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Solitons for the modified (2 + 1)-dimensional Konopelchenko–Dubrovsky equations

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Abstract

In the paper, we consider the modified (2 + 1)-dimensional Konopelchenko–Dubrovsky equations which possess high order nonlinear terms. Under the aid of Maple, we derive the exact traveling wave solutions of the mKDs by the auxiliary equation approach. Under some special conditions, Jacobi elliptic function solutions, degenerated triangular function solutions, and solitons for the mKD equations are constructed.

MSC: 35Q53; 35B

Keywords: Modified Konopelchenko–Dubrovsky equations; Auxiliary equation approach; Jacobi elliptic function solutions; Degenerated triangle function solutions; Solitons

1 Introduction

Konopelchenko and Dubrovsky [1] ever presented a (2 + 1)-dimensional Konopelchenko and Dubrovsky (KD) model as follows:

$$\begin{cases} u_t - u_{xxx} - 6b_0uu_x + \frac{3}{2}a_0^2u^2u_x - 3v_y + 3a_0u_xv = 0, \\ u_y = v_x, \end{cases} \tag{1}$$

where u and v are two analytic functions corresponding to the variables x, y, and t; a_0 and b_0 are the real parameters. If $a_0 = 0$, Eq. (1) reshapes into the well-known Kadomtsev–Petviashvili (KP) equation; If $b_0 = 0$, it turns into the modified KP equation; If $u_y = 0$, the first row of Eq. (1) recasts into the Gardner equation, the combination of KdV and modified KdV [2].

In the field of nonlinear partial differential equations (PDEs), one knows that all kinds of solutions for these equations are of particular interest to scientists [3–15]. Particularly, many researchers have made use of some effective methods to study the exact solutions of KD Eq. (1). Special and general N-soliton solutions of Eq. (1) have been obtained via Hirota's bilinear approach [16]. Bilinear-form Bäcklund transformation and single-soliton solutions have been presented by [17]. Based on the Hirota bilinear form of Eq. (1), the lump waves, solitary waves, as well as interaction between lump waves and solitary waves, have been obtained [18]. Based on the Hirota direct method and linear superposition principle, Eq. (1) was investigated to have the complexiton and resonant multiple wave solutions [19]. By means of the variable separation method and improved mapping approach,

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new types of variable-separation solutions (including solitary wave solutions, periodic wave solutions, and rational function solutions) for Eq. (1) have been successfully obtained [20]. Based on the similarity transformations approach together with Lie group theory, the KD system has been reshaped into another system of ordinary differential equations, new closed form solutions of Eq. (1) have been obtained [21]. The tanh-sech method, cosh-sinh method, and the exponential method were employed to discuss Eq. (1) by Wazwaz [22], and some exact solutions of distinct physical structures, solitons, kinks, and periodic wave solutions were derived.

Motivated by these works [1, 2, 19–22], the presented paper will study the following modified Konopelchenko and Dubrovsky (mKD) equations:

$$\begin{cases} u_t - u_{xxx} - bu^n u_x + au^{2n}u_x - 3v_y + ku_x v^n = 0, \\ u_y = v_x, \end{cases} \tag{2}$$

where a, b, k, and n are nonzero real constants. Furthermore, when $n = 1$, $b = 6b_0$, $a = \frac{3}{2}a_0^2$, and $k = 3a_0$ in Eq. (2), it turns into Eq. (1). The paper will make use of the auxiliary differential equation approach to solve Eq. (2) analytically and obtains abundant new exact solutions of Eq. (2) in general and special cases. Moreover, some Jacobi elliptic function solutions will degenerate solitons and triangular function solutions when the module approaches to 0 or 1. It shows that most of the solutions provided in this work are different to those presented in [1, 16–24].

2 Method description

The main idea of the auxiliary equation approach is described as follows. Firstly, we introduce a transform $\xi = \mu(x-ct)$ to a given partial differential equation (PDE) with dependent variable u, independent variables x and t. The PDE will be changed into an ordinary differential equation (ODE). Of course, μ and c should be nonzero. Secondly, we seek for the solution of the ODE by the following:

$$u(\xi) = \sum_{i=0}^{n} h_i \omega^i(\xi), \tag{3}$$

where the integer n and the constants $h_0, h_1, \dots, h_n$ are unknown. For nonnegative integers q and p, the highest order of $\frac{\partial^p u}{\partial \xi^p}$ and $u^q \frac{\partial^p u}{\partial \xi^p}$ are $n+p$ and $qn+p$, respectively. Balancing the highest order in ODE, n will be determined. Thirdly, we introduce an expression $\omega(\xi)$ defined as

$$\left(\frac{d\omega}{d\xi}\right)^2 = c_1 + c_2\omega^2 + \frac{c_3}{2}\omega^4, \tag{4}$$

where $c_i \in R$ $(i = 1, 2, 3)$. Substituting Eqs. (3) and (4) into ODE and then forcing the coefficients of each power of $\omega(\xi)$ to be zero, several algebraic expressions will be obtained. Solving these expressions with the aid of Maple, some exact solutions for PDE can be obtained.

