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Positive solutions of singular Hadamard-type fractional differential equations with infinite-point boundary conditions or integral boundary conditions

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Abstract

We establish the existence of positive solutions to a class of a singular nonlinear Hadamard-type fractional differential equations with infinite-point boundary conditions (BCs) or integral BCs. Our analysis is based on Leray–Schauder type continuation. Several examples are given to illustrate our results.

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1 Introduction

Our aim in this article is to study the problem of existence of continuous solutions of the following singular Hadamard fractional differential equation:

$$ {}_{\mathrm{H}}D^{\gamma} v(t) + f\big(t, v(t), {}_{\mathrm{H}}D^{\delta} v(t), v'(t)\big) = 0, \quad \text{a.e. } t \in (1,e), \tag{1}$$

together with either the functional integral boundary condition given by

$$ v(1) = 0, \qquad v(e) = v_0 + \lambda \int_1^e v\big(\phi(\xi)\big) \frac{\phi'(\xi)}{\phi(\xi)}\, d\xi, \tag{2}$$

or the infinite-point boundary conditions given by

$$ v(1) = 0, \qquad v(e) = v_0 + \lambda \sum_{j=1}^{\infty} a_j v\big(\phi(\eta_j)\big), \tag{3}$$

where $f : [1,e] \times R^+ \times R^2$ is an L^p-Carathéodory positive function, $p > \frac{1}{\gamma - 1}$, ${}_{\mathrm{H}}D^{\delta}$ is the Hadamard fractional derivative of order δ, and $1 < \gamma < 2$, $0 < \delta < 1$, $1 \le \gamma - \delta < 2$. The constants a_j, λ, and v_0 are nonnegative, the function $\phi : [1,e] \to [1,e]$, $\phi(t) \le t$ is continuous and the singularity occurring in our problems is associated with $v' \in C(1,e]$ at the left endpoint $t = 1$.

Due to the fact that fractional-order models are more accurate than integer-order models (that is, there are more degrees of freedom in fractional-order models), the subject of fractional differential equations has recently evolved into an interesting subject for many researchers due to its multiple applications in economics, engineering, physics, chemistry, mechanics. However, most of the results for fractional differential equations are concerned with the Riemann–Liouville fractional derivative or the Caputo fractional derivative (see for example Agarwal et al. [1], Akcan and Çetin [4], Bai and Qiu [5], Bai and Sun [6], Callegari and Nachman [9], Chalishajar and Kumar [10], El-Saka et al. [11], El-Sayed et al. [12], Kosmatov [19], Li and Zhang [21], Liu et al. [22], Qiao and Zhou [26], Qiu and Bai [27], Rida et al. [28], Song et al. [30], Staněk [31], Tian and Chen [33].

In 1892, Hadamard introduced another kind of fractional derivatives, i.e., Hadamard-type fractional differential equations, which differs from the preceding ones in the sense that the kernel of the integral and derivative contain logarithmic function of arbitrary exponent, which is presented as a quite different kind of weakly singular kernel. Details and properties of Hadamard fractional derivatives and integrals can be found in Kilbas et al. [18], Butzer et al. [8], Gambo et al. [13]. Recently, there were some results on Hadamard-type fractional differential equations; see Ahmad et al. [3], Ahmad and Ntouyas [2], Benchohra et al. [7], Lyons and Neugebauer [24], Matar [25], Shammakh [29], Thiramanus et al. [32], Yang [35], Zhang et al. [37], and the references cited therein.

The study of boundary value problems (BVPs) involving infinite-point BCs has become attractive recently, many significant and interesting cases of BVPs of fractional order were considered with infinite-point BCs by (for example) Gao and Han [14], Ge et al. [15], Guo et al. [16], Hu and Zhang [17], Li et al. [20], Liu et al. [23], Zhang and Zhong [38] and Zhang [39] (see also to the references cited therein). In the year 2016, Xu and Yang [34] proposed a generalization of the PID controller and studied two kinds of fractional-order differential equations arising in control theory together with the infinite-point boundary conditions. Their results can describe the corresponding control system accurately and also provide a platform for the understanding of our environment. However, investigations on the infinite-point problems for differential equations of fractional or integer order have gradually aroused people's attentions and interests, but such investigations are still few.

Motivated by the above-mentioned developments and results, we consider the BVP given by (1) and (3) or by (1) and (2). In each case, we determine sufficient conditions on f guaranteeing that these problems has a continuous positive solution. We first find the existence of positive solutions of the problem (1) subject to the multi-point boundary conditions

$$\nu(1) = 0, \qquad \nu(e) = \nu_0 + \lambda \sum_{j=1}^{m} a_j \nu\bigl(\phi(\eta_j)\bigr). \tag{4}$$

The main new features presented in this paper are as follows. First, the boundary value problem has a more general form, in which f is not continuous, but only Carathéodory and allowed to be singular at $t = 1$. Second, the nonlocal boundary conditions of the unknown function are more general cases, which include two-point, three-point, multi-point, infinite point and integral boundary conditions, and some nonlocal problems as special cases. Third, positive continuous solutions of problems (1), (3) or (1), (2) or (1), (4) are obtained.

2 Preliminaries

Throughout the paper $\|x\|_p = (\int_1^e |x(t)|^p\,dt)^{\frac{1}{p}}$ is the norm in $L^p[1,e]$, $\|u\|_0 = \max\{|u(t)| : t \in [1,e]\}$ be the norm in the space $C[1,e]$ and $AC[1,e]$ denote the space of absolute continuous functions on $[1,e]$. We denoted by $L^p_{\mathrm{loc}}(1,e]$ the space of functions on $[1,e]$ defined by

$$L^p_{\mathrm{loc}}(1,e] = \bigl\{\nu \mid \nu|_{[c,d]} \in L^p[c,d] \text{ for every compact interval } [c,d] \subset (1,e]\bigr\}.$$

We make the following assumptions:

(H_1) $\eta_j \in (1,e)$, $j = 1,2,\dots,m$, $1 < \eta_1 < \eta_2 < \cdots < \eta_m < e$, a_j, λ, and ν_0 are nonnegative, $1 < \gamma < 2$, $0 < \delta < 1$ and $1 \le \gamma - \delta < 2$.

(H_2) The function $\phi : [1,e] \to [1,e]$, $\phi(t) \le t$ is continuous.

(H_3) f is L^p-Carathéodory function on $[1,e] \times R^+ \times R^2$ i.e., for each $(\nu_1,\nu_2,\nu_3) \in R^+ \times R^2$, the function $f(\cdot,\nu_1,\nu_2,\nu_3) : [1,e] \to R$ is Lebesgue measurable and for each $t \in [1,e]$, the function $f(t,\cdot,\cdot,\cdot) : R^+ \times R^2 \to R$ is continuous.

(H_4) There exist $p(t), q(t), s(t) \in L^p[1,e]$ and $r(t) \in L^p_{\mathrm{loc}}(1,e]$ with $(\log t)^{\gamma-2} r(t) \in L^p[1,e]$, with $p > \frac{1}{\gamma-1}$ such that

$$\begin{aligned} &\bigl|f(t,\nu_1,\nu_2,\nu_3)\bigr| \le p(t)|\nu_1| + q(t)|\nu_2| + r(t)|\nu_3| + s(t), \\ &\quad \text{a.e. } t \in [1,e], \text{and all } (\nu_1,\nu_2,\nu_3) \in R^+ \times R^2. \end{aligned} \tag{5}$$

Definition 1 ([18]) The Hadamard fractional integral of order γ for a function $\nu \in L^p[1,e]$, $1 \le p < \infty$, is defined as

$${}_{\mathrm{H}}J^{\gamma}\nu(t) = \frac{1}{\Gamma(\gamma)} \int_1^t \left(\log\frac{t}{\theta}\right)^{\gamma-1} \frac{\nu(\theta)}{\theta}\,d\theta, \quad \gamma > 0, \tag{6}$$

provided the integral exists, where $\log(\cdot) = \log_e(\cdot)$.

Definition 2 ([18]) The Hadamard derivative of fractional order γ for a function $\nu : [1,\infty) \to \mathbb{R}$ is defined as

$${}_{\mathrm{H}}D^{\gamma}\nu(t) = \frac{1}{\Gamma(n-\gamma)} \left(t\frac{d}{dt}\right)^n \int_1^t \left(\log\frac{t}{\theta}\right)^{n-\gamma-1} \frac{\nu(\theta)}{\theta}\,d\theta, \quad n-1 < \gamma < n. \tag{7}$$

The relationship between fractional integration (6) and derivatives (7) is stated in the next theorem [18].

Theorem 1 *Let* $\gamma > 0$, $n-1 < \gamma < n$, *then*:

(d_1) *The Hadamard fractional differential equation* ${}_{\mathrm{H}}D^{\gamma}\nu(t) = 0$ *is valid if and only if*

$$\nu(t) = \sum_{i=1}^{n} c_i (\log t)^{\gamma-i},$$

where $c_i \in R$ $(i = 1,\dots,n)$ *are arbitrary constants.*

In particular, when $1 < \gamma < 2$, *the relation* ${}_{\mathrm{H}}D^{\gamma}\nu(t) = 0$ *holds, if and only if*

$$\nu(t) = c_1(\log t)^{\gamma-1} + c_2(\log t)^{\gamma-2} \quad \textit{for any } c_1, c_2 \in R.$$

(d_2) *The quality* ${}_{\mathrm{H}}D^{\gamma}\,{}_{\mathrm{H}}J^{\gamma}v(t) = v(t)$ *holds for every* $v \in L^p[1,e]$.

(d_3) *Let* $v \in C[1,\infty) \cap L^p[1,\infty)$. *The following formula holds*:

$$ {}_{\mathrm{H}}J^{\gamma}\,{}_{\mathrm{H}}D^{\gamma}v(t) = v(t) - \sum_{i=1}^{n} c_i(\log t)^{\gamma-i}. $$

(d_4) ${}_{\mathrm{H}}J^{\gamma}\,{}_{\mathrm{H}}J^{\beta}v(t) = {}_{\mathrm{H}}J^{\gamma+\beta}v(t)$, $\beta > 0$.

(d_5) ${}_{\mathrm{H}}J^{1-\delta}(\log t)^{\gamma-1} = \frac{\Gamma(\gamma)}{\Gamma(\gamma-\delta+1)}(\log t)^{\gamma-\delta}$, $\delta \in (0,1)$.

Our results in this article are based upon the Leray–Schauder continuation principle.

Theorem 2 (Leray–Schauder continuation principle; see e.g. [36]) *Let* X *be a Banach space and* $T : X \to X$ *be a compact map. Assume that there exists an* $D > 0$ *so that if* $v = \lambda Tv$ *for* $\lambda \in [0,1]$ *then* $\|v\|_X \le D$. *Then* $v = Tv$ *is solvable.*

Lemma 1 *Suppose that* $\rho \in L^p[1,e]$, $p > \frac{1}{\gamma-1}$, $1 \le t_1 < t_2 \le e$ *and* $\frac{1}{p} + \frac{1}{q} = 1$, *then we have*

$$ \left|\int_1^t \left(\log\frac{t}{\theta}\right)^{\gamma-2}\frac{\rho(\theta)}{\theta}\,d\theta\right| \le \left(\frac{(\log t)^b}{b}\right)^{\frac{1}{q}}\|\rho\|_p, \tag{8} $$

$$ \begin{aligned} &\left|\int_1^{t_2} \left(\log\frac{t_2}{\theta}\right)^{\gamma-2}\frac{\rho(\theta)}{\theta}\,d\theta - \int_1^{t_1} \left(\log\frac{t_1}{\theta}\right)^{\gamma-2}\frac{\rho(\theta)}{\theta}\,d\theta\right| \\ &\quad\le \left(\frac{1}{b}\right)^{\frac{1}{q}}\left((\log t_1)^b - (\log t_2)^b + \left(\log\frac{t_2}{t_1}\right)^b\right)^{\frac{1}{q}}\|\rho\|_p + \left(\frac{1}{b}\right)^{\frac{1}{q}}\left(\log\frac{t_2}{t_1}\right)^{\frac{b}{q}}\|\rho\|_p, \end{aligned} \tag{9} $$

where $b = (\gamma-2)q + 1$.

