Sawangtong and Sawangtong *Advances in Difference Equations* (2019) 2019:419
https://doi.org/10.1186/s13662-019-2350-7

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH **Open Access**

Green's function homotopy perturbation method for the initial-boundary value problems

Wannika Sawangtong[1,2] and Panumart Sawangtong[2,3*]

*Correspondence: panumart.s@sci.kmutnb.ac.th
[2]Centre of Excellence in Mathematics, PERDO, Commission on Higher Education, Ministry of Education, Bangkok, Thailand
[3]Department of Mathematics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, Bangkok, Thailand
Full list of author information is available at the end of the article

Abstract

This article deals with the novel method for finding solutions for the initial-boundary value problems (IBVPs), which is called the Sawangtong's Green function homotopy perturbation method, shortly called SGHPM. The SGHPM is a method which combines the homotopy perturbation method with Green's function method. The convergence analysis for the SGHPM is shown. Furthermore, some examples are presented to illustrate the validity of the proposed method and to ensure that SGHPM is a technique which is powerful and efficient for finding approximate analytic solutions of IBVPs.

Keywords: Sawangtong's Green function homotopy perturbation method; Green's function method; Initial-boundary value problems; Integral equation

1 Introduction

It has been known for a long time that Green's function method is a powerful classical one for analytical manipulation of solution of boundary value problems. Furthermore, the homotopy perturbation method (HPM) is a technique which is powerful and efficient for finding approximate analytic solutions of nonlinear initial value problems without the need of a linearization process. The HPM was first introduced by He in 1998 [3, 4]. In general, HPM and Green's function method have been successfully applied to solve many linear and nonlinear equations in science and engineering by many authors [2–6].

The main objective of this article is to propose a novel method for finding solutions for the initial-boundary value problems (IBVPs), which is called the Sawangtong's Green function homotopy perturbation method (SGHPM). The SGHPM is a combination of HPM and Green's function method.

The organization of the rest of the paper is as follows. The idea of SGHPM is given in Sect. 2. In Sect. 3, the solution existence and convergence analysis for SGHPM method are investigated. The applications of SGHPM for finding an analytical solution for IBVPs are verified in Sect. 4. The last section deals with the conclusions about the SGHPM technique.

2 Basic idea of SGHPM

Let a, b and T be positive constants with $0 < a < b$ and $0 < T \leq \infty$. To illustrate the basic ideas of the new method, we consider the following initial-boundary value problem

(IBVP):

$$\left.\begin{aligned}&u_t(x,t)=\tfrac{1}{r(x)}(k(x)u_x(x,t))_x+u(x,t)\quad\text{for }(x,t)\in(a,b)\times(0,T],\\&u(x,0)=u_0(x)\quad\text{for }x\in[a,b],\\&\alpha_1u(a,t)+\beta_1u_x(a,t)=0\quad\text{and}\quad\alpha_2u(b,t)+\beta_2u_x(b,t)=0\quad\text{for }0<t\le T,\end{aligned}\right\}\tag{1}$$

where u_0 is a given function, k, k' and r are continuous on $[a,b]$, $k(x)>0$ on $[a,b]$ and $r(x)>0$ on $[a,b]$, and α_i and β_i for $i=1,2$ are real constants such that $\alpha_1\alpha_2\ge 0$, $\alpha_1^2+\alpha_2^2\ne 0$, $\beta_1\beta_2\ge 0$ and $\beta_1^2+\beta_2^2\ne 0$. Let $G(x,t,\xi,\tau)$ denote Green's function corresponding to the IBVP (1). Based on the eigenfunction expansion technique [2], Green's function is defined by

$$G(x,t,\xi,\tau)=\sum_{k=0}^{\infty}\phi_k(x)\phi_k(\xi)e^{-\lambda_k(t-\tau)}\quad\text{for }0\le\tau<t\le T,\tag{2}$$

where λ_k and ϕ_k are eigenvalue and its corresponding eigenfunction given by the following regular Sturm–Liouville problem:

$$\begin{aligned}&\frac{d}{dx}\left(k(x)\frac{d\phi}{dx}\right)+\lambda r(x)\phi=0\quad\text{for }x\in(a,b),\\&\alpha_1\phi(a)+\beta_1\frac{d}{dx}\phi(a)=0\quad\text{and}\quad\alpha_2\phi(b)+\beta_2\frac{d}{dx}\phi(b)=0.\end{aligned}\tag{3}$$

Note that the following properties are well-known for the regular Sturm–Liouville problem:

1. All the eigenvalues of the Sturm–Liouville problem (3) are real.
2. All the eigenvalues of the Sturm–Liouville problem (3) are simple, that is, to each eigenvalue there corresponds only one linearly independent eigenfunction. Further, the eigenvalues form an infinite sequence and can be ordered according to increasing magnitude so that $\lambda_1<\lambda_2<\lambda_3<\cdots<\lambda_n<\cdots$.
3. Eigenfunctions ϕ_n are real and can be normalized so that $\int_a^b r(x)\phi_n^2(x)\,dx=1$.
4. If ϕ_n and ϕ_m are two eigenfunctions of the Sturm–Liouville problem (3) corresponding to eigenvalues λ_n and λ_m, respectively, and if $\lambda_n\ne\lambda_m$, then $\int_a^b r(x)\phi_n(x)\phi_m(x)\,dx=0$.
5. The set of eigenfunctions $\{\phi_n\}$ is complete.