3 Exact solutions to Eq. (2)

Introducing $\xi = \mu(x + y - ct)$ (where $c \neq 0$, $\mu \in R$) into (2), we obtain the following form:

$$\begin{cases} -cu_\xi - \mu^2 u_{\xi\xi\xi} - b(u^n)u_\xi + a(u^{2n})u_\xi - 3v_\xi + ku_\xi v^n = 0, \\ u_\xi = v_\xi. \end{cases} \tag{5}$$

Substituting $u = v$ into the first equation of (5) yields

$$-(3 + c)u' - \mu^2 u''' + a\left(u^{2n}\right)u' + (k - b)u^n u' = 0. \tag{6}$$

Using a transform $u^n = W$, we know that

$$\begin{cases} u' = \frac{1}{n-1}^{\frac{1}{n-1}-1} W', \\ u'' = \frac{1}{n-1}(\frac{1}{n-1} - 1)W^{\frac{1}{n-1}-2}(W')^2 + \frac{1}{n-1} W^{\frac{1}{n-1}-1} W'', \\ u''' = \frac{(2-n)(3-2n)}{(n-1)^3} W^{\frac{1}{n-1}-3} W'^2 + \frac{3(2-n)}{(n-1)^2} W^{\frac{1}{n-1}-2} W'W'' + \frac{1}{n-1} W^{\frac{1}{n-1}-1} W'''. \end{cases} \tag{7}$$

Eq. (6) can be rewritten as

$$\begin{aligned} &-(c + 3)W^2 W' - \mu^2\left(\frac{1}{n} - 1\right)\left(\frac{1}{n} - 2\right)W'^3 - 3\mu^2\left(\frac{1}{n} - 1\right)WW'W'' - \mu^2 W^2 W''' \\ &\quad + (k - b)W^3 W' + aW^4 W' = 0, \end{aligned} \tag{8}$$

hence $n = 1$, Eq. (8) has the following solution:

$$W(\xi) = h_0 + h_1\omega, \tag{9}$$

where $\omega(\xi)$ satisfies the sub-equation $(\frac{d\omega}{d\xi})^2 = c_1 + c_2\omega^2 + \frac{c_3}{2}\omega^4$, some of the solutions can be expressed in Table 1 (also listed in [23] and [24]).

Substituting Eq. (9) and the expression $(\frac{d\omega}{d\xi})^2 = c_1 + c_2\omega^2 + \frac{c_3}{2}\omega^4$ into Eq. (8) and equating to zero the coefficients of all powers of $\omega(\xi)$ yields

$$-ah_1^5 + \frac{\mu^2}{2}\left(\frac{1}{n} - 1\right)\left(\frac{1}{n} - 2\right)h_1^3 c_3 + 3\left(\frac{1}{n} - 1\right)\mu^2 h_1^3 c_3 + 3\mu^2 h_1^3 c_3 = 0, \tag{10}$$

$$\left[(b - k)h_1 - 2ah_0h_1\right]h_1^3 - 2ah_0h_1^4 + 3\left(\frac{1}{p} - 1\right)\mu^2 h_0 h_1^2 c_3 + 6\mu^2 h_0 h_1^2 c_3 = 0, \tag{11}$$

$$\begin{aligned} &ch_1^3 + \left[3 + (b - k)h_0 - ah_0^2\right]h_1^3 + 2\left[(b - k)h_1 - 2ah_0h_1\right]h_0h_1^2 - ah_0^2h_1^3 \\ &\quad + \mu^2\left(\frac{1}{n} - 1\right)\left(\frac{1}{n} - 2\right)h_1^3 c_2 + 3\left(\frac{1}{n} - 1\right)\mu^2 h_1^3 c_2 + 3c_3\mu^2 h_0^2 + \mu^2 c_2 h_1^3 = 0, \end{aligned} \tag{12}$$

$$\begin{aligned} &2ch_0h_1^2 + 2\left[3 + (b - k)h_0 - ah_0^2\right]h_1^2h_0 + \left[(b - k)h_1 - 2ah_0h_1\right]h_0^2h_1 \\ &\quad + 3\left(\frac{1}{n} - 1\right)\mu^2 h_1^2 h_0 c_2 + 2\mu^2 h_1^2 h_0 c_2 = 0, \end{aligned} \tag{13}$$

$$\begin{aligned} &ch_0^2h_1 + \left[3 + (b - k)h_0 - ah_0^2\right]h_0^2h_1 + \left(\frac{1}{n} - 1\right)\left(\frac{1}{n} - 2\right)\mu^2 h_1^3 c_1 \\ &\quad + \mu^2 h_0^2 h_1 c_2 = 0. \end{aligned} \tag{14}$$

