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Oscillation of fourth-order strongly noncanonical differential equations with delay argument

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Abstract

The aim of this paper is to study oscillatory properties of the fourth-order strongly noncanonical equation of the form

$$(r_3(t)(r_2(t)(r_1(t)y'(t))')')' + p(t)y(\tau(t)) = 0,$$

where $\int^\infty \frac{1}{r_i(s)}\,\mathrm{d}s < \infty$, $i = 1,2,3$. Reducing possible classes of the nonoscillatory solutions, new oscillatory criteria are established.

MSC: 34K11; 34C10

Keywords: Noncanonical operator; Fourth order differential equations; Oscillation

1 Introduction

In the paper, we consider the fourth-order delay differential equation

$$\bigl(r_3(t)\bigl(r_2(t)\bigl(r_1(t)y'(t)\bigr)'\bigr)'\bigr)' + p(t)y\bigl(\tau(t)\bigr) = 0, \qquad (E)$$

where $r_i \in C^{(4-i)}(t_0,\infty)$, $r_i(t) > 0$, $i = 1,2,3$, $p(t) \in C(t_0,\infty)$, $p(t) > 0$, $\tau(t) \in C(t_0,\infty)$, $\tau(t) \le t$, $\tau'(t) > 0$, and $\tau(t) \to \infty$ as $t \to \infty$.

By a solution of Eq. (E) we mean all continuous functions $y(t)$ for which

$$\bigl(r_3(t)\bigl(r_2(t)\bigl(r_1(t)y'(t)\bigr)'\bigr)'\bigr)' \in C\bigl([T_y,\infty)\bigr), \qquad T_y \ge t_0,$$

exist and satisfy Eq. (E) on $[T_y,\infty)$. We consider only those solutions $y(t)$ of (E) which satisfy $\sup\{|y(t)| : t \ge T\} > 0$ for all $T \ge T_y$. We assume that (E) possesses such a solution. A solution of (E) is called oscillatory if it has arbitrarily large zeros on $[T_y,\infty)$ and otherwise it is called nonoscillatory. Equation (E) is said to be oscillatory if all its solutions are oscillatory.

Throughout the paper it is supposed that Eq. (E) is strongly noncanonical, that is,

$$\int^\infty \frac{1}{r_i(s)}\,\mathrm{d}s < \infty, \qquad i = 1,2,3. \qquad (1.1)$$

Fourth-order differential equations naturally appear in models concerning physical, biological, and chemical phenomena, such as, for instance, problems of elasticity, deformation of structures, or soil settlement, see, for example, [2]. In mechanical and engineering problems, questions concerning the existence of oscillatory solutions play an important role. During the past decades, there has been a constant interest in obtaining sufficient conditions for oscillatory properties of different classes of fourth-order differential equations with deviating argument, see [2, 3, 6, 8–20].

In general, there are two approaches for the investigation of higher-order differential equations with noncanonical operators. One method requires to find a canonical representation of studied equation with closed form formulas for coefficients. For details, see [1, 4, 5, 7]. The second approach is to establish the conditions that reduce the number of possible classes of nonoscillatory solutions and consequently to find conditions for oscillation of (E). Our method belongs to the second one and yields easily verifiable oscillation criteria.

2 Preliminary results

Throughout the paper we assume that (1.1) holds, and so we can employ the notation

$$\pi_i(t) = \int_t^\infty \frac{1}{r_i(s)}\,\mathrm{d}s, \qquad \pi_{ij}(t) = \int_t^\infty \frac{1}{r_i(s)}\pi_j(s)\,\mathrm{d}s$$

and

$$\pi_{ijk}(t) = \int_t^\infty \frac{1}{r_i(s)}\pi_{jk}(s)\,\mathrm{d}s,$$

where $i, j, k \in \{1, 2, 3\}$ are mutually different. To simplify the writing of quasi-derivatives, we denote

$$L_1 y(t) = r_1(t)y'(t), \qquad L_{i+1}y(t) = r_{i+1}(t)L_i'y(t), \quad i = 1, 2, 3,$$

where formally $r_4(t) \equiv 1$. We start with the following auxiliary results which are elementary but very useful.

Lemma 1 *Let* (1.1) *hold. Then*

$$\pi_{ij}(t) + \pi_{ji}(t) = \pi_i(t)\pi_j(t).$$

Proof Since

$$\bigl(\pi_i(t)\pi_j(t)\bigr)' = -\frac{\pi_j(t)}{r_i(t)} - \frac{\pi_i(t)}{r_j(t)},$$

an integration of this equality from t to ∞ yields

$$\pi_i(t)\pi_j(t) = \int_t^\infty \frac{1}{r_i(s)}\pi_j(s)\,\mathrm{d}s + \int_t^\infty \frac{1}{r_j(s)}\pi_i(s)\,\mathrm{d}s = \pi_{ij}(t) + \pi_{ji}(t). \qquad \square$$

Lemma 2 *Let* (1.1) *hold. Then*

$$\pi_{123}(t) + \pi_{32}(t)\pi_1(t) - \pi_3(t)\pi_{12}(t) = \pi_{321}(t).$$

Proof Proof of this lemma is similar to that of Lemma 1 and so it can be omitted. □

It follows from a generalization of lemma of Kiguradze [9] that the set of positive solutions of (E) has the following structure.

