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Exact and approximate solutions for the fractional Schrödinger equation with variable coefficients

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Abstract

In this paper, by introducing the fractional derivatives in the sense of Caputo, the modified general mapping deformation method (MGMDM) and the modified fractional variational iteration method (MFVIM) are applied to obtain some exact and approximate solutions of the variable-coefficient fractional Schrödinger equation (VFNLS) with time and space fractional derivatives. With the aid of symbolic computation, a broad class of exact analytical solutions and their structure of the VFNLS are investigated. Furthermore, the approximate iterative series showed that the MFVIM is powerful, reliable and effective when compared with some traditional decomposition method in searching for the approximate solutions of the complex nonlinear partial differential equations with variable coefficients and fractional derivatives.

Keywords: Modified fractional variational iteration method; Caputo derivative; Fractional nonlinear Schrödinger equation; Exact solutions; Approximate solutions

1 Introduction

In recent years, due to the wide applications of fractional differential equations (FDEs) in nonlinear science [1], many phenomena can be described successfully by using FDEs such as chaotic oscillations [2], electrochemistry [3], engineering [4] and so on [5]. Searching for exact solutions of these FDEs plays an important and significant role in the study on the dynamics of those phenomena. Many powerful methods have been proposed to handle this subject, such as the Darboux transformation method [6], the split-step Fourier transform method [7], the fractional characteristic method [8, 9]. But because of the complexity of the nonlinear terms, most FDEs do not have exact analytic solutions, so approximate and numerical methods must be used. Many efforts have been proposed for these problems, including the homotopy analysis method (HAM) [10], the homotopy perturbation method (HPM) [11], the adomian decomposition method (ADM) [12].

The variational iteration method (VIM) and the fractional variational iteration method (FVIM) were established in [13] and [14, 15] by He, respectively, and they were thoroughly used by many researchers [16, 17]. Some new developments about the fractional derivative and its application are available in Refs. [18–20]. After giving some modification, Hong and Lu proposed the MFVIM for some complex nonlinear partial differential equations

with fractional derivative [21]. The motivation of this paper is to construct some exact and approximate solutions for the VFNLs by using the MGMDM and MFVIM.

We give some definitions and properties of the fractional calculus theory which are used further in this paper [22–24].

Definition 1 The fractional derivative is defined as the following limit form:

$$f^\alpha = \lim_{h \rightarrow 0} \frac{\Delta^\alpha [f(x) - f(0)]}{h^\alpha}.$$

Definition 2 The Riemann–Liouville fractional integral operator of order $\alpha > 0$ for a function $f(x)$ is defined as

$$J_x^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x - \xi)^{\alpha-1} f(\xi) d\xi, \quad \alpha > 0, x > 0, J_x^0 f(x) = f(x).$$

Also we have the following properties:

$$J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x), \quad J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x), \quad J^\alpha x^\gamma = \frac{\Gamma(\gamma + 1)}{\Gamma(\alpha + \gamma + 1)} x^{\alpha+\gamma}.$$

Definition 3 For $\alpha > 0, x > 0, f(x) \in C_{-1}^n$, the Caputo fractional derivative operator of order α on the whole space is defined as

$$D^\alpha f(x) = J^{n-\alpha} D^n f(x) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^x (x - \xi)^{n-\alpha-1} f^{(n)}(\xi) d\xi, & n - 1 < \alpha < n, n \in \mathbb{N}. \\ \frac{d^{(n)} f(x)}{dx^n}, & \alpha = n. \end{cases}$$

We have the following properties:

$$D^\alpha C = 0 \quad (C \text{ is a constant}), \quad D^\alpha x^\gamma = \begin{cases} \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)} x^{\gamma-\alpha}, & \gamma > \alpha - 1, \\ 0, & \gamma \leq \alpha - 1. \end{cases}$$

$$J^\alpha D^\alpha f(x) = f(x) - \sum_{k=0}^{n-1} f^{(k)}(0^+) \frac{x^k}{k!}, \quad n - 1 < \alpha < n, \quad D^\alpha J^\alpha f(x) = f(x).$$