Table 1 Different solutions of the sub-equation

No.	$z(\xi)$	c_3	c_2	c_1
1	$sn(\xi), cd(\xi)=\frac{cn(\xi)}{dn(\xi)}$	$2r^2$	$-(r^2+1)$	1
2	$cn(\xi)$	$-2r^2$	$2r^2-1$	$1-r^2$
3	$dn(\xi)$	-2	$2-r^2$	r^2-1
4	$nc(\xi)=\frac{1}{cn(\xi)}$	$2(1-r^2)$	$2r^2-1$	$-r^2$
5	$ns(\xi)=\frac{1}{sn(\xi)}, dc(\xi)=\frac{dn(\xi)}{cn(\xi)}$	2	$-(r^2+1)$	r^2
6	$nd(\xi)=\frac{1}{dn(\xi)}$	$2(r^2-1)$	$2-r^2$	-1
7	$cs(\xi)=\frac{cn(\xi)}{sn(\xi)}$	2	$2-r^2$	$1-r^2$
8	$sc(\xi)=\frac{sn(\xi)}{cn(\xi)}$	$2(1-r^2)$	$2-r^2$	1
9	$sd(\xi)=\frac{sn(\xi)}{dn(\xi)}$	$2r^2(r^2-1)$	$2r^2-1$	1
10	$ds(\xi)=\frac{dn(\xi)}{sn(\xi)}$	2	$2r^2-1$	r^4-r^2
11	$rcn(\xi)\pm dn(\xi)$	$-\frac{1}{2}$	$\frac{r^2+1}{2}$	$-\frac{(1-r^2)^2}{4}$
12	$\frac{1}{sn(\xi)}\pm\frac{cn(\xi)}{sn(\xi)}$	$\frac{1}{2}$	$\frac{-2r^2+1}{2}$	$\frac{1}{4}$
13	$\frac{1}{cn(\xi)}\pm\frac{sn(\xi)}{cn(\xi)}$	$\frac{1-r^2}{2}$	$\frac{r^2+1}{2}$	$\frac{1-r^2}{4}$
14	$\frac{1}{sn(\xi)}\pm\frac{dn(\xi)}{sn(\xi)}$	$\frac{1}{2}$	$\frac{r^2-2}{2}$	$\frac{r^4}{4}$
15	$sn(\xi)\pm icn(\xi), \frac{dn(\xi)}{\sqrt{1-r^2}sn(\xi)\pm cn(\xi)}$	$\frac{r^2}{2}$	$\frac{r^2-2}{2}$	$\frac{r^2}{4}$
16	$rsn(\xi)\pm idn(\xi), \frac{sn(\xi)}{1\pm cn(\xi)}$	$\frac{1}{2}$	$\frac{1-2r^2}{2}$	$\frac{1}{4}$
17	$\frac{sn(\xi)}{1\pm dn(\xi)}$	$\frac{r^2}{2}$	$\frac{r^2-2}{2}$	$\frac{1}{4}$
18	$\frac{dn(\xi)}{1\pm rsn(\xi)}$	$\frac{r^2-1}{2}$	$\frac{r^2+1}{2}$	$\frac{r^2-1}{4}$
19	$\frac{cn(\xi)}{1\pm sn(\xi)}$	$\frac{1-r^2}{2}$	$\frac{r^2+1}{2}$	$\frac{-r^2+1}{4}$
20	$\frac{sn(\xi)}{dn(\xi)\pm cn(\xi)}$	$\frac{(1-r^2)^2}{2}$	$\frac{r^2+1}{2}$	$\frac{1}{4}$
21	$\frac{cn(\xi)}{\sqrt{1-r^2}\pm dn(\xi)}$	$\frac{r^4}{2}$	$\frac{r^2-2}{2}$	$\frac{1}{4}$

Under the help of Maple, the above algebraic equation system can be solved as follows:

$$\begin{cases} n=1, \quad h_0=\frac{(b-k)}{2a}, \quad h_1=\pm\frac{(b-k)}{2a}\sqrt{-\frac{c_3}{c_2}}, \\ \mu=\pm\frac{(b-k)}{2}\sqrt{-\frac{1}{3ac_2}}, \quad c=-3-\frac{(b-k)^2}{6a}, \end{cases} \tag{15}$$

$$\begin{cases} n=\frac{1}{2}, \quad h_0=\frac{2(b-k)}{5a}, \quad h_1=\pm\frac{2(b-k)}{5a}\sqrt{-\frac{c_3}{c_2}}, \\ \mu=\pm\frac{(b-k)}{5}\sqrt{-\frac{2}{3ac_2}}, \quad c=-3-\frac{16(b-k)^2}{75a}, \end{cases} \tag{16}$$

$$\begin{cases} n=2, \quad \mu=\pm\left(\frac{-5c_2(b-k)^2+5\epsilon(b-k)^2\sqrt{c_2^2-2c_1c_3}}{48c_1c_3a}\right)^{\frac{1}{2}}, \\ h_0=\frac{5(b-k)}{8a}, \quad h_1=\pm\frac{\mu}{4}\sqrt{\frac{30c_3}{a}}, \quad c=-3-\frac{5(b-k)^2}{32a}-\frac{1}{4}\mu^2c_2, \end{cases} \tag{17}$$

and

$$\begin{cases} h_0=\frac{(b-k)(2n+1)}{2a(n+2)}, \quad h_1=\pm\frac{(b-k)(2n+1)}{2a(n+2)}\sqrt{-\frac{c_3}{c_2}}, \quad 2c_1c_3=c_2^2, \\ \mu=\pm\frac{(b-k)n}{n+2}\sqrt{-\frac{2n+1}{2a(n+1)c_2}}, \quad c=-3-\frac{(b-k)^2(2n+1)}{a(n+2)^2(n+1)}, \end{cases} \tag{18}$$

where $\epsilon=\pm1$.

3.1 Solutions in the case $n = 1$

Consider Eq. (15). From Table 1, some exact solutions for Eq. (2) can be obtained.

$$u_{1.1}=v_{1.1}=\frac{b-k}{2a}+\frac{\epsilon(b-k)r}{2a}\sqrt{\frac{2}{r^2+1}}sn\left[\frac{(b-k)}{2}\sqrt{\frac{1}{3a(r^2+1)}}(x+y-ct)\right], \tag{19}$$

$$u_{1.2} = v_{1.2} = \frac{b-k}{2a} + \frac{\epsilon(b-k)r}{2a}\sqrt{\frac{2}{r^2+1}}cd\left[\frac{(b-k)}{2}\sqrt{\frac{1}{3a(r^2+1)}}(x+y-ct)\right], \tag{20}$$

$$u_{1.3} = v_{1.3} = \frac{b-k}{2a} + \frac{\epsilon(b-k)r}{2a}\sqrt{\frac{2}{2r^2-1}}cn\left[\frac{b-k}{2}\sqrt{\frac{1}{3a(1-2r^2)}}(x+y-ct)\right], \tag{21}$$