Proof Using the Hölder inequality with $\frac{1}{p} + \frac{1}{q} = 1$, we get

$$ \begin{aligned} \left|\int_1^t \left(\log\frac{t}{\theta}\right)^{\gamma-2}\frac{\rho(\theta)}{\theta}\,d\theta\right| &\le \left(\int_1^t \left(\log\frac{t}{\theta}\right)^{(\gamma-2)q}\frac{d\theta}{\theta^q}\right)^{\frac{1}{q}}\left(\int_1^t |\rho(\theta)|^p\,d\theta\right)^{\frac{1}{p}} \\ &\le \left(\int_1^t \left(\log\frac{t}{\theta}\right)^{(\gamma-2)q}\frac{d\theta}{\theta}\right)^{\frac{1}{q}}\|\rho\|_p \\ &\le \left(\frac{(\log t)^b}{b}\right)^{\frac{1}{q}}\|\rho\|_p. \end{aligned} $$

Using again the Hölder inequality and the inequality $(1-y)^q \le 1 - y^q$, $|y| < 1$, we get

$$ \begin{aligned} &\left|\int_1^{t_2} \left(\log\frac{t_2}{\theta}\right)^{\gamma-2}\frac{\rho(\theta)}{\theta}\,d\theta - \int_1^{t_1} \left(\log\frac{t_1}{\theta}\right)^{\gamma-2}\frac{\rho(\theta)}{\theta}\,d\theta\right| \\ &\quad\le \int_1^{t_1}\left(\left(\log\frac{t_1}{\theta}\right)^{\gamma-2} - \left(\log\frac{t_2}{\theta}\right)^{\gamma-2}\right)\frac{|\rho(\theta)|}{\theta}\,d\theta + \int_{t_1}^{t_2}\left(\log\frac{t_2}{\theta}\right)^{\gamma-2}\frac{|\rho(\theta)|}{\theta}\,d\theta \\ &\quad\le \left(\int_1^{t_1}\left(\left(\log\frac{t_1}{\theta}\right)^{\gamma-2} - \left(\log\frac{t_2}{\theta}\right)^{\gamma-2}\right)^q\frac{d\theta}{\theta^q}\right)^{\frac{1}{q}}\left(\int_1^{t_1}|\rho(\theta)|^p\,d\theta\right)^{\frac{1}{p}} \\ &\qquad+ \left(\int_{t_1}^{t_2}\left(\log\frac{t_2}{\theta}\right)^{(\gamma-2)q}\frac{d\theta}{\theta^q}\right)^{\frac{1}{q}}\left(\int_{t_1}^{t_2}|\rho(\theta)|^p\,d\theta\right)^{\frac{1}{p}} \end{aligned} $$

$$
\begin{aligned}
&\leq \left(\int_1^{t_1} \left(\log \frac{t_1}{\theta}\right)^{(\gamma-2)q} \left(1 - \left(\frac{\log \frac{t_2}{\theta}}{\log \frac{t_1}{\theta}}\right)^{\gamma-2}\right)^q \frac{d\theta}{\theta}\right)^{\frac{1}{q}} \|\rho\|_p \\
&\quad + \left(\frac{1}{(\gamma-2)q+1}\right)^{\frac{1}{q}} \left(\log \frac{t_2}{t_1}\right)^{\frac{(\gamma-2)q+1}{q}} \|\rho\|_p \\
&\leq \left(\int_1^{t_1} \left(\left(\log \frac{t_1}{\theta}\right)^{(\gamma-2)q} - \left(\log \frac{t_2}{\theta}\right)^{(\gamma-2)q}\right) \frac{d\theta}{\theta}\right)^{\frac{1}{q}} \|\rho\|_p \\
&\quad + \left(\frac{1}{b}\right)^{\frac{1}{q}} \left(\log \frac{t_2}{t_1}\right)^{\frac{b}{q}} \|\rho\|_p \\
&= \left(\frac{1}{b}\right)^{\frac{1}{q}} \left((\log t_1)^b - (\log t_2)^b + \left(\log \frac{t_2}{t_1}\right)^b\right)^{\frac{1}{q}} \|\rho\|_p + \left(\frac{1}{b}\right)^{\frac{1}{q}} \left(\log \frac{t_2}{t_1}\right)^{\frac{b}{q}} \|\rho\|_p.
\end{aligned}
$$

Hence the inequality (9) holds for all $1 \leq t_1 < t_2 \leq e$. □

Lemma 2 *Suppose that* $\rho \in L^p[1,e]$, $p > \frac{1}{\gamma-1}$, *then we see that*

$$
\int_1^t \left(\log \frac{t}{\theta}\right)^{\gamma-2} \frac{\rho(\theta)}{\theta}\, d\theta \quad \textit{is continuous on } [1,e].
$$

Proof The result follows from the inequality (9), since $0 < b = (\gamma - 2)q + 1 < 1$, then as $t_1 \to t_2$ in (9), the left-hand side tends to 0. □

Lemma 3 *Suppose that* $\rho \in L^p[1,e]$, *then we have*:

(i) *For* $t \in [1,e]$,

$$
\frac{d}{dt} \int_1^t \left(\log \frac{t}{\theta}\right)^{\gamma-1} \frac{\rho(\theta)}{\theta}\, d\theta = \frac{(\gamma-1)}{t} \int_1^t \left(\log \frac{t}{\theta}\right)^{\gamma-2} \frac{\rho(\theta)}{\theta}\, d\theta. \tag{10}
$$

(ii) *Let* $\{\rho_k\} \subset L^p[1,e]$ *be* L^p*-convergent sequence and let* $\lim_{k\to\infty} \rho_k = \rho$. *Then*

$$
\lim_{k\to\infty} \int_1^t \left(\log \frac{t}{\theta}\right)^{\gamma-2} \frac{\rho_k(\theta)}{\theta}\, d\theta = \int_1^t \left(\log \frac{t}{\theta}\right)^{\gamma-2} \frac{\rho(\theta)}{\theta}\, d\theta.
$$

Proof (i) We have by interchanging the order of integration

$$
\int_1^t \left(\int_1^s \left(\log \frac{s}{\theta}\right)^{\gamma-2} \frac{\rho(\theta)}{\theta}\, d\theta\right) \frac{ds}{s} = \frac{1}{\gamma-1} \int_1^t \left(\log \frac{t}{\theta}\right)^{\gamma-1} \frac{\rho(\theta)}{\theta}\, d\theta.
$$

For $t \in [1,e]$, since $\int_1^s (\log \frac{s}{\theta})^{\gamma-2} \frac{\rho(\theta)}{\theta}\, d\theta$ is a continuous function by Lemma 2, by differentiating both sides, the equality (10) follows.

(ii) Using the Hölder inequality, we obtain

$$
\begin{aligned}
&\left| \int_1^t \left(\log \frac{t}{\theta}\right)^{\gamma-2} \frac{\rho_k(\theta)}{\theta}\, d\theta - \int_1^t \left(\log \frac{t}{\theta}\right)^{\gamma-2} \frac{\rho(\theta)}{\theta}\, d\theta \right| \\
&\leq \int_1^t \left(\log \frac{t}{\theta}\right)^{\gamma-2} \frac{|\rho_k(\theta) - \rho(\theta)|}{\theta}\, d\theta
\end{aligned}
$$

$$\leq \left(\int_1^t \left(\log \frac{t}{\theta}\right)^{(\gamma-2)q} \frac{d\theta}{\theta^q}\right)^{\frac{1}{q}} \left(\int_1^t \left|\rho_k(\theta) - \rho(\theta)\right|^p d\theta\right)^{\frac{1}{p}}$$

$$\leq \left(\frac{(\log t)^{(\gamma-2)q+1}}{(\gamma-2)q+1}\right)^{\frac{1}{q}} \|\rho_k - \rho\|_p \leq \left(\frac{1}{b}\right)^{\frac{1}{q}} \|\rho_k - \rho\|_p.$$

Hence for $t \in [1, e]$, we have

$$\left\| \int_1^t \left(\log \frac{t}{\theta}\right)^{\gamma-2} \frac{\rho_k(\theta)}{\theta}\, d\theta - \int_1^t \left(\log \frac{t}{\theta}\right)^{\gamma-2} \frac{\rho(\theta)}{\theta}\, d\theta \right\|_0 \leq \left(\frac{1}{b}\right)^{\frac{1}{q}} \|\rho_k - \rho\|_p,$$

and the result follows. □

3 Existence of positive solutions of problem (1), (4)

For authors convenience, denote f_v by

$$f_v(t) = f\bigl(t, v(t), {}_{\mathrm{H}}D^{\delta} v(t), v'(t)\bigr).$$

Lemma 4 *Suppose that the condition* (H_1) *holds, then for* $f_v(t) \in L^p[1, e]$ *the boundary value problem*

$$ {}_{\mathrm{H}}D^{\gamma} v(t) + f_v(t) = 0, \quad a.e.\ t \in (1, e), \tag{11}$$

subject to the multi-point boundary conditions

$$v(1) = 0, \qquad v(e) = v_0 + \lambda \sum_{j=1}^{m} a_j v\bigl(\phi(\eta_j)\bigr), \tag{12}$$

has a unique solution $v \in AC[1, e]$ *if and only if* v *is a solution of the integral equation*

$$\begin{aligned} v(t) &= \frac{(\log t)^{\gamma-1}}{(1-\sigma)\Gamma(\gamma)} \int_1^e \left(\log \frac{e}{\theta}\right)^{\gamma-1} \frac{f_v(\theta)}{\theta}\, d\theta - \frac{1}{\Gamma(\gamma)} \int_1^t \left(\log \frac{t}{\theta}\right)^{\gamma-1} \frac{f_v(\theta)}{\theta}\, d\theta \\ &\quad - \frac{\lambda(\log t)^{\gamma-1}}{(1-\sigma)\Gamma(\gamma)} \sum_{j=1}^{m} a_j \int_1^{\phi(\eta_j)} \left(\log \frac{\phi(\eta_j)}{\theta}\right)^{\gamma-1} \frac{f_v(\theta)}{\theta}\, d\theta + \frac{v_0(\log t)^{\gamma-1}}{1-\sigma} \\ &= \int_1^e G(t,\theta) \frac{f_v(\theta)}{\theta}\, d\theta + \sum_{j=1}^{m} \frac{\lambda a_j (\log t)^{\gamma-1}}{(1-\sigma)} \int_1^e G\bigl(\phi(\eta_j), \theta\bigr) \frac{f_v(\theta)}{\theta}\, d\theta \\ &\quad + \frac{v_0(\log t)^{\gamma-1}}{1-\sigma}, \end{aligned} \tag{13}$$

where

$$G(t,\theta) = \frac{1}{\Gamma(\gamma)} \begin{cases} (\log t)^{\gamma-1}(\log \frac{e}{\theta})^{\gamma-1} - (\log \frac{t}{\theta})^{\gamma-1}, & 1 \leq \theta \leq t \leq e, \\ (\log t)^{\gamma-1}(\log \frac{e}{\theta})^{\gamma-1}, & 1 \leq t \leq \theta \leq e, \end{cases} \tag{14}$$

and

$$\sigma = \lambda \sum_{j=1}^{m} a_j \bigl(\log \phi(\eta_j)\bigr)^{\gamma-1} \neq 1.$$

Proof As discussed in [18], the solution of the Hadamard-type fractional differential equation (11) can be written as

$$v(t) = -\frac{1}{\Gamma(\gamma)}\int_1^t \left(\log\frac{t}{\theta}\right)^{\gamma-1}\frac{f_v(\theta)}{\theta}\,d\theta + c_1(\log t)^{\gamma-1} + c_2(\log t)^{\gamma-2}.$$

By using the BCs (12), we have $c_2 = 0$, then

$$v(t) = -\frac{1}{\Gamma(\gamma)}\int_1^t \left(\log\frac{t}{\theta}\right)^{\gamma-1}\frac{f_v(\theta)}{\theta}\,d\theta + c_1(\log t)^{\gamma-1}. \tag{15}$$

In view of condition $v(e) = v_0 + \lambda\sum_{j=1}^m a_j v(\phi(\eta_j))$, we have

$$\begin{aligned}&-\frac{1}{\Gamma(\gamma)}\int_1^e \left(\log\frac{e}{\theta}\right)^{\gamma-1}\frac{f_v(\theta)}{\theta}\,d\theta + c_1\\ &\quad= v_0 + \lambda\sum_{j=1}^m a_j\left[-\frac{1}{\Gamma(\gamma)}\int_1^{\phi(\eta_j)}\left(\log\frac{\phi(\eta_j)}{\theta}\right)^{\gamma-1}\frac{f_v(\theta)}{\theta}\,d\theta + c_1\big(\log\phi(\eta_j)\big)^{\gamma-1}\right],\end{aligned}$$

and we get

$$\begin{aligned}c_1 &= \frac{1}{1-\sigma}\left[v_0 + \frac{1}{\Gamma(\gamma)}\int_1^e \left(\log\frac{e}{\theta}\right)^{\gamma-1}\frac{f_v(\theta)}{\theta}\,d\theta\right.\\ &\quad\left.-\frac{\lambda\sum_{j=1}^m a_j}{\Gamma(\gamma)}\int_1^{\phi(\eta_j)}\left(\log\frac{\phi(\eta_j)}{\theta}\right)^{\gamma-1}\frac{f_v(\theta)}{\theta}\,d\theta\right].\end{aligned}$$

Substituting in Eq. (15), we have the formula

$$\begin{aligned}v(t) &= \frac{(\log t)^{\gamma-1}}{(1-\sigma)\Gamma(\gamma)}\int_1^e \left(\log\frac{e}{\theta}\right)^{\gamma-1}\frac{f_v(\theta)}{\theta}\,d\theta - \frac{1}{\Gamma(\gamma)}\int_1^t \left(\log\frac{t}{\theta}\right)^{\gamma-1}\frac{f_v(\theta)}{\theta}\,d\theta\\ &\quad-\frac{\lambda(\log t)^{\gamma-1}}{(1-\sigma)\Gamma(\gamma)}\sum_{j=1}^m a_j\int_1^{\phi(\eta_j)}\left(\log\frac{\phi(\eta_j)}{\theta}\right)^{\gamma-1}\frac{f_v(\theta)}{\theta}\,d\theta + \frac{v_0(\log t)^{\gamma-1}}{1-\sigma}.\end{aligned}$$

