Zhang et al. *Advances in Difference Equations* (2019) 2019:414
https://doi.org/10.1186/s13662-019-2341-8

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH **Open Access**

A delayed e-epidemic SLBS model for computer virus

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Abstract

We propose an e-epidemic time-delay Susceptible-Latent-Breaking out-Susceptible (*SLBS*) model to study delay dynamics appearing due to antivirus software, which takes time to clean the viruses from latent and breaking-out computers. We perform nonlinear stability analysis, Hopf bifurcation analysis, and its direction and stability. Numerical simulation results (time series analysis and bifurcation diagram) give useful insights for delay dynamics. We investigate the effect of the control parameters like rate of infection of all the classes and cure rates on the model system. Our results suggest that time delay is responsible for destabilizing the system dynamics. For smooth functionality of a computer system, our results suggest the minimum use of removable storage devices like smart phones, optical discs, memory cards, external hard disk, digital cameras, and so on and use of effective antivirus softwares.

Keywords: Time delay; SLBS model; Computer virus; Stability; Hopf bifurcation

1 Introduction

With increasing popularity of the internet, increasing numbers of network-based applications enter into our everyday life, which can bring about as much as potential hazards malware for network users [1]. Understanding the virus spread dynamics is most important for defence strategies and computer security [2]. In last decades the study of widespread infection of the computers connected to internet has attracted the interest of the researchers at home and abroad. To illustrate the computer virus transmission dynamics, Murray [3] has suggested high similarities between computer and biological viruses. Kephart and White [4, 5] investigated SIS models for the spread of computer virus. Wierman [2, 6] proposed the SIR computer virus propagation model. Considering that the computer virus has a latent period, Yuan et al. [7, 8] incorporated the exposed class E (infected but not yet broken-out) to the classical SIR and SEIR computer virus model. However, the SEIR computer virus model assumes that the recovered computers have a permanent immunization period, which is not consistent with real situation. Mishra and Saini [9] proposed the SEIRS computer virus model to reveal common worm propagation. There are also some other computer virus models with vaccination [10–12], quarantine [13–16], effect of antivirus software [17], and so on.

However, most of the mentioned computer virus models assume that the infected computers that are in latency do not infect other computers. This is not consistent with reality. An infected computer that is in latency can affect other computers during file downloading or copying. Apart from this, new viruses and newer versions of old viruses may infect the cured computers, and effect of removable storage devices are also assumed. Based on these assumptions, the following computer virus model with graded infection rate has been proposed by Yang and Yang [18]:

$$\begin{aligned} \frac{dS(t)}{dt} &= \mu_1 - \bigl(\beta_1 L(t) + \beta_2 B(t)\bigr)S(t) + \gamma_1 L(t) + \gamma_2 B(t) - (\delta + \theta)S(t), \\ \frac{dL(t)}{dt} &= \mu_2 + \bigl(\beta_1 L(t) + \beta_2 B(t)\bigr)S(t) - (\gamma_1 + \alpha + \delta)L(t) + \theta S(t), \\ \frac{dB(t)}{dt} &= \alpha L(t) - (\gamma_2 + \delta)B(t), \end{aligned} \tag{1}$$

where $S(t)$, $L(t)$, and $B(t)$ denotes the numbers of uninfected, latent, and breaking-out computers at time t, β_1, β_2, μ_1, μ_2, γ_1, γ_2, δ, α, and θ are the parameters of system (1), and the meanings of all the parameters are the same as those in [18]. Yang and Yang [18] studied the local and global stability of system (1).

Time delay comes from the time sharing of the communication medium and the computation time entailed for communication processing and physical signal coding. The concept of delay comes in the 1970s when analogue controllers were replaced by digital controllers in computer networks [19]. Yang [20] demonstrated that the computational delay can cause system instability in a digital controller. Delay is an important aspect because it directly affects the speed of the digital device on an operating computer. In [18] the effect of time delay is not considered; nevertheless, delay acts crucially on system dynamics. Therefore we have deliberated time delay due to the period that antivirus software uses to clean viruses from latent and breaking-out computers. Recently, computer virus models with time delay have been investigated by some scholars [21–26]. In addition to computer virus models with time delay, there are also some other dynamical models with time delay, which have shown that time delay causes problems such as instability and restrict the conceivable performance of the control systems. For example, the predator–prey model [27–30], the epidemic model [31–35], and the neural network model [36–39]. All the mentioned works about delayed dynamical models have shown that time delay can produce complicated nonlinear phenomena with the change of time. Therefore it is important to know that at which time the delay destabilizes the system. Thus we have considered the effect of time delay on the system dynamics. Considering the effect of time delay (denoted as τ) due to the period that antivirus software uses to clean viruses, we investigate the following delayed model:

$$\begin{aligned} \frac{dS(t)}{dt} &= \mu_1 - \bigl(\beta_1 L(t) + \beta_2 B(t)\bigr)S(t) + \gamma_1 L(t-\tau) + \gamma_2 B(t-\tau) - (\delta + \theta)S(t), \\ \frac{dL(t)}{dt} &= \mu_2 + \bigl(\beta_1 L(t) + \beta_2 B(t)\bigr)S(t) - \gamma_1 L(t-\tau) - (\alpha + \delta)L(t) + \theta S(t), \\ \frac{dB(t)}{dt} &= \alpha L(t) - \gamma_2 B(t-\tau) - \delta B(t). \end{aligned} \tag{2}$$

This paper is organized as follows. Section 2 deals with linear stability and Hopf bifurcation analysis. In Sect. 3, we study the nonlinear stability analysis using the Lyapunov direct method. In Sect. 4, we obtain the stability and direction of Hopf bifurcation by using the theory of center manifold and normal form. Numerical simulation results are presented in Sect. 5. Conclusions and discussions are presented in Sect. 6.

2 Linear stability and Hopf bifurcation analysis

This section reports the stability analysis of only one existing endemic equilibrium point E^* and the critical point τ_0 for the local Hopf bifurcation with the help of transversality condition.

Now by direct computation system (2) has a unique endemic equilibrium (S^*, L^*, B^*), where

$$S^* = \frac{\mu_1 + A_1 B^*}{A_2 + A_3 B^*}, \qquad L^* = \frac{\gamma_2 + \delta}{\alpha} B^*,$$

and B^* is the unique positive root of the equation

$$m_2 \bigl(B^*\bigr)^2 + m_1 B^* + m_0 = 0 \tag{3}$$

with

$$\begin{aligned} m_0 &= -\alpha(\mu_1\theta + \mu_2\delta + \mu_2\theta) < 0, \\ m_1 &= \alpha\delta A_1 + \delta(\alpha + \gamma_2 + \delta)A_2 - \alpha(\mu_1 + \mu_2)A_3, \\ m_2 &= \delta(\alpha + \gamma_2 + \delta)A_3 > 0, \end{aligned}$$

and

$$A_1 = \frac{\gamma_1(\gamma_2 + \delta)}{\alpha} + \gamma_2, \qquad A_2 = \delta + \theta, \qquad A_3 = \frac{\beta_1(\gamma_2 + \delta)}{\alpha} + \beta_2.$$

The product of the roots of B^* is $\frac{m_0}{m_2}$, which is clearly negative, and the discriminant of Eq. (3) is $m_1^2 - 4m_0 m_2$, which is also positive since $m_0 < 0$. Thus Eq. (3) has one positive and one negative root by Vieta's theorem.

The characteristic equation of system (2) at the endemic equilibrium is

$$\lambda^3 + p_2\lambda^2 + p_1\lambda + p_0 + \bigl(q_2\lambda^2 + q_1\lambda + q_0\bigr)e^{-\lambda\tau} + (r_1\lambda + r_0)e^{-2\lambda\tau} = 0, \tag{4}$$

where

$$\begin{aligned} p_0 &= a_1(a_6a_7 - a_5a_8) + a_4(a_2a_8 - a_3a_7), \\ p_1 &= a_1(a_5 + a_8) + a_5a_8 - a_2a_4 - a_6a_7, \\ p_2 &= -(a_1 + a_5 + a_8), \\ q_0 &= a_4(a_2b_4 + a_8b_1 - a_7b_2) - a_1(a_5b_4 - a_8b_3), \\ q_1 &= b_3(a_1 + a_8) + b_4(a_1 + a_5) - a_4b_1, \\ q_2 &= -(b_3 + b_4), \end{aligned}$$

$$r_0 = b_4(a_4 b_1 - a_1 b_3),$$
$$r_1 = b_3 b_4,$$

and

$$a_1 = -(\beta_1 L^* + \beta_2 B^* + \delta + \theta), \qquad a_2 = -\beta_1 S^*,$$
$$a_3 = -\beta_2 S^*, \qquad a_4 = \beta_1 L^* + \beta_2 B^* + \theta,$$
$$a_5 = \beta_1 S^* - (\alpha + \delta), \qquad a_6 = \beta_2 S^*,$$
$$a_7 = \alpha, \qquad a_8 = -\delta, \qquad b_1 = \gamma_1,$$
$$b_2 = \gamma_2, \qquad b_3 = -\gamma_1,$$
$$b_4 = -\gamma_2.$$

For $\tau = 0$, Eq. (4) becomes

$$\lambda^3 + (p_2 + q_2)\lambda^2 + (p_1 + q_1 + r_1)\lambda + p_0 + q_0 + r_0 = 0. \tag{5}$$

Lemma 2.1 *When $\tau = 0$, all the roots of Eq. (5) have negative real parts, and the endemic point $E^*(S^*, L^*, B^*)$ of system (2) is locally asymptotically stable (LAS).*

Multiplying both sides of Eq. (4) by $e^{\lambda\tau}$, we obtain

$$q_2\lambda^2 + q_1\lambda + q_0 + (\lambda^3 + p_2\lambda^2 + p_1\lambda + p_0)e^{\lambda\tau} + (r_1\lambda + r_0)e^{-\lambda\tau} = 0. \tag{6}$$

For $\tau > 0$, let $\lambda = i\omega$ ($\omega > 0$) be the root of Eq. (6). Then

$$(p_0 + r_0 - p_2\omega^2)\cos\tau\omega - ((p_1 - r_1)\omega - \omega^3)\sin\tau\omega = q_2\omega^2 - q_0,$$
$$(p_0 - r_0 - p_2\omega^2)\sin\tau\omega + ((p_1 + r_1)\omega - \omega^3)\cos\tau\omega = -q_1\omega.$$