Definition 4 The fractional derivative of compounded functions is defined as

$$d^\alpha f = \Gamma(1 + \alpha) f.$$

Definition 5 The integral with respect to $d(x)^\alpha$ is defined as the solution of the fractional differential equation

$$dy = f(x)d(x)^\alpha, \quad x \geq 0, y(0) = 0.$$

2 Exact solution and their structure of the GFNLs

Consider the following generalized time and space fractional nonlinear Schrödinger equation with variable coefficients:

$$i \frac{\partial^\alpha u}{\partial z^\alpha} + \frac{1}{2} a(z) \frac{\partial^{2\beta} u}{\partial t^{2\beta}} + b(z) u |u|^2 - ic(z) u = 0, \quad z > 0, 0 < \alpha, \beta \leq 1, \tag{1}$$

where $u = u(z, t)$, $\frac{\partial u^\alpha}{\partial z} = D_z^\alpha u$, $\frac{\partial^{2\beta} u}{\partial t^{2\beta}} = D_t^{2\beta} (D_t^\beta u)$, when $\alpha = \beta = 1$, this equation turns to the famous nonlinear Schrödinger equations in an optical fiber [25–27]. Here $u(z, t)$ is the complex envelope of the electrical field, z is the normalized propagation distance along the fiber, t is the retarded time and the subscripts denote partial derivatives, the real analytic functions $a(z)$ and $b(z)$ are the slowly increasing dispersion coefficient and nonlinear coefficient, respectively, which represent the group velocity dispersion (GVD) and the self-phase modulation (SPM), $c(z)$ represents the heat-insulating amplification or loss. The transmission of a soliton in the real communication system of an optical soliton is described by Eq. (1) [9].

If we let $t \rightarrow x, z \rightarrow t$, Eq. (1) turns to the following form:

$$i \frac{\partial^\alpha u}{\partial t^\alpha} + \frac{1}{2} a(t) \frac{\partial^{2\beta} u}{\partial x^{2\beta}} + b(t) u |u|^2 - ic(t) u = 0, \quad t > 0, 0 < \alpha, \beta \leq 1. \tag{2}$$

We can give the complex variable transformation as follows:

$$u = A(t) \varphi(\xi) e^{i\eta}, \tag{3}$$

$$\xi = \frac{k_1 x^\beta}{\Gamma(1 + \beta)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} c_1(\tau) d\tau, \tag{4}$$

$$\eta = \frac{k_2 x^\beta}{\Gamma(1 + \beta)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} c_2(\tau) d\tau,$$

with the following consistency conditions:

$$A(t) = k e^{\frac{\alpha}{\Gamma(1+\alpha)} \int_0^t \tau^{\alpha-1} c(\tau) d\tau}, \quad c_1(t) = -k_1 k_2 a(t), \tag{5}$$

where k, k_1, k_2 are arbitrary nonzero constants.

Substituting (3), (4), (5) into (2), we obtain

$$k_1^2 a(t) \varphi_{\xi\xi}(\xi) - [2c_2(t) + k_2^2 a(t)] \varphi(\xi) + 2b(t) A^2(t) \varphi^3(\xi) = 0, \tag{6}$$

where $\varphi_{\xi\xi}(\xi) = \frac{d^2 \varphi(\xi)}{d\xi^2}$, with the idea of the homogeneous balance principle, by balancing the highest-order linear term $\varphi_{\xi\xi}(\xi)$ and the nonlinear $\varphi^3(\xi)$ in (6), we assume that Eq. (6) has the following solutions:

$$\varphi = \varphi(\xi) = A_0 + A_1 F(\xi) = A_0 + A_1 F, \tag{7}$$

$$F'^2 = \sum_{i=0}^4 a_i F^i, \tag{8}$$

where a_i ($i = 0, 1, 2, 3, 4$) are constants to be determined. We write $F = F(\xi)$ and $F' = \frac{dF(\xi)}{d\xi}$.