Conversely, let $v(t)$ be a solution of (13), we want to obtain (11) and (12). Now to obtain Eq. (11), operating on both sides of (13) by ${}_{\mathrm{H}}J^{2-\gamma}$ (using (d_4) and (d_5)), we get

$$\begin{aligned}{}_{\mathrm{H}}J^{2-\gamma}v(t) &= \frac{\log t}{(1-\sigma)}\int_1^e \left(\log\frac{e}{\theta}\right)^{\gamma-1}\frac{f_v(\theta)}{\theta}\,d\theta - \int_1^t \left(\log\frac{t}{\theta}\right)\frac{f_v(\theta)}{\theta}\,d\theta\\ &\quad-\frac{\lambda\log t}{(1-\sigma)}\sum_{j=1}^m a_j\int_1^{\phi(\eta_j)}\left(\log\frac{\phi(\eta_j)}{\theta}\right)^{\gamma-1}\frac{f_v(\theta)}{\theta}\,d\theta + \frac{v_0\Gamma(\gamma)\log t}{1-\sigma}.\end{aligned}$$

Again operating by $(t\frac{d}{dt})^2$ on both sides of the last equation, we obtain

$$\left(t\frac{d}{dt}\right)^2 {}_{\mathrm{H}}J^{2-\gamma}v(t) = -f_v(t),$$

that is,

$${}_{\mathrm{H}}D^{\gamma}v(t) + f_v(t) = 0.$$

Now to check the conditions in (12) are satisfied, we can easy from (13) show that $\nu(1) = 0$. Also to verify $\nu(e) = \nu_0 + \lambda\sum_{j=1}^{m} a_j\nu(\phi(\eta_j))$, and we have by a simple calculation using Eq. (13)

$$\nu(e) - \nu_0 = \frac{\sigma}{(1-\sigma)\Gamma(\gamma)}\int_1^e\left(\log\frac{e}{\theta}\right)^{\gamma-1}\frac{f_\nu(\theta)}{\theta}\,d\theta - \frac{\lambda}{(1-\sigma)\Gamma(\gamma)}\sum_{j=1}^{m}a_j\int_1^{\phi(\eta_j)}\left(\log\frac{\phi(\eta_j)}{\theta}\right)^{\gamma-1}\frac{f_\nu(\theta)}{\theta}\,d\theta + \frac{\sigma\nu_0}{1-\sigma}$$

and

$$\begin{aligned}\lambda\sum_{j=1}^{m}a_j\nu\big(\phi(\eta_j)\big) &= \frac{\sigma}{(1-\sigma)\Gamma(\gamma)}\int_1^e\left(\log\frac{e}{\theta}\right)^{\gamma-1}\frac{f_\nu(\theta)}{\theta}\,d\theta \\ &\quad - \frac{\lambda}{\Gamma(\gamma)}\sum_{j=1}^{m}a_j\int_1^{\phi(\eta_j)}\left(\log\frac{\phi(\eta_j)}{\theta}\right)^{\gamma-1}\frac{f_\nu(\theta)}{\theta}\,d\theta \\ &\quad - \frac{\lambda\sigma}{(1-\sigma)\Gamma(\gamma)}\sum_{j=1}^{m}a_j\int_1^{\phi(\eta_j)}\left(\log\frac{\phi(\eta_j)}{\theta}\right)^{\gamma-1}\frac{f_\nu(\theta)}{\theta}\,d\theta + \frac{\sigma\nu_0}{1-\sigma},\end{aligned}$$

and then we get $\nu(e) = \nu_0 + \lambda\sum_{j=1}^{m} a_j\nu(\phi(\eta_j))$.

This complete the proof of the equivalent between the problem (11)–(12) and the integral equation (13).

Now to construct the function $G(t,\theta)$, from the relation $\frac{1}{1-\sigma} = 1 + \frac{\sigma}{1-\sigma}$, we have

$$\begin{aligned}\nu(t) &= \bigg[\frac{(\log t)^{\gamma-1}}{\Gamma(\gamma)}\int_1^e\left(\log\frac{e}{\theta}\right)^{\gamma-1}\frac{f_\nu(\theta)}{\theta}\,d\theta \\ &\quad + \frac{\lambda(\log t)^{\gamma-1}\sum_{j=1}^{m}a_j(\log\phi(\eta_j))^{\gamma-1})}{(1-\sigma)\Gamma(\gamma)}\int_1^e\left(\log\frac{e}{\theta}\right)^{\gamma-1}\frac{f_\nu(\theta)}{\theta}\,d\theta\bigg] \\ &\quad - \frac{1}{\Gamma(\gamma)}\int_1^t\left(\log\frac{t}{\theta}\right)^{\gamma-1}\frac{f_\nu(\theta)}{\theta}\,d\theta \\ &\quad - \frac{\lambda(\log t)^{\gamma-1}}{(1-\sigma)\Gamma(\gamma)}\sum_{j=1}^{m}a_j\int_1^{\phi(\eta_j)}\left(\log\frac{\phi(\eta_j)}{\theta}\right)^{\gamma-1}\frac{f_\nu(\theta)}{\theta}\,d\theta + \frac{\nu_0(\log t)^{\gamma-1}}{1-\sigma} \\ &= \frac{1}{\Gamma(\gamma)}\int_1^t\left[(\log t)^{\gamma-1}\left(\log\frac{e}{\theta}\right)^{\gamma-1} - \left(\log\frac{t}{\theta}\right)^{\gamma-1}\right]\frac{f_\nu(\theta)}{\theta}\,d\theta \\ &\quad + \frac{1}{\Gamma(\gamma)}\int_t^e(\log t)^{\gamma-1}\left(\log\frac{e}{\theta}\right)^{\gamma-1}\frac{f_\nu(\theta)}{\theta}\,d\theta + \frac{\nu_0(\log t)^{\gamma-1}}{1-\sigma} \\ &\quad + \frac{\lambda(\log t)^{\gamma-1}}{(1-\sigma)\Gamma(\gamma)}\sum_{j=1}^{m}a_j\int_1^{\phi(\eta_j)}\left[\big(\log\phi(\eta_j)\big)^{\gamma-1}\left(\log\frac{e}{\theta}\right)^{\gamma-1} - \left(\log\frac{\phi(\eta_j)}{\theta}\right)^{\gamma-1}\right] \\ &\quad \times\frac{f_\nu(\theta)}{\theta}\,d\theta + \frac{\lambda(\log t)^{\gamma-1}}{(1-\sigma)\Gamma(\gamma)}\sum_{j=1}^{m}a_j\int_{\phi(\eta_j)}^e\big(\log\phi(\eta_j)\big)^{\gamma-1}\left(\log\frac{e}{\theta}\right)^{\gamma-1}\frac{f_\nu(\theta)}{\theta}\,d\theta \\ &= \int_1^e G(t,\theta)\frac{f_\nu(\theta)}{\theta}\,d\theta + \sum_{j=1}^{m}\frac{\lambda a_j(\log t)^{\gamma-1}}{(1-\sigma)}\int_1^e G\big(\phi(\eta_j),\theta\big)\frac{f_\nu(\theta)}{\theta}\,d\theta + \frac{\nu_0(\log t)^{\gamma-1}}{1-\sigma}.\end{aligned}$$

Note that Lemma 3(i) guarantees that $\int_1^t(\log\frac{t}{\theta})^{\gamma-1}\frac{f_\nu(\theta)}{\theta}\,d\theta \in C^1[1,e]$.

Therefore ν' exists for a.e. $t \in [1,e]$ and, on differentiating (13), we obtain

$$\begin{aligned} \nu'(t) &= \frac{(\log t)^{\gamma-2}}{t(1-\sigma)\Gamma(\gamma-1)} \int_1^e \left(\log\frac{e}{\theta}\right)^{\gamma-1} \frac{f_\nu(\theta)}{\theta}\,d\theta \\ &\quad - \frac{1}{t\Gamma(\gamma-1)} \int_1^t \left(\log\frac{t}{\theta}\right)^{\gamma-2} \frac{f_\nu(\theta)}{\theta}\,d\theta + \frac{\nu_0(\gamma-1)(\log t)^{\gamma-2}}{t(1-\sigma)} \\ &\quad - \frac{\lambda(\log t)^{\gamma-2}}{t(1-\sigma)\Gamma(\gamma-1)} \sum_{j=1}^m a_j \int_1^{\phi(\eta_j)} \left(\log\frac{\phi(\eta_j)}{\theta}\right)^{\gamma-1} \frac{f_\nu(\theta)}{\theta}\,d\theta, \end{aligned} \tag{16}$$

and we have $\nu' \in C(1,e]$ (cf. Lemma 2). Finally, we prove that $\nu \in AC[1,e]$.

For $f_\nu(t) \in L^p[1,e]$, we have

$$\begin{aligned} &\int_1^e \left|\nu'(t)\right|\,dt \\ &\quad\le \int_1^e \frac{(\log t)^{\gamma-2}}{(1-\sigma)\Gamma(\gamma-1)} \frac{dt}{t} \left(\int_1^e \left(\log\frac{e}{\theta}\right)^{\gamma-1} \frac{|f_\nu(\theta)|}{\theta}\,d\theta\right) \\ &\qquad + \frac{1}{\Gamma(\gamma-1)} \int_1^e \left(\int_1^t \left(\log\frac{t}{\theta}\right)^{\gamma-2} \frac{|f_\nu(\theta)|}{\theta}\,d\theta\right)\frac{dt}{t} + \int_1^e \frac{\nu_0(\gamma-1)(\log t)^{\gamma-2}}{(1-\sigma)}\frac{dt}{t} \\ &\qquad + \int_1^e \frac{\lambda(\log t)^{\gamma-2}}{(1-\sigma)\Gamma(\gamma-1)} \frac{dt}{t} \sum_{j=1}^m a_j \int_1^{\phi(\eta_j)} \left(\log\frac{\phi(\eta_j)}{\theta}\right)^{\gamma-1} \frac{|f_\nu(\theta)|}{\theta}\,d\theta \\ &\quad= \frac{1}{(1-\sigma)\Gamma(\gamma)} \int_1^e \left(\log\frac{e}{\theta}\right)^{\gamma-1} \frac{|f_\nu(\theta)|}{\theta}\,d\theta \\ &\qquad + \frac{1}{\Gamma(\gamma-1)} \int_1^e \left(\int_\theta^e \left(\log\frac{t}{\theta}\right)^{\gamma-2} \frac{dt}{t}\right) \frac{|f_\nu(\theta)|}{\theta}\,d\theta + \frac{\nu_0}{1-\sigma} \\ &\qquad + \frac{\lambda}{(1-\sigma)\Gamma(\gamma)} \sum_{j=1}^m a_j \int_1^{\phi(\eta_j)} \left(\log\frac{\phi(\eta_j)}{\theta}\right)^{\gamma-1} \frac{|f_\nu(\theta)|}{\theta}\,d\theta \\ &\quad\le \frac{1}{(1-\sigma)\Gamma(\gamma)} \left(\int_1^e \left(\log\frac{e}{\theta}\right)^{(\gamma-1)q} \frac{d\theta}{\theta}\right)^{\frac{1}{q}} \|f_\nu\|_p \\ &\qquad + \frac{1}{\Gamma(\gamma)} \left(\int_1^e \left(\log\frac{e}{\theta}\right)^{(\gamma-1)q} \frac{d\theta}{\theta}\right)^{\frac{1}{q}} \|f_\nu\|_p + \frac{\nu_0}{1-\sigma} \\ &\qquad + \frac{\lambda}{(1-\sigma)\Gamma(\gamma)} \sum_{j=1}^m a_j \left(\int_1^{\phi(\eta_j)} \left(\log\frac{\phi(\eta_j)}{\theta}\right)^{(\gamma-1)q} \frac{d\theta}{\theta}\right)^{\frac{1}{q}} \|f_\nu\|_p \\ &\quad\le \frac{2-\sigma+\lambda\sum_{j=1}^m a_j}{(1-\sigma)[(\gamma-1)q+1]^{\frac{1}{q}}\Gamma(\gamma)} \|f_\nu\|_p + \frac{\nu_0}{1-\sigma}, \end{aligned}$$

and we have

$$\left\|\nu'\right\|_{L^1} \le \frac{2-\sigma+\lambda\sum_{j=1}^m a_j}{(1-\sigma)[(\gamma-1)q+1]^{\frac{1}{q}}\Gamma(\gamma)} \|f_\nu\|_p + \frac{\nu_0}{1-\sigma}.$$

So ν is an absolutely continuous function. Thus ν' exists, for a.e. $t \in [1,e]$. □

Lemma 5 *The function $G(t,\theta)$ defined by (14) satisfies the following properties*:

(a) $G(t,\theta) \geq 0$, $G(t,\theta) \in C([1,e] \times [1,e])$ *and* $G(1,\theta) = G(e,\theta) = 0$ *for* $\theta \in [1,e]$.

(b) $\max\{G(t,\theta) : (t,\theta) \in [1,e] \times [1,e]\} = \mathbb{E}$, *where* $\mathbb{E} = \frac{1}{\Gamma(\gamma)}(\frac{1}{4})^{\gamma-1}$.

Proof (a) It is clear that G is continuous on $[1,e] \times [1,e]$ and $G(1,\theta) = G(e,\theta) = 0$ for $\theta \in [1,e]$.