Thus

$$\begin{aligned} \cos\tau\omega &= \frac{s_{14}\omega^4 + s_{12}\omega^2 + s_{10}}{\omega^6 + s_{04}\omega^4 + s_{02}\omega^2 + s_{00}}, \\ \sin\tau\omega &= \frac{s_{15}\omega^5 + s_{13}\omega^3 + s_{11}\omega}{\omega^6 + s_{04}\omega^4 + s_{02}\omega^2 + s_{00}}, \end{aligned} \tag{7}$$

where

$$s_{00} = p_0^2 - r_0^2, \qquad s_{02} = p_1^2 - r_1^2 - 2p_0p_2, \qquad s_{04} = p_2^2 - 2p_1, \qquad s_{10} = -(p_0 - r_0)q_0,$$
$$s_{12} = (p_0 - r_0)q_2 - (p_1 - r_1)q_1 + p_2q_0, \qquad s_{14} = q_1 - p_2q_2,$$
$$s_{11} = (p_1 + r_1)q_0 - (p_0 + r_0)q_1,$$
$$s_{13} = p_2q_1 - q_0 - (p_1 + r_1)q_2, \qquad s_{15} = q_2.$$

Therefore we obtain the equation

$$\omega^{12} + s_5\omega^{10} + s_4\omega^8 + s_3\omega^6 + s_2\omega^4 + s_1\omega^2 + s_0 = 0 \tag{8}$$

with

$$\begin{aligned}
s_0 &= s_{00}^2 - s_{10}^2,\\
s_1 &= 2(s_{00}s_{02} - s_{10}s_{12}) - s_{11}^2,\\
s_2 &= s_{02}^2 - s_{12}^2 + 2(s_{00}s_{04} - s_{10}s_{14} - s_{11}s_{13}),\\
s_3 &= 2(s_{00} + s_{02}s_{04} - s_{12}s_{14} - s_{11}s_{15}) - s_{13}^2,\\
s_4 &= s_{04}^2 - s_{14}^2 + 2(s_{02} - s_{13}s_{15}),\\
s_5 &= 2s_{04} - s_{15}^2.
\end{aligned}$$

Now we have (H_1): Eq. (8) has at least one positive root ω_0. Thus from Eq. (7) we have

$$\tau_0 = \frac{1}{\omega_0}\cos^{-1}\left\{\frac{s_{14}\omega_0^4 + s_{12}\omega_0^2 + s_{10}}{\omega_0^6 + s_{04}\omega_0^4 + s_{02}\omega_0^2 + s_{00}}\right\}.$$

Differentiating Eq. (6) with respect to τ, we obtain

$$\left[\frac{d\lambda}{d\tau}\right]^{-1} = \frac{2q_2\lambda + q_1 + (3\lambda^2 + 2p_2\lambda + p_1)e^{\lambda\tau} + r_1e^{-\lambda\tau}}{(r_1\lambda^2 + r_0\lambda)e^{-\lambda\tau} - (\lambda^4 + p_2\lambda^3 + p_1\lambda^2 + p_0\lambda)e^{\lambda\tau}} - \frac{\tau}{\lambda}.$$

Thus we get

$$\Re\left[\frac{d\lambda}{d\tau}\right]_{\tau=\tau_0}^{-1} = \frac{P_RQ_R + P_IQ_I}{Q_R^2 + Q_I^2},$$

where

$$\begin{aligned}
P_R &= \left(p_1 + r_1 - 3\omega_0^2\right)\cos\tau_0\omega_0 - 2p_2\omega_0\sin\tau_0\omega_0 + q_1,\\
P_I &= \left(p_1 - r_1 - 3\omega_0^2\right)\sin\tau_0\omega_0 + 2p_2\omega_0\cos\tau_0\omega_0 + 2q_2\omega_0,\\
Q_R &= \left(p_1\omega_0^2 - r_1\omega_0^2 - \omega_0^4\right)\cos\tau_0\omega_0 - \left(p_2\omega_0^3 - p_0\omega_0 + r_0\omega_0\right)\sin\tau_0\omega_0,\\
Q_I &= \left(p_1\omega_0^2 + r_1\omega_0^2 - \omega_0^4\right)\sin\tau_0\omega_0 + \left(p_2\omega_0^3 - p_0\omega_0 + r_0\omega_0\right)\cos\tau_0\omega_0.
\end{aligned}$$

The transversality condition holds if (H_2): $P_RQ_R + P_IQ_I \neq 0$. We have the following result [25].

Theorem 2.2 *Let an endemic point E^* of system (2) exist, and let conditions (H_1) and (H_2) be satisfied. Then it is LAS at $\tau \in [0, \tau_0)$ and unstable for $\tau > \tau_0$. Furthermore, system (2) undergoes Hopf bifurcation at E^* when $\tau = \tau_0$, and a family of periodic solutions bifurcate from $E^*(S^*, L^*, B^*)$ near $\tau = \tau_0$.*

3 Global stability analysis

This section deals with the nonlinear or global stability analysis by constructing suitable Lyapunov function for the endemic equilibrium point of the delayed model system (2).

Theorem 3.1 *If* $\min\{l_1, l_2, l_3\} > 0$ *with*

$$\begin{aligned} l_1 &= \frac{1}{M_1 S^*}(\mu_1 + \gamma_1 L^* + \gamma_2 B^*) - \left(1 + \frac{\gamma_1 M_2 \tau}{m_2}\right)\left(\beta_1 + \frac{\beta_2 B^*}{m_2} + \frac{\theta}{m_2}\right), \\ l_2 &= \left(\beta_1 - \frac{\gamma_1}{m_1}\right) - \frac{2\gamma_1^2 M_2 \tau}{m_2^2} + \frac{1}{L^*}\left(\frac{1}{M_2} - \frac{\gamma_1 M_2 \tau}{m_2^2}\right)(\mu_2 + \beta_2 B^* S^* + \theta S^*) \\ &\quad - \frac{\alpha}{m_3}\left(1 + \frac{\gamma_2 M_3 \tau}{m_3}\right), \\ l_3 &= \left(\frac{\gamma_2}{m_1} - \beta_2\right) + \frac{\beta_2 M_1}{m_2}\left(1 + \frac{\gamma_1 M_2 \tau}{m_2}\right) - \frac{\alpha L^*}{B^*}\left(\frac{1}{M_3} - \frac{\gamma_2 M_3 \tau}{m_3^2}\right) + \frac{2\gamma_2^2 M_3 \tau}{m_3^2}, \end{aligned} \tag{9}$$

where $m_1 < S(t) < M_1$, $m_2 < L(t) < M_2$, *and* $m_3 < B(t) < M_3$ *for* $t > 0$, *then the endemic equilibrium* $E^*(S^*, L^*, B^*)$ *of system* (2) *is globally asymptotically stable* (*GAS*).

Proof First, we construct a proper Lyapunov function to derive a sufficient condition that guarantees that the endemic point E^* of the model system (2) is globally asymptotically stable. Let

$$S(t) = S^* e^{u(t)}, \qquad L(t) = L^* e^{v(t)}, \qquad B(t) = B^* e^{w(t)}. \tag{10}$$

Now the endemic equilibrium E^* transforms into trivial equilibrium $u(t) = v(t) = w(t) = 0$ for all $t > 0$. Thus system (2) is reduced as follows:

$$\begin{aligned} \frac{du}{dt} &= -\frac{1}{S}(\mu_1 + \gamma_1 L^* + \gamma_2 B^*)\left(e^{u(t)} - 1\right) - \beta_1 L^*\left(e^{v(t)} - 1\right) - \beta_2 B^*\left(e^{w(t)} - 1\right) \\ &\quad + \gamma_1 \frac{L^*}{S}\left(e^{v(t-\tau)} - 1\right) + \gamma_2 \frac{B^*}{S}\left(e^{w(t-\tau)} - 1\right), \end{aligned} \tag{11}$$

$$\begin{aligned} \frac{dv}{dt} &= S^*\left(\beta_1 + \beta_2 B^* S^* + \frac{\theta}{L}\right)\left(e^{u(t)} - 1\right) - \frac{1}{L}(\mu_2 + \beta_2 B^* S^* - \gamma_1 L^* + \theta S^*)\left(e^{v(t)} - 1\right) \\ &\quad + \beta_2 \frac{B^* S(t)}{L}\left(e^{w(t)} - 1\right) - \gamma_1 \frac{L^*}{L}\left(e^{v(t-\tau)} - 1\right), \end{aligned} \tag{12}$$

$$\frac{dw}{dt} = \alpha \frac{L^*}{B}\left(e^{v(t)} - 1\right) - \alpha \frac{L^*}{B}\left(e^{w(t)} - 1\right) - \gamma_2 \frac{B^*}{B}\left(e^{w(t-\tau)} - 1\right) + \gamma_2 \frac{B^*}{B}\left(e^{w(t)} - 1\right). \tag{13}$$

Let $V_1(t) = |u(t)|$. Now calculate the derivative of $V_1(t)$ with the solution of (2). It follows from Eq. (11) that

$$\begin{aligned} D^+ V_1 \leq &- \frac{1}{M_1}(\mu_1 + \gamma_1 L^* + \gamma_2 B^*)\left|e^{u(t)} - 1\right| - \beta_1 L^*\left|e^{v(t)} - 1\right| - \beta_2 B^*\left|e^{w(t)} - 1\right| \\ &+ \gamma_1 \frac{L^*}{m_1}\left|e^{v(t-\tau)} - 1\right| + \gamma_2 \frac{B^*}{m_1}\left|e^{w(t-\tau)} - 1\right|. \end{aligned} \tag{14}$$

Again, due to form of (14), we consider the functional

$$V_{11}(t) \le V_1(t) + \frac{\gamma_1 L^*}{m_1}\int_{t-\tau}^{t}\left|e^{v(s)}-1\right|ds + \frac{\gamma_2 B^*}{m_1}\int_{t-\tau}^{t}\left|e^{w(s)}-1\right|ds,$$

whose derivative along the solution of system (2) is given by

$$\begin{aligned} D^+V_{11}(t) \le{}& D^+V_1(t) + \frac{\gamma_1 L^*}{m_1}\left(\left|e^{v(t)}-1\right| - \left|e^{v(t-\tau)}-1\right|\right) \\ &+ \frac{\gamma_2 B^*}{m_1}\left(\left|e^{w(t)}-1\right| - \left|e^{w(t-\tau)}-1\right|\right) \\ \le{}& -\frac{1}{M_1}\left(\mu_1+\gamma_1 L^*+\gamma_2 B^*\right)\left|e^{u(t)}-1\right| - \beta_1 L^*\left|e^{v(t)}-1\right| \\ &- \beta_2 B^*\left|e^{w(t)}-1\right| \\ &+ \frac{\gamma_1 L^*}{m_1}\left|e^{v(t)}-1\right| + \frac{\gamma_2 B^*}{m_1}\left|e^{w(t)}-1\right| \\ ={}& -\frac{1}{M_1}\left(\mu_1+\gamma_1 L^*+\gamma_2 B^*\right)\left|e^{u(t)}-1\right| + L^*\left(\frac{\gamma_1}{m_1}-\beta_1\right)\left|e^{v(t)}-1\right| \\ &+ B^*\left(\frac{\gamma_2}{m_1}-\beta_2\right)\left|e^{w(t)}-1\right|. \end{aligned} \tag{15}$$