Using Green's second identity, the IBVP (1) can be transformed into the associated integral equation

$$u(x,t)=\int_a^b G(x,t,\xi,0)u_0(\xi)\,d\xi+\int_0^t\int_a^b G(x,t,\xi,\tau)u(\xi,\tau)\,d\xi\,d\tau.\tag{4}$$

By the homotopy perturbation technique [3, 4], we construct a homotopy $v(x,t;p):[a,b]\times[0,T]\times[0,1]\to R$ which satisfies

$$\begin{aligned}H\big(v(x,t;p);p\big)&=(1-p)\big(v(x,t;p)-\widetilde{v}_0(x,t)\big)+p\bigg(v(x,t;p)-\int_a^b G(x,t,\xi,0)u_0(\xi)\,d\xi\\&\quad+\int_0^t\int_a^b G(x,t,\xi,\tau)v(\xi,\tau;p)\,d\xi\,d\tau\bigg)\\&=0,\end{aligned}\tag{5}$$

where $p \in [0,1]$ is an embedding parameter and $\widetilde{v}_0(x,t)$ is an initial guess for (5), which satisfies initial and boundary conditions that can be freely chosen [1]. Equation (5) is called a homotopy equation. Equivalently, it can be written as follows:

$$\begin{aligned} v(x,t;p) = \widetilde{v}_0 + p\biggl(&-\widetilde{v}_0 + \int_a^b G(x,t,\xi,0)u_0(\xi)\,d\xi \\ &+ \int_0^t \int_a^b G(x,t,\xi,\tau)v(\xi,\tau;p)\,d\xi\,d\tau\biggr). \end{aligned} \tag{6}$$

Obviously, we have

$$\begin{aligned} p = 0 \quad &\Rightarrow \quad H\bigl(v(x,t;0);0\bigr) = v(x,t;0) - \widetilde{v}_0(x,t) = 0, \\ p = 1 \quad &\Rightarrow \quad H\bigl(v(x,t;1);1\bigr) = v(x,t;1) - \int_a^b G(x,t,\xi,0)u_0(\xi)\,d\xi \\ &\qquad\qquad - \int_0^t \int_a^b G(x,t,\xi,\tau)v(\xi,\tau;1)\,d\xi\,d\tau = 0. \end{aligned}$$

By the HPM technique, the solution $v(x,t;p)$ in Eq. (6) is presented by the infinite series

$$v(x,t;p) = \sum_{n=0}^{\infty} p^n v_n(x,t). \tag{7}$$

By substituting Eq. (7) into Eq. (6), we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} p^n v_n = \widetilde{v}_0 + p\biggl(&-\widetilde{v}_0 + \int_a^b G(x,t,\xi,0)u_0(\xi)\,d\xi \\ &+ \int_0^t \int_a^b G(x,t,\xi,\tau)\biggl(\sum_{n=0}^{\infty} p^n v_n(\xi,\tau)\biggr)\,d\xi\,d\tau\biggr). \end{aligned}$$

By equating the coefficients of the corresponding powers of p one can find an approximate solution $v_n(x,t)$ for $n = 0,1,2,\ldots$ of Eq. (7). We then get the recurrence relation as given below:

$$\left.\begin{aligned} &v_0(x,t) = \widetilde{v}_0(x,t), \\ &v_1(x,t) = -\widetilde{v}_0 + \textstyle\int_a^b G(x,t,\xi,0)u_0(\xi)\,d\xi + \int_0^t \int_a^b G(x,t,\xi,\tau)v_0(\xi,\tau)\,d\xi\,d\tau, \\ &v_{n+1}(x,t) = \textstyle\int_0^t \int_a^b G(x,t,\xi,\tau)v_n(\xi,\tau)\,d\xi\,d\tau \quad \text{for } n \geq 1. \end{aligned}\right\} \tag{8}$$

From Eq. (7), the solution $v(x,t;p)$ is

$$v(x,t;p) = v_0(x,t) + pv_1(x,t) + p^2 v_2(x,t) + p^3 v_3(x,t) + \cdots.$$

As p converges to 1, the approximate analytical solution $u(x,t)$ of IBVP (1) can be expressed as

$$u(x,t) = v(x,t;1) = v_0(x,t) + v_1(x,t) + v_2(x,t) + v_3(x,t) + \cdots.$$

3 Solution existence and convergence analysis

In order to obtain the existence result for problem (1), let us introduce the Banach space $C([a,b]\times[0,T])$. The $C([a,b]\times[0,T])$ is the space of all continuous functions on $[a,b]\times[0,T]$, and its norm is defined by $\|u\| = \max_{(x,t)\in[a,b]\times[0,T]} |u(\xi,\tau)|$ for any $u \in C([a,b]\times[0,T])$. The existence of the solution for problem (1) is established in the following theorem.

Theorem 1 *Assume that* $\int_0^T\int_a^b G(x,t,\xi,\tau)\,d\xi\,d\tau < 1$ *for any* $(x,t)\in[a,b]\times[0,T]$. *Then, problem* (1) *has a unique solution* $u \in C([a,b]\times[0,T])$.

Proof Let us consider the corresponding integral equation (4) to problem (1):

$$u(x,t) = \int_a^b G(x,t,\xi,0)u_0(\xi)\,d\xi + \int_0^t\int_a^b G(x,t,\xi,\tau)u(\xi,\tau)\,d\xi\,d\tau.$$

Let F be an operator such that $F : C([a,b]\times[0,T]) \to C([a,b]\times[0,T])$ and

$$F\bigl(u(x,t)\bigr) = \int_a^b G(x,t,\xi,0)u_0(\xi)\,d\xi + \int_0^t\int_a^b G(x,t,\xi,\tau)u(\xi,\tau)\,d\xi\,d\tau.$$

We next will show that operator F is contractive. Let u and v be in $C([a,b]\times[0,T])$. Then by the positivity G and for any $(x,t)\in[a,b]\times[0,T]$,

$$\begin{aligned}
\bigl|F\bigl(u(x,t)\bigr) - F\bigl(v(x,t)\bigr)\bigr| &= \left|\int_0^t\int_a^b G(x,t,\xi,\tau)\bigl(u(\xi,\tau)-v(\xi,\tau)\bigr)\,d\xi\,d\tau\right| \\
&\le \int_0^t\int_a^b G(x,t,\xi,\tau)\,d\xi\,d\tau \max_{(x,t)\in[a,b]\times[0,T]}\bigl|u(\xi,\tau)-v(\xi,\tau)\bigr| \\
&\le \int_0^T\int_a^b G(x,t,\xi,\tau)\,d\xi\,d\tau\,\|u-v\|.
\end{aligned}$$

This implies that $\|Fu - Fv\| \le \int_0^T\int_a^b G(x,t,\xi,\tau)\,d\xi\,d\tau\,\|u-v\|$, i.e., F is a contraction mapping. Therefore, the Banach fixed point theorem yields that the integral equation (4) has a unique solution, or equivalently, problem (1) has a unique solution $u \in C([a,b]\times[0,T])$. □

The convergence of the SGHPM is described in the theorem below.