By definition of the function G, for all $(t,\theta) \in [1,e] \times [1,e]$, if $\theta \leq t$, it can be written as

$$\begin{aligned} G(t,\theta) &= \frac{1}{\Gamma(\gamma)}\left[(\log t)^{\gamma-1}\left(\log\frac{e}{\theta}\right)^{\gamma-1} - \left(\log\frac{t}{\theta}\right)^{\gamma-1}\right] \\ &= \frac{1}{\Gamma(\gamma)}(\log t)^{\gamma-1}\left[(1-\log\theta)^{\gamma-1} - \left(1-\frac{\log\theta}{\log t}\right)^{\gamma-1}\right] \\ &\geq \frac{1}{\Gamma(\gamma)}(\log t)^{\gamma-1}\left[(1-\log\theta)^{\gamma-1} - (1-\log\theta)^{\gamma-1}\right] = 0. \end{aligned}$$

Furthermore, we conclude that

$$\begin{aligned} G\bigl(\phi(\eta_j),\theta\bigr) &= \frac{1}{\Gamma(\gamma)}\left[\bigl(\log\phi(\eta_j)\bigr)^{\gamma-1}\left(\log\frac{e}{\theta}\right)^{\gamma-1} - \left(\log\frac{\phi(\eta_j)}{\theta}\right)^{\gamma-1}\right] \\ &\geq \frac{1}{\Gamma(\gamma)}\bigl(\log\phi(\eta_j)\bigr)^{\gamma-1}\left[(1-\log\theta)^{\gamma-1} - (1-\log\theta)^{\gamma-1}\right] = 0. \end{aligned}$$

If $t \leq \theta$, it is obvious that $G(t,\theta)$ and $G(\phi(\eta_j),\theta) \geq 0$. Therefore, we can deduce that

$$G(t,\theta) \geq 0 \quad \text{for all } (t,\theta) \in [1,e] \times [1,e].$$

(b) Let $L(t,\theta) := \frac{1}{\Gamma(\gamma)}(\log t)^{\gamma-1}(\log\frac{e}{\theta})^{\gamma-1}$, $1 \leq t \leq \theta \leq e$, then $L(\cdot,\theta)$ is non-decreasing function on $[1,e]$.

Let $K(t,\theta) := \frac{1}{\Gamma(\gamma)}[(\log t)^{\gamma-1}(\log\frac{e}{\theta})^{\gamma-1} - (\log\frac{t}{\theta})^{\gamma-1}]$, $1 \leq \theta \leq t \leq e$. Then

$$\begin{aligned} \frac{\partial K(t,\theta)}{\partial t} &= \frac{1}{t\Gamma(\gamma-1)}\left[(\log t)^{\gamma-2}\left(\log\frac{e}{\theta}\right)^{\gamma-1} - \left(\log\frac{t}{\theta}\right)^{\gamma-2}\right] \\ &= \frac{1}{t\Gamma(\gamma-1)}(\log t)^{\gamma-2}\left[\left(\log\frac{e}{\theta}\right)^{\gamma-1} - \left(1-\frac{\log\theta}{\log t}\right)^{\gamma-2}\right] \\ &\leq \frac{1}{t\Gamma(\gamma-1)}(\log t)^{\gamma-2}\left[(1-\log\theta)^{\gamma-1} - (1-\log\theta)^{\gamma-2}\right] \leq 0, \end{aligned}$$

which implies that $K(\cdot,\theta)$ is non-increasing, for all $\theta \in [1,e]$, hence, we obtain

$$K(t,\theta) \leq K(\theta,\theta) \quad \text{for all } 1 \leq \theta \leq t \leq e,$$

and we have

$$\begin{aligned} \max\bigl\{G(t,\theta) : (t,\theta) \in [1,e] \times [1,e]\bigr\} &= \frac{1}{\Gamma(\gamma)}\max\left\{(\log\theta)^{\gamma-1}\left(\log\frac{e}{\theta}\right)^{\gamma-1} : \theta \in [1,e]\right\} \\ &= \frac{1}{\Gamma(\gamma)}\left(\frac{1}{4}\right)^{\gamma-1} = \mathbb{E}. \end{aligned}$$

□

Lemma 6 *Suppose that* (H_1), (H_3) *hold, then we have*

$$\begin{aligned}{}_{\mathrm{H}}D^{\delta}v(t) &= \frac{(\log t)^{\gamma-\delta-1}}{(1-\sigma)\Gamma(\gamma-\delta)}\int_1^e\left(\log\frac{e}{\theta}\right)^{\gamma-1}\frac{f_v(\theta)}{\theta}\,d\theta \\ &\quad-\frac{1}{\Gamma(\gamma-\delta)}\int_1^t\left(\log\frac{t}{\theta}\right)^{\gamma-\delta-1}\frac{f_v(\theta)}{\theta}\,d\theta+\frac{v_0\Gamma(\gamma)(\log t)^{\gamma-\delta-1}}{(1-\sigma)\Gamma(\gamma-\delta)} \\ &\quad-\frac{\lambda(\log t)^{\gamma-\delta-1}}{(1-\sigma)\Gamma(\gamma-\delta)}\sum_{j=1}^m a_j\int_1^{\phi(\eta_j)}\left(\log\frac{\phi(\eta_j)}{\theta}\right)^{\gamma-1}\frac{f_v(\theta)}{\theta}\,d\theta\in C[1,e]. \end{aligned}\tag{17}$$

Proof By applying Definition 2, Eq. (13), and Theorem 1(d_4), (d_5), we obtain

$$\begin{aligned}{}_{\mathrm{H}}D^{\delta}v(t) &= \left(t\frac{d}{dt}\right){}_{\mathrm{H}}J^{1-\delta}v(t), \\ &= \left(t\frac{d}{dt}\right){}_{\mathrm{H}}J^{1-\delta}\Bigg(\frac{(\log t)^{\gamma-1}}{(1-\sigma)\Gamma(\gamma)}\int_1^e\left(\log\frac{e}{\theta}\right)^{\gamma-1}\frac{f_v(\theta)}{\theta}\,d\theta \\ &\quad-\frac{1}{\Gamma(\gamma)}\int_1^t\left(\log\frac{t}{\theta}\right)^{\gamma-1}\frac{f_v(\theta)}{\theta}\,d\theta+\frac{v_0(\log t)^{\gamma-1}}{1-\sigma} \\ &\quad-\frac{\lambda(\log t)^{\gamma-1}}{(1-\sigma)\Gamma(\gamma)}\sum_{j=1}^m a_j\int_1^{\phi(\eta_j)}\left(\log\frac{\phi(\eta_j)}{\theta}\right)^{\gamma-1}\frac{f_v(\theta)}{\theta}\,d\theta\Bigg) \\ &= \left(t\frac{d}{dt}\right)\Bigg(\frac{(\log t)^{\gamma-\delta}}{(1-\sigma)\Gamma(\gamma-\delta+1)}\int_1^e\left(\log\frac{e}{\theta}\right)^{\gamma-1}\frac{f_v(\theta)}{\theta}\,d\theta \\ &\quad-\frac{1}{\Gamma(\gamma-\delta+1)}\int_1^t\left(\log\frac{t}{\theta}\right)^{\gamma-\delta}\frac{f_v(\theta)}{\theta}\,d\theta+\frac{v_0\Gamma(\gamma)(\log t)^{\gamma-\delta}}{(1-\sigma)\Gamma(\gamma-\delta+1)} \\ &\quad-\frac{\lambda(\log t)^{\gamma-\delta}}{(1-\sigma)\Gamma(\gamma-\delta+1)}\sum_{j=1}^m a_j\int_1^{\phi(\eta_j)}\left(\log\frac{\phi(\eta_j)}{\theta}\right)^{\gamma-1}\frac{f_v(\theta)}{\theta}\,d\theta\Bigg) \\ &= \frac{(\log t)^{\gamma-\delta-1}}{(1-\sigma)\Gamma(\gamma-\delta)}\int_1^e\left(\log\frac{e}{\theta}\right)^{\gamma-1}\frac{f_v(\theta)}{\theta}\,d\theta \\ &\quad-\frac{1}{\Gamma(\gamma-\delta)}\int_1^t\left(\log\frac{t}{\theta}\right)^{\gamma-\delta-1}\frac{f_v(\theta)}{\theta}\,d\theta+\frac{v_0\Gamma(\gamma)(\log t)^{\gamma-\delta-1}}{(1-\sigma)\Gamma(\gamma-\delta)} \\ &\quad-\frac{\lambda(\log t)^{\gamma-\delta-1}}{(1-\sigma)\Gamma(\gamma-\delta)}\sum_{j=1}^m a_j\int_1^{\phi(\eta_j)}\left(\log\frac{\phi(\eta_j)}{\theta}\right)^{\gamma-1}\frac{f_v(\theta)}{\theta}\,d\theta. \end{aligned}$$

Hence ${}_{\mathrm{H}}D^{\delta}v(t)\in C[1,e]$ (notice that $\int_1^t(\log\frac{t}{\theta})^{\gamma-\delta-1}\frac{f_v(\theta)}{\theta}\,d\theta\in C[1,e]$). □

Consider the Banach space

$$\begin{aligned}V=\Big\{&v\in C[1,e]\cap C^1(1,e]:v(t)\geq 0,{}_{\mathrm{H}}D^{\delta}v(t)\in C[1,e]\text{ and} \\ &\lim_{t\to 1^+}(\log t)^{2-\gamma}v'(t)\text{ exists}\Big\},\end{aligned}$$

with the weighted norm $\|v\|=\|v\|_0+\|{}_{\mathrm{H}}D^{\delta}v\|_0+\|v'\|_1$, where $\|v'\|_1=\sup_{t\in[1,e]}|(\log t)^{2-\gamma}\times v'(t)|$.

For $v \in V$, we define a nonlinear operator N by

$$(Nv)(t) = f\big(t, v(t), {}_{\mathrm{H}}D^{\delta}v(t), v'(t)\big), \quad t \in [1,e].$$

From (H_4), we conclude that $N : V \to L^p$ is well defined. In fact

$$\begin{aligned} \|Nv\|_p &= \big\|f\big(t, v(t), {}_{\mathrm{H}}D^{\delta}v(t), v'(t)\big)\big\|_p \\ &\le \|pv\|_p + \big\|q\,{}_{\mathrm{H}}D^{\delta}v\big\|_p + \big\|rv'\big\|_p + \|s\|_p \\ &\le \|p\|_p\|v\|_0 + \|q\|_p\big\|{}_{\mathrm{H}}D^{\delta}v\big\|_0 + \big\|(\log t)^{\gamma-2}r\big\|_p\big\|v'\big\|_1 + \|s\|_p \\ &< \infty. \end{aligned} \tag{18}$$

Let $f_v \in L^p[1,e]$, $p > \frac{1}{\gamma-1}$ for a.e. $t \in [1,e]$, then we have the following lemma.

Lemma 7 *Suppose that the assumption* (H_1)–(H_3) *hold. Then the functions* (13), (16) *and* (17) *satisfy*

$$\|v\|_0 \le \mathcal{A}\|f_v\|_p + \frac{v_0}{1-\sigma}, \qquad \big\|v'\big\|_1 \le \mathcal{B}\|f_v\|_p + \frac{v_0}{1-\sigma}, \tag{19}$$

and

$$\big\|{}_{\mathrm{H}}D^{\delta}v\big\|_0 \le \mathcal{C}\|f_v\|_p + \frac{v_0\Gamma(\gamma)}{(1-\sigma)\Gamma(\gamma-\delta)}, \tag{20}$$

where

$$\mathcal{A} = \frac{2 + \lambda\sum_{j=1}^{m} a_j}{\Gamma(\gamma)(1-\sigma)[q(\gamma-1)+1]^{\frac{1}{q}}}, \tag{21}$$

$$\mathcal{B} = \frac{1}{\Gamma(\gamma-1)}\left[\frac{1 + \lambda\sum_{j=1}^{m} a_j}{(1-\sigma)[q(\gamma-1)+1]^{\frac{1}{q}}} + \frac{1}{[q(\gamma-2)+1]^{\frac{1}{q}}}\right], \tag{22}$$

and

$$\mathcal{C} = \frac{1}{\Gamma(\gamma-\delta)}\left[\frac{1 + \lambda\sum_{j=1}^{m} a_j}{(1-\sigma)[q(\gamma-1)+1]^{\frac{1}{q}}} + \frac{1}{[q(\gamma-\delta-1)+1]^{\frac{1}{q}}}\right]. \tag{23}$$

Proof Again by Hölder's inequality and under the assumption (H_1), for all $t \in [1,e]$, we have

$$\begin{aligned} \big|v(t)\big| &\le \frac{1}{\Gamma(\gamma)}\left[\frac{1}{(1-\sigma)}\left(\int_1^e \left(\log\frac{e}{\theta}\right)^{q(\gamma-1)}\frac{d\theta}{\theta}\right)^{\frac{1}{q}} + \left(\int_1^t \left(\log\frac{t}{\theta}\right)^{q(\gamma-1)}\frac{d\theta}{\theta}\right)^{\frac{1}{q}}\right. \\ &\quad \left. + \frac{\lambda}{(1-\sigma)}\sum_{j=1}^{m} a_j\left(\int_1^{\phi(\eta_j)}\left(\log\frac{\phi(\eta_j)}{\theta}\right)^{q(\gamma-1)}\frac{d\theta}{\theta}\right)^{\frac{1}{q}}\right]\|f_v\|_p + \frac{v_0}{1-\sigma} \\ &= \frac{1}{\Gamma(\gamma)}\left[\frac{1}{(1-\sigma)[q(\gamma-1)+1]^{\frac{1}{q}}} + \frac{(\log t)^{\frac{q(\gamma-1)+1}{q}}}{[q(\gamma-1)+1]^{\frac{1}{q}}}\right. \end{aligned}$$

$$
\begin{aligned}
&+\frac{\lambda\sum_{j=1}^{m}a_j(\log(\phi(\eta_j)))^{\frac{q(\gamma-1)+1}{q}}}{(1-\sigma)[q(\gamma-1)+1]^{\frac{1}{q}}}\Bigg]\|f_\nu\|_p+\frac{\nu_0}{1-\sigma}\\
&\le\frac{2-\sigma+\lambda\sum_{j=1}^{m}a_j(\log\eta_j)^{\frac{q(\gamma-1)+1}{q}}}{\Gamma(\gamma)(1-\sigma)[q(\gamma-1)+1]^{\frac{1}{q}}}\|f_\nu\|_p+\frac{\nu_0}{1-\sigma}\\
&\le\frac{2+\lambda\sum_{j=1}^{m}a_j}{\Gamma(\gamma)(1-\sigma)[q(\gamma-1)+1]^{\frac{1}{q}}}\|f_\nu\|_p+\frac{\nu_0}{1-\sigma}.
\end{aligned}
$$