Substituting (7) and (8) into (6), and setting the coefficients of $F^i(\xi), i = 0, 1, 2, \dots$, to zero yield an ODE with respect to the unknowns $A_0, A_1, a_0, a_1, a_2, a_3, a_4, k_1, k_2, a(t), b(t)$ and $c_2(t)$. After solving the ODE by Mathematica software we could determine the following solutions:

Case 1

$$A_0 = a_1 = a_3 = 0, \quad A_1 = \text{const}, \quad c_2(t) = \frac{1}{2}(a_2k_1^2 - k_2^2)a(t),$$

$$b(t) = -a_4k_1^2A_1^{-2}a(t)A^{-2}(t).$$

By using the general mapping deformation method [28], we can obtain the following solutions of the corresponding Eq. (2):

$$u_{1_j} = kA_1 e^{\frac{\alpha}{\Gamma(1+\alpha)} \int_0^t \tau^{\alpha-1} c(\tau) d\tau + i[\frac{k_2 x^\beta}{\Gamma(1+\beta)} + \frac{1}{2\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} (a_2k_1^2 - k_2^2)a(\tau) d\tau]} F_{1_j}(\xi_{1_j}),$$

$$\xi_{1_j} = \frac{k_1 x^\beta}{\Gamma(1+\beta)} - \frac{k_1 k_2}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} a(\tau) d\tau, \quad j = 1, 2, 3, \dots$$

We have the consistency conditions

$$b(t) = -a_4k_1^2A_1^{-2}a(t)A^{-2}(t).$$

Here F_{1_j} is an arbitrary solution of the equation $F_{1_j}'' = a_{0_j} + a_{2_j}F_{1_j}^2 + a_{4_j}F_{1_j}^4$, we can obtain the 52 classes of exact solutions F_{1_j} from Ref. [29]; for example, if we let $a_{0_1} = 1 - m^2$, $a_{2_1} = 2m^2 - 1$, $a_{4_1} = -m^2$, $F_{1_1} = cn\xi$, we have

$$u_{1_1} = kA_1 e^{\frac{\alpha}{\Gamma(1+\alpha)} \int_0^t \tau^{\alpha-1} c(\tau) d\tau + i[\frac{k_2 x^\beta}{\Gamma(1+\beta)} + \frac{1}{2\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} ((2m^2-1)k_1^2 - k_2^2)a(\tau) d\tau]} cn(\xi_{1_j}),$$

with the consistency conditions $b(t) = m^2k_1^2A_1^{-2}a(t)A^{-2}(t)$, $A(t) = ke^{\frac{\alpha}{\Gamma(1+\alpha)} \int_0^t \tau^{\alpha-1} c(\tau) d\tau}$, if we let $a_{0_2} = 1$, $a_{2_2} = -1 - m^2$, $a_{4_2} = m^2$, $F_{1_2} = sn\xi$, we have

$$u_{1_2} = kA_1 e^{\frac{\alpha}{\Gamma(1+\alpha)} \int_0^t \tau^{\alpha-1} c(\tau) d\tau + i[\frac{k_2 x^\beta}{\Gamma(1+\beta)} + \frac{1}{2\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} ((-m^2-1)k_1^2 - k_2^2)a(\tau) d\tau]} sn(\xi_{1_j}),$$

with the consistency conditions $b(t) = -m^2k_1^2A_1^{-2}a(t)A^{-2}(t)$, if we let $a_{0_3} = m^2 - 1$, $a_{2_3} = 2 - m^2$, $a_{4_3} = -1$, $F_{1_3} = dn\xi$, we derive

$$u_{1_3} = kA_1 e^{\frac{\alpha}{\Gamma(1+\alpha)} \int_0^t \tau^{\alpha-1} c(\tau) d\tau + i[\frac{k_2 x^\beta}{\Gamma(1+\beta)} + \frac{1}{2\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} ((2-m^2)k_1^2 - k_2^2)a(\tau) d\tau]} dn(\xi_{1_j}),$$

with the consistency conditions $b(t) = k_1^2A_1^{-2}a(t)A^{-2}(t)$. And so on.