Now Eq. (12) can be written as

$$\begin{aligned} \frac{dv}{dt} ={}& S^*\left(\beta_1+\frac{\beta_2 B^*}{L}+\frac{\theta}{L}\right)\left(e^{u(t)}-1\right) - \frac{1}{L}\left(\mu_2+\beta_2 B^* S^* - \gamma_1 L^* + \theta S^*\right)\left(e^{v(t)}-1\right) \\ &- \gamma_1\frac{L^*}{L}\left(e^{v(t)} - \int_{t-\tau}^{t} e^{v(s)}\frac{dv}{ds}\,ds - 1\right) + \beta_2\frac{B^* S(t)}{L}\left(e^{w(t)}-1\right) \\ ={}& S^*\left(\beta_1+\frac{\beta_2 B^*}{L}+\frac{\theta}{L}\right)\left(e^{u(t)}-1\right) - \frac{1}{L}\left(\mu_2+\beta_2 B^* S^* - \gamma_1 L^* + \theta S^*\right)\left(e^{v(t)}-1\right) \\ &- \gamma_1\frac{L^*}{L}\left(e^{v(t)}-1\right) + \beta_2\frac{B^* S(t)}{L}\left(e^{w(t)}-1\right) + \gamma_1\frac{L^*}{L}\int_{t-\tau}^{t} e^{v(s)}\frac{dv}{ds}\,ds \\ ={}& S^*\left(\beta_1+\frac{\beta_2 B^*}{L}+\frac{\theta}{L}\right)\left(e^{u(t)}-1\right) - \frac{1}{L}\left(\mu_2+\beta_2 B^* S^* + \theta S^*\right)\left(e^{v(t)}-1\right) \\ &+ \beta_2\frac{B^* S(t)}{L}\left(e^{w(t)}-1\right) + \gamma_1\frac{L^*}{L}\int_{t-\tau}^{t} e^{v(s)}\left\{S^*\left(\beta_1+\frac{\beta_2 B^*}{L}+\frac{\theta}{L}\right)\left(e^{u(s)}-1\right)\right. \\ &- \frac{1}{L}\left(\mu_2+\beta_2 B^* S^* - \gamma_1 L^* + \theta S^*\right)\left(e^{v(s)}-1\right) + \beta_2\frac{B^* S(t)}{L}\left(e^{w(s)}-1\right) \\ &\left. - \gamma_1\frac{L^*}{L}\left(e^{v(s-\tau)}-1\right)\right\}ds, \end{aligned} \tag{16}$$

where we used the following relation:

$$e^{v(t-\tau)} = e^{v(t)} - \int_{t-\tau}^{t} e^{v(s)}\frac{dv}{ds}\,ds.$$

Let $V_2(t) = |v(t)|$. Computing the derivative of $V_2(t)$ along the solution of (2), from Eq. (16) it follows that

$$\begin{aligned}
D^+V_2 \le{}& S^*\left(\beta_1 + \frac{\beta_2 B^*}{m_2} + \frac{\theta}{m_2}\right)\left|e^{u(t)} - 1\right| - \frac{1}{M_2}\left(\mu_2 + \beta_2 B^* S^* + \theta S^*\right)\left|e^{v(t)} - 1\right| \\
&+ \beta_2 \frac{B^* M_1}{m_2}\left|e^{w(t)} - 1\right| + \gamma_1 \frac{L^*}{m_2}\int_{t-\tau}^{t} e^{v(s)}\left\{S^*\left(\beta_1 + \frac{\beta_2 B^*}{m_2} + \frac{\theta}{m_2}\right)\left|e^{u(s)} - 1\right|\right. \\
&+ \frac{1}{m_2}\left(\mu_2 + \beta_2 B^* S^* + \gamma_1 L^* + \theta S^*\right)\left|e^{v(s)} - 1\right| + \beta_2 \frac{B^* M_1}{m_2}\left|e^{w(s)} - 1\right| \\
&\left.+ \gamma_1 \frac{L^*}{m_2}\left|e^{v(s-\tau)} - 1\right|\right\} ds.
\end{aligned}$$

We find that there exists $t_1 > 0$ such that $L^* e^{v(t)} < M_2$ for all $t > t_1$, and for $t > t_1 + \tau$, we have

$$\begin{aligned}
D^+V_2 \le{}& S^*\left(\beta_1 + \frac{\beta_2 B^*}{m_2} + \frac{\theta}{m_2}\right)\left|e^{u(t)} - 1\right| - \frac{1}{M_2}\left(\mu_2 + \beta_2 B^* S^* + \theta S^*\right)\left|e^{v(t)} - 1\right| \\
&+ \beta_2 \frac{B^* M_1}{m_2}\left|e^{w(t)} - 1\right| + \gamma_1 \frac{M_2}{m_2}\int_{t-\tau}^{t} \{S^*\left(\beta_1 + \frac{\beta_2 B^*}{m_2} + \frac{\theta}{m_2}\right)\left|e^{u(s)} - 1\right| \\
&+ \frac{1}{m_2}\left(\mu_2 + \beta_2 B^* S^* + \gamma_1 L^* + \theta S^*\right)\left|e^{v(s)} - 1\right| + \beta_2 \frac{B^* M_1}{m_2}\left|e^{w(s)} - 1\right| \\
&+ \gamma_1 \frac{L^*}{m_2}\left|e^{v(s-\tau)} - 1\right|\} ds. \qquad (17)
\end{aligned}$$

Again, due to the form of (17), we consider the functional

$$\begin{aligned}
V_{22}(t) \le{}& V_2(t) + \gamma_1 \frac{M_2}{m_2}\int_{t-\tau}^{t}\int_{L}^{t}\left\{S^*\left(\beta_1 + \frac{\beta_2 B^*}{m_2} + \frac{\theta}{m_2}\right)\left|e^{u(s)} - 1\right|\right. \\
&+ \frac{1}{m_2}\left(\mu_2 + \beta_2 B^* S^* + \gamma_1 L^* + \theta S^*\right)\left|e^{v(s)} - 1\right| + \beta_2 \frac{B^* M_1}{m_2}\left|e^{w(s)} - 1\right| \\
&\left.+ \gamma_1 \frac{L^*}{m_2}\left|e^{v(s-\tau)} - 1\right|\right\} ds\, dL + \frac{\gamma_1^2 L^* M_2 \tau}{m_2^2}\int_{t-\tau}^{t}\left|e^{v(s)} - 1\right| ds,
\end{aligned}$$

whose right derivative along the solution of system (2) is given by

$$\begin{aligned}
D^+V_{22}(t) \le{}& D^+V_2(t) + \frac{\gamma_1 M_2 \tau}{m_2}\left\{S^*\left(\beta_1 + \frac{\beta_2 B^*}{m_2} + \frac{\theta}{m_2}\right)\left|e^{u(t)} - 1\right| + \frac{\beta_2 B^* M_1}{m_2}\left|e^{w(t)} - 1\right|\right. \\
&\left.+ \frac{1}{m_2}\left(\mu_2 + \beta_2 B^* S^* + \gamma_1 L^* + \theta S^*\right)\left|e^{v(t)} - 1\right| + \frac{\gamma_1 L^*}{m_2}\left|e^{v(t-\tau)} - 1\right|\right\} \\
&- \frac{\gamma_1 M_2}{m_2}\int_{t-\tau}^{t}\left\{S^*\left(\beta_1 + \frac{\beta_2 B^*}{m_2} + \frac{\theta}{m_2}\right)\left|e^{u(s)} - 1\right| + \frac{\beta_2 B^* M_1}{m_2}\left|e^{w(s)} - 1\right|\right. \\
&\left.+ \frac{1}{m_2}\left(\mu_2 + \beta_2 B^* S^* + \gamma_1 L^* + \theta S^*\right)\left|e^{v(s)} - 1\right| + \frac{\gamma_1 L^*}{m_2}\left|e^{v(s-\tau)} - 1\right|\right\} ds \\
&+ \frac{\gamma_1^2 L^* M_2 \tau}{m_2^2}\left(\left|e^{v(t)} - 1\right| - \left|e^{v(t-\tau)} - 1\right|\right) \\
\le{}& S^*\left(\beta_1 + \frac{\beta_2 B^*}{m_2} + \frac{\theta}{m_2}\right)\left|e^{u(t)} - 1\right| - \frac{1}{M_2}\left(\mu_2 + \beta_2 B^* S^* + \theta S^*\right)\left|e^{v(t)} - 1\right|
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\beta_2 B^* M_1}{m_2}\left|e^{w(t)}-1\right|+\frac{\gamma_1 M_2 \tau}{m_2}\left\{S^*\left(\beta_1+\frac{\beta_2 B^*}{m_2}+\frac{\theta}{m_2}\right)\left|e^{u(t)}-1\right|\right.\\
&+\frac{\beta_2 B^* M_1}{m_2}\\
&\left.\times\left|e^{w(t)}-1\right|+\frac{1}{m_2}\left(\mu_2+\beta_2 B^* S^*+\gamma_1 L^*+\theta S^*\right)\left|e^{v(t)}-1\right|\right\}\\
&+\frac{\gamma_1^2 L^* M_2 \tau}{m_2^2}\left|e^{v(t)}-1\right|\\
={}& S^*\left(1+\frac{\gamma_1 M_2 \tau}{m_2}\right)\left(\beta_1+\frac{\beta_2 B^*}{m_2}+\frac{\theta}{m_2}\right)\left|e^{u(t)}-1\right|+\frac{2\gamma_1^2 L^* M_2 \tau}{m_2^2}\left|e^{v(t)}-1\right|\\
&+\left(\frac{\gamma_1 M_2 \tau}{m_2^2}-\frac{1}{M_2}\right)\left(\mu_2+\beta_2 B^* S^*+\theta S^*\right)\left|e^{v(t)}-1\right|\\
&+\frac{\beta_2 B^* M_1}{m_2}\left(1+\frac{\gamma_1 M_2 \tau}{m_2}\right)\left|e^{w(t)}-1\right|. \qquad (18)
\end{aligned}
$$