Lemma 3 *Assume that* $y(t)$ *is a positive solution of* (E). *Then* $y(t)$ *satisfies one of the following conditions*:

(N_1): $L_1y(t) > 0$, $L_2y(t) > 0$, $L_3y(t) > 0$, $L_4y(t) < 0$,
(N_2): $L_1y(t) > 0$, $L_2y(t) > 0$, $L_3y(t) < 0$, $L_4y(t) < 0$,
(N_3): $L_1y(t) > 0$, $L_2y(t) < 0$, $L_3y(t) < 0$, $L_4y(t) < 0$,
(N_4): $L_1y(t) > 0$, $L_2y(t) < 0$, $L_3y(t) > 0$, $L_4y(t) < 0$,
(N_5): $L_1y(t) < 0$, $L_2y(t) > 0$, $L_3y(t) > 0$, $L_4y(t) < 0$,
(N_6): $L_1y(t) < 0$, $L_2y(t) < 0$, $L_3y(t) > 0$, $L_4y(t) < 0$,
(N_A): $L_1y(t) < 0$, $L_2y(t) > 0$, $L_3y(t) < 0$, $L_4y(t) < 0$,
(N_B): $L_1y(t) < 0$, $L_2y(t) < 0$, $L_3y(t) < 0$, $L_4y(t) < 0$.

The first two results are intended to reduce the number of classes that will be investigated.

Theorem 1 *If*

$$\int_{t_1}^{\infty} \pi_{32}(s)p(s)\,\mathrm{d}s = \infty, \tag{2.1}$$

then a positive solution $y(t)$ *of* (E) *does not satisfy* (N_1)–(N_4) *of Lemma* 3.

Proof Assume on the contrary that $y(t)$ is an eventually positive solution of (E) satisfying condition (N_1) or (N_4) of Lemma 3 for $t \geq t_1 \geq t_0$. Since $y(t)$ is positive and nondecreasing, there exists a positive constant $k > 0$ such that $y(t) \geq k$ for $t \geq t_1$.

Integrating (E) from t_1 to ∞, we get

$$L_3y(t_1) \geq \int_{t_1}^{\infty} p(s)y(\tau(s))\,\mathrm{d}s \geq k\int_{t_1}^{\infty} p(s)\,\mathrm{d}s,$$

which is a contradiction with respect to (2.1).

Now, we assume that $y(t)$ is an eventually positive solution of (E) satisfying condition (N_2) of Lemma 3 for $t \geq t_1$. Integrating (E) from t_1 to t and using that $y(t)$ is positive and nondecreasing, we get

$$-L_3y(t) \geq k\int_{t_1}^{t} p(s)\,\mathrm{d}s. \tag{2.2}$$

Integrating the above inequality from t_1 to ∞, we obtain

$$L_2y(t_1) \geq k\int_{t_1}^{\infty} \frac{1}{r_3(u)}\int_{t_1}^{u} p(s)\,\mathrm{d}s\,\mathrm{d}u = k\int_{t_1}^{\infty} \pi_3(u)p(u)\,\mathrm{d}u,$$

which contradicts (2.1).

Finally, we assume that $y(t)$ is an eventually positive solution of (E) satisfying condition (N_3) of Lemma 3 for $t \geq t_1$. Similarly as above, we are led to (2.2). Integrating this from t_1 to t, we obtain

$$-L_2 y(t) \geq k \int_{t_1}^{t} \frac{1}{r_3(u)} \int_{t_1}^{u} p(s)\,\mathrm{d}s\,\mathrm{d}u.$$

An integration from t_1 to ∞ yields

$$\begin{aligned} L_1 y(t_1) &\geq k \int_{t_1}^{\infty} \frac{1}{r_2(v)} \int_{t_1}^{v} \frac{1}{r_3(u)} \int_{t_1}^{u} p(s)\,\mathrm{d}s\,\mathrm{d}u\,\mathrm{d}v \\ &= k \int_{t_1}^{\infty} \frac{1}{r_3(u)} \int_{t_1}^{u} p(s)\,\mathrm{d}s \int_{u}^{\infty} \frac{1}{r_2(v)}\,\mathrm{d}v\,\mathrm{d}u \\ &= k \int_{t_1}^{\infty} \frac{1}{r_3(u)} \pi_2(u) \int_{t_1}^{u} p(s)\,\mathrm{d}s\,\mathrm{d}u = k \int_{t_1}^{\infty} \pi_{32}(s) p(s)\,\mathrm{d}s, \end{aligned}$$

which is a contradiction and the proof is finished. □

Theorem 2 *If*

$$\int_{t_1}^{\infty} \pi_{12}\big(\tau(s)\big) p(s)\,\mathrm{d}s = \infty, \tag{2.3}$$

then the positive solution $y(t)$ of (E) does not satisfy (N_5), (N_6) of Lemma 3.