Remark 1 Let us take

$$\alpha = \beta = 1, \quad kA_1 = c_3 \sqrt{\frac{-k_2 m^2}{k_4(2m^2 - 1)}},$$

$$k_2 = c_2, \quad k_1 = c_1 \sqrt{\frac{k_2}{(2m^2 - 1)}}, \quad a(t) \rightarrow \beta(t), \quad b(t) \rightarrow \delta(t),$$

$$c(t) \rightarrow \alpha(t), \quad t \rightarrow z, \quad x \rightarrow t.$$

We find that $u_{1,1}$ turns to the solutions u_{31} in Ref. [25],

$$u_{1,1} = c_3 \sqrt{\frac{-k_2 m^2}{k_4(2m^2 - 1)}} e^{\int_0^z \alpha(\tau) d\tau + i[c_2 t + \frac{1}{2} \int_0^z (c_1^2 k_2 - c_2^2) \beta(\tau) d\tau]} \times cn \left[c_1 \sqrt{\frac{k_2}{2m^2 - 1}} (t - c_2) \int_0^z \beta(\tau) d\tau \right].$$

Let us take

$$\begin{aligned} \alpha = \beta = 1, \quad m = 1, \quad kA_1 = cR, \quad k_2 = C_1, \\ k_1 = \sqrt{R}C_2, \quad a(t) \rightarrow \beta(t), \quad b(t) \rightarrow \delta(t), \quad c(t) \rightarrow \alpha(t), \\ t \rightarrow z, \quad t \rightarrow x. \end{aligned}$$

We find that $u_{1,1}$ has degenerated to the famous bright-soliton solutions u_{31} in Ref. [26].

$$u_{1,1,2} = cR e^{\int_0^z \alpha(\tau) d\tau + i[C_1 t + \frac{1}{2} \int_0^z (C_2^2 R - C_1^2) \beta(\tau) d\tau]} \operatorname{sech} h \left\{ \sqrt{R} \left[C_2 t - C_1 C_2 \int_0^z \beta(\tau) d\tau \right] \right\}.$$

Let us take

$$\begin{aligned} \alpha = \beta = 1, \quad m = 1, \quad k = \sqrt{-\frac{c_2}{2c_4}}, \quad k_2 = A_3, \quad k_1 = \sqrt{\frac{2c_2}{-m^2 - 1}} A_2, \\ a(t) \rightarrow \beta(t), \quad b(t) \rightarrow \alpha(t), \\ c(t) \rightarrow \gamma(t), \quad t \rightarrow z, \quad t \rightarrow x. \end{aligned}$$

We find that $u_{1,2}$ has degenerated to the famous dark-soliton solutions

$$u_{1,2,1} = A_1 \sqrt{-\frac{c_2}{2c_4}} e^{\int_0^z \gamma(\tau) d\tau + i[A_3 t + \frac{2A_2^2 c_2 - A_3^2}{2} \int_0^z \beta(\tau) d\tau]} \times \tanh \left[A_2 \sqrt{-c_2} \left(t - A_3 \int_0^z \beta(\tau) d\tau \right) \right], \quad c_2 < 0, c_4 > 0.$$

Case 2

$$\begin{aligned} A_0 = \text{const}, \quad A_1 = \frac{4a_4 A_0}{a_3}, \quad 8a_1 a_4^2 = 4a_2 a_3 a_4 - a_3^3, \quad a_3 a_4 \neq 0, \\ c_2(t) = \left(\frac{a_1 a_4 k_1^2}{a_3} - \frac{a_3^2 k_1^2}{16a_4} - \frac{k_2^2}{2} \right) a(t), \quad b(t) = -\frac{a_3^2 k_1^2}{16a_4 A_0^2} a(t) A^{-2}(t). \end{aligned}$$

