Science, Guangdong Ocean University, Zhangjiang, P.R. China

Abstract

For the nonlinear difference equations of the form

$$w(z+1)w(z-1) = h(z)w^m(z),$$

where $h(z)$ is a nonzero rational function and $m = \pm 2, \pm 1, 0$, we show that its transcendental meromorphic solution is mainly determined by its zeros, 1-value points and poles except for some special cases. Examples for the sharpness of these results are given.

MSC: 30D35; 39B32

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1 Introduction

For a given meromorphic function $f(z)$, we use the standard notation of the Nevanlinna theory (see e.g. [2, 4, 10]), such as $T(r,f), m(r,f), N(r,f), \rho(f), \lambda(f)$ and $\lambda(1/f)$. And we say that a meromorphic function $a(z)$ is a small function of $f(z)$, if $T(r,a) = o(T(r,f)) = S(r,f)$. Denote the set of all small functions of $f(z)$ by S_f.

Let $f(z)$ and $g(z)$ be two meromorphic functions, $a \in S_f \cap S_g$. We say $f(z)$ and $g(z)$ share a IM (CM), if $f(z) - a$ and $g(z) - a$ have the same zeros ignoring multiplicities (counting multiplicities). And we say $f(z)$ and $g(z)$ share ∞ IM (CM), if they have the same poles ignoring multiplicities (counting multiplicities).

Our aim in the paper is to investigate the uniqueness of meromorphic solutions of nonlinear difference equations, which are given by Ronkainen in [8], in the form

$$w(z+1)w(z-1) = h(z)w^m(z), \tag{1.1}$$

where $h(z)$ is a nonzero rational function and $m = \pm 2, \pm 1, 0$. This idea is partly due to the investigation of the uniqueness of meromorphic solutions of some differential equations (see e.g. [1, 9, 12]), and partly due to some recent research on the uniqueness of meromorphic solutions of several kinds of difference equations (see e.g. [3, 6, 7]). One of these results reads as follows.

Theorem A ([3]) *Let $f(z)$ be a finite order transcendental meromorphic solution of the equation*

$$P_1(z)f(z+1)+P_2(z)f(z)=P_3(z),$$

where $P_1(z), P_2(z), P_3(z)$ are nonzero polynomials such that $P_1(z)+P_2(z)\not\equiv 0$. If a meromorphic function $g(z)$ shares $0,1,\infty$ CM with $f(z)$, then one of the following cases holds:

(i) $f(z)\equiv g(z)$;

(ii) $f(z)+g(z)=f(z)g(z)$;

(iii) *there exist a polynomial $\beta(z)=a_0z+b_0$ and a constant a_0 satisfying $e^{a_0}\neq e^{b_0}$, such that*

$$f(z)=\frac{1-e^{\beta(z)}}{e^{\beta(z)}(e^{a_0-b_0}-1)},\qquad g(z)=\frac{1-e^{\beta(z)}}{1-e^{b_0-a_0}},$$

where $a_0\neq 0, b_0$ are constants.

Considering Theorem A and Eq. (1.1), we prove the following results.

Theorem 1.1 *Let $w(z)$ be a finite order transcendental meromorphic solution of Eq. (1.1), where $m=-2,-1,0,1$. If a meromorphic function $u(z)$ shares $0,1,\infty$ CM with $w(z)$, then $w(z)\equiv u(z)$.*

Theorem 1.2 *Let $w(z)$ be a finite order transcendental meromorphic solution of Eq. (1.1), where $m=2$, and $h(z)$ satisfies*

$$\lim_{z\to\infty} h(z)\neq 1. \tag{1.2}$$

If a meromorphic function $u(z)$ shares $0,1,\infty$ CM with $w(z)$, then $w(z)\equiv u(z)$.

The following examples show that the numbers of shared values in Theorem 1.1 and Theorem 1.2 cannot be reduced.