Proof Assume on the contrary that $y(t)$ is an eventually positive solution of (E) satisfying condition (N_5) of Lemma 3 for $t \geq t_1 \geq t_0$. Since $L_2 y(t)$ is a positive and increasing function there exists a positive constant $k > 0$ such that

$$L_2 y(t) \geq k$$

for $t \geq t_1$. Integrating the previous inequality from t to ∞, we have

$$-r_1(t) y'(t) \geq k \int_{t}^{\infty} \frac{1}{r_2(s)}\,\mathrm{d}s.$$

After integration from $\tau(t)$ to ∞, we get

$$y\big(\tau(t)\big) \geq k \int_{\tau(t)}^{\infty} \frac{1}{r_1(u)} \int_{u}^{\infty} \frac{1}{r_2(s)}\,\mathrm{d}s\,\mathrm{d}u = k\pi_{12}\big(\tau(t)\big). \tag{2.4}$$

On the other hand, in view of (2.4), an integration of (E) from t_1 to ∞ yields

$$L_3 y(t_1) \geq \int_{t_1}^{\infty} p(s) y\big(\tau(s)\big)\,\mathrm{d}s \geq k \int_{t_1}^{\infty} p(s) \pi_{12}\big(\tau(s)\big)\,\mathrm{d}s,$$

which contradicts (2.3).

Now, we assume that $y(t)$ is an eventually positive solution of (E) satisfying condition (N_6) of Lemma 3 for $t \geq t_1 \geq t_0$. Seeing that $L_1(y)$ is a negative and decreasing function, there exists a constant $k > 0$ such that

$$L_1 y(t) = r_1(t)y'(t) \le -k$$

for $t \ge t_1$, and integrating this inequality from $\tau(t)$ to ∞, we have

$$y\bigl(\tau(t)\bigr) \ge k \int_{\tau(t)}^{\infty} \frac{1}{r_1(s)}\,\mathrm{d}s. \tag{2.5}$$

Integrating (E) from t_1 to ∞ and using (2.5), we obtain

$$\begin{aligned} L_3 y(t_1) &\ge \int_{t_1}^{\infty} p(s)y\bigl(\tau(s)\bigr)\,\mathrm{d}s \ge k \int_{t_1}^{\infty} p(s) \int_{\tau(s)}^{\infty} \frac{1}{r_1(u)}\,\mathrm{d}u\,\mathrm{d}s \\ &= k \int_{t_1}^{\infty} p(s)\pi_1\bigl(\tau(s)\bigr)\,\mathrm{d}s, \end{aligned}$$

which is a contradiction to (2.3). The proof is completed. □

Theorems 2.1 and 2.3 reduce the number of possible nonoscillatory solutions of (E) only to (N_A) or (N_B), which essentially simplifies examination of (E).

3 Main results

Now we provide useful monotonic properties of nonoscillatory solutions of (E) satisfying conditions (N_A) or (N_B) of Lemma 3. We begin with the following auxiliary result.

Lemma 4 *Assume that $y(t)$ is an eventually positive solution of (E) satisfying condition (N_A) of Lemma 3 and*

$$\int_{t_0}^{\infty} p(s)\pi_3(s)\pi_1\bigl(\tau(s)\bigr)\,\mathrm{d}s = \infty. \tag{3.1}$$

Then

$$\lim_{t\to\infty} r_1(t)y'(t) = \lim_{t\to\infty} y(t) = 0. \tag{3.2}$$

Proof Assume that $y(t)$ is an eventually positive solution of (E) satisfying condition (N_A) of Lemma 3 for $t \ge t_1 \ge t_0$.

Since $y(t)$ is positive and decreasing, there exists $\lim_{t\to\infty} y(t) = \ell \ge 0$. We claim that $\ell = 0$. If not, then $y(\tau(t)) \ge \ell > 0$, eventually, let us say for $t \ge t_1$. An integration of (E) from t_1 to t yields

$$-\bigl(L_2 y(t)\bigr)' \ge \frac{1}{r_3(t)} \int_{t_1}^{t} p(s)y\bigl(\tau(s)\bigr)\,\mathrm{d}s \ge \frac{\ell}{r_3(t)} \int_{t_1}^{t} p(s)\,\mathrm{d}s. \tag{3.3}$$

Integrating from t_1 to ∞, we obtain

$$\bigl(r_2(r_1 y')\bigr)'(t_1) \ge \ell \int_{t_1}^{\infty} \frac{1}{r_3(u)} \int_{t_1}^{u} p(s)\,\mathrm{d}s\,\mathrm{d}u = \ell \int_{t_1}^{\infty} p(s)\pi_3(s)\,\mathrm{d}s,$$

which contradicts (3.1), and we conclude that $y(t) \to 0$ as $t \to \infty$.