Now Eq. (13) can be rewritten as

$$
\begin{aligned}
\frac{dw}{dt} ={}& \alpha\frac{L^*}{B}\left(e^{v(t)}-1\right)-\alpha\frac{L^*}{B}\left(e^{w(t)}-1\right)-\gamma_2\frac{B^*}{B}\left(e^{w(t)}-\int_{t-\tau}^{t} e^{w(s)}\frac{dw}{ds}\,ds-1\right)\\
&+\gamma_2\frac{B^*}{B}\left(e^{w(t)}-1\right)\\
={}& \alpha\frac{L^*}{B}\left(e^{v(t)}-1\right)-\alpha\frac{L^*}{B}\left(e^{w(t)}-1\right)+\gamma_2\frac{B^*}{B}\int_{t-\tau}^{t} e^{w(s)}\frac{dw}{ds}\,ds\\
={}& \alpha\frac{L^*}{B}\left(e^{v(t)}-1\right)-\alpha\frac{L^*}{B}\left(e^{w(t)}-1\right)+\gamma_2\frac{B^*}{B}\int_{t-\tau}^{t} e^{w(s)}\left\{\alpha\frac{L^*}{B}\left(e^{v(s)}-1\right)\right.\\
&\left.+\left(\gamma_2\frac{B^*}{B}-\alpha\frac{L^*}{B}\right)\left(e^{w(s)}-1\right)-\gamma_2\frac{B^*}{B}\left(e^{w(s-\tau)}-1\right)\right\}ds, \qquad (19)
\end{aligned}
$$

where we used the relation

$$
e^{w(t-\tau)} = e^{w(t)} - \int_{t-\tau}^{t} e^{w(s)}\frac{dw}{ds}\,ds.
$$

Let $V_3(t) = |w(t)|$. Computing the right derivative of $V_3(t)$ along the solution of (2), from Eq. (19) it follows that

$$
\begin{aligned}
D^+V_3(t) ={}& \frac{\alpha L^*}{m_3}\left|e^{v(t)}-1\right|-\frac{\alpha L^*}{M_3}\left|e^{w(t)}-1\right|+\frac{\gamma_2 B^*}{m_3}\int_{t-\tau}^{t} e^{w(s)}\left\{\frac{\alpha L^*}{m_3}\left|e^{v(s)}-1\right|\right.\\
&\left.+\left(\frac{\gamma_2 B^*}{m_3}+\frac{\alpha L^*}{m_3}\right)\left|e^{w(s)}-1\right|+\frac{\gamma_2 B^*}{m_3}\left|e^{w(s-\tau)}-1\right|\right\}ds.
\end{aligned}
$$

We find that there exists $t_1 > 0$ such that $B^* e^{w(t)} < M_3$ for all $t > t_1$. and for $t > t_1 + \tau$, we have

$$
\begin{aligned}
D^+V_3(t) \le{}& \frac{\alpha L^*}{m_3}\left|e^{v(t)}-1\right|-\frac{\alpha L^*}{M_3}\left|e^{w(t)}-1\right|+\frac{\gamma_2 M_3}{m_3}\int_{t-\tau}^{t}\left\{\frac{\alpha L^*}{m_3}\left|e^{v(s)}-1\right|\right.\\
&\left.+\left(\frac{\gamma_2 B^*}{m_3}+\frac{\alpha L^*}{m_3}\right)\left|e^{w(s)}-1\right|+\frac{\gamma_2 B^*}{m_3}\left|e^{w(s-\tau)}-1\right|\right\}ds. \qquad (20)
\end{aligned}
$$

Again, due to the structure of (20), we consider the functional

$$V_{33}(t) = V_3(t) + \frac{\gamma_2 M_3}{m_3}\int_{t-\tau}^{t}\int_{B}^{t}\left\{\frac{\alpha L^*}{m_3}\left|e^{v(s)}-1\right| + \left(\frac{\gamma_2 B^*}{m_3}+\frac{\alpha L^*}{m_3}\right)\left|e^{w(s)}-1\right| + \frac{\gamma_2 B^*}{m_3}\left|e^{w(s-\tau)}-1\right|\right\}\,ds\,dB + \frac{\gamma_2^2 M_3 B^*\tau}{m_3^2}\int_{t-\tau}^{t}\left|e^{w(s)}-1\right|\,ds,$$

whose upper right derivative along the solution of system (2) is given by

$$\begin{aligned}
D^+V_{33}(t) &\le D^+V_3(t) + \frac{\gamma_2 M_3\tau}{m_3}\left\{\frac{\alpha L^*}{m_3}\left|e^{v(t)}-1\right| + \left(\frac{\gamma_2 B^*}{m_3}+\frac{\alpha L^*}{m_3}\right)\left|e^{w(t)}-1\right|\right. \\
&\quad \left. + \frac{\gamma_2 B^*}{m_3}\left|e^{w(t-\tau)}-1\right|\right\} - \frac{\gamma_2 M_3}{m_3}\int_{t-\tau}^{t}\left\{\frac{\alpha L^*}{m_3}\left|e^{v(s)}-1\right| + \left(\frac{\gamma_2 B^*}{m_3}+\frac{\alpha L^*}{m_3}\right)\right. \\
&\quad \left.\times\left|e^{w(s)}-1\right| + \frac{\gamma_2 B^*}{m_3}\left|e^{w(s-\tau)}-1\right|\right\}ds \\
&\quad + \frac{\gamma_2^2 M_3 B^*\tau}{m_3^2}\left(\left|e^{w(t)}-1\right| - \left|e^{w(t-\tau)}-1\right|\right) \\
&\le \frac{\alpha L^*}{m_3}\left|e^{v(t)}-1\right| - \frac{\alpha L^*}{M_3}\left|e^{w(t)}-1\right| \\
&\quad + \frac{\gamma_2 M_3\tau}{m_3}\left\{\frac{\alpha L^*}{m_3}\left|e^{v(t)}-1\right| + \left(\frac{\gamma_2 B^*}{m_3}+\frac{\alpha L^*}{m_3}\right)\left|e^{w(t)}-1\right|\right\} \\
&\quad + \frac{\gamma_2^2 M_3 B^*\tau}{m_3^2}\left|e^{w(t)}-1\right| \\
&= \frac{\alpha L^*}{m_3}\left(1+\frac{\gamma_2 M_3\tau}{m_3}\right)\left|e^{v(t)}-1\right| - \alpha L^*\left(\frac{1}{M_3}-\frac{\gamma_2 M_3\tau}{m_3^2}\right)\left|e^{w(t)}-1\right| \\
&\quad + \frac{2\gamma_2^2 M_3 B^*\tau}{m_3^2}\left|e^{w(t)}-1\right|.
\end{aligned} \tag{21}$$

Let us define the Lyapunov functional

$$V(t) = V_{11}(t) + V_{22}(t) + V_{33} > \left|u(t)\right| + \left|v(t)\right| + \left|w(t)\right|.$$

Computing the upper right derivative of $V(t)$ along the solution of the system (2),] and using (15), (18), and (21), we obtain

$$\begin{aligned}
D^+V(t) &= D^+V_{11}(t) + D^+V_{22}(t) + D^+V_{33}(t) \\
&\le -\frac{1}{M_1}\left(\mu_1+\gamma_1 L^*+\gamma_2 B^*\right)\left|e^{u(t)}-1\right| + L^*\left(\frac{\gamma_1}{m_1}-\beta_1\right)\left|e^{v(t)}-1\right| \\
&\quad + B^*\left(\frac{\gamma_2}{m_1}-\beta_2\right)\left|e^{w(t)}-1\right| + S^*\left(1+\frac{\gamma_1 M_2\tau}{m_2}\right)\left(\beta_1+\frac{\beta_2 B^*}{m_2}+\frac{\theta}{m_2}\right)\left|e^{u(t)}-1\right| \\
&\quad + \frac{2\gamma_1^2 L^* M_2\tau}{m_2^2}\left|e^{v(t)}-1\right| + \left(\frac{\gamma_1 M_2\tau}{m_2^2}-\frac{1}{M_2}\right)\left(\mu_2+\beta_2 B^* S^*+\theta S^*\right)\left|e^{v(t)}-1\right| \\
&\quad + \frac{\beta_2 B^* M_1}{m_2}\left(1+\frac{\gamma_1 M_2\tau}{m_2}\right)\left|e^{w(t)}-1\right| + \frac{\alpha L^*}{m_3}\left(1+\frac{\gamma_2 M_3\tau}{m_3}\right)\left|e^{v(t)}-1\right|
\end{aligned}$$

$$
\begin{aligned}
&\quad - \alpha L^*\left(\frac{1}{M_3} - \frac{\gamma_2 M_3 \tau}{m_3^2}\right)\left|e^{w(t)} - 1\right| + \frac{2\gamma_2^2 M_3 B^* \tau}{m_3^2}\left|e^{w(t)} - 1\right| \\
&= -S^*\left\{\frac{\mu_1 + \gamma_1 L^* + \gamma_2 B^*}{M_1 S^*} - \left(1 + \frac{\gamma_1 M_2 \tau}{m_2}\right)\left(\beta_1 + \frac{\beta_2 B^*}{m_2} + \frac{\theta}{m_2}\right)\right\}\left|e^{u(t)} - 1\right| \\
&\quad - L^*\left\{\left(\beta_1 - \frac{\gamma_1}{m_1}\right) - \frac{2\gamma_1^2 M_2 \tau}{m_2^2} + \frac{1}{L^*}\left(\frac{1}{M_2} - \frac{\gamma_1 M_2 \tau}{m_2^2}\right)\left(\mu_2 + \beta_2 B^* S^* + \theta S^*\right)\right. \\
&\quad \left. - \frac{\alpha}{m_3}\left(1 + \frac{\gamma_2 M_3 \tau}{m_3}\right)\right\}\left|e^{v(t)} - 1\right| - B^*\left\{\left(\frac{\gamma_2}{m_1} - \beta_2\right) + \frac{\beta_2 M_1}{m_2}\left(1 + \frac{\gamma_1 M_2 \tau}{m_2}\right)\right. \\
&\quad \left. - \frac{\alpha L^*}{B^*}\left(\frac{1}{M_3} - \frac{\gamma_2 M_3 \tau}{m_3^2}\right) + \frac{2\gamma_2^2 M_3 \tau}{m_3^2}\right\}\left|e^{w(t)} - 1\right| \\
&= -S^* l_1\left|e^{u(t)} - 1\right| - L^* l_2\left|e^{v(t)} - 1\right| - B^* l_3\left|e^{w(t)} - 1\right|,
\end{aligned}
$$

where l_1, l_2, and l_3 are defined in (9).