Example 1 In the following examples, $w_j(z)$ and $u_j(z)\equiv -w_j(z)$ share $0,\infty$ CM ($j=1,\ldots,5$):

(1) $w_1(z)=z\tan(\pi z/2)$ satisfies the difference equation

$$w(z+1)w(z-1)=(z+1)(z-1)z^2w^{-2}(z);$$

(2) $w_2(z)=z\tan^2(\pi z/3)\tan^2(\pi z/3-\pi/6)$ satisfies the difference equation

$$w(z+1)w(z-1)=(z+1)(z-1)zw^{-1}(z);$$

(3) $w_3(z)=z\tan(\pi z/4)$ satisfies the difference equation

$$w(z+1)w(z-1)=-(z+1)(z-1);$$

(4) $w_4(z) = z\tan(\pi z/6)\tan(\pi z/6 - \pi/6)$ satisfies the difference equation

$$w(z+1)w(z-1) = -\frac{(z+1)(z-1)}{z}w(z);$$

(5) $w_5(z) = e^{z^2}\tan(\pi z)$ satisfies the difference equation

$$w(z+1)w(z-1) = e^2w^2(z).$$

Remark 1 We have tried hard but failed to find examples for the sharpness of the "CM" shared condition in Theorem 1.1 and Theorem 1.2 until now.

The following example shows that the condition (1.2) in Theorem 1.2 is necessary.

Example 2 $w(z) = e^z$ and $u(z) = e^{-z}$ share $0, 1, \infty$ CM, and $w(z)$ satisfies the difference equation

$$w(z+1)w(z-1) = w^2(z).$$

Here $h(z) \equiv 1$ and $w(z) \not\equiv u(z)$.

2 Some lemmas

From the results of Lan and Chen [5] and Zhang and Yang [11], we have the following.

Lemma 2.1 ([5, 11]) *Let $w(z)$ be a finite order transcendental meromorphic solution of Eq.* (1.1), *where $m = -2, \pm1, 0$. Then $\lambda(w - a) = \lambda(1/w) = \rho(w) \ge 1$, where a is an arbitrary constant.*

We need the following result.

Lemma 2.2 *Let $\theta_1 \ne \theta_2 \in [-\pi, \pi)$ be two given real numbers. Then, for any given integer $k \ge 1$, there exist some $\theta_3, \theta_4 \in [-\pi, \pi)$ such that*

$$\operatorname{Re} e^{i(\theta_1 + k\theta_3)} > 0 > \operatorname{Re} e^{i(\theta_2 + k\theta_3)}, \qquad \operatorname{Re} e^{i(\theta_2 + k\theta_4)} > 0 > \operatorname{Re} e^{i(\theta_1 + k\theta_4)}.$$

Proof Since $\theta_1 \ne \theta_2 \in [-\pi, \pi)$, we have $\theta_1 - \theta_2 \ne 0, 2\pi$, and hence $-1 \le \cos(\theta_1 - \theta_2) < 1$. If $\theta_1 + \theta_2 \in (-2\pi, 0]$, we choose a point $\alpha = -(\pi + \theta_1 + \theta_2)/2k \in [-\pi, \pi)$, and we have

$$\begin{aligned} 2\cos(\theta_1 + k\alpha)\cos(\theta_2 + k\alpha) &= \cos(\theta_1 + \theta_2 + 2k\alpha) + \cos(\theta_1 - \theta_2) \\ &= \cos(-\pi) + \cos(\theta_1 - \theta_2) = -1 + \cos(\theta_1 - \theta_2) < 0. \end{aligned}$$

Without loss of generality, assume that $\cos(\theta_1 + k\alpha) > 0$, then $\cos(\theta_2 + k\alpha) < 0$, and we can denote $\theta_3 = \alpha$. What is more, if $k\alpha < 0$, denote $\theta_4 = \alpha + \pi/k$; if $k\alpha \ge 0$, denote $\theta_4 = \alpha - \pi/k$, then we have $\cos(\theta_2 + k\theta_4) > 0 > \cos(\theta_1 + k\theta_4)$.