On the other hand, since $-r_1y'$ is positive and decreasing, there exists

$$\lim_{t\to\infty} -r_1y'(t) = \ell \geq 0.$$

We assume on the contrary that $\ell > 0$. Then

$$-r_1y'(t) > \ell, \quad t \geq t_1.$$

Integrating from t to ∞, one gets

$$y(t) \geq \ell\pi_1(t),$$

which setting into (3.3) yields

$$-\bigl(L_2y(t)\bigr)' \geq \frac{\ell}{r_3(t)}\int_{t_1}^{t} p(s)\pi_1\bigl(\tau(s)\bigr)\,\mathrm{d}s.$$

An integration from t_1 to ∞ yields

$$\bigl(r_2\bigl(r_1y'\bigr)\bigr)'(t_1) \geq \ell\int_{t_1}^{\infty}\frac{1}{r_3(u)}\int_{t_1}^{u} p(s)\pi_1\bigl(\tau(s)\bigr)\,\mathrm{d}s\,\mathrm{d}u = \ell\int_{t_1}^{\infty} p(s)\pi_3(s)\pi_1\bigl(\tau(s)\bigr)\,\mathrm{d}s.$$

This is a contradiction, and the proof is complete now. □

Theorem 3 *Let* (3.1) *hold. Assume that* $y(t)$ *is an eventually positive solution of* (*E*) *satisfying condition* (N_A) *of Lemma* 3. *Then*

$$\frac{y(t)}{\pi_{12}(t)} \quad \text{is decreasing,} \tag{3.4}$$

$$\frac{y(t)}{\pi_{123}(t)} \quad \text{is increasing.} \tag{3.5}$$

Proof Assume that $y(t)$ is an eventually positive solution of (*E*) satisfying condition (N_A) of Lemma 3 for $t \geq t_1 \geq t_0$. At first, we shall show that $\frac{y(t)}{\pi_{12}(t)}$ is decreasing. Employing (3.2) and using that $L_2y(t)$ is positive and decreasing, we have

$$-r_1(t)y'(t) = \int_t^{\infty}\frac{L_2y(s)}{r_2(s)}\,\mathrm{d}s \leq L_2y(t)\pi_2(t),$$

which implies

$$\left(\frac{r_1(t)y'(t)}{\pi_2(t)}\right)' = \frac{L_2y(t)\pi_2(t) + r_1(t)y'(t)}{r_2(t)\pi_2^2(t)} \geq 0.$$

Thus, $\frac{r_1(t)y'(t)}{\pi_2(t)}$ is increasing, and in view of (3.2), we get

$$-y(t) = \int_t^{\infty}\frac{r_1(s)y'(s)}{\pi_2(s)}\frac{\pi_2(s)}{r_1(s)}\,\mathrm{d}s \geq \frac{r_1(t)y'(t)}{\pi_2(t)}\pi_{12}(t),$$

which yields

$$\left(\frac{y(t)}{\pi_{12}(t)}\right)' = \frac{r_1(t)y'(t)\pi_{12}(t) + y(t)\pi_2(t)}{r_1(t)\pi_{12}^2(t)} \le 0,$$

and we conclude that $\frac{y(t)}{\pi_{12}(t)}$ is a decreasing function.

Now, we shall prove that $\frac{y(t)}{\pi_{123}(t)}$ is an increasing function. Employing that $L_3y(t)$ is a negative and decreasing function, we have

$$-L_2y(t) \le \int_t^\infty \frac{L_3y(s)}{r_3(s)}\,\mathrm{d}s \le L_3y(t)\pi_3(t),$$

which yields

$$\left(\frac{L_2y(t)}{\pi_3(t)}\right)' = \frac{L_3y(t)\pi_3(t) + L_2y(t)}{r_3(t)\pi_3^2(t)} \ge 0$$

and $\frac{L_2y(t)}{\pi_3(t)}$ is increasing. Therefore,

$$-r_1(t)y'(t) = \int_t^\infty \frac{L_2y(s)}{\pi_3(s)}\frac{\pi_3(s)}{r_2(s)}\,\mathrm{d}s \ge \frac{L_2y(t)}{\pi_3(t)}\pi_{23}(t).$$

This inequality implies that $\frac{r_1(t)y'(t)}{\pi_{23}(t)}$ is increasing. Finally,

$$-y(t) = \int_t^\infty \frac{r_1(s)y'(s)}{\pi_{23}(s)}\frac{\pi_{23}(s)}{r_1(s)}\,\mathrm{d}s \le \frac{r_1(t)y'(t)\pi_{123}(t)}{\pi_{23}(t)},$$

which implies

$$\left(\frac{y(t)}{\pi_{123}(t)}\right)' = \frac{r_1(t)y'(t)\pi_{123}(t) + y(t)\pi_{23}(t)}{r_1(t)\pi_{123}^2(t)} \ge 0,$$

and we conclude that $\frac{y(t)}{\pi_{123}(t)}$ is increasing. □

Theorem 4 *Assume that $y(t)$ is an eventually positive solution of (E) satisfying condition (N_B) of Lemma 3. Then*