Theorem 2 *Assume that* $\int_0^T\int_a^b G(x,t,\xi,\tau)\,d\xi\,d\tau < 1$ *for any* $(x,t)\in[a,b]\times[0,T]$. *Let* $\{v_n\}_{n=0}^\infty$ *be a sequence in a Banach space* $C([a,b]\times[0,T])$ *given by* (8). *If there exists a positive constant* σ *with* $0<\sigma<1$ *and* $v_n(x,t)\le\sigma v_{n-1}(x,t)$ *for any* $(x,t)\in[a,b]\times[0,T]$ *and* $n = 1,2,3,\ldots$, *then the infinite series* $\sum_{n=0}^\infty v_n$ *converges to* u, *where* u *is the solution of problem* (1).

Proof Let S_n be the nth partial sum of the series $\sum_{n=0}^\infty v_n$. Firstly, we will show that the sequence $\{S_n\}_{n=0}^\infty$ be a Cauchy sequence in $C([a,b]\times[0,T])$.

Let $m,l\in N$ be such that $m>l$. Then for any $(x,t)\in[a,b]\times[0,T]$,

$$\begin{aligned}
\bigl|S_m(x,t)-S_{m-1}(x,t)\bigr| &= \bigl|v_m(x,t)\bigr| \le \sigma\bigl|v_{m-1}(x,t)\bigr| \le \sigma^2\bigl|v_{m-2}(x,t)\bigr| \\
&\le\cdots\le \sigma^m\bigl|v_0(x,t)\bigr| \le \sigma^m\|v_0\|.
\end{aligned}$$

Thus,

$$\|S_m - S_{m-1}\| = \max_{(x,t)\in[a,b]\times[0,T]} \left|S_m(x,t) - S_{m-1}(x,t)\right| \le \sigma^m \|v_0\|. \tag{9}$$

By (9), we obtain that for any $(x,t) \in [a,b] \times [0,T]$,

$$\left|S_m(x,t) - S_l(x,t)\right| \le \sum_{k=0}^{m-l-1} |S_{l+k+1} - S_{l+k}| \le \sum_{k=0}^{m-l-1} \sigma^{l+k+1}\|v_0\| = \sigma^{l+1}\frac{1-\sigma^{m-l}}{1-\sigma}\|v_0\|.$$

It follows from $0 < \sigma < 1$ that $\|S_m - S_l\| \le \frac{\sigma^{l+1}}{1-\sigma}\|v_0\|$. As $l \to \infty$, we can conclude that sequence $\{S_n\}_{n=0}^{\infty}$ is a Cauchy sequence in $C([a,b] \times [0,T])$. Let $\widetilde{u} = \lim_{n\to\infty} S_n$. Since u is the solution of problem (1), u satisfies

$$u(x,t) = \int_a^b G(x,t,\xi,0)u_0(\xi)\,d\xi + \int_0^t \int_a^b G(x,t,\xi,\tau)u(\xi,\tau)\,d\xi\,d\tau.$$

Suppose that $\widetilde{u} \ne u$. Then by the positivity of G and for any $(x,t) \in [a,b] \times [0,T]$,

$$\begin{aligned}
\left|\widetilde{u}(x,t) - u(x,t)\right| &= \left|\lim_{n\to\infty}\sum_{k=0}^{n-1} v_k - u(x,t)\right| \\
&\le \int_0^t \int_a^b G(x,t,\xi,\tau)\left|\left(\lim_{n\to\infty}\sum_{k=0}^{n-2} v_k\right) - u(\xi,\tau)\right| d\xi\,d\tau \\
&\le \int_0^T \int_a^b G(x,t,\xi,\tau)\,d\xi\,d\tau\,\|\widetilde{u} - u\|.
\end{aligned}$$

This means that $\|\widetilde{u} - u\| \le \int_0^T \int_a^b G(x,t,\xi,\tau)\,d\xi\,d\tau\,\|\widetilde{u} - u\|$. Since $\|\widetilde{u} - u\| \ne 0$, we obtain that $\int_0^T \int_a^b G(x,t,\xi,\tau)\,d\xi\,d\tau \ge 1$ for any $(x,t) \in [a,b] \times [0,T]$. This contradicts the assumption that $\int_0^T \int_a^b G(x,t,\xi,\tau)\,d\xi\,d\tau < 1$ for any $(x,t) \in [a,b] \times [0,T]$. Hence, the series $\sum_{n=0}^{\infty} v_n$ converges to u, which is the solution of problem (1). □

Theorem 3 *Let $\widetilde{\lambda}$ be the principal eigenvalue of the regular Sturm–Liouville problem (3) with $\widetilde{\lambda} > 0$ and let $\widetilde{\phi}$ be the principal eigenfunction associated with the principal eigenvalue.*

If $u_0 = \widetilde{\phi}$, then the analytical solution of problem (1) is the of form $u(x,t) = (b - a)\widetilde{\phi}(x)e^{-(\widetilde{\lambda}-1)t}$ for any $(x,t) \in [a,b] \times [0,T]$.