$$u_{1.4} = v_{1.4} = \frac{b-k}{2a} + \frac{\epsilon(b-k)}{2a}\sqrt{\frac{2}{2-r^2}}dn\left[\frac{b-k}{2}\sqrt{\frac{1}{3a(r^2-2)}}(x+y-ct)\right], \tag{22}$$

$$u_{1.5} = v_{1.5} = \frac{b-k}{2a} + \frac{\epsilon(b-k)}{2a}\sqrt{\frac{2(r^2-1)}{2r^2-1}}nc\left[\frac{b-k}{2}\sqrt{\frac{1}{3a(1-2r^2)}}(x+y-ct)\right], \tag{23}$$

$$u_{1.6} = v_{1.6} = \frac{b-k}{2a} + \frac{\epsilon(b-k)}{2a}\sqrt{\frac{2}{r^2+1}}ns\left[\frac{(b-k)}{2}\sqrt{\frac{1}{3a(r^2+1)}}(x+y-ct)\right], \tag{24}$$

$$u_{1.7} = v_{1.7} = \frac{b-k}{2a} + \frac{\epsilon(b-k)}{2a}\sqrt{\frac{2}{r^2+1}}dc\left[\frac{(b-k)}{2}\sqrt{\frac{1}{3a(r^2+1)}}(x+y-ct)\right], \tag{25}$$

$$u_{1.8} = v_{1.8} = \frac{b-k}{2a} + \frac{\epsilon(b-k)}{2a}\sqrt{\frac{2(r^2-1)}{r^2-2}}nd\left[\frac{b-k}{2}\sqrt{\frac{1}{3a(r^2-2)}}(x+y-ct)\right], \tag{26}$$

$$u_{1.9} = v_{1.9} = \frac{(b-k)}{2a} + \frac{\epsilon(b-k)}{2a}\sqrt{\frac{2}{r^2-2}}cs\left[\frac{(b-k)}{2}\sqrt{\frac{1}{3a(r^2-2)}}(x+y-ct)\right], \tag{27}$$

$$u_{1.10} = v_{1.10} = \frac{b-k}{2a} + \frac{\epsilon(b-k)}{2a}\sqrt{\frac{2(1-r^2)}{r^2-2}}sc\left[\frac{b-k}{2}\sqrt{\frac{1}{3a(r^2-2)}}(x+y-ct)\right], \tag{28}$$

$$u_{1.11} = v_{1.11} = \frac{b-k}{2a} + \frac{\epsilon(b-k)r}{2a}\sqrt{\frac{2(r^2-1)}{1-2r^2}}sd\left[\frac{b-k}{2}\sqrt{\frac{1}{3a(1-2r^2)}}(x+y-ct)\right], \tag{29}$$

$$u_{1.12} = v_{1.12} = \frac{b-k}{2a} + \frac{\epsilon(b-k)}{2a}\sqrt{\frac{2}{1-2r^2}}ds\left[\frac{b-k}{2}\sqrt{\frac{1}{3a(1-2r^2)}}(x+y-ct)\right], \tag{30}$$

$$\begin{aligned} u_{1.13} &= v_{1.13} \\ &= \frac{b-k}{2a} + \frac{\epsilon(b-k)}{2a}\sqrt{\frac{1}{1+r^2}}\left\{rcn\left[\frac{(b-k)}{2}\sqrt{-\frac{2}{3a(r^2+1)}}(x+y-ct)\right]\right. \\ &\quad \left.\pm nd\left[\frac{(b-k)}{2}\sqrt{-\frac{2}{3a(r^2+1)}}(x+y-ct)\right]\right\}, \end{aligned} \tag{31}$$

$$\begin{aligned} u_{1.14} &= v_{1.14} \\ &= \frac{b-k}{2a} + \frac{\epsilon(b-k)}{2a}\sqrt{\frac{1}{2r^2-1}}\left\{ns\left[\frac{(b-k)}{2}\sqrt{\frac{2}{3a(2r^2-1)}}(x+y-ct)\right]\right. \\ &\quad \left.\pm cs\left[\frac{(b-k)}{2}\sqrt{-\frac{2}{3a(1-2r^2)}}(x+y-ct)\right]\right\}, \end{aligned} \tag{32}$$

$$\begin{aligned}
u_{1.15} &= v_{1.15} \\
&= \frac{b-k}{2a} + \frac{\epsilon(b-k)}{2da}\sqrt{\frac{r^2-1}{1+r^2}}\left\{nc\left[\frac{(b-k)}{2}\sqrt{-\frac{2}{3a(r^2+1)}}(x+y-ct)\right]\right. \\
&\quad \left.\pm sc\left[\frac{(b-k)}{2}\sqrt{-\frac{2}{3a(r^2+1)}}(x+y-ct)\right]\right\},
\end{aligned} \tag{33}$$

$$\begin{aligned}
u_{1.16} &= v_{1.16} \\
&= \frac{b-k}{2a} + \frac{\epsilon(b-k)}{2a}\sqrt{\frac{1}{2-r^2}}\left\{ns\left[\frac{(b-k)}{2}\sqrt{-\frac{2}{3a(r^2-2)}}(x+y-ct)\right]\right. \\
&\quad \left.\pm ds\left[\frac{(b-k)}{2}\sqrt{-\frac{2}{3a(r^2-2)}}(x+y-ct)\right]\right\},
\end{aligned} \tag{34}$$

$$\begin{aligned}
u_{1.17} &= v_{1.17} \\
&= \frac{(b-k)}{2a} + \frac{\epsilon(b-k)r}{2a}\sqrt{\frac{1}{2-r^2}}\left\{sn\left[\frac{\beta}{2}\sqrt{-\frac{2}{3a(r^2-2)}}(x+y-ct)\right]\right. \\
&\quad \left.\pm icn\left[\frac{(b-k)}{2}\sqrt{-\frac{2}{3a(r^2-2)}}(x+y-ct)\right]\right\},
\end{aligned} \tag{35}$$