Since the model system (2) is positive invariant, for all $t > t_1^*$, we have

$$
\begin{aligned}
&S^* e^{u(t)} = S(t) > \underline{S}, \\
&L^* e^{v(t)} = L(t) > \underline{L}, \\
&B^* e^{w(t)} = B(t) > \underline{B}.
\end{aligned}
$$

Using the mean value theorem, we have

$$
\begin{aligned}
&S^*\left|e^{u(t)} - 1\right| = S^* e^{\theta_1(t)}\left|u(t)\right| > m_1\left|u(t)\right|, \\
&L^*\left|e^{v(t)} - 1\right| = L^* e^{\theta_2(t)}\left|v(t)\right| > m_2\left|v(t)\right|, \\
&B^*\left|e^{w(t)} - 1\right| = B^* e^{\theta_3(t)}\left|w(t)\right| > m_3\left|w(t)\right|,
\end{aligned}
$$

where $S^* e^{\theta_1(t)}$ lies between S^* and $S(t)$, $L^* e^{\theta_2(t)}$ lies between L^* and $L(t)$, and $B^* e^{\theta_3(t)}$ lies between B^* and $B(t)$. Therefore

$$
\begin{aligned}
D^+ V(t) &\le -l_1 \underline{S}\left|u(t)\right| - l_2 \underline{L}\left|v(t)\right| - l_3 \underline{B}\left|w(t)\right| \\
&\le -l\left(\left|u(t)\right| + \left|v(t)\right| + \left|w(t)\right|\right), \quad \text{where } l = \min\{l_1\underline{S}, l_2\underline{L}, l_3\underline{B}\}.
\end{aligned}
\tag{22}
$$

Note that $V(t) > |u(t)| + |v(t)| + |w(t)|$. Hence from theory of global stability and Eq. (22) we conclude that the zero solution of the reduced system (11)–(13) is GAS. Therefore the endemic equilibrium E^* of model system (2) is GAS. □

4 Stability and direction of Hopf bifurcation

In this section, we discuss the stability and direction of Hopf bifurcation using theory of normal form and center manifold [40] of the delayed system (2). Let $x_1 = S - S^*$, $x_2 = I - I^*$, $x_3 = T - T^*$, and $x_i(t) = x_i(\tau t)$ for $i = 1, 2, 3$. Then the delay system (2) is converted to the following functional differential equation in $\mathbb{C} = \mathbb{C}([-1, 0], \mathbb{R}^3)$:

$$
\dot{x} = L_\mu(x_t) + F(\mu, x_t), \tag{23}
$$

where $x(t) = (x_1(t), x_2(t), x_3(t))^T \in \mathbb{C}$, $x_t(\theta) = x(t+\theta)$, $\theta \in [-1,0]$, and $L_\mu : \mathbb{C} \to \mathbb{R}^3$, $F : \mathbb{R} \times \mathbb{C} \to \mathbb{R}^3$ are given by

$$L_\mu(\phi) = (\tau_0 + \mu)\big[J_1\phi(0) + J_2\phi(-1)\big] \tag{24}$$

with

$$J_1 = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ 0 & a_7 & a_8 \end{pmatrix},$$
$$J_2 = \begin{pmatrix} 0 & b_1 & b_2 \\ 0 & b_3 & 0 \\ 0 & 0 & b_4 \end{pmatrix}, \tag{25}$$
$$f(\mu,\phi) = (\tau_0 + \mu)\begin{pmatrix} -\phi_1(0)(\beta_1\phi_2(0) + \beta_2\phi_3(0)) \\ \phi_1(0)(\beta_1\phi_2(0) + \beta_2\phi_3(0)) \\ 0 \end{pmatrix}.$$

By the Riesz representation theorem there exists a 3×3 matrix-valued function $\eta(\theta,\mu)$ with components of bounded variation for $\theta \in [-1,0]$ such that

$$L_\mu\phi = \int_{-1}^{0} d\eta(\theta,\mu)\phi(\theta), \quad \forall\phi \in \mathbb{C}.$$

By considering Eq. (24) we can choose

$$\eta(\theta,\mu) = (\tau_0 + \mu)\big[J_1\delta(\theta) + J_2\delta(\theta+1)\big],$$

where $\delta(\theta)$ is the Dirac delta function.

For $\phi \in \mathbb{C}^1([-1,0],\mathbb{R}^3)$, define

$$A(\mu)\phi(\theta) = \begin{cases} \frac{d\phi(\theta)}{d\theta}, & \theta \in [-1,0), \\ \int_{-1}^{0} d\eta(s,\mu)\phi(s) = L_\mu\phi, & \theta = 0, \end{cases} \tag{26}$$

and

$$R(\mu)\phi(\theta) = \begin{cases} 0, & \theta \in [-1,0), \\ F(\mu,\phi), & \theta = 0. \end{cases}$$

Now system (23) becomes

$$\dot{x}_t = A(\mu)x_t + R(\mu)x_t, \tag{27}$$

where $x_t(\theta) = x(t+\theta)$ for $\theta \in [-1,0]$.

For $\psi \in \mathbb{C}^1([0,1],(\mathbb{R}^3)^*)$, define

$$A^*(\mu)\psi(s) = \begin{cases} -\frac{d\psi(s)}{ds}, & s \in (0,1], \\ \int_{-1}^{0} d\eta(t,0)\psi(-t), & s = 0, \end{cases}$$

and the bilinear inner product

$$\big\langle \psi(s), \phi(\theta)\big\rangle = \bar{\psi}(0).\phi(0) - \int_{\theta=-1}^{0}\int_{\xi=0}^{\theta} \bar{\psi}^{T}(\xi-\theta)\,d\eta(\theta)\phi(\xi)\,d\xi, \tag{28}$$

where $\eta(\theta) = \eta(\theta,0)$. Thus $A(0)$ and $A^*(0)$ are adjoint operators, and $\pm i\omega_0\tau_0$ are their eigenvalues of $A(0)$. Let $v(\theta) = (1, v_1, v_2)^T e^{i\omega_0\tau_0\theta}$ and $v^*(s) = P(1, v_1^*, v_2^*)^T e^{i\omega_0\tau_0 s}$ be the eigenvectors of $A(0)$ and $A^*(0)$ at $\theta = 0$ corresponding to the eigenvalues $i\omega_0\tau_0$ and $-i\omega_0\tau_0$ respectively.

Thus we obtain

$$\begin{aligned}
v_1 &= \frac{a_4(a_8 + b_4e^{-i\omega_0\tau_0} - i\omega_0)}{a_6a_7 - (a_5 + b_3e^{-i\omega_0\tau_0} - i\omega_0)(a_8 + b_4e^{-i\omega_0\tau_0} - i\omega_0)},\\
v_2 &= \frac{-a_7v_1}{a_8 + b_4e^{-i\omega_0\tau_0} - i\omega_0},\\
v_1^* &= -\frac{a_1 + i\omega_0}{a_4},\\
v_2^* &= -\frac{a_2 + b_1e^{i\omega_0\tau_0} + (a_5 + b_3e^{i\omega_0\tau_0} + i\omega_0)v_1^*}{a_7}.
\end{aligned}$$

From Eq. (28) we have

$$\big\langle v^*(s), v(\theta)\big\rangle = \bar{P}\big\{\big(1 + v_1\bar{v_1}^* + v_2\bar{v_2}^*\big) + \tau_0\big(v_1\big(b_1 + b_3\bar{v_1}^*\big) + v_2\big(b_2 + b_4\bar{v_2}^*\big)\big)e^{-i\omega_0\tau_0}\big\}.$$

Using the normalization condition $\langle v^*(s), v(\theta)\rangle = 1$, we obtain

$$\bar{P} = \big[\big(1 + v_1\bar{v_1}^* + v_2\bar{v_2}^*\big) + \tau_0\big(v_1\big(b_1 + b_3\bar{v_1}^*\big) + v_2\big(b_2 + b_4\bar{v_2}^*\big)\big)e^{-i\omega_0\tau_0}\big]^{-1}.$$

By the same method we can easily prove that $\langle v^*, \bar{v}\rangle = 0$, thus omitting it. Now we obtain v and v^*.

Following the same steps as in [40], we obtain the following expressions:

$$\begin{aligned}
g_{20} &= 2\tau_0\bar{P}\big(\bar{v_1^*} - 1\big)(\beta_1v_1 + \beta_2v_2),\\
g_{11} &= 2\tau_0\bar{P}\big(\bar{v_1^*} - 1\big)\big(\beta_1(v_1 + \bar{v_1}) + \beta_2(v_2 + \bar{v_2})\big),\\
g_{02} &= 2\tau_0\bar{P}\big(\bar{v_1^*} - 1\big)(\beta_1\bar{v_1} + \beta_2\bar{v_2}),\\
g_{21} &= \tau_0\bar{P}\big(\bar{v_1^*} - 1\big)((\beta_1\bar{v_1} + \beta_2\bar{v_2})W_{20}^{(1)}(0) + 2(\beta_1v_1 + \beta_2v_2)W_{11}^{(1)}(0)\\
&\quad + \beta_1\big(W_{20}^{(2)}(0) + 2W_{11}^{(2)}(0)\big) + \beta_2\big(W_{20}^{(3)}(0) + 2W_{11}^{(3)}(0)\big)
\end{aligned} \tag{29}$$

with

$$\begin{aligned}
W_{20}(\theta) &= \frac{ig_{20}}{\omega_0\tau_0}v(\theta) + \frac{i\bar{g}_{02}}{3\omega_0\tau_0}\bar{v}(\theta) + E_1e^{2i\omega_0\tau_0\theta},\\
W_{11}(\theta) &= -\frac{ig_{11}}{\omega_0\tau_0}v(\theta) + \frac{i\bar{g}_{11}}{\omega_0\tau_0}\bar{v}(\theta) + E_2,
\end{aligned} \tag{30}$$

where $E_1 = (E_1^{(1)}, E_1^{(2)}, E_1^{(3)})^T$ and $E_2 = (E_2^{(1)}, E_2^{(2)}, E_2^{(3)})^T \in \mathbb{R}^3$ are further given constant vectors.

Solving this system for E_1, we obtain

$$\begin{aligned}
E_1^{(1)} &= \frac{2}{\tilde{A}}\begin{vmatrix} -h_1 & -a_2 - b_1 e^{-2i\omega_0\tau_0} & h_2 \\ h_1 & 2i\omega_0 - a_5 - b_3 e^{-2i\omega_0\tau_0} & -a_6 \\ 0 & -a_7 & h_3 \end{vmatrix}, \\
E_1^{(2)} &= \frac{2}{\tilde{A}}\begin{vmatrix} 2i\omega_0 - a_1 & -h_1 & h_2 \\ -a_4 & h_1 & -a_6 \\ 0 & 0 & h_3 \end{vmatrix}, \\
E_1^{(3)} &= \frac{2}{\tilde{A}}\begin{vmatrix} 2i\omega_0 - a_1 & -a_2 - b_1 e^{-2i\omega_0\tau_0} & -h_1 \\ -a_4 & 2i\omega_0 - a_5 - b_3 e^{-2i\omega_0\tau_0} & h_1 \\ 0 & -a_7 & 0 \end{vmatrix}, \\
\tilde{A} &= \begin{vmatrix} 2i\omega_0 - a_1 & -a_2 - b_1 e^{-2i\omega_0\tau_0} & h_2 \\ -a_4 & 2i\omega_0 - a_5 - b_3 e^{-2i\omega_0\tau_0} & -a_6 \\ 0 & -a_7 & h_3 \end{vmatrix},
\end{aligned}$$

with

$$\begin{aligned}
&h_1 = \beta_1 \nu_1 + \beta_2 \nu_2, \qquad h_2 = -a_3 - b_2 e^{-2i\omega_0\tau_0}, \\
&h_3 = 2i\omega_0 - a_8 - b_4 e^{-2i\omega_0\tau_0}. \\
&E_2^{(1)} = -\frac{2}{\tilde{B}}\begin{vmatrix} -h_4 & a_2 + b_1 & a_3 + b_2 \\ h_4 & a_5 + b_3 & a_6 \\ 0 & a_7 & a_8 + b_4 \end{vmatrix}, \\
&E_2^{(2)} = -\frac{2}{\tilde{B}}\begin{vmatrix} a_1 & -h_4 & a_3 + b_2 \\ a_4 & h_4 & a_6 \\ 0 & 0 & a_8 + b_4 \end{vmatrix}, \\
&E_2^{(3)} = -\frac{2}{\tilde{B}}\begin{vmatrix} a_1 & a_2 + b_1 & -h_4 \\ a_4 & a_5 + b_3 & h_4 \\ 0 & a_7 & 0 \end{vmatrix}, \\
&\tilde{B} = -\begin{vmatrix} a_1 & a_2 + b_1 & a_3 + b_2 \\ a_4 & a_5 + b_3 & a_6 \\ 0 & a_7 & a_8 + b_4 \end{vmatrix},
\end{aligned}$$

with

$$h_4 = \beta_1 \Re\{\nu_1\} + \beta_2 \Re\{\nu_2\}.$$

Thus we can compute the following values:

$$\begin{aligned}
c_1(0) &= \frac{i}{2\omega_0\tau_0}\left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right) + \frac{g_{21}}{2}, \\
\mu &= -\frac{\Re\{c_1(0)\}}{\Re\{\lambda'(\tau_0)\}},
\end{aligned}$$

$$\beta = 2\Re\{c_1(0)\},$$
$$T = -\frac{\Im\{c_1(0)\} + \mu_2\Im\{\lambda'(\tau_0)\}}{\omega_0\tau_0},$$

which determine the properties of a bifurcating periodic solution at the critical value τ_0. The notations μ, β, and T determine the direction of Hopf bifurcation, stability, and period of the bifurcating periodic solutions, respectively. Now we state the results in the following theorem.