Proof By Green's second identity, the IBVP (1) can be transformed into the integral equation

$$u(x,t) = \int_a^b G(x,t,\xi,0)\widetilde{\phi}(\xi)\,d\xi + \int_0^t \int_a^b G(x,t,\xi,\tau)u(\xi,\tau)\,d\xi\,d\tau,$$

where $G(x,t,\xi,\tau)$ is the corresponding Green's function and

$$G(x,t,\xi,\tau) = \sum_{k=0}^{\infty} \phi_k(x)\phi_k(\xi)e^{-\lambda_k(t-\tau)} \quad \text{for } 0 \le \tau < t \le T.$$

By HPM technique, the homotopy $v(x,t;p):[a,b]\times[0,T]\times[0,1]\to R$ is defined by

$$\begin{aligned} v(x,t;p) &= \widetilde{v}_0 + p\biggl(-\widetilde{v}_0 + \int_a^b G(x,t,\xi,0)\widetilde{\phi}(\xi)\,d\xi \\ &\quad + \int_0^t\int_a^b G(x,t,\xi,\tau)v(\xi,\tau;p)\,d\xi\,d\tau\biggr), \end{aligned} \tag{10}$$

where $p\in[0,1]$ is an embedding parameter, $\widetilde{v}_0$ is an initial function which can be chosen freely, and $\widetilde{v}_0$ satisfies the initial and boundary conditions of problem (1). We see that

$$\begin{aligned} p=0 \quad &\Rightarrow \quad v(x,t;0)=\widetilde{v}_0(x,t), \\ p=1 \quad &\Rightarrow \quad v(x,t;1)=\int_a^b G(x,t,\xi,0)\widetilde{\phi}(\xi)\,d\xi + \int_0^t\int_a^b G(x,t,\xi,\tau)v(\xi,\tau;1)\,d\xi\,d\tau. \end{aligned}$$

The case of $p=1$ means that $v(x,t;1)$ satisfies the corresponding integral equation to problem (1), or equivalently $v(x,t;1)$ is the analytical solution of problem (1). By the HPM technique, we assume $v(x,t;p)=\sum_{n=0}^{\infty}p^n v_n(x,t)$ and then substitute $v(x,t;p)=\sum_{n=0}^{\infty}p^n v_n(x,t)$ into (10). We then have that

$$\begin{aligned} \sum_{n=0}^{\infty}p^n v_n(x,t) &= \widetilde{v}_0 + p\biggl(-\widetilde{v}_0 + \int_a^b G(x,t,\xi,0)\widetilde{\phi}(\xi)\,d\xi \\ &\quad + \int_0^t\int_a^b G(x,t,\xi,\tau)\biggl(\sum_{n=0}^{\infty}p^n v_n(\xi,\tau)\biggr)\,d\xi\,d\tau\biggr), \end{aligned}$$

or

$$\begin{aligned} v_0(x,t) &= \widetilde{v}_0(x,t), \\ v_1(x,t) &= -\widetilde{v}_0 + \int_a^b G(x,t,\xi,0)\widetilde{\phi}(\xi)\,d\xi + \int_0^t\int_a^b G(x,t,\xi,\tau)v_0(\xi,\tau)\,d\xi\,d\tau, \\ v_{n+1}(x,t) &= \int_0^t\int_a^b G(x,t,\xi,\tau)v_n(\xi,\tau)\,d\xi\,d\tau \quad \text{for } n\geq 1. \end{aligned}$$

First, we set $\widetilde{v}_0(x,t)=(\frac{b-a}{2})\widetilde{\phi}(x)e^{-\widetilde{\lambda}t}$. Then $v_0(x,t)=\widetilde{v}_0(x,t)=(\frac{b-a}{2})\widetilde{\phi}(x)e^{-\widetilde{\lambda}t}$. We next consider that

$$\begin{aligned} v_1(x,t) &= -\widetilde{v}_0 + \int_a^b G(x,t,\xi,0)\widetilde{\phi}(\xi)\,d\xi + \int_0^t\int_a^b G(x,t,\xi,\tau)v_0(\xi,\tau)\,d\xi\,d\tau \\ &= -\biggl(\frac{b-a}{2}\biggr)\widetilde{\phi}(x)e^{-\widetilde{\lambda}t} + \int_a^b\sum_{k=0}^{\infty}\phi_k(x)\phi_k(\xi)e^{-\lambda_k t}\widetilde{\phi}(\xi)\,d\xi \\ &\quad + \int_0^t\int_a^b\sum_{k=0}^{\infty}\phi_k(x)\phi_k(\xi)e^{-\lambda_k(t-\tau)}\biggl(\frac{b-a}{2}\biggr)\widetilde{\phi}(\xi)e^{-\widetilde{\lambda}\tau}\,d\xi\,d\tau. \end{aligned}$$

From the orthogonality of eigenfunctions we get

$$\begin{aligned} v_1(x,t) &= -\left(\frac{b-a}{2}\right)\widetilde{\phi}(x)e^{-\widetilde{\lambda}t} + \widetilde{\phi}(x)e^{-\widetilde{\lambda}t} \\ &\quad + \left(\frac{b-a}{2}\right)\widetilde{\phi}(x)\int_0^t e^{-\widetilde{\lambda}(t-\tau)}e^{-\widetilde{\lambda}\tau}\,d\tau \\ &= \left(\frac{b-a}{2}\right)\widetilde{\phi}(x)e^{-\widetilde{\lambda}t} + \left(\frac{b-a}{2}\right)\widetilde{\phi}(x)e^{-\widetilde{\lambda}t}t. \end{aligned}$$