If $\theta_1 + \theta_2 \in (0, 2\pi)$, choose a point $\beta = (\pi - \theta_1 - \theta_2)/2k \in (-\pi, \pi)$, then

$$2\cos(\theta_1 + k\beta)\cos(\theta_2 + k\beta) = \cos(\pi) + \cos(\theta_1 - \theta_2) = -1 + \cos(\theta_1 - \theta_2) < 0.$$

From the equation above, we can similarly obtain θ_3 and θ_4, which we need.

Finally, note that $\operatorname{Re} e^{i\theta} = \cos\theta$, and we finish our proof. □

3 Proof of Theorem 1.1

Since $w(z)$ and $u(z)$ are meromorphic functions and share $0, 1, \infty$ CM, from the second main theorem of Nevanlinna theory, we have

$$\begin{aligned} T(r,u) &\le N(r,u) + N\left(r, \frac{1}{u}\right) + N\left(r, \frac{1}{u-1}\right) + S(r,u) \\ &\le N(r,w) + N\left(r, \frac{1}{w}\right) + N\left(r, \frac{1}{w-1}\right) + S(r,u) \\ &\le 3T(r,w) + S(r,u). \end{aligned}$$

This indicates that $\rho(u) \le \rho(w)$, and hence $u(z)$ is also of finite order.

Now from the assumption that $w(z)$ and $u(z)$ share $0, 1, \infty$ CM again, we get

$$\frac{u}{w} = e^{p(z)}, \tag{3.1}$$

$$\frac{u-1}{w-1} = e^{q(z)}, \tag{3.2}$$

where $p(z), q(z)$ are polynomials such that $\deg p(z) = l, \deg q(z) = s$.

We claim that $e^{p(z)} \equiv e^{q(z)}$, then we get $w(z) \equiv u(z)$, which follows from (3.1) and (3.2) immediately.

Otherwise, $e^{p(z)} \not\equiv e^{q(z)}$, then $e^{p(z)} \not\equiv 1$ and $e^{q(z)} \not\equiv 1$. Now (3.1) and (3.2) give

$$w(z) = \frac{1 - e^{q(z)}}{e^{p(z)} - e^{q(z)}} \tag{3.3}$$

and

$$w(z) - 1 = \frac{1 - e^{p(z)}}{e^{p(z)} - e^{q(z)}}. \tag{3.4}$$

From (3.3) and (3.4), we see that

$$N\left(r, \frac{1}{w}\right) \le N\left(r, \frac{1}{1-e^q}\right) \le T\left(r, 1-e^q\right) + O(1) \le T\left(r, e^q\right) + O$$

and

$$N\left(r, \frac{1}{w-1}\right) \le N\left(r, \frac{1}{1-e^q}\right) \le T\left(r, 1-e^p\right) + O(1) \le T\left(r, e^p\right) + O(1).$$

Thus, we have

$$\lambda(w) \le \rho\left(e^q\right) = s, \qquad \lambda(w-1) \le \rho\left(e^p\right) = l. \tag{3.5}$$

If $s > l$, then

$$N\left(r, \frac{1}{1-e^p}\right) \le T\left(r, e^p\right) + O(1) = S\left(r, e^q\right). \tag{3.6}$$