$$\begin{aligned}
u_{1.18} &= v_{1.18} \\
&= \frac{(b-k)}{2a} + \frac{\epsilon(b-k)r}{2a}\sqrt{\frac{1}{2-r^2}} \\
&\quad \times \frac{dn[\frac{(b-k)}{2}\sqrt{-\frac{2}{3a(r^2-2)}}(x+y-ct)]}{\sqrt{1-r^2}sn[\frac{(b-k)}{2}\sqrt{-\frac{2}{3a(r^2-2)}}(x+y-ct)] \pm cn[\frac{(b-k)}{2}\sqrt{-\frac{2}{3a(r^2-2)}}(x+y-ct)]},
\end{aligned} \tag{36}$$

$$\begin{aligned}
u_{1.19} &= v_{1.19} \\
&= \frac{(b-k)}{2a} + \frac{\epsilon(b-k)}{2a}\sqrt{\frac{1}{2r^2-1}}\left\{rsn\left[\frac{(b-k)}{2}\sqrt{-\frac{2}{3a(1-2r^2)}}(x+y-ct)\right]\right. \\
&\quad \left.\pm idn\left[\frac{(b-k)}{2}\sqrt{-\frac{2}{3a(1-2r^2)}}(x+y-ct)\right]\right\},
\end{aligned} \tag{37}$$

$$\begin{aligned}
u_{1.20} &= v_{1.20} \\
&= \frac{(b-k)}{2a} + \frac{\epsilon(b-k)}{2a}\sqrt{\frac{1}{2r^2-1}} \cdot \frac{sn[\frac{(b-k)}{2}\sqrt{-\frac{1}{3a(1-2r^2)}}(x+y-ct)]}{1 \pm cn[\frac{(b-k)}{2}\sqrt{-\frac{2}{3a(1-2r^2)}}(x+y-ct)]},
\end{aligned} \tag{38}$$

$$\begin{aligned}
u_{1.21} &= v_{1.21} \\
&= \frac{(b-k)}{2a} + \frac{\epsilon(b-k)}{2a}\sqrt{\frac{r^2}{2-r^2}} \cdot \frac{sn[\frac{(b-k)}{2}\sqrt{-\frac{1}{3a(r^2-2)}}(x+y-ct)]}{1 \pm dn[\frac{(b-k)}{2}\sqrt{-\frac{2}{3a(r^2-2)}}(x+y-ct)]},
\end{aligned} \tag{39}$$

$$\begin{aligned} u_{1.22} &= v_{1.22} \\ &= \frac{(b-k)}{2a} + \frac{\epsilon(b-k)}{2a}\sqrt{\frac{1-r^2}{r^2+1}} \cdot \frac{dn[\frac{(b-k)}{2}\sqrt{-\frac{2}{3a(r^2+1)}}(x+y-ct)]}{1 \pm rsn[\frac{(b-k)}{2}\sqrt{-\frac{2}{3a(r^2+1)}}(x+y-ct)]}, \end{aligned} \tag{40}$$

$$\begin{aligned} u_{1.23} &= v_{1.23} \\ &= \frac{(b-k)}{2a} + \frac{\epsilon(b-k)}{2a}\sqrt{\frac{r^2-1}{1+r^2}} \cdot \frac{cn[\frac{(b-k)}{2}\sqrt{-\frac{2}{3a(r^2+1)}}(x+y-ct)]}{1 \pm sn[\frac{(b-k)}{2}\sqrt{-\frac{2}{3a(r^2+1)}}(x+y-ct)]}, \end{aligned} \tag{41}$$

$$\begin{aligned} u_{1.24} &= v_{1.24} \\ &= \frac{(b-k)}{2a} + \frac{\epsilon(b-k)(1-r^2)}{2a}\sqrt{-\frac{1}{r^2+1}} \\ &\quad \times \frac{sn[\frac{(b-k)}{2}\sqrt{-\frac{2}{3a(r^2+1)}}(x+y-ct)]}{dn[\frac{(b-k)}{2}\sqrt{-\frac{2}{3a(r^2+1)}}(x+y-ct)] \pm cn[\frac{(b-k)}{2}\sqrt{-\frac{2}{3a(r^2+1)}}(x+y-ct)]}, \end{aligned} \tag{42}$$

$$\begin{aligned} u_{1.25} &= v_{1.25} \\ &= \frac{b-k}{2a} + \frac{\epsilon(b-k)r^2}{2a}\sqrt{\frac{1}{2-r^2}} \frac{cn[\frac{b-k}{2}\sqrt{-\frac{2}{3a(r^2-2)}}(x+y-ct)]}{\sqrt{1-r^2} \pm dn[\frac{b-k}{2}\sqrt{-\frac{2}{3a(r^2-2)}}(x+y-ct)]}, \end{aligned} \tag{43}$$

where $c = -3 - \frac{(b-k)^2}{6a}$.

When r approaches 0 or 1, sn, cn, and dn degenerate into triangular or hyperbolic functions. Therefore, in the case $n = 1$, we can deduce Jacobi elliptic function solutions, solitons, and triangular function solutions for Eq. (2).