Next we have that

$$\begin{aligned} v_2(x,t) &= \int_0^t\int_a^b G(x,t,\xi,\tau)v_1(\xi,\tau)\,d\xi\,d\tau \\ &= \int_0^t\int_a^b \sum_{k=0}^{\infty}\phi_k(x)\phi_k(\xi)e^{-\lambda_k(t-\tau)} \\ &\quad \times\left[\left(\frac{b-a}{2}\right)\widetilde{\phi}(\xi)e^{-\widetilde{\lambda}\tau} + \left(\frac{b-a}{2}\right)\widetilde{\phi}(\xi)e^{-\widetilde{\lambda}\tau}\tau\right]d\xi\,d\tau \\ &= \left(\frac{b-a}{2}\right)\widetilde{\phi}(x)e^{-\widetilde{\lambda}t}t + \left(\frac{b-a}{2}\right)\widetilde{\phi}(x)e^{-\widetilde{\lambda}t}\frac{t^2}{2!} \end{aligned}$$

and

$$\begin{aligned} v_3(x,t) &= \int_0^t\int_a^b G(x,t,\xi,\tau)v_2(\xi,\tau)\,d\xi\,d\tau \\ &= \left(\frac{b-a}{2}\right)\widetilde{\phi}(x)e^{-\widetilde{\lambda}t}\frac{t^2}{2!} + \left(\frac{b-a}{2}\right)\widetilde{\phi}(x)e^{-\widetilde{\lambda}t}\frac{t^3}{3!}. \end{aligned}$$

From the above calculations, we obtain that

$$v_n(x,t) = \left(\frac{b-a}{2}\right)\widetilde{\phi}(x)e^{-\widetilde{\lambda}t}\frac{t^{n-1}}{(n-1)!} + \left(\frac{b-a}{2}\right)\widetilde{\phi}(x)e^{-\widetilde{\lambda}t}\frac{t^n}{n!}$$

for any $n \geq 1$. Since $v(x,t;p) = \sum_{n=0}^{\infty} p^n v_n(x,t)$, we have

$$\begin{aligned} v(x,t;p) &= v_0(x,t) + pv_1(x,t) + p^2v_2(x,t) + p^3v_3(x,t) + \cdots \\ &= \left(\frac{b-a}{2}\right)\widetilde{\phi}(x)e^{-\widetilde{\lambda}t} + p\left[\left(\frac{b-a}{2}\right)\widetilde{\phi}(x)e^{-\widetilde{\lambda}t} + \left(\frac{b-a}{2}\right)\widetilde{\phi}(x)e^{-\widetilde{\lambda}t}t\right] \\ &\quad + p^2\left[\left(\frac{b-a}{2}\right)\widetilde{\phi}(x)e^{-\widetilde{\lambda}t}t + \left(\frac{b-a}{2}\right)\widetilde{\phi}(x)e^{-\widetilde{\lambda}t}\frac{t^2}{2!}\right] \\ &\quad + p^3\left[\left(\frac{b-a}{2}\right)\widetilde{\phi}(x)e^{-\widetilde{\lambda}t}\frac{t^2}{2!} + \left(\frac{b-a}{2}\right)\widetilde{\phi}(x)e^{-\widetilde{\lambda}t}\frac{t^3}{3!}\right] \\ &\quad + \cdots + p^n\left[\left(\frac{b-a}{2}\right)\widetilde{\phi}(x)e^{-\widetilde{\lambda}t}\frac{t^{n-1}}{(n-1)!} + \left(\frac{b-a}{2}\right)\widetilde{\phi}(x)e^{-\widetilde{\lambda}t}\frac{t^n}{n!}\right] + \cdots. \end{aligned}$$

As p converges to 1, the analytic solution $u(x,t)$ of problem (1) is given by

$$\begin{aligned} u(x,t) &= v(x,t;1) \\ &= (b-a)\widetilde{\phi}(x)e^{-\widetilde{\lambda}t} + (b-a)\widetilde{\phi}(x)e^{-\widetilde{\lambda}t}t \\ &\quad + (b-a)\widetilde{\phi}(x)e^{-\widetilde{\lambda}t}\frac{t^2}{2!} + (b-a)\widetilde{\phi}(x)e^{-\widetilde{\lambda}t}\frac{t^3}{3!} + \cdots \\ &= (b-a)\widetilde{\phi}(x)e^{-\widetilde{\lambda}t}\left[1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right] \\ &= (b-a)\widetilde{\phi}(x)e^{-(\widetilde{\lambda}-1)t}. \end{aligned}$$

Hence, problem (1) has an analytic solution $u(x,t) = (b-a)\widetilde{\phi}(x)e^{-(\widetilde{\lambda}-1)t}$ where $\widetilde{\lambda}$ is the principal eigenvalue and $\widetilde{\phi}$ is its corresponding principal eigenfunction of the regular Strum–Liouville problem defined by (3). □

Note that if the principal eigenvalue $\widetilde{\lambda}$ is zero, then we let the initial function $u_0 = \phi_1$, which is the eigenfunction ϕ_k with $k = 1$. We apply the method in Theorem 3 and then obtain the analytical solution of the IBVP (1) in the following form: $u(x,t) = (b-a)\phi_1(x)e^{-(\lambda_1-1)t}$ for any $(x,t) \in [a,b] \times [0,T]$ and λ_1 being the eigenvalue corresponding to the eigenfunction ϕ_1.

4 Applications

To illustrate the SGHPM for solving the IBVPs, we consider the following examples.