From the second main theorem of Nevanlinna theory again, we have

$$\begin{aligned} T(r,e^q) &\le N(r,e^q) + N\left(r,\frac{1}{e^q}\right) + N\left(r,\frac{1}{e^q-1}\right) + S(r,e^q) \\ &= N\left(r,\frac{1}{e^q-1}\right) + S(r,e^q), \end{aligned}$$

which leads to

$$N\left(r,\frac{1}{e^q-1}\right) = T(r,e^q) + S(r,e^q). \tag{3.7}$$

Since the common zeros of $1-e^{p-q}$ and $1-e^q$ should be the zeros of $1-e^p$, from (3.3), (3.6) and (3.7), we can find that

$$\begin{aligned} N\left(r,\frac{1}{w}\right) &= N\left(r,\frac{e^p(1-e^{p-q})}{1-e^q}\right) \\ &\ge N\left(r,\frac{1}{1-e^q}\right) - N\left(r,\frac{1}{1-e^p}\right) = T(r,e^q) + S(r,e^q), \end{aligned}$$

and hence $\lambda(w) \ge \rho(e^q) = s$. Thus, from Lemma 2.1, we get $\lambda(w-1) = \lambda(w) \ge s > l$, which contradicts the second conclusion in (3.5).

If $s < l$, then with a similar reasoning we can deduce a similar contradiction to the first conclusion in (3.5). Therefore, we prove that $s = l$.

If $\deg(q(z)-p(z)) < l$, then

$$N\left(r,\frac{1}{1-e^{p-q}}\right) \le T(r,e^{p-q}) + O(1) = S(r,e^q).$$

From this equation, (3.3) and (3.7), we see that

$$N\left(r,\frac{1}{w}\right) \ge N\left(r,\frac{1}{1-e^q}\right) - N\left(r,\frac{1}{1-e^{p-q}}\right) = T(r,e^q) + S(r,e^q),$$

which implies that $\lambda(w) \ge \rho(e^q) = s = l$. Then from Lemma 2.1 and (3.3), we can deduce the contradiction that

$$l \le \lambda(w) = \lambda(1/w) \le \lambda\left(1-e^{q-p}\right) = \rho\left(e^{q-p}\right) < l.$$

Thus, $\deg(q(z)-p(z)) = l \ge 1$, and hence if we set

$$p(z) = a_l z^l + a_{l-1}z^{l-1} + \cdots + a_0$$

and

$$q(z) = b_l z^l + b_{l-1}z^{l-1} + \cdots + b_0,$$

then $a_l b_l \ne 0$ and $a_l \ne b_l$. Denote $a_l = r_1 e^{i\theta_1}, b_l = r_2 e^{i\theta_2}$ where $\theta_1, \theta_2 \in [-\pi, \pi)$.

Next, we discuss four cases step by step and give the relative contradictions.

Case 1: $m = 0$. From (1.1) and (3.3), we get

$$\frac{1 - e^{q(z+1)}}{e^{p(z+1)} - e^{q(z+1)}} \frac{1 - e^{q(z-1)}}{e^{p(z-1)} - e^{q(z-1)}} = w(z+1)w(z-1) = h(z). \tag{3.8}$$

Subcase 1.1: $\theta_1 = \theta_2$. Now $|a_l| = r_1 \neq r_2 = |b_l|$. If $r_1 < r_2$, then, for all $z = re^{i\theta_3}$ such that $\theta_1 + l\theta_3 = 0$, we have

$$a_l z^l = r_1 r^l e^{i(\theta_1 + l\theta_3)} = r_1 r^l < r_2 r^l = r_2 r e^{i(\theta_1 + l\theta_3)} = b_l z^l. \tag{3.9}$$

From (3.8) and (3.9), we get

$$\begin{aligned} \lim_{r\to\infty} h\bigl(re^{i\theta_3}\bigr) &= \lim_{r\to\infty} \frac{1 - e^{q(re^{i\theta_3}+1)}}{e^{p(re^{i\theta_3}+1)} - e^{q(re^{i\theta_3}+1)}} \frac{1 - e^{q(re^{i\theta_3}-1)}}{e^{p(re^{i\theta_3}-1)} - e^{q(re^{i\theta_3}-1)}} \\ &= \lim_{r\to\infty} \frac{1 - e^{r_2 r^l (1+o(1))}}{e^{r_1 r^l (1+o(1))} - e^{r_2 r^l (1+o(1))}} \frac{1 - e^{r_2 r^l (1+o(1))}}{e^{r_1 r^l (1+o(1))} - e^{r_2 r^l (1+o(1))}} = 1. \end{aligned} \tag{3.10}$$