$$\frac{y(t)}{\pi_1(t)} \quad \textit{is increasing.} \tag{3.6}$$

Proof Assume that $y(t)$ is an eventually positive solution of (E) satisfying condition (N_B) of Lemma 3 for $t \ge t_1 \ge t_0$. Applying the monotonic property of $r_1(t)y'(t)$, we get

$$-y(t) = \int_t^\infty \frac{r_1(s)y'(s)}{r_1(s)}\,\mathrm{d}s \le r_1(t)y'(t)\pi_1(t),$$

which gives

$$\left(\frac{y(t)}{\pi_1(t)}\right)' = \frac{r_1(t)y'(t)\pi_1(t) + y(t)}{r_1(t)\pi_1^2(t)} \ge 0,$$

and we conclude that $\frac{y(t)}{\pi_1(t)}$ is increasing. □

Now, we are prepared for establishing the criteria for the essential classes (N_A) and (N_B) to be empty.

Theorem 5 *Let* (3.1) *hold. If*

$$\limsup_{t\to\infty}\left\{\frac{\pi_{123}(\tau(t))}{\pi_{12}(\tau(t))}\int_{t_1}^{\tau(t)} p(s)\pi_{12}\big(\tau(s)\big)\,\mathrm{d}s+\frac{1}{\pi_{12}(\tau(t))}\int_{\tau(t)}^{t} p(s)G\big(s,\tau(t)\big)\pi_{12}\big(\tau(s)\big)\,\mathrm{d}s\right.$$
$$\left.+\frac{1}{\pi_{123}(\tau(t))}\int_t^{\infty} p(s)G\big(s,\tau(t)\big)\pi_{123}\big(\tau(s)\big)\,\mathrm{d}s\right\}>1, \tag{3.7}$$

where

$$G(s,t)=\pi_{321}(s)+\pi_{12}(t)\pi_3(s)-\pi_1(t)\pi_{32}(s),$$

then the class (N_A) *of Lemma* 3 *is empty.*

Proof Assume on the contrary that $y(t)$ is an eventually positive solution of (E) satisfying condition (N_A) of Lemma 3 for $t\ge t_1\ge t_0$. Integrating (E) twice from t_1 to t and from t to ∞, we obtain

$$L_2y(t)\ge\int_t^{\infty}\frac{1}{r_3(u)}\int_{t_1}^{u} p(s)y\big(\tau(s)\big)\,\mathrm{d}s\,\mathrm{d}u.$$

Changing the order of integrating in the previous inequality, we see

$$\big(r_1(t)y'(t)\big)'\ge\frac{\pi_3(t)}{r_2(t)}\int_{t_1}^{t} p(s)y\big(\tau(s)\big)\,\mathrm{d}s+\frac{1}{r_2(t)}\int_t^{\infty} p(s)y\big(\tau(s)\big)\pi_3(s)\,\mathrm{d}s.$$

Integrating the above inequality from t to ∞, one gets

$$\begin{aligned}-r_1(t)y'(t)&\ge\int_t^{\infty}\frac{\pi_3(u)}{r_2(u)}\int_{t_1}^{u} p(s)y\big(\tau(s)\big)\,\mathrm{d}s\,\mathrm{d}u+\int_t^{\infty}\frac{1}{r_2(u)}\int_u^{\infty} p(s)y\big(\tau(s)\big)\pi_3(s)\,\mathrm{d}s\,\mathrm{d}u\\&=\pi_{23}(t)\int_{t_1}^{t} p(s)y\big(\tau(s)\big)\,\mathrm{d}s+\int_t^{\infty}\pi_{23}(s)p(s)y\big(\tau(s)\big)\,\mathrm{d}s\\&\quad+\int_t^{\infty} p(s)y\big(\tau(s)\big)\pi_3(s)\big[\pi_2(t)-\pi_2(s)\big]\,\mathrm{d}s.\end{aligned}$$

It follows from Lemma 1 that $\pi_{23}(s)+\pi_{32}(s)=\pi_2(s)\pi_3(s)$, and so

$$\begin{aligned}-y'(t)&=\frac{\pi_{23}(t)}{r_1(t)}\int_{t_1}^{t} p(s)y\big(\tau(s)\big)\,\mathrm{d}s+\frac{\pi_2(t)}{r_1(t)}\int_t^{\infty} p(s)y\big(\tau(s)\big)\pi_3(s)\,\mathrm{d}s\\&\quad-\frac{1}{r_1(t)}\int_t^{\infty} p(s)y\big(\tau(s)\big)\pi_{32}(s)\,\mathrm{d}s.\end{aligned}$$