Example 1 Consider the heat equation problem with the Dirichlet boundary condition:

$$\left.\begin{aligned} &u_t(x,t) = u_{xx}(x,t) + u(x,t) \quad \text{for } (x,t) \in (0,1) \times (0,T], \\ &u(x,0) = \sin(\pi x) \quad \text{for } x \in [0,1], \\ &u(0,t) = 0 \quad \text{and} \quad u(1,t) = 0 \quad \text{for } 0 < t \le T. \end{aligned}\right\} \tag{11}$$

The regular Sturm–Liouville problem corresponding to problem (11) is

$$\begin{aligned} &\phi''(x) + \lambda\phi(x) = 0 \quad \text{for } x \in (0,1), \\ &\phi(0) = 0 \quad \text{and} \quad \phi(1) = 0. \end{aligned} \tag{12}$$

It is well-known that the regular Sturm–Liouville problem (12) has eigenvalues and eigenfunctions given by

$$\lambda_k = (k\pi)^2 \quad \text{and} \quad \phi_k(x) = \sin(k\pi x) \quad \text{for } k = 1,2,3,\ldots.$$

We see that these eigenfunctions are orthogonal, and that the set $\{\sqrt{2}\sin(k\pi x)\}_{k=1}^{\infty}$ consists of orthonormal eigenfunctions. Furthermore, the corresponding Green's function of the IBVP (11) is defined by

$$G(x,t,\xi,\tau) = 2\sum_{k=1}^{\infty}\sin(k\pi x)\sin(k\pi\xi)e^{-k^2\pi^2(t-\tau)} \quad \text{for } 0 \le \tau < t \le T. \tag{13}$$

For this BVP (11), we see that the principal eigenvalue $\widetilde{\lambda} = \pi^2$ is not zero and its corresponding eigenfunction is $\widetilde{\phi}(x) = \sin(\pi x)$. It then follows from Theorem 3 that the analytical solution of IBVP (11) is given by $u(x,t) = \sin(\pi x)e^{-(\pi^2-1)t}$ for any $(x,t) \in [0,1] \times [0,T]$.

Example 2 Consider the heat equation problem with the Neumann boundary condition:

$$\left.\begin{aligned} &u_t(x,t) = u_{xx}(x,t) + u(x,t) \quad \text{for } (x,t) \in (0,1) \times (0,T], \\ &u(x,0) = \cos(\pi x) \quad \text{for } x \in [0,1], \\ &u_x(0,t) = 0 \quad \text{and} \quad u_x(1,t) = 0 \quad \text{for } 0 < t \leq T. \end{aligned}\right\} \tag{14}$$

The regular Sturm–Liouville problem associated with problem (14) is defined by

$$\begin{aligned} &\phi''(x) + \lambda\phi(x) = 0 \quad \text{for } x \in (0,1), \\ &\phi'(0) = 0 \quad \text{and} \quad \phi'(1) = 0. \end{aligned} \tag{15}$$

The regular Sturm–Liouville problem (12) has eigenvalues and eigenfunctions given by

$$\lambda_k = (k\pi)^2 \quad \text{and} \quad \phi_k(x) = \cos(k\pi x) \quad \text{for } k = 0,1,2,3,\ldots,$$

respectively. We then have that these eigenfunctions are orthogonal, and that the set $\{1\} \cup \{\sqrt{2}\cos(k\pi x)\}_{k=1}^{\infty}$ consists of orthonormal eigenfunctions. Furthermore, the corresponding Green's function of IBVP (14) is defined by

$$G(x,t,\xi,\tau) = 2 + 2\sum_{k=1}^{\infty} \cos(k\pi x)\cos(k\pi\xi)e^{-k^2\pi^2(t-\tau)} \quad \text{for } 0 \leq \tau < t \leq T.$$

In this example, we see that the principal eigenvalue $\widetilde{\lambda}$ is zero. Then we assume the initial function u_0 by $u_0(x) = \phi_1(x) = \cos(\pi x)$. By applying Theorem 3, the analytical solution of IBVP (14) is defined by $u(x,t) = \cos(\pi x)e^{-(\pi^2-1)t}$ for any $(x,t) \in [0,1] \times [0,T]$.

Example 3 Consider the heat equation problem with the periodic boundary condition:

$$\left.\begin{aligned} &u_t(x,t) = u_{xx}(x,t) + u(x,t) \quad \text{for } (x,t) \in (0,1) \times (0,T], \\ &u(x,0) = \sin(2\pi x) \quad \text{for } x \in [0,1], \\ &u(0,t) = u(1,t) \quad \text{and} \quad u_x(0,t) = u_x(1,t) \quad \text{for } 0 < t \leq T. \end{aligned}\right\} \tag{16}$$

The regular Sturm–Liouville problem of the problem (16) is the following:

$$\begin{aligned} &\phi''(x) + \lambda\phi(x) = 0 \quad \text{for } x \in (0,1), \\ &\phi(0) = \phi(1) \quad \text{and} \quad \phi'(0) = \phi'(1). \end{aligned} \tag{17}$$

It's well-known that the eigenvalues and eigenfunctions of the regular Sturm–Liouville problem (17) are

$$\lambda_k = (2k\pi)^2 \quad \text{and} \quad \phi_k(x) = \sin(2k\pi x) \quad \text{for } k = 1,2,3,\ldots,$$

respectively. The set $\{\sqrt{2}\sin(2k\pi x)\}_{k=1}^{\infty}$ forms an orthonormal set. Moreover, the Green's function of IBVP (16) is of the form:

$$G(x,t,\xi,\tau) = 2\sum_{k=1}^{\infty}\sin(2k\pi x)\sin(2k\pi\xi)e^{-4k^2\pi^2(t-\tau)} \quad \text{for } 0 \le \tau < t \le T.$$

It follows from Theorem 3 that $u(x,t) = \sin(2\pi x)e^{-(4\pi^2-1)t}$ for any $(x,t) \in [0,1] \times [0,T]$ is the analytic solution of problem (16).