Hence

$$
\|\nu\|_0\le\frac{2+\lambda\sum_{j=1}^{m}a_j}{\Gamma(\gamma)(1-\sigma)[q(\gamma-1)+1]^{\frac{1}{q}}}\|f_\nu\|_p+\frac{\nu_0}{1-\sigma}=\mathcal{A}\|f_\nu\|_p+\frac{\nu_0}{1-\sigma}.
$$

Similarly (cf. (16)), we have as before

$$
\begin{aligned}
\big|(\log t)^{2-\gamma}\nu'(t)\big|&\le\frac{1}{(1-\sigma)\Gamma(\gamma-1)}\left(\int_1^e\left(\log\frac{e}{\theta}\right)^{q(\gamma-1)}\frac{1}{\theta}\,d\theta\right)^{\frac{1}{q}}\|f_\nu\|_p\\
&\quad+\frac{1}{\Gamma(\gamma-1)}\left(\int_1^t\left(\log\frac{t}{\theta}\right)^{q(\gamma-2)}\frac{1}{\theta}\,d\theta\right)^{\frac{1}{q}}\|f_\nu\|_p+\frac{\nu_0(\gamma-1)}{(1-\sigma)}\\
&\quad+\frac{\lambda}{(1-\sigma)\Gamma(\gamma-1)}\sum_{j=1}^{m}a_j\left(\int_1^{\phi(\eta_j)}\left(\log\frac{\phi(\eta_j)}{\theta}\right)^{q(\gamma-1)}\frac{1}{\theta}\,d\theta\right)^{\frac{1}{q}}\|f_\nu\|_p\\
&\le\frac{1}{\Gamma(\gamma-1)}\Bigg[\frac{1}{(1-\sigma)[q(\gamma-1)+1]^{\frac{1}{q}}}+\frac{1}{[q(\gamma-2)+1]^{\frac{1}{q}}}\\
&\quad+\frac{\lambda\sum_{j=1}^{m}a_j}{(1-\sigma)[q(\gamma-1)+1]^{\frac{1}{q}}}\Bigg]\|f_\nu\|_p+\frac{\nu_0}{1-\sigma}\\
&\le\frac{1}{\Gamma(\gamma-1)}\Bigg[\frac{1+\lambda\sum_{j=1}^{m}a_j}{(1-\sigma)[q(\gamma-1)+1]^{\frac{1}{q}}}+\frac{1}{[q(\gamma-2)+1]^{\frac{1}{q}}}\Bigg]\|f_\nu\|_p+\frac{\nu_0}{1-\sigma}.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
\|\nu'\|_1&\le\frac{1}{\Gamma(\gamma-1)}\Bigg[\frac{1+\lambda\sum_{j=1}^{m}a_j}{(1-\sigma)[q(\gamma-1)+1]^{\frac{1}{q}}}+\frac{1}{[q(\gamma-2)+1]^{\frac{1}{q}}}\Bigg]\|f_\nu\|_p+\frac{\nu_0}{1-\sigma}\\
&=\mathcal{B}\|f_\nu\|_p+\frac{\nu_0}{1-\sigma}.
\end{aligned}
$$

Similarly, for $t\in[1,e]$, we obtain

$$
\begin{aligned}
\big|{}_{\mathrm{H}}D^{\delta}\nu(t)\big|&\le\frac{1}{(1-\sigma)\Gamma(\gamma-\delta)}\int_1^e\left(\log\frac{e}{\theta}\right)^{\gamma-1}\frac{f_\nu(\theta)}{\theta}\,d\theta\\
&\quad+\frac{1}{\Gamma(\gamma-\delta)}\int_1^t\left(\log\frac{t}{\theta}\right)^{\gamma-\delta-1}\frac{f_\nu(\theta)}{\theta}\,d\theta+\frac{\nu_0\Gamma(\gamma)}{(1-\sigma)\Gamma(\gamma-\delta)}\\
&\quad+\frac{\lambda}{(1-\sigma)\Gamma(\gamma-\delta)}\sum_{j=1}^{m}a_j\int_1^{\phi(\eta_j)}\left(\log\frac{\phi(\eta_j)}{\theta}\right)^{\gamma-1}\frac{f_\nu(\theta)}{\theta}\,d\theta
\end{aligned}
$$

$$\begin{aligned}&\leq \frac{1}{\Gamma(\gamma-\delta)}\left[\frac{1+\lambda\sum_{j=1}^{m}a_j}{(1-\sigma)[q(\gamma-1)+1]^{\frac{1}{q}}}+\frac{1}{[q(\gamma-\delta-1)+1]^{\frac{1}{q}}}\right]\|f_\nu\|_p\\&\quad+\frac{\nu_0\Gamma(\gamma)}{(1-\sigma)\Gamma(\gamma-\delta)}.\end{aligned}$$

Hence

$$\begin{aligned}\left\|{}_{\mathrm{H}}D^{\delta}\nu\right\|_0 &\leq \frac{1}{\Gamma(\gamma-\delta)}\left[\frac{1+\lambda\sum_{j=1}^{m}a_j}{(1-\sigma)[q(\gamma-1)+1]^{\frac{1}{q}}}+\frac{1}{[q(\gamma-\delta-1)+1]^{\frac{1}{q}}}\right]\|f_\nu\|_p\\&\quad+\frac{\nu_0\Gamma(\gamma)}{(1-\sigma)\Gamma(\gamma-\delta)}=\mathcal{C}\|f_\nu\|_p+\frac{\nu_0\Gamma(\gamma)}{(1-\sigma)\Gamma(\gamma-\delta)}.\end{aligned}$$ □

Now, in order to prove problem (1), (4) has a positive solution, we define an integral operator T on V by the formula

$$\begin{aligned}(T\nu)(t)&=\int_1^e G(t,\theta)\frac{f_\nu(\theta)}{\theta}\,d\theta+\sum_{j=1}^{m}\frac{\lambda a_j(\log t)^{\gamma-1}}{(1-\sigma)}\int_1^e G\bigl(\phi(\eta_j),\theta\bigr)\frac{f_\nu(\theta)}{\theta}\,d\theta\\&\quad+\frac{\nu_0(\log t)^{\gamma-1}}{1-\sigma}\\&=\frac{(\log t)^{\gamma-1}}{(1-\sigma)\Gamma(\gamma)}\int_1^e\left(\log\frac{e}{\theta}\right)^{\gamma-1}\frac{f_\nu(\theta)}{\theta}\,d\theta-\frac{1}{\Gamma(\gamma)}\int_1^t\left(\log\frac{t}{\theta}\right)^{\gamma-1}\frac{f_\nu(\theta)}{\theta}\,d\theta\\&\quad-\frac{\lambda(\log t)^{\gamma-1}}{(1-\sigma)\Gamma(\gamma)}\sum_{j=1}^{m}a_j\int_1^{\phi(\eta_j)}\left(\log\frac{\phi(\eta_j)}{\theta}\right)^{\gamma-1}\frac{f_\nu(\theta)}{\theta}\,d\theta\\&\quad+\frac{\nu_0(\log t)^{\gamma-1}}{1-\sigma}.\end{aligned}\tag{24}$$

We have

$$\begin{aligned}(T\nu)'(t)&=\frac{(\log t)^{\gamma-2}}{t(1-\sigma)\Gamma(\gamma-1)}\int_1^e\left(\log\frac{e}{\theta}\right)^{\gamma-1}\frac{f_\nu(\theta)}{\theta}\,d\theta\\&\quad-\frac{1}{t\Gamma(\gamma-1)}\int_1^t\left(\log\frac{t}{\theta}\right)^{\gamma-2}\frac{f_\nu(\theta)}{\theta}\,d\theta+\frac{\nu_0(\gamma-1)(\log t)^{\gamma-2}}{t(1-\sigma)}\\&\quad-\frac{\lambda(\log t)^{\gamma-2}}{t(1-\sigma)\Gamma(\gamma-1)}\sum_{j=1}^{m}a_j\int_1^{\phi(\eta_j)}\left(\log\frac{\phi(\eta_j)}{\theta}\right)^{\gamma-1}\frac{f_\nu(\theta)}{\theta}\,d\theta\end{aligned}\tag{25}$$

and

$$\begin{aligned}\bigl({}_{\mathrm{H}}D^{\delta}T\nu\bigr)(t)&=\frac{(\log t)^{\gamma-\delta-1}}{(1-\sigma)\Gamma(\gamma-\delta)}\int_1^e\left(\log\frac{e}{\theta}\right)^{\gamma-1}\frac{f_\nu(\theta)}{\theta}\,d\theta\\&\quad-\frac{1}{\Gamma(\gamma-\delta)}\int_1^t\left(\log\frac{t}{\theta}\right)^{\gamma-\delta-1}\frac{f_\nu(\theta)}{\theta}\,d\theta+\frac{\nu_0\Gamma(\gamma)(\log t)^{\gamma-\delta-1}}{(1-\sigma)\Gamma(\gamma-\delta)}\\&\quad-\frac{\lambda(\log t)^{\gamma-\delta-1}}{(1-\sigma)\Gamma(\gamma-\delta)}\sum_{j=1}^{m}a_j\int_1^{\phi(\eta_j)}\left(\log\frac{\phi(\eta_j)}{\theta}\right)^{\gamma-1}\frac{f_\nu(\theta)}{\theta}\,d\theta.\end{aligned}\tag{26}$$

The properties of the operator T are given in the next lemma.

Lemma 8 *Let* (H_1)–(H_3) *hold. Then* $T : V \to V$ *and* T *is a completely continuous operator.*

Proof Let $v \in V$ and let $f_v(t) = f(t, v(t), {}_{\mathrm{H}}D^{\delta}v(t), v'(t))$ for a.e. $t \in [1, e]$. Then $f_v \in L^p[1, e]$ because f satisfies (H_3) and f_v is positive. Since $\int_1^t (\log\frac{t}{\theta})^{\gamma-1}\frac{f_v(\theta)}{\theta}\,d\theta \in C[1, e]$ and from $G \geq 0$ by Lemma 5, it follows from the equality (24) that $Tv \in C[1, e]$ and $Tv \geq 0$ for $t \in [1, e]$. Next using the equalities (25) and (26), we have ${}_{\mathrm{H}}D^{\delta}Tv \in C[1, e]$ and $\lim_{t\to 1^+}(\log t)^{2-\gamma}(Tv)'(t)$ exists and is continuous.

Consequently, $T : V \to V$.

As in the proof of Lemma 7, for all $v \in V$ and a.e. $t \in [1, e]$, we get

$$\|Tv\|_0 \leq \mathcal{A}\|f_v\|_p + \frac{v_0}{1-\sigma}, \qquad \|(Tv)'\|_1 \leq \mathcal{B}\|f_v\|_p + \frac{v_0}{1-\sigma},$$

and

$$\|{}_{\mathrm{H}}D^{\delta}Tv\|_0 \leq \mathcal{C}\|f_v\|_p + \frac{v_0\Gamma(\gamma)}{(1-\sigma)\Gamma(\gamma-\delta)}.$$

Thus, we see that the set $\{Tv\}$ is uniformly bounded in $C[1, e] \cap C^1(1, e]$.

In order to prove that T is a continuous operator, let $\{v_n\} \subset V$ be a convergent sequence and let $\lim_{n\to\infty}\|v_n - v\| = 0$. Then $v \in V$ and $\|v_n\|_0 \leq S$ for $n \in N$, where S is a positive constant.