When $k = l = \frac{2b-a}{2a}\sqrt{\frac{2}{1+r^2}}$ and $w = k(3 + \frac{(b-k)^2}{6a})$, the solutions u_1, v_1, u_2, v_2, u_3, v_3, u_4, v_4 in [25] are the solutions $u_{1.1}$, $u_{1.2}$, $u_{1.6}$, $u_{1.7}$, respectively. When $k = l = \frac{2b-a}{2a}\sqrt{\frac{2}{1-2r^2}}$ and $w = k(3 + \frac{(b-k)^2}{6a})$, the solutions $u_{1.5}$, $u_{1.12}$ are the same as u_5, v_5, u_8, v_8 in [25]. When $k = l = \frac{2b-a}{2a}\sqrt{\frac{2}{r^2-2}}$ and $w = k(3 + \frac{(b-k)^2}{6a})$, the expressions u_6, v_6, u_7, v_7 are also obtained by Zhang [25].

When the modulus $r \to 1$, choosing $b = 6b_0$, $a = \frac{3}{2}a_0^2$, and $k = 3a_0$ in the solutions $u_{1.1}$ and $u_{2.3}$, we obtain

$$u_{1.1.1} = v_{1.1.1} = \frac{2b_0 - a_0}{a_0^2}\left(1 \pm \tanh\left[\frac{2b_0 - a_0}{2a_0}\left(x + y + \frac{4(a_0^2 + b_0^2 - a_0 b_0)}{a_0^2}t\right)\right]\right) \tag{44}$$

and

$$u_{1.6.1} = v_{1.6.1} = \frac{2b_0 - a_0}{a_0^2}\left(1 \pm \coth\left[\frac{2b_0 - a_0}{2a_0}\left(x + y + \frac{4(a_0^2 + b_0^2 - a_0 b_0)}{a_0^2}t\right)\right]\right), \tag{45}$$

which have also been established in Wazwaz [22].

3.2 Solutions in the case $n = \frac{1}{2}$

From Eq. (16) and Table 1, we obtain solutions of Eq. (2) when $n = \frac{1}{2}$:

$$\begin{aligned} u_{2.1} &= v_{2.1} \\ &= \left\{\frac{2(b-k)}{5a} \pm \frac{2(b-k)r}{5a}\sqrt{\frac{2}{r^2+1}}sn\left[\frac{(b-k)}{5}\sqrt{\frac{2}{3a(r^2+1)}}(x+y-ct)\right]\right\}^2, \end{aligned} \tag{46}$$

$$\begin{aligned} u_{2.2} &= v_{2.2} \\ &= \left\{\frac{2(b-k)}{5a} \pm \frac{2(b-k)r}{5a}\sqrt{\frac{2}{r^2+1}}cd\left[\frac{(b-k)}{5}\sqrt{\frac{2}{3a(r^2+1)}}(x+y-ct)\right]\right\}^2, \end{aligned} \tag{47}$$

$$\begin{aligned} u_{2.3} &= v_{2.3} \\ &= \left\{\frac{2(b-k)}{5a} \pm \frac{2(b-k)r}{5a}\sqrt{\frac{2}{2r^2-1}}cn\left[\frac{(b-k)}{5}\sqrt{\frac{2}{3a(1-2r^2)}}(x+y-ct)\right]\right\}^2, \end{aligned} \tag{48}$$

$$\begin{aligned} u_{2.4} &= v_{2.4} \\ &= \left\{\frac{2(b-k)}{5a} \pm \frac{2(b-k)}{5a}\sqrt{\frac{2}{2-r^2}}dn\left[\frac{(b-k)}{5}\sqrt{-\frac{2}{3a(2-r^2)}}(x+y-ct)\right]\right\}^2, \end{aligned} \tag{49}$$

$$\begin{aligned} u_{2.5} &= v_{2.5} \\ &= \left\{\frac{2(b-k)}{5a} \pm \frac{2(b-k)}{5a}\sqrt{\frac{2r^2-2}{2r^2-1}}nc\left[\frac{(b-k)}{5}\sqrt{\frac{2}{3a(1-2r^2)}}(x+y-ct)\right]\right\}^2, \end{aligned} \tag{50}$$

$$\begin{aligned} u_{2.6} &= v_{2.6} \\ &= \left\{\frac{2(b-k)}{5a} \pm \frac{2(b-k)}{5a}\sqrt{\frac{2}{r^2+1}}ns\left[\frac{(b-k)}{5}\sqrt{\frac{2}{3a(r^2+1)}}(x+y-ct)\right]\right\}^2, \end{aligned} \tag{51}$$

$$\begin{aligned} u_{2.7} &= v_{2.7} \\ &= \left\{\frac{2(b-k)}{5a} \pm \frac{2(b-k)}{5a}\sqrt{\frac{2}{r^2+1}}dc\left[\frac{\beta}{5}\sqrt{\frac{2}{3a(r^2+1)}}(x+y-ct)\right]\right\}^2, \end{aligned} \tag{52}$$

$$\begin{aligned} u_{2.8} &= v_{2.8} \\ &= \left\{\frac{2(b-k)}{5a} \pm \frac{2(b-k)}{5a}\sqrt{\frac{2-2r^2}{2-r^2}}nd\left[\frac{(b-k)}{5}\sqrt{\frac{2}{3a(r^2-2)}}(x+y-ct)\right]\right\}^2, \end{aligned} \tag{53}$$

$$\begin{aligned} u_{2.9} &= v_{2.9} \\ &= \left\{\frac{2(b-k)}{5a} \pm \frac{2(b-k)}{5a}\sqrt{\frac{2}{r^2-2}}cs\left[\frac{(b-k)}{5}\sqrt{\frac{2}{3a(r^2-2)}}(x+y-ct)\right]\right\}^2, \end{aligned} \tag{54}$$