Theorem 4.1 *If* $\mu > 0$ ($\mu < 0$), *then the Hopf bifurcation is supercritical* (*subcritical*); *if* $\beta < 0$ ($\beta > 0$), *then the bifurcating periodic solutions are stable* (*unstable*); *if* $T > 0$ ($T < 0$), *then the periodic solutions increase* (*decrease*).

5 Numerical simulations

Numerical simulation confirms that the delayed system dynamics exhibits a periodic solution for $\tau > 8.40567$. The initial condition throughout the simulation is taken as [10, 5, 5]. The unique endemic equilibrium $E^*(16.0404, 29.3064, 14.6532)$ of system (32) can be obtained by means of Mathematica for the following set of parameter values:

$$\begin{aligned} &\mu_1 = 4, \qquad \mu_2 = 2, \qquad \alpha = 0.2, \\ &\beta_1 = 0.01, \qquad \beta_2 = 0.02, \\ &\gamma_1 = 0.1, \qquad \gamma_2 = 0.3, \\ &\delta = 0.1, \qquad \theta = 0.02. \end{aligned} \tag{31}$$

System (2) becomes

$$\begin{aligned} \frac{dS}{dt} &= 4 - (0.01L + 0.02B)S + 0.1L(t-\tau) + 0.3B(t-\tau) - 0.12S, \\ \frac{dL}{dt} &= 2 + (0.01L + 0.02B)S - 0.1L(t-\tau) - 0.3L + 0.02S, \\ \frac{dB}{dt} &= 0.2L - 0.3B(t-\tau) - 0.1B. \end{aligned} \tag{32}$$

From Eq. (5) at $\tau = 0$, we have the characteristic polynomial $\lambda^3 + A_2\lambda^2 + A_1\lambda + A_0 = 0$, where $A_0 = 0.0395353 > 0$, $A_1 = 0.519925$, $A_2 = 1.34572 > 0$, and $A_1A_2 - A_0 = 0.66014 > 0$. Hence from the set of parameter values given in Eq. (31) we obtain that the endemic equilibrium point E^* is locally asymptotically stable, which is confirmed by the Routh–Hurwitz criterion.

In Fig. 1, time series and Lyapunov characteristic exponent (LCE) are performed for exploring system dynamics with and without delay. In absence of delay, the system dynamics is stable for the chosen set of parameters shown in Fig. 1(a). A Lyapunov characteristic exponent (LCE) diagram is plotted using Wolf algorithm [41] in absence of delay in Fig. 1(b). The value of LCEs (–0.10025, –0.61767, –0.62175) indicates that the system dynamics is stable for the model system (1).

By computing we obtain $\omega_0 = 0.262283$ and $\tau_0 = 8.40567$. The delayed model system shows stable behavior for $\tau = 7.54$ and 8.01, shown in Figs. 1(c) and 1(e), respectively.

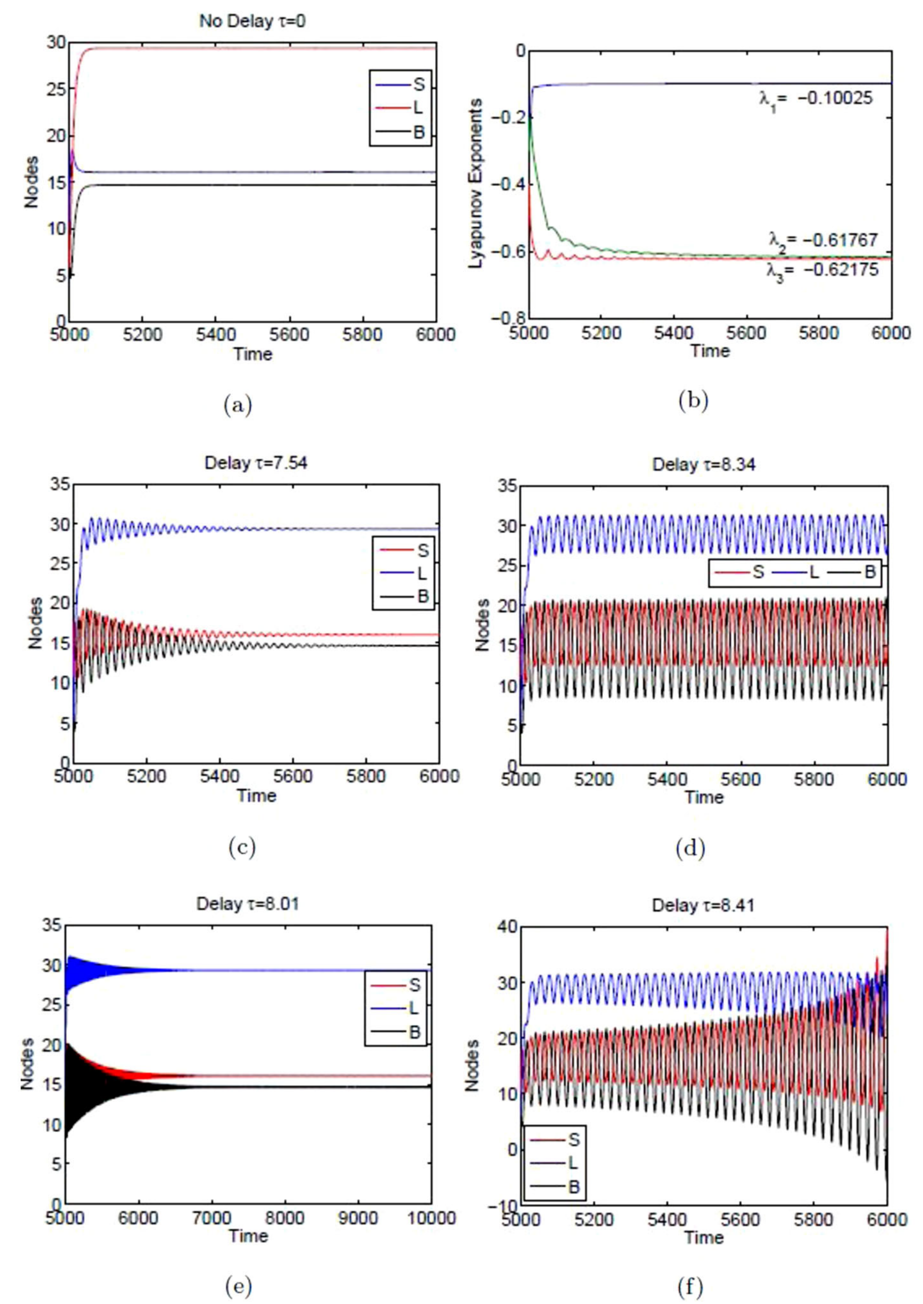

Figure 1 (**a**) Time series without delay, (**b**) LCE test for (**a**). Others shows time series of all three classes for different values of time delay. The values of other parameters are same as in (31)

Oscillatory behavior can be seen for the values of time delay $\tau = 8.34$ and 8.41 in Figs. 1(d) and 1(f), respectively. From Fig. 1(d) we can notice that $\tau = 8.34 < \tau_0$; however, it shows oscillatory behavior.

Bifurcation diagram illustrated in Fig. 2 also confirms that $\tau = 8.40567$ is the critical value for the proposed delayed system (2). When the value of τ is below $\tau_0 = 8.40567$, the endemic point $E^*(16.0404, 29.3064, 14.6532)$ is asymptotically stable by Theorem 2.2. In this case the computer viruses can be controlled easily. However, once the value

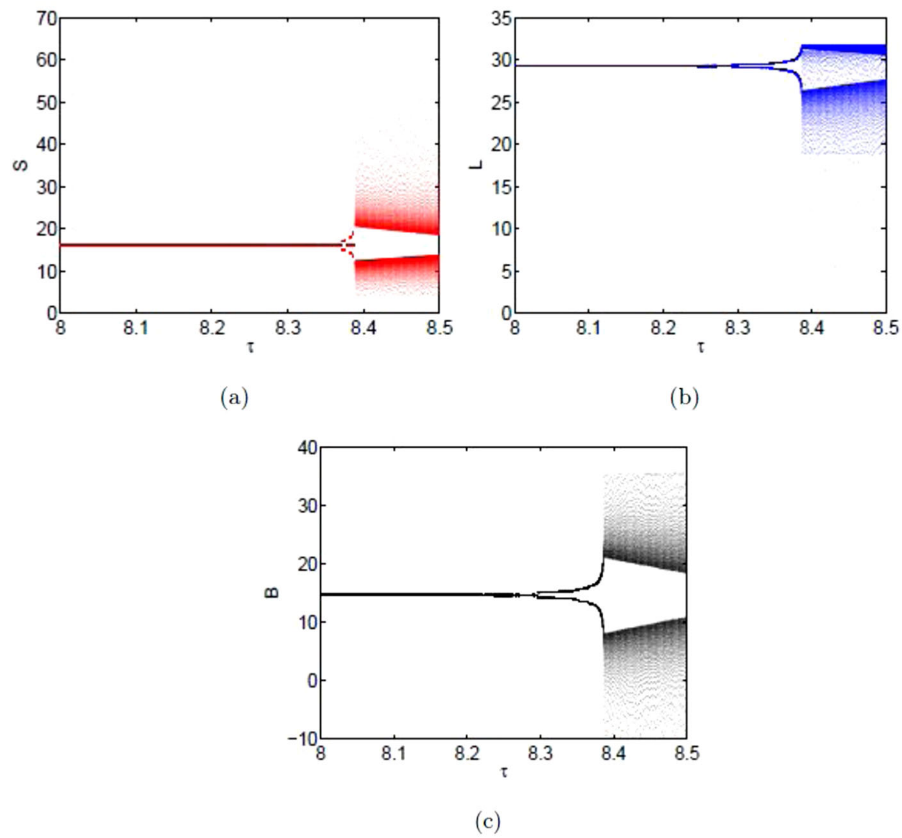

Figure 2 The bifurcation diagram of (**a**) uninfected/susceptible node, (**b**) latent node, and (**c**) breaking-out node with respect to τ. The values of other parameters are as in (31)

of τ passes through $\tau_0 = 8.40567$, a local Hopf bifurcation occurs, which means that the computer viruses will be out of control and system dynamics will become unstable.