We acquire the following exact solutions of Eq. (2) by using Appendix B in Ref. [28]:

$$u_{2,j} = kA_0 \left(1 + \frac{4a_4}{a_3} \right) e^{\frac{\alpha}{\Gamma(1+\alpha)} \int_0^t \tau^{\alpha-1} c(\tau) d\tau + i \left[\frac{k_2 x^\beta}{\Gamma(1+\beta)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \left(\frac{a_1 a_4 k_1^2}{a_3} - \frac{a_3^2 k_1^2}{16a_4} - \frac{k_2^2}{2} \right) a(\tau) d\tau \right]} \times F_{2,j} \left[\frac{k_1 x^\beta}{\Gamma(1+\beta)} - \frac{k_1 k_2}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} a(\tau) d\tau \right],$$

$$j = 1, 2, 3, \dots, 9.$$

If we let $a_{0_4} = 1, a_{1_4} = -4, a_{2_4} = 8 - 4m^2, a_{3_4} = 4 - 4m^2, a_{4_4} = 8m^2 - 8, F_4 = \frac{cn\xi}{cn\xi \pm sn\xi dn\xi}$, we get the general Jacobi elliptic functions solution.

$$u_{2_4} = -7kA_0 e^{\frac{\alpha}{\Gamma(1+\alpha)} \int_0^t \tau^{\alpha-1} c(\tau) d\tau + i \left[\frac{k_2 x^\beta}{\Gamma(1+\beta)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} (8k_1^2 - \frac{(m^2-1)k_1^2}{8} - \frac{k_2^2}{2}) a(\tau) d\tau \right]} \frac{cn\xi}{cn\xi \pm sn\xi dn\xi},$$

$$\xi = \frac{k_1 x^\beta}{\Gamma(1+\beta)} - \frac{k_1 k_2}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} a(\tau) d\tau.$$

We have the consistency conditions $b(t) = -\frac{(m^2-1)k_1^2}{8A_0^2} a(t)A^{-2}(t)$.

If we let $a_{0_7} = \frac{1}{4}, a_{1_7} = 1, a_{2_7} = 2 - m^2, a_{3_7} = 2 - 2m^2, a_{4_7} = 1 - m^2, F_7 = \frac{sn\xi}{1 - sn\xi + cn\xi}$, we have

$$u_{2_7} = 3kA_0 e^{\frac{\alpha}{\Gamma(1+\alpha)} \int_0^t \tau^{\alpha-1} c(\tau) d\tau + i \left[\frac{k_2 x^\beta}{\Gamma(1+\beta)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \left(\frac{k_1^2}{2} - \frac{(1-m^2)k_1^2}{4} - \frac{k_2^2}{2} \right) a(\tau) d\tau \right]}$$

$$\times \left(sn \left[\frac{k_1 x^\beta}{\Gamma(1+\beta)} - \frac{k_1 k_2}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} a(\tau) d\tau \right] \right)$$

$$/ \left(1 - sn \left[\frac{k_1 x^\beta}{\Gamma(1+\beta)} - \frac{k_1 k_2}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} a(\tau) d\tau \right] \right)$$

$$+ cn \left[\frac{k_1 x^\beta}{\Gamma(1+\beta)} - \frac{k_1 k_2}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} a(\tau) d\tau \right],$$

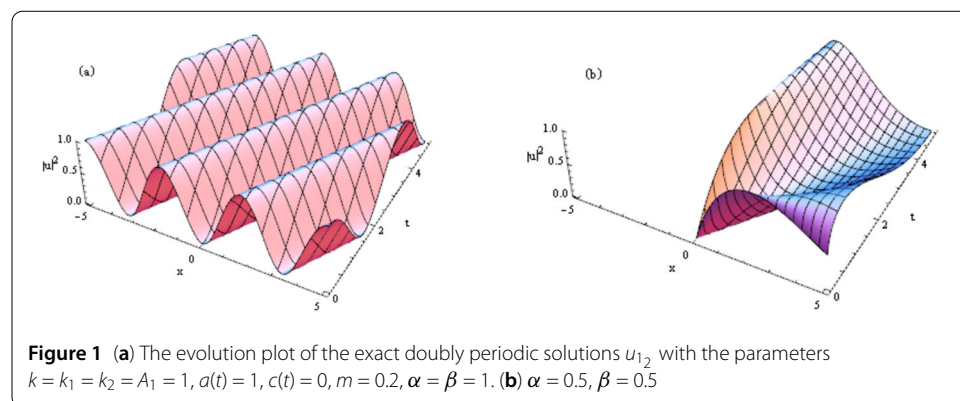
with the consistency conditions $b(t) = -\frac{(1-m^2)k_1^2}{4A_0^2} a(t)A^{-2}(t)$. And so on.