Since f is an L^p-Carathéodory function we have

$$\lim_{n\to\infty} f\bigl(t, v_n(t), {}_{\mathrm{H}}D^{\delta}v_n(t), v_n'(t)\bigr) = f\bigl(t, v(t), {}_{\mathrm{H}}D^{\delta}v(t), v'(t)\bigr) \quad \text{for a.e. } t \in [1, e].$$

By (5) and the dominated convergent theorem in L^p-space,

$$\lim_{n\to\infty}\bigl\|f\bigl(t, v_n, {}_{\mathrm{H}}D^{\delta}v_n, v_n'\bigr) - f\bigl(t, v, {}_{\mathrm{H}}D^{\delta}v, v'\bigr)\bigr\|_p = 0.$$

Put

$$f_{v,n}(t) = f\bigl(t, v_n(t), {}_{\mathrm{H}}D^{\delta}v_n(t), v_n'(t)\bigr),$$

then we have

$$\lim_{n\to\infty} f_{v,n}(t) = f_v(t) \quad \text{and} \quad \lim_{n\to\infty}\|f_{v,n} - f_v\|_p = 0 \quad \text{for a.e. } t \in [1, e].$$

Now we deduce from (24) that

$$\begin{aligned}\bigl|(Tv_n)(t) - (Tv)(t)\bigr| &\leq \frac{2-\sigma+\lambda\sum_{j=1}^m a_j}{(1-\sigma)\Gamma(\gamma)}\int_1^e \bigl|f_{v,n}(\theta) - f_v(\theta)\bigr|\,d\theta \\ &\leq \frac{2+\lambda\sum_{j=1}^m a_j}{(1-\sigma)\Gamma(\gamma)}\int_1^e \bigl|f_{v,n}(\theta) - f_v(\theta)\bigr|\,d\theta,\end{aligned}$$

and from (25), we have

$$\begin{aligned}&\Gamma(\gamma-1)\bigl|(\log t)^{2-\gamma}(Tv_n)'(t) - (\log t)^{2-\gamma}(Tv)'(t)\bigr| \\ &\quad\leq \frac{1}{(1-\sigma)}\int_1^e \bigl|f_{v,n}(\theta) - f_v(\theta)\bigr|\,d\theta + \int_1^t \left(\log\frac{t}{\theta}\right)^{\gamma-2}\bigl|f_{v,n}(\theta) - f_v(\theta)\bigr|\,d\theta\end{aligned}$$

$$
\begin{aligned}
&\quad + \frac{\lambda}{1-\sigma}\sum_{j=1}^{m} a_j \int_1^e \bigl|f_{\nu,n}(\theta) - f_\nu(\theta)\bigr|\,d\theta \\
&\leq \frac{1+\lambda\sum_{j=1}^m a_j}{(1-\sigma)} \int_1^e \bigl|f_{\nu,n}(\theta) - f_\nu(\theta)\bigr|\,d\theta + \frac{1}{[q(\gamma-2)+1]^{\frac{1}{q}}}\|f_{\nu,n} - f_\nu\|_{L^p}.
\end{aligned}
$$

Similarly, from (26) we have

$$
\bigl|({}_{\mathrm{H}}D^\delta T\nu_n)(t) - ({}_{\mathrm{H}}D^\delta T\nu)(t)\bigr| \leq \frac{2+\lambda\sum_{j=1}^m a_j}{(1-\sigma)\Gamma(\gamma-\delta)} \int_1^e \bigl|f_{\nu,n}(\theta) - f_\nu(\theta)\bigr|\,d\theta.
$$

Thus $\lim_{n\to\infty}\|T\nu_n - T\nu\| = 0$, which proves that T is a continuous operator.

Now, we need to prove that $\{T\nu\}$ be equicontinuous. For $1 \leq t_1 < t_2 \leq e$, we have the relation (cf. (24)) in a similar way to Lemma 1

$$
\begin{aligned}
&\bigl|(T\nu)(t_2) - (T\nu)(t_1)\bigr| \\
&\leq \frac{[(\log t_2)^{\gamma-1} - (\log t_1)^{\gamma-1}]}{(1-\sigma)\Gamma(\gamma)} \int_1^e \left(\log\frac{e}{\theta}\right)^{\gamma-1} \frac{|f_\nu(\theta)|}{\theta}\,d\theta \\
&\quad + \frac{1}{\Gamma(\gamma)} \int_1^{t_1} \left[\left(\log\frac{t_2}{\theta}\right)^{\gamma-1} - \left(\log\frac{t_1}{\theta}\right)^{\gamma-1}\right] \frac{|f_\nu(\theta)|}{\theta}\,d\theta \\
&\quad + \frac{1}{\Gamma(\gamma)} \int_{t_1}^{t_2} \left(\log\frac{t_2}{\theta}\right)^{\gamma-1} \frac{|f_\nu(\theta)|}{\theta}\,d\theta + \frac{\nu_0[(\log t_2)^{\gamma-1} - (\log t_1)^{\gamma-1}]}{1-\sigma} \\
&\quad + \frac{\lambda[(\log t_2)^{\gamma-1} - (\log t_1)^{\gamma-1}]}{(1-\sigma)\Gamma(\gamma)} \sum_{j=1}^m a_j \int_1^{\phi(\eta_j)} \left(\log\frac{\phi(\eta_j)}{\theta}\right)^{\gamma-1} \frac{|f_\nu(\theta)|}{\theta}\,d\theta \\
&\leq \frac{[(\log t_2)^{\gamma-1} - (\log t_1)^{\gamma-1}]}{d^{\frac{1}{q}}(1-\sigma)\Gamma(\gamma)}\|f_\nu\|_p \\
&\quad + \frac{1}{d^{\frac{1}{q}}\Gamma(\gamma)}\left((\log t_2)^d - (\log t_1)^d - \left(\log\frac{t_2}{t_1}\right)^d\right)^{\frac{1}{q}}\|f_\nu\|_p \\
&\quad + \frac{1}{d^{\frac{1}{q}}\Gamma(\gamma)}\left(\log\frac{t_2}{t_1}\right)^{\frac{d}{q}}\|f_\nu\|_p + \frac{\nu_0[(\log t_2)^{\gamma-1} - (\log t_1)^{\gamma-1}]}{1-\sigma} \\
&\quad + \frac{\lambda[(\log t_2)^{\gamma-1} - (\log t_1)^{\gamma-1}]\sum_{j=1}^m a_j}{d^{\frac{1}{q}}(1-\sigma)\Gamma(\gamma)}\|f_\nu\|_p,
\end{aligned}
$$

where $d = (\gamma-1)q + 1$. Similarly, it follows from (25) that

$$
\begin{aligned}
&\bigl|(\log t_2)^{2-\gamma}(T\nu)'(t_2) - (\log t_1)^{2-\gamma}(T\nu)'(t_1)\bigr| \\
&\leq \frac{1}{d^{\frac{1}{q}}(1-\sigma)\Gamma(\gamma-1)}\left[\frac{1}{t_2} - \frac{1}{t_1}\right]\|f_\nu\|_p \\
&\quad + \left|-\frac{(\log t_2)^{2-\gamma}}{t_2\Gamma(\gamma-1)} \int_1^{t_1} \left(\log\frac{t_2}{\theta}\right)^{\gamma-2} \frac{f_\nu}{\theta}\,d\theta + \frac{(\log t_2)^{2-\gamma}}{t_2\Gamma(\gamma-1)} \int_1^{t_1} \left(\log\frac{t_1}{\theta}\right)^{\gamma-2} \frac{f_\nu}{\theta}\,d\theta\right| \\
&\quad + \left|-\frac{(\log t_2)^{2-\gamma}}{t_2\Gamma(\gamma-1)} \int_1^{t_1} \left(\log\frac{t_1}{\theta}\right)^{\gamma-2} \frac{f_\nu}{\theta}\,d\theta + \frac{(\log t_1)^{2-\gamma}}{t_1\Gamma(\gamma-1)} \int_1^{t_1} \left(\log\frac{t_1}{\theta}\right)^{\gamma-2} \frac{f_\nu}{\theta}\,d\theta\right|
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{(\log t_2)^{2-\gamma}}{t_2\Gamma(\gamma-1)} \int_{t_1}^{t_2} \left(\log\frac{t_2}{\theta}\right)^{\gamma-2} \frac{|f_\nu|}{\theta}\,d\theta + \frac{\nu_0(\gamma-1)}{(1-\sigma)}\left[\frac{1}{t_2}-\frac{1}{t_1}\right] \\
&+ \frac{\lambda\sum_{j=1}^m a_j}{d^{\frac{1}{q}}(1-\sigma)\Gamma(\gamma-1)}\left[\frac{1}{t_2}-\frac{1}{t_1}\right]\|f_\nu\|_p \\
\leq{}& \frac{1}{d^{\frac{1}{q}}(1-\sigma)\Gamma(\gamma-1)}\left[\frac{1}{t_2}-\frac{1}{t_1}\right]\|f_\nu\|_p \\
&+ \frac{(\log t_2)^{2-\gamma}}{b^{\frac{1}{q}} t_2\Gamma(\gamma-1)}\left((\log t_1)^b - (\log t_2)^b + \left(\log\frac{t_2}{t_1}\right)^b\right)^{\frac{1}{q}}\|f_\nu\|_p \\
&+ \left[\frac{(\log t_2)^{2-\gamma}}{b^{\frac{1}{q}} t_2\Gamma(\gamma-1)} - \frac{(\log t_1)^{2-\gamma}}{b^{\frac{1}{q}} t_1\Gamma(\gamma-1)}\right]\|f_\nu\|_p + \frac{(\log t_2)^{2-\gamma}}{b^{\frac{1}{q}} t_2\Gamma(\gamma-1)}\left(\log\frac{t_2}{t_1}\right)^{\frac{b}{q}}\|f_\nu\|_p \\
&+ \frac{\nu_0(\gamma-1)}{(1-\sigma)}\left[\frac{1}{t_2}-\frac{1}{t_1}\right] + \frac{\lambda\sum_{j=1}^m a_j}{d^{\frac{1}{q}}(1-\sigma)\Gamma(\gamma-1)}\left[\frac{1}{t_2}-\frac{1}{t_1}\right]\|f_\nu\|_p.
\end{aligned}
$$

Also (cf. (26)), by putting $h = (\gamma-\delta-1)q+1$, we get

$$
\begin{aligned}
&\left|\left({}_{\mathrm{H}}D^\delta T\nu\right)(t_2) - \left({}_{\mathrm{H}}D^\delta T\nu\right)(t_1)\right| \\
&\leq \frac{[(\log t_2)^{\gamma-\delta-1} - (\log t_1)^{\gamma-\delta-1}]}{(1-\sigma)\Gamma(\gamma-\delta)} \int_1^e \left(\log\frac{e}{\theta}\right)^{\gamma-1}\frac{|f_\nu(\theta)|}{\theta}\,d\theta \\
&\quad+ \frac{1}{\Gamma(\gamma-\delta)}\int_1^{t_1}\left[\left(\log\frac{t_2}{\theta}\right)^{\gamma-\delta-1} - \left(\log\frac{t_1}{\theta}\right)^{\gamma-\delta-1}\right]\frac{|f_\nu(\theta)|}{\theta}\,d\theta \\
&\quad+ \frac{1}{\Gamma(\gamma-\delta)}\int_{t_1}^{t_2}\left(\log\frac{t_2}{\theta}\right)^{\gamma-\delta-1}\frac{|f_\nu(\theta)|}{\theta}\,d\theta \\
&\quad+ \frac{\nu_0\Gamma(\gamma)[(\log t_2)^{\gamma-\delta-1} - (\log t_1)^{\gamma-\delta-1}]}{(1-\sigma)\Gamma(\gamma-\delta)} \\
&\quad+ \frac{\lambda[(\log t_2)^{\gamma-\delta-1} - (\log t_1)^{\gamma-\delta-1}]}{(1-\sigma)\Gamma(\gamma-\delta)}\sum_{j=1}^m a_j\int_1^{\phi(\eta_j)}\left(\log\frac{\phi(\eta_j)}{\theta}\right)^{\gamma-1}\frac{|f_\nu(\theta)|}{\theta}\,d\theta \\
&\leq \frac{[(\log t_2)^{\gamma-\delta-1} - (\log t_1)^{\gamma-\delta-1}]}{d^{\frac{1}{q}}(1-\sigma)\Gamma(\gamma-\delta)}\|f_\nu\|_p \\
&\quad+ \frac{1}{h^{\frac{1}{q}}\Gamma(\gamma-\delta)}\left((\log t_2)^h - (\log t_1)^h - \left(\log\frac{t_2}{t_1}\right)^h\right)^{\frac{1}{q}}\|f_\nu\|_p \\
&\quad+ \frac{1}{h^{\frac{1}{q}}\Gamma(\gamma-\delta)}\left(\log\frac{t_2}{t_1}\right)^{\frac{h}{q}}\|f_\nu\|_p + \frac{\nu_0\Gamma(\gamma)[(\log t_2)^{\gamma-\delta-1} - (\log t_1)^{\gamma-\delta-1}]}{(1-\sigma)\Gamma(\gamma-\delta)} \\
&\quad+ \frac{\lambda[(\log t_2)^{\gamma-\delta-1} - (\log t_1)^{\gamma-\delta-1}]\sum_{j=1}^m a_j}{d^{\frac{1}{q}}(1-\sigma)\Gamma(\gamma-\delta)}\|f_\nu\|_p, \quad \text{if } \gamma > \delta+1,
\end{aligned}
$$

and

$$
\left|\left({}_{\mathrm{H}}D^\delta T\nu\right)(t_2) - \left({}_{\mathrm{H}}D^\delta T\nu\right)(t_1)\right| \leq \int_{t_1}^{t_2}\left|f_\nu(\theta)\right|d\theta, \quad \text{if } \gamma = \delta+1.
$$

As $t_2 \to t_1$, the right-hand side of the above four inequalities tends to zero. Therefore $\{Tv\}$ is equicontinuous.

Since the set of functions $\{Tv\}$, $\{t^{2-\gamma}(Tv)'\}$ and $\{{}_{\mathrm{H}}D^{\delta}Tv\}$ are bounded in $C[1,e]$ and equicontinuous on $[1,e]$, T is relatively compact in V by the Arzelà–Ascoli theorem. Combining this fact with the continuity of T we see that T is a completely continuous operator. □

Our main result of this section is as follows.

Theorem 3 *Assume that* (H_1)–(H_4) *hold and let* $\sigma < 1$. *Suppose that the functions* p, q *and* r *satisfy*

$$\mathcal{A}\|p\|_p + \mathcal{B}\|q\|_p + \mathcal{C}\big\|(\log t)^{\gamma-2}r\big\|_p < 1, \tag{27}$$

where the constants $\mathcal{A}$, $\mathcal{B}$ *and* $\mathcal{C}$ *are given by* (21)–(23), *respectively.*

Then the multi-point boundary value problem (1), (4) *has at least one positive solution.*

Proof From Lemma 4, we know that $v \in V$ is a solution of (1), (4) if and only if

$$Tv = v. \tag{28}$$

By Lemma 8, we can apply the Leray–Schauder continuation theorem to obtain the existence of a solution for (28) in V.