As $h(z)$ is a rational function, for all $\theta \in [-\pi, \pi)$, we can get from (3.10)

$$\lim_{r\to\infty} h\bigl(re^{i\theta}\bigr) = 1. \tag{3.11}$$

However, for the θ_4 such that $\theta_1 + l\theta_4 = -\pi$, we can deduce that

$$\begin{aligned} \lim_{r\to\infty} h\bigl(re^{i\theta_4}\bigr) &= \lim_{r\to\infty} \frac{1 - e^{q(re^{i\theta_4}+1)}}{e^{p(re^{i\theta_4}+1)} - e^{q(re^{i\theta_4}+1)}} \frac{1 - e^{q(re^{i\theta_4}-1)}}{e^{p(re^{i\theta_4}-1)} - e^{q(re^{i\theta_4}-1)}} \\ &= \lim_{r\to\infty} \frac{1 - e^{-r_1 r^l (1+o(1))}}{e^{-r_2 r^l (1+o(1))} - e^{-r_1 r^l (1+o(1))}} \frac{1 - e^{-r_1 r^l (1+o(1))}}{e^{-r_2 r^l (1+o(1))} - e^{-r_1 r^l (1+o(1))}} = \infty, \end{aligned}$$

a contradiction to (3.11).

If $r_1 > r_2$, we can easily get a similar contradiction.

Subcase 1.2: $\theta_1 \neq \theta_2$. By Lemma 2.2, there exist some $\theta_5, \theta_6 \in [-\pi, \pi)$ such that

$$\operatorname{Re} e^{i(\theta_1 + k\theta_5)} > 0 > \operatorname{Re} e^{i(\theta_2 + k\theta_5)}, \qquad \operatorname{Re} e^{i(\theta_2 + k\theta_6)} > 0 > \operatorname{Re} e^{i(\theta_1 + k\theta_6)}.$$

This means that, for $j = 0, 1, 2$, and $r_3 = r_1 \operatorname{Re} e^{i(\theta_1 + l\theta_5)}$, $r_4 = r_2 \operatorname{Re} e^{i(\theta_2 + l\theta_6)}$, we have

$$p\bigl(re^{i\theta_5} + j\bigr) = e^{r_3 r^l (1+o(1))}, \qquad q\bigl(re^{i\theta_5} + j\bigr) = o(1) \tag{3.12}$$

and

$$q\bigl(re^{i\theta_6} + j\bigr) = e^{r_4 r^l (1+o(1))}, \qquad p\bigl(re^{i\theta_6} + j\bigr) = o(1), \tag{3.13}$$

as $r \to \infty$.

We can get from (3.12) and (3.13)

$$\lim_{r\to\infty} h\bigl(re^{i\theta_5}\bigr) = \lim_{r\to\infty} \frac{1 - e^{q(re^{i\theta_5}+1)}}{e^{p(re^{i\theta_5}+1)} - e^{q(re^{i\theta_5}+1)}} \frac{1 - e^{q(re^{i\theta_5}-1)}}{e^{p(re^{i\theta_5}-1)} - e^{q(re^{i\theta_5}-1)}}$$

$$= \lim_{r\to\infty} \frac{1-o(1)}{e^{r_3 r^l(1+o(1))}-o(1)} \frac{1-o(1)}{e^{r_3 r^l(1+o(1))}-o(1)} = 0 \tag{3.14}$$