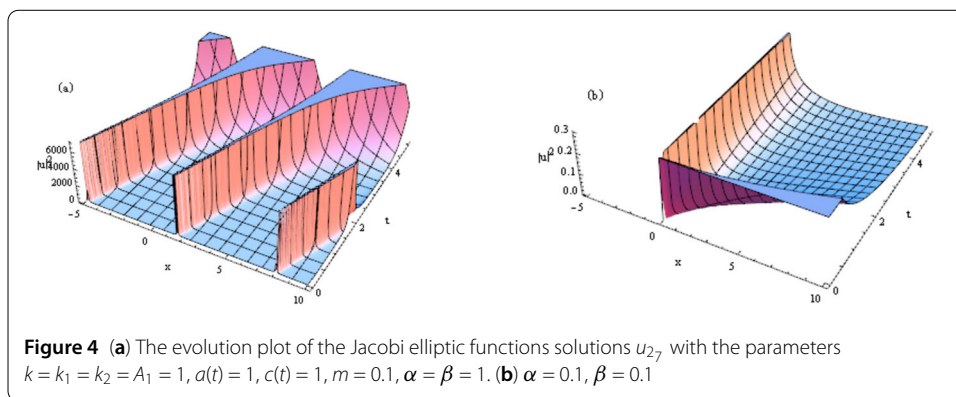
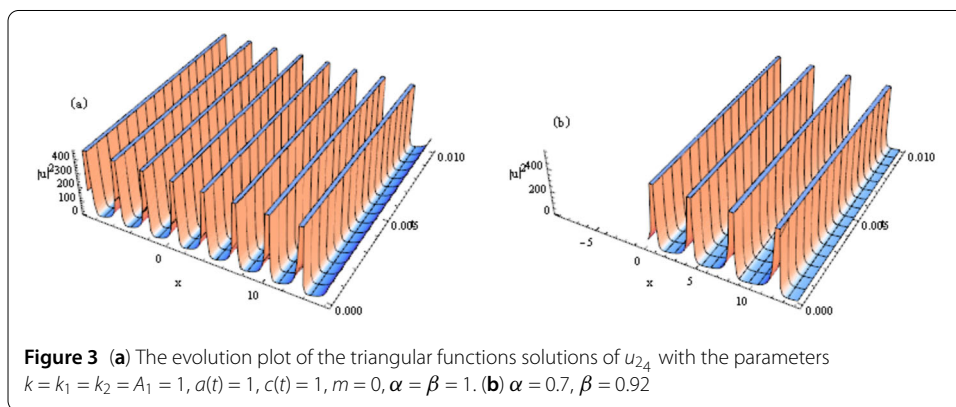
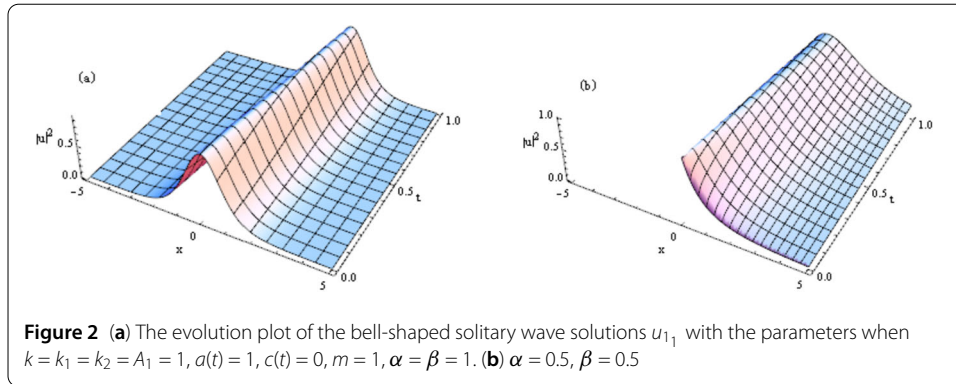
Remark 2 The solutions u_{1_j} and u_{2_j} are new exact solutions for Eq. (2) to the best of our knowledge. When the modulus of these Jacobi elliptic functions solutions has degenerated to 1 or 0, we can obtain the corresponding solitary-like solutions and triangular-like functions solutions. Some structure of these solutions of Eq. (1) are simulated in Fig. 1–Fig. 4