$$\begin{aligned} u_{2.10} &= v_{2.10} \\ &= \left\{\frac{2(b-k)}{5a} \pm \frac{2(b-k)}{5a}\sqrt{\frac{2r^2-2}{2-r^2}}sc\left[\frac{(b-k)}{5}\sqrt{\frac{2}{3a(r^2-2)}}(x+y-ct)\right]\right\}^2, \end{aligned} \tag{55}$$

$$
\begin{aligned}
u_{2.11} &= v_{2.11} \\
&= \left\{ \frac{2(b-k)}{5a} \pm \frac{2(b-k)}{5a}\sqrt{\frac{2r^2(1-r^2)}{2r^2-1}}\, sd\left[\frac{b-k}{5}\sqrt{\frac{2}{3a-6ar^2}}(x+y-ct)\right]\right\}^2,
\end{aligned}
\tag{56}
$$

$$
\begin{aligned}
u_{2.12} &= v_{2.12} \\
&= \left\{ \frac{2(b-k)}{5a} \pm \frac{2(b-k)}{5a}\sqrt{\frac{2}{1-2r^2}}\, ds\left[\frac{b-k}{5}\sqrt{\frac{2}{3a(1-2r^2)}}(x+y-ct)\right]\right\}^2,
\end{aligned}
\tag{57}
$$

$$
\begin{aligned}
u_{2.13} &= v_{2.13} \\
&= \left\{ \frac{2(b-k)}{5a} \pm \frac{2(b-k)}{5a}\sqrt{\frac{1}{r^2+1}}\left(rcn\left[\frac{(b-k)}{5}\sqrt{-\frac{4}{3a(r^2+1)}}(x+y-ct)\right]\right.\right. \\
&\quad \left.\left. \pm\, dn\left[\frac{\beta}{5}\sqrt{-\frac{4}{3a(r^2+1)}}(x+y-ct)\right]\right)\right\}^2,
\end{aligned}
\tag{58}
$$

$$
\begin{aligned}
u_{2.14} &= v_{2.14} \\
&= \left\{ \frac{2(b-k)}{5a} \pm \frac{2(b-k)}{5a}\sqrt{-\frac{1}{1-2r^2}}\left(ns\left[\frac{(b-k)}{5}\sqrt{-\frac{4}{3a(-2r^2+1)}}(x+y-ct)\right]\right.\right. \\
&\quad \left.\left. \pm\, cs\left[\frac{(b-k)}{5}\sqrt{-\frac{4}{3a(-2r^2+1)}}(x+y-ct)\right]\right)\right\}^2,
\end{aligned}
\tag{59}
$$

$$
\begin{aligned}
u_{2.15} &= v_{2.15} \\
&= \left\{ \frac{2(b-k)}{5a} \pm \frac{2(b-k)}{5a}\sqrt{-\frac{1-r^2}{r^2+1}}\left(nc\left[\frac{(b-k)}{5}\sqrt{-\frac{4}{3a(r^2+1)}}(x+y-ct)\right]\right.\right. \\
&\quad \left.\left. \pm\, sc\left[\frac{(b-k)}{5}\sqrt{-\frac{4}{3a(r^2+1)}}(x+y-ct)\right]\right)\right\}^2,
\end{aligned}
\tag{60}
$$

$$
\begin{aligned}
u_{2.16} &= v_{2.16} \\
&= \left\{ \frac{2(b-k)}{5a} \pm \frac{2(b-k)}{5a}\sqrt{-\frac{1}{r^2-2}}\left(ns\left[\frac{(b-k)}{5}\sqrt{-\frac{4}{3a(r^2-2)}}(x+y-ct)\right]\right.\right. \\
&\quad \left.\left. \pm\, ds\left[\frac{(b-k)}{5}\sqrt{-\frac{4}{3a(r^2-2)}}(x+y-ct)\right]\right)\right\}^2,
\end{aligned}
\tag{61}
$$

$$
\begin{aligned}
u_{2.17} &= v_{2.17} \\
&= \left\{ \frac{2(b-k)}{5a} \pm \frac{2(b-k)r}{5a}\sqrt{-\frac{1}{r^2-2}}\left(sn\left[\frac{(b-k)}{5}\sqrt{-\frac{4}{3a(r^2-2)}}(x+y-ct)\right]\right.\right. \\
&\quad \left.\left. \pm\, icn\left[\frac{(b-k)}{5}\sqrt{-\frac{4}{3a(r^2-2)}}(x+y-ct)\right]\right)\right\}^2,
\end{aligned}
\tag{62}
$$

$$
\begin{aligned}
u_{2.18} &= v_{2.18} \\
&= \left\{ \frac{2(b-k)}{5a} \pm \frac{2(b-k)r}{5a} \sqrt{-\frac{1}{r^2-2}} \right. \\
&\quad \times \left. \frac{dn[\frac{(b-k)}{5}\sqrt{-\frac{4}{3a(r^2-2)}}(x+y-ct)]}{\sqrt{1-r^2}sn[\frac{(b-k)}{5}\sqrt{-\frac{4}{3a(r^2-2)}}(x+y-ct)] \pm cn[\frac{(b-k)}{5}\sqrt{-\frac{4}{3a(r^2-2)}}(x+y-ct)]} \right\}^2,
\end{aligned} \tag{63}
$$

$$
\begin{aligned}
u_{2.19} &= v_{2.19} \\
&= \left\{ \frac{2(b-k)}{5a} \pm \frac{2(b-k)}{5a} \sqrt{\frac{1}{2r^2-1}} \frac{sn[\frac{(b-k)}{5}\sqrt{\frac{4}{3a(2r^2-1)}}(x+y-ct)]}{1 \pm cn[\frac{(b-k)}{5}\sqrt{\frac{4}{3a(2r^2-1)}}(x+y-ct)]} \right\}^2,
\end{aligned} \tag{64}
$$