Effect of infected and cure rates on the delayed model system (32) are noticed to investigate the computer network performance in Fig. 3. The infected rates $\beta_1 = (0.01, 0.03, 0.05, 0.7)$ and $\beta_2 = (0.2, 0.4, 0.6, 0.8)$ vary, and we observe that as the infected rate increases, the susceptible class decreases, and the number of latent and breaking-out nodes increases, as shown in Figs. 3(a)–3(f). The cure rate $\gamma_1 = (0.2, 0.1, 0.02, 0.01)$, at which latent node cured, is varied, and we observe that as the cure rate of latent computer decreases, susceptible and breaking-out computer also decreases; however, the latent computer increases, as shown in Figs. 4(a)–4(c). The cure rate $\gamma_2 = (0.2, 0.1, 0.02, 0.01)$ at which breaking-out computer cured is varied, and we observe that as the cure rate decreases, the number of susceptible and latent computers decreases, and that of breaking-out computers increases, as shown in Figs. 4(d)–4(f). The effect of infected rate at which uninfected computers are infected due to the influence of infected removable storage media (θ) is performed in Fig. 5. As the infected rate $\theta = (0.02, 0.09, 0.16, 0.23)$ of susceptible computers increases, the number of uninfected computers decreases, but the number of latent and breaking-out computers increases.

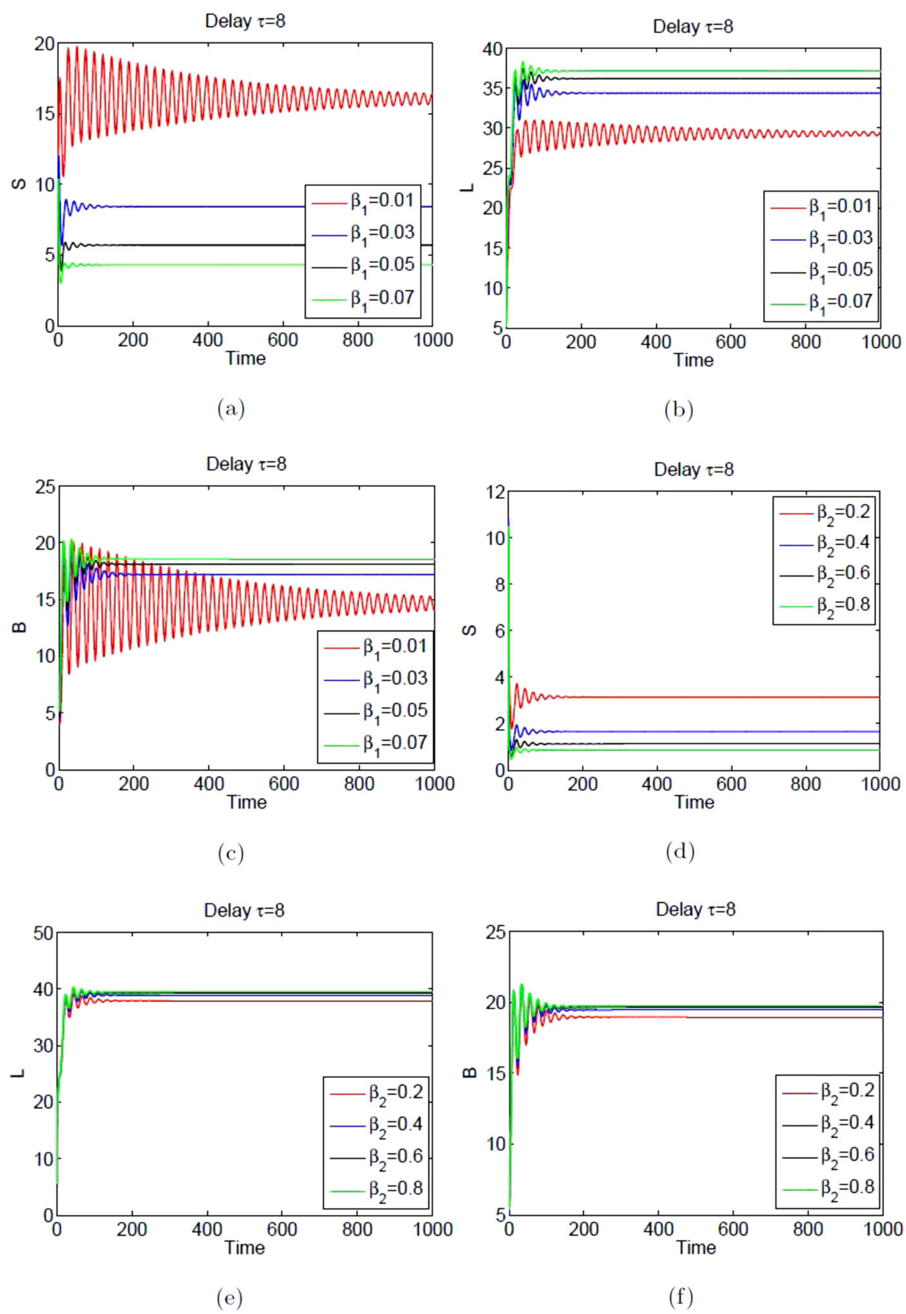

Figure 3 Effect of infected rates (β_1, β_2) on system dynamics (**a**) uninfected/susceptible node, (**b**) latent node, and (**c**) breaking-out node with $\tau = 8$. The values of other parameters are as in (31)

In Figs. 6 and 7 a sensitive to initial condition (SIC) test is performed for all three classes without and with delay, respectively. When we consider the initial condition coinciding with the equilibrium point, the system dynamics indicates a chaotic scenario; however, the system behavior is stable. Two different initial conditions [16.04, 29.3087, 14.6513] and [16.00, 29.2587, 14.5513] are considered for execution of SIC test for the nondelay system in Figs. 6(a)–6(c) and the delay system in Figs. 7(a)–7(c). A slight change in the initial

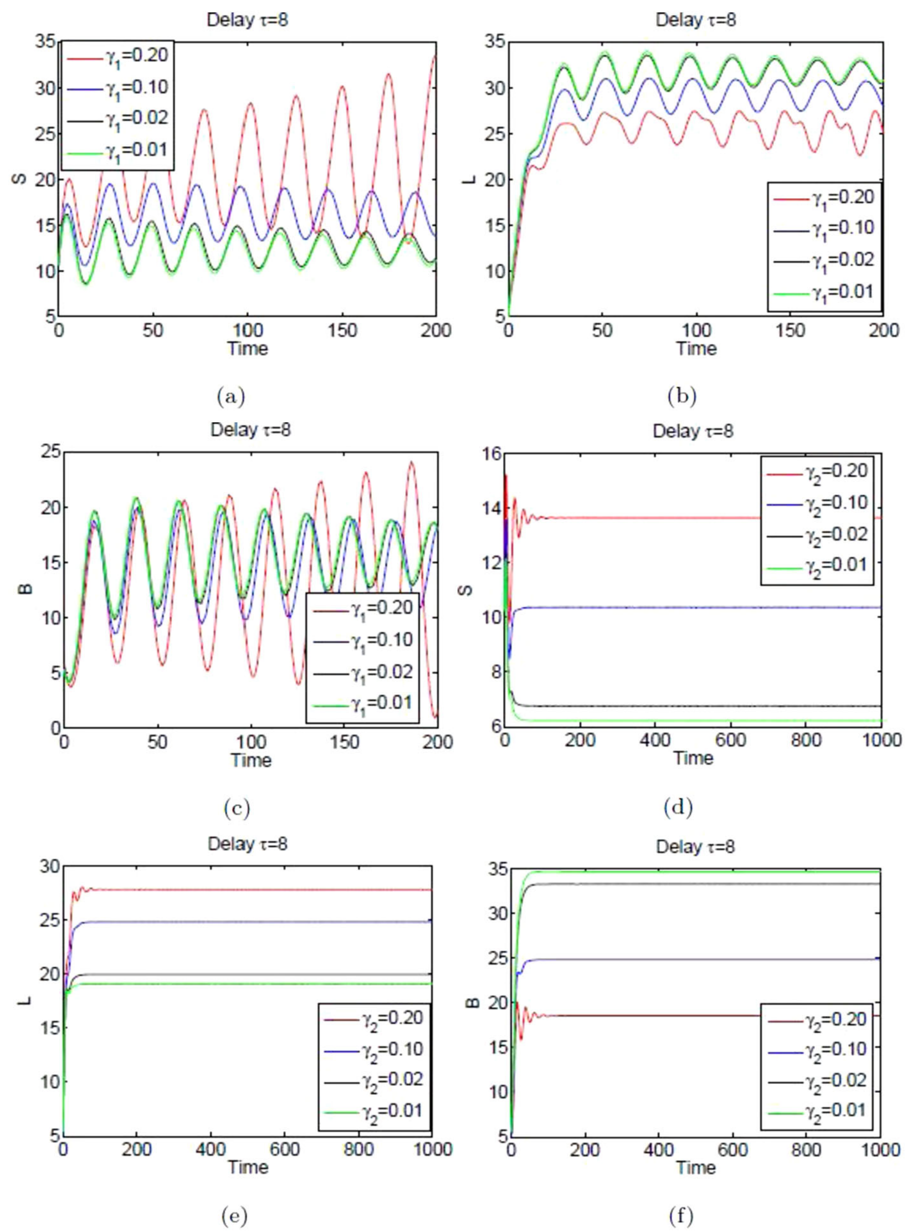

Figure 4 Effect of cure rates (γ_1, γ_2) at which latent and breaking-out computers cured on system dynamics (**a**) uninfected/susceptible node, (**b**) latent node, and (**c**) breaking-out node with $\tau = 8$. The values of other parameters are same as in (31)

condition leads to a new trajectory and confirms that the system dynamics is chaotic in absence and presence of delay for the proposed system (32).