3 The MFVIM and approximate solutions for the VFNLS

According to the idea of FVIM [17, 20] and MFVIM [21], we can build a correction functional for Eq. (2) as follows:

$$u_{n+1} = u_n + J_t^\alpha \left\{ \lambda(t, x) \left[i \frac{\partial^\alpha u_n}{\partial t^\alpha} + \frac{1}{2} a(t) \frac{\partial^{2\beta} \tilde{u}_n}{\partial x^{2\beta}} + b(t) \tilde{u}_n |\tilde{u}_n|^2 - ic(t) \tilde{u}_n \right] \right\}. \tag{9}$$





Here $u_0 = u(x, 0) = f(x)$, $\lambda(t, x)$ is a general Lagrange multiplier, which can be identified optimally with the variational theory. The function \tilde{u}_n is a restricted variation, which means $\delta \tilde{u}_n = 0$. Therefore, we first determine the Lagrange multiplier $\lambda(t, x)$ that will be identified optimally via integration by parts. The successive approximations $u_{n+1}, n \geq 0$, of the solution $u(x, t)$ will be readily obtained upon using the obtained $\lambda(t, x)$ and any selective function u_0 . The initial values are usually used for choosing the zeroth approximation u_0 . Consequently, the exact solution may be procured by using $u = \lim_{n \rightarrow \infty} u_n$.

In the following, we will apply the MFVIM to a model about Eq. (2) to illustrate the strength of the method.

Example We now consider the following time-space fractional NLS equation [30–32]:

$$i \frac{\partial^\alpha u}{\partial t^\alpha} + a \frac{\partial^{2\beta} u}{\partial x^{2\beta}} + bu|u|^2 - icu = 0, \quad t > 0, 0 < \alpha, \beta \leq 1, \quad u(x, 0) = Ae^{ix}. \tag{10}$$

Here a, b, c are constants. This equation occurs in various kinds of theoretical physics, such as nonlinear optics, superconductivity and plasma physics, which also can represent some dynamical system in quantum mechanics, fluid dynamics and nonlinear dynamics [31].

The correction functional for (10) reads

$$u_{n+1} = u_n + J_t^\alpha \left\{ \lambda(t, x) \left[i \frac{\partial^\alpha u_n}{\partial t^\alpha} + a \frac{\partial^{2\beta} \tilde{u}_n}{\partial x^{2\beta}} + b\tilde{u}_n|\tilde{u}_n|^2 - ic\tilde{u}_n \right] \right\}. \tag{11}$$

Making the above correction functional stationary,

$$\delta u_{n+1} = \delta u_n + \lambda(t, x) i \delta u_n - J_t^\alpha \left\{ [i\lambda^\alpha(t, x) \delta u_n] \right\}. \tag{12}$$

After getting the coefficients of δu_n to zero we can determine the Lagrange multiplier

$$\lambda = i. \tag{13}$$

We produces the iteration formulation as follows:

$$u_{n+1} = u_n + iJ_t^\alpha \left[i \frac{\partial^\alpha u_n}{\partial t^\alpha} + a \frac{\partial^{2\beta} u_n}{\partial x^{2\beta}} + bu_n|u_n|^2 - icu_n \right], \quad |u|^2 = |u_0|^2. \tag{14}$$