Integrating once more from t to ∞ and employing Lemma 2, we have

$$\begin{aligned}y(t)&\ge\pi_{123}(t)\int_{t_1}^{t} p(s)y\big(\tau(s)\big)\,\mathrm{d}s+\int_t^{\infty} p(s)y\big(\tau(s)\big)\pi_{123}(s)\,\mathrm{d}s\\&\quad+\int_t^{\infty} p(s)y\big(\tau(s)\big)\pi_3(s)\big[\pi_{12}(t)-\pi_{12}(s)\big]\,\mathrm{d}s\end{aligned}$$

$$
\begin{aligned}
&- \int_t^\infty p(s)y\big(\tau(s)\big)\pi_{32}(s)\big[\pi_1(t)-\pi_1(s)\big]\,\mathrm{d}s \\
&= \pi_{123}(t)\int_{t_1}^t p(s)y\big(\tau(s)\big)\,\mathrm{d}s + \int_t^\infty p(s)y\big(\tau(s)\big)\pi_{321}(s)\,\mathrm{d}s \\
&\quad + \pi_{12}(t)\int_t^\infty p(s)y\big(\tau(s)\big)\pi_3(s)\,\mathrm{d}s - \pi_1(t)\int_t^\infty p(s)y\big(\tau(s)\big)\pi_{32}(s)\,\mathrm{d}s \\
&= \pi_{123}(t)\int_{t_1}^t p(s)y\big(\tau(s)\big)\,\mathrm{d}s + \int_t^\infty p(s)y\big(\tau(s)\big)G(s,t)\,\mathrm{d}s.
\end{aligned}
$$

Then

$$
\begin{aligned}
y\big(\tau(t)\big) &\ge \pi_{123}\big(\tau(t)\big)\int_{t_1}^{\tau(t)} p(s)y\big(\tau(s)\big)\,\mathrm{d}s + \int_{\tau(t)}^\infty p(s)y\big(\tau(s)\big)G\big(s,\tau(t)\big)\,\mathrm{d}s \\
&= \pi_{123}\big(\tau(t)\big)\int_{t_1}^{\tau(t)} p(s)y\big(\tau(s)\big)\,\mathrm{d}s + \int_{\tau(t)}^t p(s)y\big(\tau(s)\big)G\big(s,\tau(t)\big)\,\mathrm{d}s \\
&\quad + \int_t^\infty p(s)y\big(\tau(s)\big)G\big(s,\tau(t)\big)\,\mathrm{d}s.
\end{aligned}
$$

Using that $\frac{y(t)}{\pi_{12}(t)}$ is decreasing and $\frac{y(t)}{\pi_{123}(t)}$ is increasing, the last inequality yields

$$
\begin{aligned}
y\big(\tau(t)\big) &\ge \pi_{123}\big(\tau(t)\big)\frac{y(\tau(t))}{\pi_{12}(\tau(t))}\int_{t_1}^{\tau(t)} p(s)\pi_{12}\big(\tau(s)\big)\,\mathrm{d}s \\
&\quad + \frac{y(\tau(t))}{\pi_{12}(\tau(t))}\int_{\tau(t)}^t p(s)G\big(s,\tau(t)\big)\pi_{12}\big(\tau(s)\big)\,\mathrm{d}s \\
&\quad + \frac{y(\tau(t))}{\pi_{123}(\tau(t))}\int_t^\infty p(s)G\big(s,\tau(t)\big)\pi_{123}\big(\tau(s)\big)\,\mathrm{d}s.
\end{aligned}
$$

Then

$$
\begin{aligned}
1 \ge \Bigg\{ &\frac{\pi_{123}(\tau(t))}{\pi_{12}(\tau(t))}\int_{t_1}^{\tau(t)} p(s)\pi_{12}\big(\tau(s)\big)\,\mathrm{d}s + \frac{1}{\pi_{12}(\tau(t))}\int_{\tau(t)}^t p(s)G\big(s,\tau(t)\big)\pi_{12}\big(\tau(s)\big)\,\mathrm{d}s \\
&+ \frac{1}{\pi_{123}(\tau(t))}\int_t^\infty p(s)G\big(s,\tau(t)\big)\pi_{123}\big(\tau(s)\big)\,\mathrm{d}s \Bigg\},
\end{aligned}
$$

which contradicts (3.7). □

Theorem 6 *If*

$$
\begin{aligned}
\limsup_{t\to\infty} \Bigg\{ &\int_{t_1}^{\tau(t)} p(s)H\big(s,\tau(t)\big)\,\mathrm{d}s + \int_{\tau(t)}^t p(s)\pi_{321}(s)\,\mathrm{d}s \\
&+ \frac{1}{\pi_1(\tau(t))}\int_t^\infty p(s)\pi_{321}(s)\pi_1\big(\tau(s)\big)\,\mathrm{d}s \Bigg\} > 1,
\end{aligned}
\tag{3.8}
$$

where

$$
H(s,t) = \pi_{32}(s)\pi_1(t) + \pi_3(s)\pi_{12}(t) + \pi_{123}(t),
$$

then the class (N_B) *of Lemma* 3 *is empty.*

Proof Assume on the contrary that $y(t)$ is an eventually positive solution of (E) satisfying condition (N_B) of Lemma 3 for $t \geq t_1 \geq t_0$. Integrating (E) twice from t_1 to t and thereafter switching the order of integration, we obtain

$$-\big(r_1(t)y'(t)\big)' \geq \frac{1}{r_2(t)} \int_{t_1}^{t} p(s)y\big(\tau(s)\big)\big(\pi_3(s) - \pi_3(t)\big)\,\mathrm{d}s.$$