To do this it is suffices to verify that the set of all possible solutions of the family of problems

$$\begin{aligned} &{}_{\mathrm{H}}D^{\gamma}v(t) + \lambda f\big(t, v(t), {}_{\mathrm{H}}D^{\delta}v(t), v'(t)\big) = 0, \quad \text{a.e. } t \in (1,e), \\ &v(1) = 0, \qquad v(e) = v_0 + \lambda \sum_{j=1}^{m} a_j v\big(\phi(\eta_j)\big), \end{aligned}$$

is, a priori, bounded in V by a constant independent of $\lambda \in [0,1]$. Then for $t \in [1,e]$ we have from Lemma 4

$$\begin{aligned} &\big|v(t)\big| \\ &\quad = \left| \frac{(\log t)^{\gamma-1}}{(1-\sigma)\Gamma(\gamma)} \int_1^e \left(\log\frac{e}{\theta}\right)^{\gamma-1} \frac{\lambda(Nv)(\theta)}{\theta}\,d\theta - \frac{1}{\Gamma(\gamma)} \int_1^t \left(\log\frac{t}{\theta}\right)^{\gamma-1} \frac{\lambda(Nv)(\theta)}{\theta}\,d\theta \right. \\ &\qquad \left. - \frac{\lambda(\log t)^{\gamma-1}}{(1-\sigma)\Gamma(\gamma)} \sum_{j=1}^{m} a_j \int_1^{\phi(\eta_j)} \left(\log\frac{\phi(\eta_j)}{\theta}\right)^{\gamma-1} \frac{\lambda(Nv)(\theta)}{\theta}\,d\theta + \frac{v_0(\log t)^{\gamma-1}}{1-\sigma} \right| \\ &\quad = \left| \frac{(\log t)^{\gamma-1}}{(1-\sigma)\Gamma(\gamma)} \int_1^e \left(\log\frac{e}{\theta}\right)^{\gamma-1} \frac{(-{}_{\mathrm{H}}D^{\gamma}v(\theta))}{\theta}\,d\theta \right. \\ &\qquad - \frac{1}{\Gamma(\gamma)} \int_1^t \left(\log\frac{t}{\theta}\right)^{\gamma-1} \frac{(-{}_{\mathrm{H}}D^{\gamma}v(\theta))}{\theta}\,d\theta \\ &\qquad \left. - \frac{\lambda(\log t)^{\gamma-1}}{(1-\sigma)\Gamma(\gamma)} \sum_{j=1}^{m} a_j \int_1^{\phi(\eta_j)} \left(\log\frac{\phi(\eta_j)}{\theta}\right)^{\gamma-1} \frac{(-{}_{\mathrm{H}}D^{\gamma}v(\theta))}{\theta}\,d\theta + \frac{v_0(\log t)^{\gamma-1}}{1-\sigma} \right|, \end{aligned}$$

which implies that, in similar way to Lemma 7,

$$\|v\|_0 \le \mathcal{A}\left\|{}_{\mathrm{H}}D^{\gamma} v\right\|_p + \frac{v_0}{1-\sigma}, \qquad \left\|v'\right\|_1 \le \mathcal{B}\left\|{}_{\mathrm{H}}D^{\gamma} v\right\|_p + \frac{v_0}{1-\sigma}, \tag{29}$$

and

$$\left\|{}_{\mathrm{H}}D^{\delta} v\right\|_0 \le \mathcal{C}\left\|{}_{\mathrm{H}}D^{\gamma} v\right\|_p + \frac{v_0 \Gamma(\gamma)}{(1-\sigma)\Gamma(\gamma-\delta)}. \tag{30}$$

Using (H_4), and Eqs. (18), (29) and (30), it follows that

$$\begin{aligned}
\left\|{}_{\mathrm{H}}D^{\gamma} v\right\|_p &= \lambda \left\|f\left(t, v, {}_{\mathrm{H}}D^{\delta} v, v'\right)\right\|_p \\
&\le \|p\|_p \|v\|_0 + \|q\|_p \left\|{}_{\mathrm{H}}D^{\delta} v\right\|_0 + \left\|(\log t)^{\gamma-2} r\right\|_p \left\|v'\right\|_1 + \|s\|_p \\
&\le \mathcal{A}\|p\|_p \left\|{}_{\mathrm{H}}D^{\gamma} v\right\|_p + \mathcal{B}\|q\|_p \left\|{}_{\mathrm{H}}D^{\gamma} v\right\|_p + \mathcal{C}\left\|(\log t)^{\gamma-2} r\right\|_p \left\|{}_{\mathrm{H}}D^{\gamma} v\right\|_p \\
&\quad + \|s\|_p + \frac{v_0(\|p\|_p + \|q\|_p)}{(1-\sigma)} + \frac{v_0 \|(\log t)^{\gamma-2} r\|_p \Gamma(\gamma)}{(1-\sigma)\Gamma(\gamma-\delta)},
\end{aligned}$$

for $t \in [1, e]$. Thus

$$\begin{aligned}
\left\|{}_{\mathrm{H}}D^{\gamma} v\right\|_p &\le \left[\mathcal{A}\|p\|_p + \mathcal{B}\|q\|_p + \mathcal{C}\left\|(\log t)^{\gamma-2} r\right\|_p\right] \left\|{}_{\mathrm{H}}D^{\gamma} v\right\|_p + \|s\|_p \\
&\quad + \frac{v_0(\|p\|_p + \|q\|_p)}{(1-\sigma)} + \frac{v_0 \|(\log t)^{\gamma-2} r\|_p \Gamma(\gamma)}{(1-\sigma)\Gamma(\gamma-\delta)}.
\end{aligned}$$

It follows from the assumption (27) that there is a constant $\mathcal{D}$, independent of $\lambda \in [0, 1]$, such that

$$\left\|{}_{\mathrm{H}}D^{\gamma} v\right\|_p \le \mathcal{D}. \tag{31}$$

This together with (29) and (30) implies that

$$\begin{aligned}
&\|v\|_0 \le \mathcal{A}\mathcal{D} + \frac{v_0}{1-\sigma} \quad \text{and} \quad \left\|v'\right\|_1 \le \mathcal{B}\mathcal{D} + \frac{v_0}{1-\sigma}, \\
&\left\|{}_{\mathrm{H}}D^{\delta} v\right\|_0 \le \mathcal{C}\mathcal{D} + \frac{v_0 \Gamma(\gamma)}{(1-\sigma)\Gamma(\gamma-\delta)}.
\end{aligned}$$

Therefore,

$$\|v\| \le [\mathcal{A} + \mathcal{B} + \mathcal{C}]\mathcal{D} + \frac{v_0[2\Gamma(\gamma-\delta) + \Gamma(\gamma)]}{(1-\sigma)\Gamma(\gamma-\delta)}.$$

This completes the proof of the theorem. □

4 Positive solutions for boundary value problem (1), (2)

Let $v \in AC[1, e]$ be the solution of the multi-point problem given by (1) and (4). Then we have the following theorem.

Theorem 4 *Suppose that the assumptions* (H_3) *and* (H_4) *are satisfied. If*

$$\mathcal{A}_1 \|p\|_p + \mathcal{B}_1 \|q\|_p + \mathcal{C}_1 \left\|(\log t)^{\gamma-2} r\right\|_p < 1,$$

and $\phi(t):[1,e]\to[1,e]$ *is a deviated and continuous differentiable function in* $[1,e]$ *with* $\phi'(t)>0$ *or* $\phi(t)$ *is a deviated and monotonically increasing function.*

Then there exists a positive solution $v\in AC[1,e]$ *of the problem* (1) *with integral boundary condition* (2) *represented by*

$$\begin{aligned} v(t) &= \frac{(\log t)^{\gamma-1}}{(1-\sigma_1)\Gamma(\gamma)}\int_1^e\left(\log\frac{e}{\theta}\right)^{\gamma-1}\frac{f_v(\theta)}{\theta}\,d\theta-\frac{1}{\Gamma(\gamma)}\int_1^t\left(\log\frac{t}{\theta}\right)^{\gamma-1}\frac{f_v(\theta)}{\theta}\,d\theta \\ &\quad-\frac{\lambda(\log t)^{\gamma-1}}{(1-\sigma_1)\Gamma(\gamma)}\int_1^e\frac{\phi'(\xi)}{\phi(\xi)}\int_1^{\phi(\xi)}\left(\log\frac{\phi(\xi)}{\theta}\right)^{\gamma-1}\frac{f_v(\theta)}{\theta}\,d\theta\,d\xi \\ &\quad+\frac{v_0(\log t)^{\gamma-1}}{1-\sigma_1}, \end{aligned} \tag{32}$$

where

$$\begin{aligned} \mathcal{A}_1 &= \frac{2+\lambda[\log(\phi(e))-\log(\phi(1))]}{\Gamma(\gamma)(1-\sigma_1)[q(\gamma-1)+1]^{\frac{1}{q}}}, \\ \mathcal{B}_1 &= \frac{1}{\Gamma(\gamma-1)}\left[\frac{1+\lambda[\log(\phi(e))-\log(\phi(1))]}{(1-\sigma_1)[q(\gamma-1)+1]^{\frac{1}{q}}}+\frac{1}{[q(\gamma-2)+1]^{\frac{1}{q}}}\right], \\ \mathcal{C}_1 &= \frac{1}{\Gamma(\gamma-\delta)}\left[\frac{1+\lambda[\log(\phi(e))-\log(\phi(1))]}{(1-\sigma_1)[q(\gamma-1)+1]^{\frac{1}{q}}}+\frac{1}{[q(\gamma-\delta-1)+1]^{\frac{1}{q}}}\right], \end{aligned}$$

and

$$\sigma_1=\frac{\lambda[[\log(\phi(e))]^\gamma-[\log(\phi(1))]^\gamma]}{\gamma}<1.$$

Proof Let $v\in AC[1,e]$ be a solution of the multi-point boundary value problem (1) and (4) given by (13).

Let $a_j=\frac{(t_j-t_{j-1})\phi'(\eta_j)}{\phi(\eta_j)}$, $\eta_j\in(t_{j-1},t_j)\subset(1,e)$ and $1=t_0<t_1<t_2<\cdots<t_m=e$. Then the multi-point boundary conditions in (4) will be

$$v(1)=0,\qquad v(e)=v_0+\lambda\sum_{j=1}^m\frac{(t_j-t_{j-1})\phi'(\eta_j)}{\phi(\eta_j)}v(\phi(\eta_j)).$$

From the continuity of the solution v of (1), (4), we can obtain

$$v(1)=0,\qquad v(e)=v_0+\lambda\lim_{m\to\infty}\sum_{j=1}^m\frac{(t_j-t_{j-1})\phi'(\eta_j)}{\phi(\eta_j)}v(\phi(\eta_j)),$$

that is, the nonlocal condition (4) is transformed to the integral condition

$$v(1)=0,\qquad v(e)=v_0+\lambda\int_1^e v(\phi(\xi))\frac{\phi'(\xi)}{\phi(\xi)}\,d\xi.$$

The constant σ will be as $m \to \infty$

$$\begin{aligned}\lim_{m\to\infty} \sigma = \sigma_1 &= \lambda \lim_{m\to\infty} \sum_{j=1}^{m} a_j\bigl(\log \phi(\eta_j)\bigr)^{\gamma-1} \\ &= \lambda \lim_{m\to\infty} \sum_{j=1}^{m} \frac{(t_j - t_{j-1})\phi'(\eta_j)}{\phi(\eta_j)}\bigl(\log \phi(\eta_j)\bigr)^{\gamma-1} \\ &= \lambda \int_1^e \bigl(\log \phi(\xi)\bigr)^{\gamma-1} \frac{\phi'(\xi)}{\phi(\xi)}\, d\xi = \frac{\lambda[[\log(\phi(e))]^{\gamma} - [\log(\phi(1))]^{\gamma}]}{\gamma}.\end{aligned}$$

Also, the constants $\mathcal{A}, \mathcal{B}, \mathcal{C}$ will be

$$\begin{aligned}\lim_{m\to\infty} \mathcal{A} = \mathcal{A}_1 &= \frac{2 + \lambda \sum_{j=1}^{\infty} \frac{(t_j-t_{j-1})\phi'(\eta_j)}{\phi(\eta_j)}}{\Gamma(\gamma)(1-\sigma_1)[q(\gamma-1)+1]^{\frac{1}{q}}} \\ &= \frac{2 + \lambda \int_1^e \frac{\phi'(\xi)}{\phi(\xi)}\, d\xi}{\Gamma(\gamma)(1-\sigma_1)[q(\gamma-1)+1]^{\frac{1}{q}}} = \frac{2 + \lambda[\log(\phi(e)) - \log(\phi(1))]}{\Gamma(\gamma)(1-\sigma_1)[q(\gamma-1)+1]^{\frac{1}{q}}}.\end{aligned}$$

Similarly

$$\mathcal{B}_1 = \frac{1}{\Gamma(\gamma-1)}\left[\frac{1 + \lambda[\log(\phi(e)) - \log(\phi(1))]}{(1-\sigma_1)[q(\gamma-1)+1]^{\frac{1}{q}}} + \frac{1}{[q(\gamma-2)+1]^{\frac{1}{q}}}\right]$$

and

$$\mathcal{C}_1 = \frac{1}{\Gamma(\gamma-\delta)}\left[\frac{1 + \lambda[\log(\phi(e)) - \log(\phi(1))]}{(1-\sigma_1)[q(\gamma-1)+1]^{\frac{1}{q}}} + \frac{1}{[q(\gamma-\delta-1)+1]^{\frac{1}{q}}}\right].$$