and

$$\begin{aligned}\lim_{r\to\infty} h\bigl(re^{i\theta_6}\bigr) &= \lim_{r\to\infty} \frac{1-e^{q(re^{i\theta_6}+1)}}{e^{p(re^{i\theta_6}+1)}-e^{q(re^{i\theta_6}+1)}} \frac{1-e^{q(re^{i\theta_6}-1)}}{e^{p(re^{i\theta_6}-1)}-e^{q(re^{i\theta_6}-1)}} \\ &= \lim_{r\to\infty} \frac{1-e^{r_4 r^l(1+o(1))}}{o(1)-e^{r_4 r^l(1+o(1))}} \frac{1-e^{r_4 r^l(1+o(1))}}{o(1)-e^{r_4 r^l(1+o(1))}} = 1,\end{aligned} \tag{3.15}$$

respectively. Since $h(z)$ is a rational function, we can get a contradiction from (3.14) and (3.15).

Case 2: $m = 1$. Now (1.1) is of the form

$$w(z+1)w(z-1) = h(z)w(z),$$

which gives

$$w(z+3)w(z) = h(z+2)h(z+1). \tag{3.16}$$

From this and (3.3), we get

$$\frac{1-e^{q(z+3)}}{e^{p(z+3)}-e^{q(z+1)}} \frac{1-e^{q(z)}}{e^{p(z+3)}-e^{q(z)}} = w(z+3)w(z) = h(z+2)h(z+1).$$

Notice that $h(z+2)h(z+1)$ is still a rational function. With (3.16), we can deduce some similar contradictions as in Case 1 again.

Case 3: $m = -1$. From (1.1) and (3.3), we get

$$\frac{1-e^{q(z+1)}}{e^{p(z+1)}-e^{q(z+1)}} \frac{1-e^{q(z-1)}}{e^{p(z-1)}-e^{q(z-1)}} \frac{1-e^{q(z)}}{e^{p(z)}-e^{q(z)}} = w(z+1)w(z-1)w(z) = h(z). \tag{3.17}$$

With (3.17), we can also deduce some similar contradictions as in the Case 1.

Case 4: $m = -2$. From (1.1) and (3.3), we get

$$\frac{1-e^{q(z+1)}}{e^{p(z+1)}-e^{q(z+1)}} \frac{1-e^{q(z-1)}}{e^{p(z-1)}-e^{q(z-1)}} \left(\frac{1-e^{q(z)}}{e^{p(z)}-e^{q(z)}}\right)^2 = w(z+1)w(z-1)w^2(z) = h(z),$$

which enables us to get some similar contradictions as in Case 1. This finishes our proof.

4 Proof of Theorem 1.2

Obviously, we can use (3.1)–(3.3) for this case directly. Moreover, we may begin our proof with assuming that $e^{p(z)} \not\equiv e^{q(z)}$, then $e^{p(z)} \not\equiv 1$ and $e^{q(z)} \not\equiv 1$. Now (3.3) also holds. Thus, we can get from (1.1) and (3.3)

$$\frac{1-e^{q(z+1)}}{e^{p(z+1)}-e^{q(z+1)}} \frac{1-e^{q(z-1)}}{e^{p(z-1)}-e^{q(z-1)}} \left(\frac{e^{p(z)}-e^{q(z)}}{1-e^{q(z)}}\right)^2 = h(z), \tag{4.1}$$

where $p(z)$ and $q(z)$ are polynomials such that

$$p(z) = a_l z^l + a_{l-1} z^{l-1} + \cdots + a_0$$

and

$$q(z) = b_s z^s + b_{s-1} z^{s-1} + \cdots + b_0,$$

where $a_l b_s \neq 0$. Denote $a_l = r_1 e^{i\theta_1}$, $b_s = r_2 e^{i\theta_2}$ where $\theta_1, \theta_2 \in [-\pi, \pi)$.