Example 4 Consider the following heat equation problem with the Dirichlet condition:

$$\left.\begin{aligned} &u_t(x,t) = u_{xx}(x,t) + 3u_x(x,t) + u(x,t) \quad \text{for } (x,t) \in (0,1) \times (0,T], \\ &u(x,0) = e^{-\frac{3x}{2}}\sin(\pi x) \quad \text{for } x \in [0,1], \\ &u(0,t) = 0 \quad \text{and} \quad u(1,t) = 0 \quad \text{for } 0 < t \le T. \end{aligned}\right\} \tag{18}$$

Let us consider the equation: $u_t(x,t) = u_{xx}(x,t) + 3u_x(x,t) + u(x,t)$. It can be rewritten in the form:

$$u_t(x,t) = \frac{1}{e^{3x}}\left(e^{3x}u_x\right)_x + u(x,t) \quad \text{for } (x,t) \in (0,1) \times (0,T].$$

Thus, the regular Sturm–Liouville problem corresponding to problem (18) is

$$\begin{aligned} &\phi''(x) + 3\phi'(x) + \lambda\phi(x) = 0 \quad \text{for } x \in (0,1), \\ &\phi(0) = 0 \quad \text{and} \quad \phi(1) = 0. \end{aligned} \tag{19}$$

The characteristic equation of Eq. (19) is

$$r^2 + 3r + \lambda = 0,$$

with zeroes

$$r_1 = \frac{-3 + \sqrt{9-4\lambda}}{2} \quad \text{and} \quad r_2 = \frac{-3 - \sqrt{9-4\lambda}}{2}.$$

If $\lambda < \frac{9}{4}$, then r_1 and r_2 are real and distinct, so the general solution of the differential equation in Eq. (19) is

$$\phi(x) = c_1e^{r_1x} + c_2e^{r_2x},$$

where c_1 and c_2 are arbitrary constants. The boundary conditions require that $c_1 + c_2 = 0$ and $c_1e^{r_1} + c_2e^{r_2} = 0$. Since the determinant of this system is $e^{r_2} - e^{r_1} \neq 0$, the system has only the trivial solution. Therefore λ isn't an eigenvalue of Eq. (19).

If $\lambda = \frac{9}{4}$, then $r_1 = r_2 = -3/2$, so the general solution of the differential equation in Eq. (19) is

$$\phi(x) = (c_1 + c_2x)e^{r_1x},$$

where c_1 and c_2 are arbitrary constants. The boundary condition $\phi(0) = 0$ requires that $c_1 = 0$, so $\phi(x) = c_2 x e^{r_1 x}$ and the boundary condition $\phi(1) = 0$ requires that $c_2 = 0$. Therefore $\lambda = 9/4$ isn't an eigenvalue of Eq. (19).

If $\lambda > \frac{9}{4}$ then

$$r_1 = -\frac{3}{2} + i\omega \quad \text{and} \quad r_2 = -\frac{3}{2} - i\omega$$

with

$$\omega = \frac{\sqrt{4\lambda - 9}}{2}, \quad \text{or equivalently} \quad \lambda = \frac{4\omega^2 + 9}{4}.$$

In this case the general solution of the differential equation in Eq. (19) is

$$\phi(x) = e^{-\frac{3x}{2}}(c_1 \cos \omega x + c_2 \sin \omega x),$$

where c_1 and c_2 are arbitrary constants. The boundary condition $\phi(0) = 0$ requires that $c_1 = 0$, so $\phi(x) = c_2 e^{-\frac{3x}{2}} \sin \omega x$. Furthermore, the boundary condition $\phi(1) = 0$ holds with $c_2 \neq 0$ if and only if $\omega = k\pi$ for any $k = 1, 2, 3, \ldots$. Then the eigenvalues are $\lambda_k = k^2\pi^2 + \frac{9}{4}$, with associated eigenfunctions $\phi_k(x) = c_2 e^{-\frac{3x}{2}} \sin(k\pi x)$ for any $k = 1, 2, 3, \ldots$. We then have that eigenfunctions are orthogonal with respect to the weight function e^{3x}, and that the set $\{\sqrt{2}e^{-\frac{3x}{2}} \sin(k\pi x)\}_{k=1}^{\infty}$ consists of orthonormal eigenfunctions. Thus, the corresponding Green's function of problem (18) is defined by

$$G(x, t, \xi, \tau) = 2\sum_{k=1}^{\infty} e^{-3x} \sin(k\pi x) \sin(k\pi \xi) e^{-(k^2\pi^2 + \frac{9}{4})(t-\tau)} \quad \text{for } 0 \leq \tau < t \leq T.$$

Therefore, the analytical solution of IBVP (18) is given by

$$u(x, t) = e^{-\frac{3x}{2}} \sin(k\pi x) e^{-(\pi^2 + \frac{5}{4})t} \quad \text{for any } (x, t) \in [0, 1] \times [0, T].$$

Example 5 Consider the parabolic partial differential equation with variable coefficients:

$$\left.\begin{aligned} &u_t(x, t) = x^2 u_{xx}(x, t) + x u_x(x, t) + u(x, t) \quad \text{for } (x, t) \in (1, 2) \times (0, T], \\ &u(x, 0) = \sin(\tfrac{\pi}{\ln 2} \ln x) \quad \text{for } x \in [1, 2], \\ &u(1, t) = 0 \quad \text{and} \quad u(2, t) = 0 \quad \text{for } 0 < t \leq T. \end{aligned}\right\} \tag{20}$$

Let us consider the equation $u_t(x, t) = x^2 u_{xx}(x, t) + x u_x(x, t) + u(x, t)$. It can be rewritten in the form:

$$u_t(x, t) = x\big(x u_x(x, t)\big)_x + u(x, t) \quad \text{for } (x, t) \in (1, 2) \times (0, T].$$

The regular Sturm–Liouville problem corresponding to problem (20) is defined by

$$\begin{aligned} &x\big(x\phi'(x)\big)' + \lambda\phi(x) = 0 \quad \text{for } x \in (1, 2), \\ &\phi(1) = 0 \quad \text{and} \quad \phi(2) = 0. \end{aligned} \tag{21}$$

If $\lambda = 0$, the differential equation in Eq. (21) reduces to $x(x\phi'(x))' = 0$, so $x\phi'(x) = c_1$,

$$\phi'(x) = \frac{c_1}{x} \quad \text{and} \quad \phi(x) = c_1 \ln x + c_2,$$

where c_1 and c_2 are arbitrary constants. The boundary condition $\phi(1) = 0$ requires that $c_2 = 0$, so $\phi(x) = c_1 \ln x$. The boundary condition $\phi(2) = 0$ requires that $c_1 \ln x = 0$, so $c_1 = 0$. Therefore, zero isn't an eigenvalue of Eq. (21).