Now from the continuity of the solution v (cf. (13)), we have

$$\begin{aligned}v(t) &= \frac{(\log t)^{\gamma-1}}{(1-\sigma_1)\Gamma(\gamma)} \int_1^e \left(\log \frac{e}{\theta}\right)^{\gamma-1} \frac{f_v(\theta)}{\theta}\, d\theta - \frac{1}{\Gamma(\gamma)} \int_1^t \left(\log \frac{t}{\theta}\right)^{\gamma-1} \frac{f_v(\theta)}{\theta}\, d\theta \\ &\quad - \frac{\lambda(\log t)^{\gamma-1}}{(1-\sigma_1)\Gamma(\gamma)} \sum_{j=1}^{\infty} \frac{(t_j - t_{j-1})\phi'(\eta_j)}{\phi(\eta_j)} \int_1^{\phi(\eta_j)} \left(\log \frac{\phi(\eta_j)}{\theta}\right)^{\gamma-1} \frac{f_v(\theta)}{\theta}\, d\theta \\ &\quad + \frac{v_0(\log t)^{\gamma-1}}{1-\sigma_1} \\ &= \frac{(\log t)^{\gamma-1}}{(1-\sigma_1)\Gamma(\gamma)} \int_1^e \left(\log \frac{e}{\theta}\right)^{\gamma-1} \frac{f_v(\theta)}{\theta}\, d\theta - \frac{1}{\Gamma(\gamma)} \int_1^t \left(\log \frac{t}{\theta}\right)^{\gamma-1} \frac{f_v(\theta)}{\theta}\, d\theta \\ &\quad - \frac{\lambda(\log t)^{\gamma-1}}{(1-\sigma_1)\Gamma(\gamma)} \int_1^e \frac{\phi'(\xi)}{\phi(\xi)} \int_1^{\phi(\xi)} \left(\log \frac{\phi(\xi)}{\theta}\right)^{\gamma-1} \frac{f_v(\theta)}{\theta}\, d\theta\, d\xi + \frac{v_0(\log t)^{\gamma-1}}{1-\sigma_1}.\end{aligned}$$

Hence, the continuous positive solution of integral boundary problem (1), (2) is given by (32). □

5 Positive solutions for infinite-point boundary problem (1), (3)

Our second main result of this paper is presented as an existence result for problem (1), (3).

We have the following theorem.

Theorem 5 *Let the assumptions* (H_2)–(H_4) *and the following conditions hold*:

(H_5) $1 < \eta_1 < \eta_2 < \cdots < \eta_j < \cdots < e, j = 1, 2, \ldots$ *and* $\sigma_2 = \lambda \sum_{j=1}^{\infty} a_j (\log \phi(\eta_j))^{\gamma-1} < 1$.

(H_6) $\mathcal{A}_2 \|p\|_p + \mathcal{B}_2 \|q\|_p + \mathcal{C}_2 \|(\log t)^{\gamma-2} r\|_p < 1$.

(H_7) *The series* $\sum_{j=1}^{\infty} a_j < \infty$ *is convergent.*

Then there exists a positive solution $v \in AC[0,1]$ *of the infinite-point boundary problem* (1) *with infinite-point boundary condition* (3) *given by the following integral equation*:

$$v(t) = \frac{(\log t)^{\gamma-1}}{(1-\sigma_2)\Gamma(\gamma)} \int_1^e \left(\log \frac{e}{\theta}\right)^{\gamma-1} \frac{f_v(\theta)}{\theta}\, d\theta - \frac{1}{\Gamma(\gamma)} \int_1^t \left(\log \frac{t}{\theta}\right)^{\gamma-1} \frac{f_v(\theta)}{\theta}\, d\theta \\ - \frac{\lambda (\log t)^{\gamma-1}}{(1-\sigma_2)\Gamma(\gamma)} \sum_{j=1}^{\infty} a_j \int_1^{\phi(\eta_j)} \left(\log \frac{\phi(\eta_j)}{\theta}\right)^{\gamma-1} \frac{f_v(\theta)}{\theta}\, d\theta + \frac{v_0 (\log t)^{\gamma-1}}{1-\sigma_2}, \tag{33}$$

where

$$\mathcal{A}_2 = \frac{2 + \lambda \sum_{j=1}^{\infty} a_j}{\Gamma(\gamma)(1-\sigma_2)[q(\gamma-1)+1]^{\frac{1}{q}}},$$

$$\mathcal{B}_2 = \frac{1}{\Gamma(\gamma-1)} \left[\frac{1 + \lambda \sum_{j=1}^{\infty} a_j}{(1-\sigma_2)[q(\gamma-1)+1]^{\frac{1}{q}}} + \frac{1}{[q(\gamma-2)+1]^{\frac{1}{q}}}\right],$$

and

$$\mathcal{C}_2 = \frac{1}{\Gamma(\gamma-\delta)} \left[\frac{1 + \lambda \sum_{j=1}^{\infty} a_j}{(1-\sigma_2)[q(\gamma-1)+1]^{\frac{1}{q}}} + \frac{1}{[q(\gamma-\delta-1)+1]^{\frac{1}{q}}}\right].$$

Proof Let $v \in AC[1, e]$ be a solution of the multi-point boundary value problem (1) and (4) given by (13). We have

$$\left|a_j v(\phi(\eta_j))\right| \le a_j \|v\|_0,$$

$$\left|a_j (\log \phi(\eta_j))^{\gamma-1}\right| \le a_j, \quad \phi(\eta_j) \le \eta_j < e,$$

and

$$\left| a_j \int_1^{\phi(\eta_j)} \left(\log \frac{\phi(\eta_j)}{\theta}\right)^{\gamma-1} \frac{f_v(\theta)}{\theta}\, d\theta \right| \le \frac{a_j}{d^{\frac{1}{q}}} \|f_v\|_p.$$

By the comparison test, we see that the three series in (3), $\lambda \sum_{j=1}^{\infty} a_j (\log \phi(\eta_j))^{\gamma-1}$ and $\sum_{j=1}^{\infty} a_j \int_1^{\phi(\eta_j)} (\log \frac{\phi(\eta_j)}{\theta})^{\gamma-1} \frac{f_v(\theta)}{\theta}\, d\theta$ are convergent. Thus, by taking the limit as $m \to \infty$ in (13) and by applying the properties of the Riemann sum for continuous functions, we obtain

$$v(t) = \frac{(\log t)^{\gamma-1}}{(1-\sigma_2)\Gamma(\gamma)} \int_1^e \left(\log \frac{e}{\theta}\right)^{\gamma-1} \frac{f_v(\theta)}{\theta}\, d\theta - \frac{1}{\Gamma(\gamma)} \int_1^t \left(\log \frac{t}{\theta}\right)^{\gamma-1} \frac{f_v(\theta)}{\theta}\, d\theta \\ - \frac{\lambda (\log t)^{\gamma-1}}{(1-\sigma_2)\Gamma(\gamma)} \sum_{j=1}^{\infty} a_j \int_1^{\phi(\eta_j)} \left(\log \frac{\phi(\eta_j)}{\theta}\right)^{\gamma-1} \frac{f_v(\theta)}{\theta}\, d\theta$$

$$+\frac{\nu_0(\log t)^{\gamma-1}}{1-\sigma_2}, \tag{34}$$

which, satisfies the differential equation (1). Furthermore, from (33) and the relation $\frac{1}{\sigma_2} = 1 + \frac{\sigma_2}{1-\sigma_2}$, we have $\nu(1) = 0$ and

$$\begin{aligned}
&\nu_0 + \lambda \sum_{j=1}^{\infty} a_j \nu\big(\phi(\eta_j)\big) \\
&= \nu_0 + \lambda \sum_{j=1}^{\infty} a_j \Bigg[\frac{(\log \phi(\eta_j))^{\gamma-1}}{(1-\sigma_2)\Gamma(\gamma)} \int_1^e \left(\log \frac{e}{\theta}\right)^{\gamma-1} \frac{f_\nu(\theta)}{\theta}\, d\theta \\
&\quad - \frac{1}{\Gamma(\gamma)} \int_1^{\phi(\eta_j)} \left(\log \frac{\phi(\eta_j)}{\theta}\right)^{\gamma-1} \frac{f_\nu(\theta)}{\theta}\, d\theta \\
&\quad - \frac{\lambda(\log \phi(\eta_j))^{\gamma-1}}{(1-\sigma_2)\Gamma(\gamma)} \sum_{j=1}^{\infty} a_j \int_1^{\phi(\eta_j)} \left(\log \frac{\phi(\eta_j)}{\theta}\right)^{\gamma-1} \frac{f_\nu(\theta)}{\theta}\, d\theta \\
&\quad + \frac{\nu_0(\log \phi(\eta_j))^{\gamma-1}}{1-\sigma_2} \Bigg] \\
&= \frac{\sigma_2}{(1-\sigma_2)\Gamma(\gamma)} \int_1^e \left(\log \frac{e}{\theta}\right)^{\gamma-1} \frac{f_\nu(\theta)}{\theta}\, d\theta + \frac{\nu_0}{1-\sigma_2} \\
&\quad - \frac{\lambda}{(1-\sigma_2)\Gamma(\gamma)} \sum_{j=1}^{\infty} a_j \int_1^{\phi(\eta_j)} \left(\log \frac{\phi(\eta_j)}{\theta}\right)^{\gamma-1} \frac{f_\nu(\theta)}{\theta}\, d\theta \\
&= \nu(e).
\end{aligned}$$

This proves that the integral equation (33) satisfies the problem given by (1) under infinite-point BCs (3). □

6 Application

Example 1 Let us consider the singular Hadamard-type fractional differential problem:

$$ {}_{\mathrm{H}}D^{3/2}\nu(t) + \frac{1}{5(t-1)^{2/7}} \frac{\nu^2}{(1+|\nu|)} + \frac{{}_{\mathrm{H}}D^{1/4}\nu(t)}{10(t-1)^{1/4}} + \frac{(\log t)^{1/2}}{60}\nu'(t) + \frac{1}{(t-1)^{1/7}} = 0, \tag{35}$$

$$\nu(1) = 0, \qquad \nu(e) = \nu_0 + \frac{1}{100}\int_1^e \nu(\xi)\frac{d\xi}{\xi}. \tag{36}$$

Here, $\gamma = 3/2$, $\delta = 1/4$, $p = 3$, $q = 3/2$, $\phi(\xi) = \xi$, $\lambda = \frac{1}{100}$ and

$$f(t,\nu_1,\nu_2,\nu_3) = \frac{1}{5(t-1)^{2/7}} \frac{\nu_1^2}{1+|\nu_1|} + \frac{1}{10(t-1)^{1/4}}\nu_2 + \frac{(\log t)^{1/2}}{60}\nu_3 + \frac{1}{(t-1)^{1/7}}.$$

Clearly

$$\big|f(t,\nu_1,\nu_2,\nu_3)\big| \le p(t)|\nu_1| + q(t)|\nu_2| + r(t)|\nu_3| + s(t),$$

where $p(t) = \frac{1}{5(t-1)^{2/7}}$, $q(t) = \frac{1}{10(t-1)^{1/4}}$, $r(t) = \frac{(\log t)^{1/2}}{60}$, $s(t) = \frac{1}{(t-1)^{1/7}}$.

Indeed we have $\|p\|_3 \approx 0.39268$, $\|q\|_3 \approx 0.16606$, $\|(\log t)^{-1/2} r\|_3 \approx 0.019962$, $\sigma_1 \approx 0.00667 < 1$, $\mathcal{A}_1 \approx 1.57228$, $\mathcal{B}_1 \approx 1.816696$, $\mathcal{C}_1 \approx 1.6647$;

$$\mathcal{A}_1 \|p\|_3 + \mathcal{B}_1 \|q\|_3 + \mathcal{C}_1 \left\| (\log t)^{\gamma-2} r \right\|_3 \approx 0.95231 < 1.$$

All the assumptions of Theorem 4 hold, therefore the singular Hadamard-type fractional differential problem (35), (36) has a continuous positive solution.

Example 2 Let γ, δ, p, q, λ and $f(t, v_1, v_2, v_3)$, be as in the previous example and consider the deviated function $\phi(\eta_j) = \eta_j = e^{\frac{1}{j}} \in [1, e]$, and let $a_j = \frac{1}{j^{5/2}}$.

Then the infinite-point boundary condition 3 becomes

$$v(1) = 0, \qquad v(e) = v_0 + \frac{1}{100} \sum_{j=1}^{\infty} \frac{1}{j^{5/2}} v\big(e^{\frac{1}{j}}\big). \tag{37}$$

It follows that $\sum_{j=1}^{\infty} a_j \approx 1.35556$, $\sum_{j=1}^{\infty} a_j (\log \phi(\eta_j))^{\gamma-1} = \sum_{j=1}^{\infty} \frac{1}{j^3} \approx 1.20205$, $\sigma_2 = 0.012021 < 1$, $\mathcal{A}_2 \approx 1.58359$, $\mathcal{B}_2 \approx 1.426999$, $\mathcal{C}_2 \approx 1.671622$. Then

$$\mathcal{A}_2 \|p\|_3 + \mathcal{B}_2 \|q\|_3 + \mathcal{C}_2 \left\| (\log t)^{\gamma-2} r \right\|_3 \approx 0.89218 < 1.$$

Therefore, all the assumptions of Theorem 5 hold and the singular Hadamard-type fractional differential problem (35), (37) has a continuous positive solution.

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