Integrating the above inequality again from t_1 to t and changing the order of integration, we get

$$\begin{aligned} -r_1(t)y'(t) &\geq \int_{t_1}^{t} \frac{1}{r_2(u)} \int_{t_1}^{u} p(s)y\big(\tau(s)\big)\big(\pi_3(s) - \pi_3(t)\big)\,\mathrm{d}s\,\mathrm{d}u \\ &= \int_{t_1}^{t} p(s)y\big(\tau(s)\big)\pi_3(s)\big[\pi_2(s) - \pi_2(t)\big]\,\mathrm{d}s \\ &\quad - \int_{t_1}^{t} p(s)y\big(\tau(s)\big)\big[\pi_{23}(s) - \pi_{23}(t)\big]\,\mathrm{d}s. \end{aligned}$$

Applying Lemma 1, we can write

$$-r_1(t)y'(t) \geq \int_{t_1}^{t} p(s)y\big(\tau(s)\big)\big[\pi_{32}(s) - \pi_3(s)\pi_2(t) + \pi_{23}(t)\big]\,\mathrm{d}s.$$

Integrating the previous inequality from t to ∞ and consequently switching the order of integration, we obtain

$$\begin{aligned} y(t) &\geq \int_{t}^{\infty} \frac{1}{r_1(u)} \int_{t_1}^{u} p(s)y\big(\tau(s)\big)\big[\pi_{32}(s) - \pi_3(s)\pi_2(u) + \pi_{23}(u)\big]\,\mathrm{d}s\,\mathrm{d}u \\ &= \int_{t_1}^{t} p(s)y\big(\tau(s)\big)\pi_{32}(s)\pi_1(t)\,\mathrm{d}s - \int_{t_1}^{t} p(s)y\big(\tau(s)\big)\pi_3(s)\pi_{12}(t)\,\mathrm{d}s \\ &\quad + \int_{t_1}^{t} p(s)y\big(\tau(s)\big)\pi_{123}(t)\,\mathrm{d}s + \int_{t}^{\infty} p(s)y\big(\tau(s)\big)\pi_{32}(s)\pi_1(s)\,\mathrm{d}s \\ &\quad - \int_{t}^{\infty} p(s)y\big(\tau(s)\big)\pi_3(s)\pi_{12}(s)\,\mathrm{d}s + \int_{t}^{\infty} p(s)y\big(\tau(s)\big)\pi_{123}(s)\,\mathrm{d}s. \end{aligned}$$

Employing the equality $\pi_{123}(t) + \pi_{32}(t)\pi_1(t) - \pi_3(t)\pi_{12}(t) = \pi_{321}(t)$, we can rewrite the above inequality into a simpler form

$$\begin{aligned} y(t) &\geq \int_{t_1}^{t} p(s)y\big(\tau(s)\big)\big[\pi_{32}(s)\pi_1(t) - \pi_3(s)\pi_{12}(t) + \pi_{123}(t)\big]\,\mathrm{d}s \\ &\quad + \int_{t}^{\infty} p(s)y\big(\tau(s)\big)\pi_{321}(s)\,\mathrm{d}s. \end{aligned}$$

Using notation for $H(s,t)$

$$y(t) \geq \int_{t_1}^{t} p(s)y\big(\tau(s)\big)H(s,t)\,\mathrm{d}s + \int_{t}^{\infty} p(s)y\big(\tau(s)\big)\pi_{321}(s)\,\mathrm{d}s, \tag{3.9}$$

then

$$
\begin{aligned}
y(\tau(t)) &\geq \int_{t_1}^{\tau(t)} p(s)y(\tau(s))H(s,\tau(t))\,\mathrm{d}s + \int_{\tau(t)}^{\infty} p(s)y(\tau(s))\pi_{321}(s)\,\mathrm{d}s \\
&= \int_{t_1}^{\tau(t)} p(s)y(\tau(s))H(s,\tau(t))\,\mathrm{d}s + \int_{\tau(t)}^{t} p(s)y(\tau(s))\pi_{321}(s)\,\mathrm{d}s \\
&\quad + \int_{t}^{\infty} p(s)y(\tau(s))\pi_{321}(s)\,\mathrm{d}s.
\end{aligned}
$$

Applying the monotonic properties of $y(t)$ (decreasing) and $\frac{y(t)}{\pi_1(t)}$ (increasing), we have

$$
\begin{aligned}
y(\tau(t)) &\geq y(\tau(t)) \int_{t_1}^{\tau(t)} p(s)H(s,\tau(t))\,\mathrm{d}s + y(\tau(t)) \int_{\tau(t)}^{t} p(s)\pi_{321}(s)\,\mathrm{d}s \\
&\quad + \frac{y(\tau(t))}{\pi_1(\tau(t))} \int_{t}^{\infty} p(s)\pi_1(\tau(s))\pi_{321}(s)\,\mathrm{d}s,
\end{aligned}
$$

which implies

$$
\begin{aligned}
1 &\geq \int_{t_1}^{\tau(t)} p(s)H(s,\tau(t))\,\mathrm{d}s + \int_{\tau(t)}^{t} p(s)\pi_{321}(s)\,\mathrm{d}s \\
&\quad + \frac{1}{\pi_1(\tau(t))} \int_{t}^{\infty} p(s)\pi_1(\tau(s))\pi_{321}(s)\,\mathrm{d}s.
\end{aligned}
$$

This contradicts (3.8), and the proof is complete. □

The following result is intended to avoid evaluation of function $H(s,t)$ and to simplify criterion (3.7).