If $\lambda < 0$, we write $\lambda = -\omega^2$ with $\omega > 0$, so Eq. (21) becomes

$$x^2\phi''(x) + x\phi'(x) - \omega^2\phi(x) = 0,$$

an Euler equation with indicial equation

$$r^2 - \omega^2 = (r - \omega)(r + \omega) = 0.$$

Therefore, $\phi(x) = c_1x^{\omega} + c_2x^{-\omega}$, where c_1 and c_2 are arbitrary constants. The boundary conditions require that $c_1 + c_2 = 0$ and $2^{\omega}c_1 + 2^{-\omega}c_2 = 0$. Since the determinant of this system is $2^{-\omega} + 2^{\omega} \neq 0$, $c_1 = c_2 = 0$. Therefore, Eq. (21) has no negative eigenvalues.

If $\lambda > 0$, we write $\lambda = \omega^2$ with $\omega > 0$. Then Eq. (21) becomes

$$x^2\phi''(x) + x\phi'(x) + \omega^2\phi(x) = 0,$$

an Euler equation with indicial equation

$$r^2 + \omega^2 = (r - i\omega)(r + i\omega) = 0.$$

Thus, $\phi(x) = c_1 \cos(\omega \ln x) + c_2 \sin(\omega \ln x)$, where c_1 and c_2 are arbitrary constants. The boundary condition $\phi(1) = 0$ requires that $c_1 = 0$. Therefore, $\phi(x) = c_2 \sin(\omega \ln x)$. Since $\phi(2) = 0$, we obtain that $c_2 \sin(\omega \ln 2) = 0$. This holds with $c_2 \neq 0$ if and only if $\omega = \frac{k\pi}{\ln 2}$ for any $k = 1, 2, 3, \ldots$. Hence, the eigenvalues of Eq. (21) are $\lambda_k = (\frac{k\pi}{\ln 2})^2$, with associated eigenfunctions $\phi_k(x) = c_2 \sin(\frac{k\pi}{\ln 2} \ln x)$ for any $k = 1, 2, 3, \ldots$. We then have that eigenfunctions are orthogonal with respect to the weight function x^{-1}, and that the set $\{\sqrt{\frac{2}{\ln 2}} \sin(\frac{k\pi}{\ln 2} \ln x)\}_{k=1}^{\infty}$ consists of orthonormal eigenfunctions. Moreover, the corresponding Green's function of problem (20) is given by

$$G(x, t, \xi, \tau) = \frac{2}{\ln 2} \sum_{k=1}^{\infty} \sin\left(\frac{k\pi}{\ln 2} \ln x\right) \sin\left(\frac{k\pi}{\ln 2} \ln \xi\right) e^{-(\frac{k\pi}{\ln 2})^2 (t-\tau)} \quad \text{for } 0 \leq \tau < t \leq T.$$

Therefore, the analytical solution of IBVP (20) is given by

$$u(x, t) = \sin\left(\frac{\pi}{\ln 2} \ln x\right) e^{-((\frac{\pi}{\ln 2})^2 - 1)t} \quad \text{for any } (x, t) \in [0, 1] \times [0, T].$$

5 Conclusion

This research paper deals with the new method used to find the solutions for IBVPs. This method is named the Sawangtong's Green function homotopy perturbation method (SGHPM). The SGHPM is the method that combines the Green's function method with

the homotopy perturbation method. For the SGHPM technique, the boundary conditions are not used in the calculation process for finding the analytical solution of the problem. But the property of the boundary conditions still is included in the property of the Green's function. This makes the SGHPM process simple, easy, and effective. Therefore, the SGHPM is a technique which is powerful and efficient for finding approximate analytic solutions of IBVPs as SGHPM applications are presented in Sect. 4.

Acknowledgements
This research was funded by King Mongkut's University of Technology North Bangkok, contract no. KMUTNB-62-KNOW-34 and this research was partially supported by the Centre of Excellence in Mathematics, PERDO, Commission on Higher Education, Ministry of Education, Thailand.

Funding
This research was funded by King Mongkut's University of Technology North Bangkok, contract no. KMUTNB-62-KNOW-34 and this research was partially supported by the Centre of Excellence in Mathematics, PERDO, Commission on Higher Education, Ministry of Education, Thailand.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
PS investigated the SGHPM method. WS and PS proposed the research idea for solving IBVPs by SGHPM. PS wrote this paper. All authors contributed to editing and revising the manuscript. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, Faculty of Science, Mahidol University, Bangkok, Thailand. [2]Centre of Excellence in Mathematics, PERDO, Commission on Higher Education, Ministry of Education, Bangkok, Thailand. [3]Department of Mathematics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, Bangkok, Thailand.

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 23 July 2019 Accepted: 25 September 2019 Published online: 02 October 2019

References

1. Baholian, E., Azizi, A., Saeidian, J.: Some notes on using the homotopy perturbation method for solving time-dependent differential equations. Math. Comput. Model. **50**, 213–224 (2009)
2. Chan, C.Y., Chan, W.Y.: Existence of classical solutions for degenerate semilinear parabolic problems. Appl. Math. Comput. **101**, 125–149 (1999)
3. He, J.: Homotopy perturbation technique. Comput. Methods Appl. Mech. Eng. **178**, 1–12 (1999)
4. He, J.: A coupling method of a homotopy technique and a perturbation technique for non-linear problems. Int. J. Non-Linear Mech. **35**, 37–43 (2000)
5. Kirk, C.M., Roberts, C.A.: A quenching problem for the heat equation. J. Integral Equ. Appl. **14**, 53–72 (2002)
6. Olmstead, W.E.: Critical speed for the avoidance of blow-up in a reactive–diffusive medium. Z. Angew. Math. Phys. **48**, 701–710 (1997)