If $l > s$, it is easy to find that there exists some ray $\theta = \theta_3$ such that $\theta_1 + l\theta_3 = 0$, for $z = re^{i\theta_3}$ and $j = -1, 0, 1$,

$$p\bigl(re^{i\theta_3} + j\bigr) = r_1 r^l \bigl(1 + o(1)\bigr), \qquad q\bigl(re^{i\theta_3 + j}\bigr) = o\bigl(r^l\bigr),$$

as $r \to \infty$. Then we can get

$$\begin{aligned} \lim_{r\to\infty} h\bigl(re^{i\theta_3}\bigr) &= \lim_{r\to\infty} \frac{1 - e^{q(re^{i\theta_3}+1)}}{e^{p(re^{i\theta_3}+1)} - e^{q(re^{i\theta_3}+1)}} \frac{1 - e^{q(re^{i\theta_3}-1)}}{e^{p(re^{i\theta_3}-1)} - e^{q(re^{i\theta_3}-1)}} \left(\frac{e^{p(re^{i\theta_3})} - e^{q(re^{i\theta_3})}}{1 - e^{q(re^{i\theta_3})}} \right)^2 \\ &= \lim_{r\to\infty} \frac{1 - e^{o(r^l)}}{e^{r_1 r^l (1+o(1))} - e^{o(r^l)}} \frac{1 - e^{o(r^l)}}{e^{r_1 r^l (1+o(1))} - e^{o(r^l)}} \left(\frac{e^{r_1 r^l (1+o(1))} - e^{o(r^l)}}{1 - e^{o(r^l)}} \right)^2 = 1, \end{aligned}$$

a contradiction to (1.2). Thus, $l \leq s$. However, with a similar reasoning above, we see that $l < s$ is also impossible. This indicates that $l = s$.

Next, we complete our proof by driving some contradictions for two cases.

Case 1: $\theta_1 = \theta_2$. If $r_1 > r_2$, then, for all $z = re^{i\theta_4}$ such that $\theta_1 + l\theta_4 = 0$, we have

$$a_l z^l = r_1 r^l e^{i(\theta_1 + l\theta_4)} = r_1 r^l > b_l z^l = r_2 r e^{i(\theta_1 + l\theta_4)} = r_2 r^l. \tag{4.2}$$

From (4.1) and (4.2), we can prove that

$$\lim_{r\to\infty} h\bigl(re^{i\theta_4}\bigr) = 1, \tag{4.3}$$

which is also a contradiction to (1.2).

Similarly, we can prove that $r_1 < r_2$ is impossible. Thus, $r_1 = r_2$. Since $e^{p(z)} \not\equiv e^{q(z)}$, now for $j = 0, 1, 2$ and the θ_4 given before, we have

$$e^{q(re^{i\theta_4}+j)} = e^{r_1 r^l (1+o(1))}, \qquad e^{p(re^{i\theta_4}+j)} - e^{q(re^{i\theta_4}+j)} = e^{r_1 r^l} \bigl(1 + o(1)\bigr).$$

This and (4.1) yield the same limit (4.3), a contradiction to (1.2).

Case 2: $\theta_1 \neq \theta_2$. By Lemma 2.2, there exists some $\theta_5 \in [-\pi, \pi)$ such that

$$\operatorname{Re} e^{i(\theta_1 + l\theta_5)} > 0 > \operatorname{Re} e^{i(\theta_2 + l\theta_5)}.$$

This means that, for $j = 0, 1, 2$, and $r_3 = r_1 \operatorname{Re} e^{i(\theta_1 + l\theta_5)}$, as $r \to \infty$, we have

$$p\bigl(re^{i\theta_5} + j\bigr) = e^{r_3 r^l (1+o(1))}, \qquad q\bigl(re^{i\theta_5} + j\bigr) = o(1). \tag{4.4}$$

It is easy to deduce from (4.1) and (4.4) that

$$\lim_{r\to\infty} h\bigl(re^{i\theta_5}\bigr) = 1,$$

a contradiction to (1.2).

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Availability of data and materials
All details of our paper, including datasets, have been present in it.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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