6 Conclusions and discussions

In this paper, we have mainly focused on an e-epidemic delayed *SLBS* model for computer viruses using bilinear incidence rates. Linear and nonlinear stabilities are performed by means of the Lyapunov method. The stability and direction of Hopf bifurcation is performed using center manifold and normal form theory. Numerical exper-

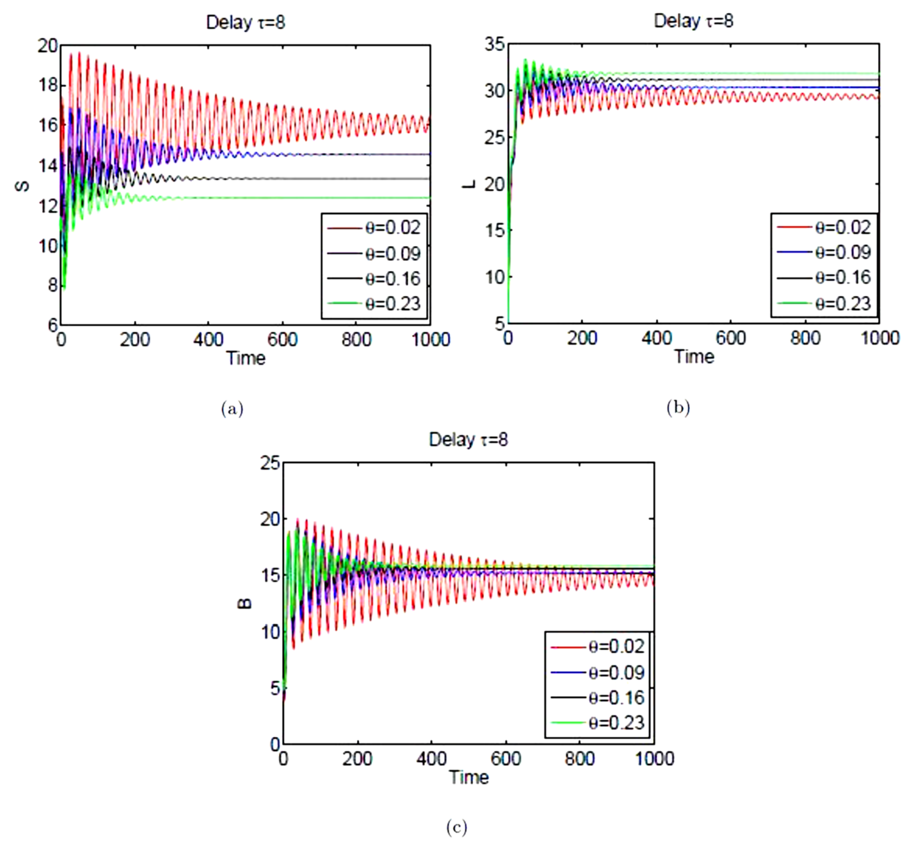

Figure 5 Effect of infected rate at which uninfected computers are infected due to the influence of infected removable storage media (θ) (**a**) uninfected/susceptible node, (**b**) latent node, and (**c**) breaking-out node with $\tau = 8$. The values of other parameters are as in (31)

iments are executed to investigate the dynamics of system (2) and verify the analytical findings for a set of parameter values. Our main results are summarized as follows:

(i) An e-epidemic delayed *SLBS* model for computer virus has been extended to explore the system dynamics. Mainly, the effect of time delay due to the period that the antivirus software uses to clean the viruses in the latent and breaking-out computers has been examined.

(ii) For a given set of parameter values, we obtain $A_0 = 0.0395353 > 0$, $A_1 = 0.519925$, $A_2 = 1.34572 > 0$, and $A_2A_1 - A_0 = 0.66014 > 0$. Hence we concluded that the endemic equilibrium point E^* (16.0404, 29.3064, 14.6532) is locally asymptotically stable by the Routh–Hurwitz criterion in absence of delay. The value of Lyapunov exponent confirms the stable system dynamics of the model system (1).

(iii) The bifurcation diagram confirms that $\tau_0 = 8.40567$ is the critical value for system (32). When $\tau < \tau_0$, the endemic point E^* is asymptotically stable, and the system is unstable for $\tau > \tau_0$.

(iv) We obtain $c_1(0) = 0.00080926 - 0.00164203i$, $\mu = -0.143792 < 0$, $\beta = 0.00161852 > 0$, and $T = -0.00082122 < 0$ by some complicated computations. Thus, according to Theorem 4.1, we conclude that the Hopf bifurcation is

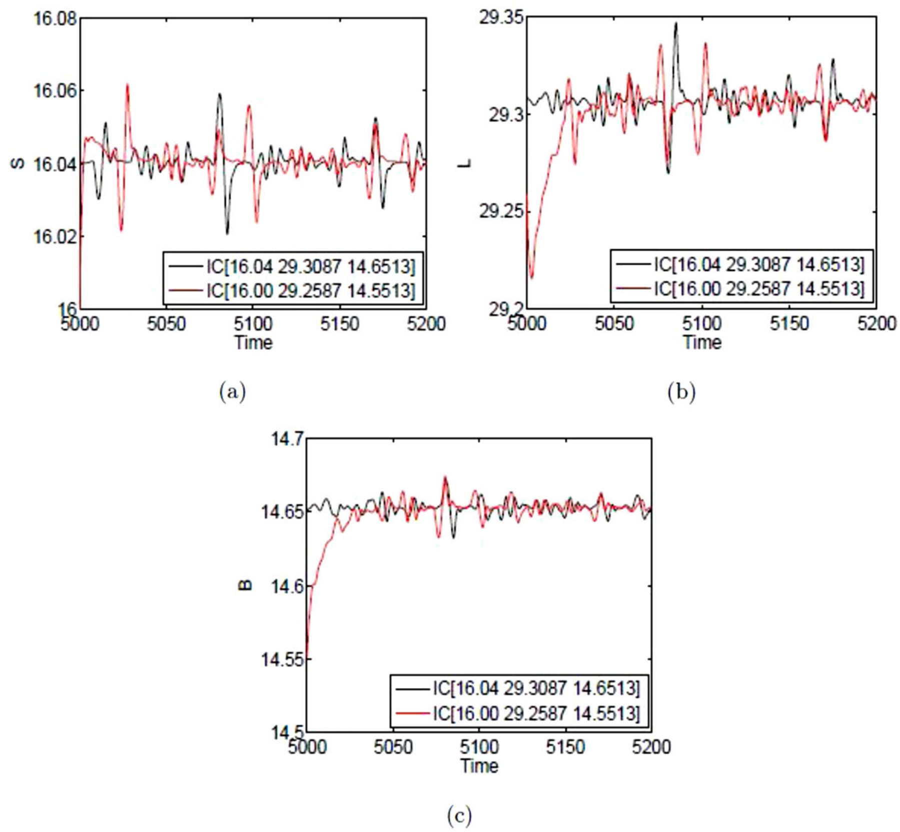

Figure 6 Time series and SIC test of all three classes without delay. The values of other parameters are as in (31)

subcritical and the bifurcating periodic solutions are unstable with decreasing period.

(v) From Fig. 5 we observe that as the infected rate at which uninfected computers are infected due to the influence of infected removable storage media (θ) increases, the number of latent and breaking-out computers increases, which is not consistent for smooth functionality of a computer system. Thus we should minimize the use of removable storage media and if necessary, to use it as minimum time as possible because a delay increases the infection probability.

Numerical investigations confirm that the virus transmission or infected and cure rates at which latent and breaking-out computers get cured have important contributions in reducing or eradicating the viruses from the computer network. Since susceptible node decreases and latent and breaking-out increases as infected rate increases, the number of susceptible and latent computers decreases, and that of breaking-out computers increases as cure rate decreases, and we can see that the model we propose will be useful to analyze the efficiency of antivirus software. The antivirus software will be efficient if the infected rate is small and the cure rate is high. Thus this model will be helpful in developing antivirus software with good quality. In our future work, we will focus on optimal control of such a removable storage device.

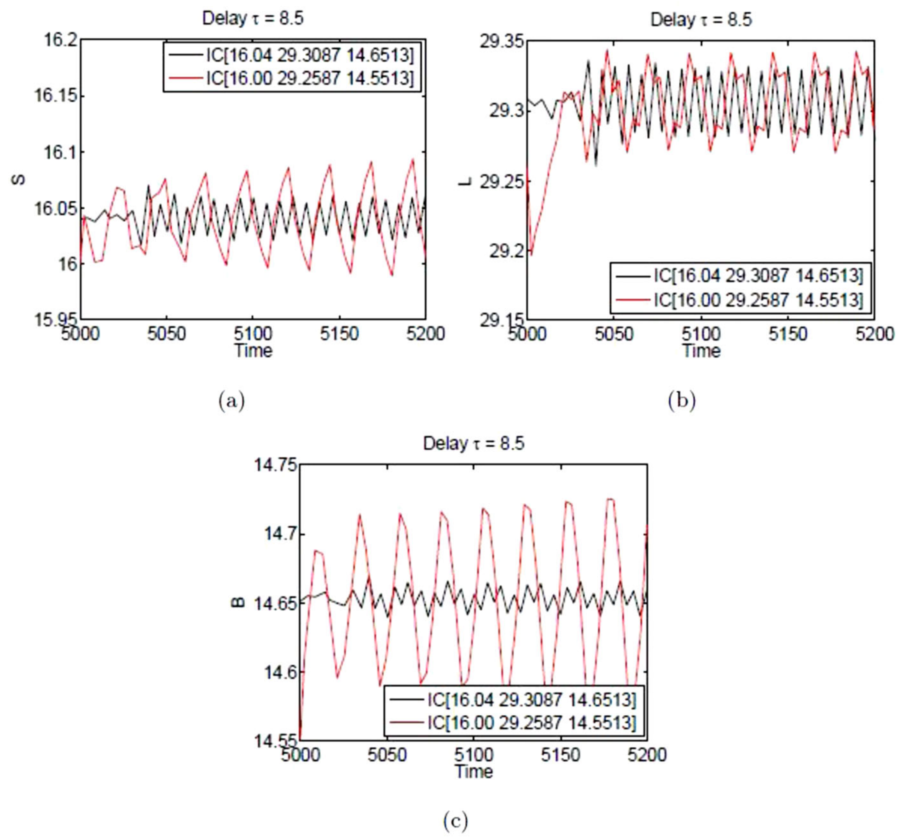

Figure 7 Time series and SIC test of (**a**) uninfected/susceptible node, (**b**) latent node, and (**c**) breaking-out node at $\tau = 8.5$. The values of other parameters are as in (31)

Acknowledgements
The authors are grateful to the editor and anonymous referees for their valuable comments and suggestions on the paper.

Funding
This research was supported by Project of Support Program for Excellent Youth Talent in Colleges and Universities of Anhui Province (No. gxyqZD2018044) and the Natural Science Foundation of the Higher Education Institutions of Anhui Province (Nos. KJ2019A0655, KJ2019A0656, KJ2019A0662).

Availability of data and materials
All the authors declare that all the data can be accessed in our manuscript in the numerical simulation section.

Competing interests
The authors declare that there is no conflict of interests.

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Publisher's Note

Received: 19 July 2019 Accepted: 16 September 2019 Published online: 27 September 2019

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