As stated before, we can select $u_0 = u(x, 0) = Ae^{ix}$, using the iteration (14) and the Mathematica software, we obtain the following successive approximations:

$$\begin{aligned} u_0 &= Ae^{ix}, \\ u_1 &= Ae^{ix} + aAe^{i(x+\pi\beta)} \frac{it^\alpha}{\Gamma(1+\alpha)} + bA^3e^{ix} \frac{it^\alpha}{\Gamma(1+\alpha)} - icAe^{ix} \frac{it^\alpha}{\Gamma(1+\alpha)} \\ &= Ae^{ix} \left(1 + \frac{c_1 it^\alpha}{\Gamma(1+\alpha)} \right), \quad c_1 = ae^{i\pi\beta} + bA^2 - ic, \\ u_2 &= u_1 - Ae^{ix} \frac{c_1 it^\alpha}{\Gamma(1+\alpha)} + aAe^{i(x+\pi\beta)} \left[\frac{it^\alpha}{\Gamma(1+\alpha)} + \frac{c_1 i^2 t^{2\alpha}}{\Gamma(1+2\alpha)} \right] \\ &\quad + bA^3e^{ix} \left[\frac{it^\alpha}{\Gamma(1+\alpha)} + \frac{c_1 i^2 t^{2\alpha}}{\Gamma(1+2\alpha)} \right] \\ &\quad - icAe^{ix} \left[\frac{it^\alpha}{\Gamma(1+\alpha)} + \frac{c_1 i^2 t^{2\alpha}}{\Gamma(1+2\alpha)} \right] \\ &= Ae^{ix} \left[1 + \frac{c_1 it^\alpha}{\Gamma(1+\alpha)} + \frac{c_1^2 i^2 t^{2\alpha}}{\Gamma(1+2\alpha)} \right], \\ u_3 &= Ae^{ix} \left[1 + \frac{c_1 it^\alpha}{\Gamma(1+\alpha)} + \frac{c_1^2 i^2 t^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{c_1^3 i^3 t^{3\alpha}}{\Gamma(1+3\alpha)} \right], \end{aligned}$$

$$\dots,$$

$$u_n = Ae^{ix} \sum_{k=0}^n \frac{1}{\Gamma(1+k\alpha)} [c_1 it^\alpha]^k.$$

The exact solution of Eq. (10) is

$$u_{\text{exact}} = Ae^{ix} \lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{1}{\Gamma(1+k\alpha)} [c_1 it^\alpha]^k = Ae^{ix} E_\alpha [(ae^{i\pi\beta} + bA^2 - ic)it^\alpha]. \tag{15}$$

Here $E_\alpha [(ae^{i\pi\beta} + bA^2 - ic)it^\alpha]$ is the Mittag-Leffler function.

Remark 3 If we select $A = 1, a = k, b = 2k, c = 0, \alpha = 1$, the solution (15) contains the result (49) in Ref. [30], the result (3.18) and (3.21) in Ref. [31], the result (27) in Ref. [32], but we can find that this iteration is much more simple, standard and powerful than the HAM, the ADM and the VIM mentioned in Refs. [30–32]. The solution (15) is a new exact solution for Eq. (10) to the best of our knowledge.

4 Conclusion

In this paper, the modified general mapping deformation method and the MFVIM are used for finding exact and approximate solutions of the GFNLS equation with the Caputo derivative. The obtained results indicate that the MFVIM is an effective, and a convenient and powerful method for solving nonlinear fractional complex differential equations when compared with some other traditional asymptotic decomposition method such as HAM, VIM and ADM. We believe that these two methods should play an important role for finding exact and approximate solutions in mathematical physics.

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Availability of data and materials

The authors affirmed the availability of data and material in deriving the solutions mentioned in this manuscript.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

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