Corollary 1 *If*

$$
\begin{aligned}
\limsup_{t\to\infty} \biggl\{ &\pi_{321}(\tau(t)) \int_{t_1}^{\tau(t)} p(s)\,\mathrm{d}s + \int_{\tau(t)}^{t} p(s)\pi_{321}(s)\,\mathrm{d}s \\
&+ \frac{1}{\pi_1(\tau(t))} \int_{t}^{\infty} p(s)\pi_{321}(s)\pi_1(\tau(s))\,\mathrm{d}s \biggr\} > 1,
\end{aligned}
\tag{3.10}
$$

then the class (N_B) *of Lemma 3 is empty.*

Proof Assume on the contrary that $y(t)$ is an eventually positive solution of (E) satisfying condition (N_B) of Lemma 3 for $t \geq t_1 \geq t_0$. Proceeding similarly as in the proof of Theorem 6, we get (3.9). It follows from monotonic properties of $H(s,t)$ and Lemma 2 that

$$
H(s,t) \geq H(t,t) = \pi_{321}(t) \quad \text{for } s \in \langle t_1, t\rangle. \tag{3.11}
$$

Using (3.11) in (3.9), we obtain

$$
y(t) \geq \pi_{321}(t) \int_{t_1}^{t} p(s)y(\tau(s))\,\mathrm{d}s + \int_{t}^{\infty} p(s)y(\tau(s))\pi_{321}(s)\,\mathrm{d}s.
$$

Then

$$y(\tau(t)) \geq \pi_{321}(\tau(t)) \int_{t_1}^{\tau(t)} p(s)y(\tau(s))\,\mathrm{d}s + \int_{\tau(t)}^{t} p(s)y(\tau(s))\pi_{321}(s)\,\mathrm{d}s + \int_{t}^{\infty} p(s)y(\tau(s))\pi_{321}(s)\,\mathrm{d}s.$$

Taking into account that $y(t)$ is decreasing and $\frac{y(t)}{\pi_1(t)}$ is increasing finally, we have

$$1 \geq \pi_{321}(\tau(t)) \int_{t_1}^{\tau(t)} p(s)\,\mathrm{d}s + \int_{\tau(t)}^{t} p(s)\pi_{321}(s)\,\mathrm{d}s + \frac{1}{\pi_1(\tau(t))} \int_{t}^{\infty} p(s)\pi_{321}(s)\pi_1(\tau(s))\,\mathrm{d}s,$$

which contradicts the assumption of the corollary. □

Picking up the previous results, we can establish easily verifiable oscillatory criteria.

Theorem 7 *Let* (2.1), (3.1), (3.7), (3.8) *hold. Then* (*E*) *is oscillatory.*

Theorem 8 *Let* (2.1), (3.1), (3.7), (3.10) *hold. Then* (*E*) *is oscillatory.*

We support our results with an illustrative example, in which also some comparison with existing latest ones is made.

Example 1 Let us consider noncanonical fourth-order delay differential equation in the form

$$\left(\left(t^2\left(t^2\left(t^2 y'(t)\right)'\right)'\right)' + at^2 y(\lambda t) = 0, \quad t \geq t_0 > 0, \tag{E_x}$$

where $a > 0$, $\lambda \in (0,1)$, $\pi_i(t) = 1/t$, $\pi_{ij}(t) = 1/(2t^2)$, $\pi_{123}(t) = \pi_{321}(t) = 1/(6t^3)$. It is easy to verify that (2.1), (3.1) hold.

Condition (3.7) takes the form

$$\frac{a}{36\lambda^2}\left(9 + 9\lambda - \lambda^2 + 18\ln\frac{1}{\lambda}\right) > 1. \tag{3.12}$$

Condition (3.8) takes the form

$$a\left(\frac{17}{6} + \ln\frac{1}{\lambda}\right) > 6. \tag{3.13}$$

By Theorem 7, Eq. (E_x) is oscillatory provided that both (3.12) and (3.13) hold. In particular case where $\lambda = 0.8$, conditions (3.12) and (3.13) reduce to $a > 1.963$. The oscillatory criterion obtained by a different technique presented in paper [4] gives oscillation of (E_x) if $a > 7.913$.

On the other hand, for $\lambda = 0.9$, Theorem 7 guarantees oscillation of Eq. (E_x) provided that $a > 1.77$, while the best criterion from [7] requires $a > 3.50$.

So our results are more efficient than the previous ones.

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