$$
\begin{aligned}
u_{2.20} &= v_{2.20} \\
&= \left\{ \frac{2(b-k)}{5a} \pm \frac{2(b-k)}{5a} \sqrt{\frac{1}{2r^2-1}} \left(rsn\left[\frac{(b-k)}{5} \sqrt{-\frac{4}{3a(1-2r^2)}}(x+y-ct) \right] \right. \right. \\
&\quad \left. \left. \pm idn\left[\frac{(b-k)}{5} \sqrt{-\frac{4}{3a(1-2r^2)}}(x+y-ct) \right] \right) \right\}^2,
\end{aligned} \tag{65}
$$

$$
\begin{aligned}
u_{2.21} &= v_{2.21} \\
&= \left\{ \frac{2(b-k)}{5a} \pm \frac{2(b-k)r}{5a} \sqrt{\frac{1}{2-r^2}} \frac{sn[\frac{b-k}{5}\sqrt{\frac{4}{3a(r^2-2)}}(x+y-ct)]}{1 \pm dn[\frac{(b-k)}{5}\sqrt{\frac{4}{3a(r^2-2)}}(x+y-ct)]} \right\}^2,
\end{aligned} \tag{66}
$$

$$
u_{2.22} = v_{2.22} = \left\{ \frac{2(b-k)}{5a} \pm \frac{2(b-k)}{5a} \sqrt{\frac{1-r^2}{r^2+1}} \frac{dn[\frac{b-k}{5}\sqrt{\frac{-4}{3a(r^2+1)}}(x+y-ct)]}{1 \pm rsn[\frac{b-k}{5}\sqrt{\frac{-4}{3a(r^2+1)}}(x+y-ct)]} \right\}^2, \tag{67}
$$

$$
u_{2.23} = v_{2.23} = \left\{ \frac{2(b-k)}{5a} \pm \frac{2(b-k)}{5a} \sqrt{\frac{r^2-1}{r^2+1}} \frac{cn[\frac{b-k}{5}\sqrt{\frac{-4}{3a(r^2+1)}}(x+y-ct)]}{1 \pm sn[\frac{b-k}{5}\sqrt{\frac{-4}{3a(r^2+1)}}(x+y-ct)]} \right\}^2, \tag{68}
$$

$$
\begin{aligned}
u_{2.24} &= v_{2.24} \\
&= \left\{ \frac{2(b-k)}{5a} \pm \frac{2(b-k)(1-r^2)}{5a} \sqrt{-\frac{1}{r^2+1}} \right. \\
&\quad \times \left. \frac{sn[\frac{(b-k)}{5}\sqrt{-\frac{4}{3a(r^2+1)}}(x+y-ct)]}{dn[\frac{(b-k)}{5}\sqrt{-\frac{4}{3a(r^2+1)}}(x+y-ct)] \pm cn[\frac{(b-k)}{5}\sqrt{-\frac{4}{3a(r^2+1)}}(x+y-ct)]} \right\}^2,
\end{aligned} \tag{69}
$$

$$
\begin{aligned}
u_{2.25} &= v_{2.25} \\
&= \left\{ \frac{2(b-k)}{5a} \pm \frac{2(b-k)r^2}{5a} \sqrt{\frac{1}{2-r^2}} \right. \\
&\quad \times \left. \frac{cn[\frac{(b-k)}{5}\sqrt{-\frac{4}{3a(r^2-2)}}(x+y-ct)]}{\sqrt{1-r^2} \pm dn[\frac{(b-k)}{5}\sqrt{-\frac{4}{3a(r^2-2)}}(x+y-ct)]} \right\}^2,
\end{aligned} \tag{70}
$$

where $c = -3 - \frac{16(b-k)^2}{75a}$.

Therefore, when $n = \frac{1}{2}$, we can obtain exact solutions for Eq. (2), some of them will degenerate solitons and triangular function solutions when r approaches 0 or 1.

3.3 Solutions in the case $n = 2$

From the expressions of Eq. (17), we get

$$u_{3.1} = v_{3.1} = \left\{\frac{5(b-k)}{8a} + \frac{\epsilon\mu r}{2}\sqrt{\frac{15}{a}}sn\left[\mu\left(x+y+3t+\frac{5(b-k)^2}{32a}t - \frac{t}{4}(r^2+1)\mu^2\right)\right]\right\}^{\frac{1}{2}},$$
$$\mu = \pm\frac{(b-k)}{4r}\left(\frac{-5(r^2+1)+5\epsilon(r^2-1)}{6a}\right)^{\frac{1}{2}}, \tag{71}$$

$$u_{3.2} = v_{3.2} = \left\{\frac{5(b-k)}{8a} + \frac{\epsilon\mu r}{2}\sqrt{\frac{15}{a}}cd\left[\mu\left(x+y+3t+\frac{5(b-k)^2}{32a}t - \frac{t}{4}(r^2+1)\mu^2\right)\right]\right\}^{\frac{1}{2}},$$
$$\mu = \pm\frac{(b-k)}{4r}\left(\frac{-5(r^2+1)+5\epsilon(r^2-1)}{6a}\right)^{\frac{1}{2}}, \tag{72}$$

$$u_{3.3} = v_{3.3}$$
$$= \left\{\frac{5(b-k)}{8a} + \frac{\epsilon\mu r}{2}\sqrt{-\frac{15}{a}}cn\left[\mu\left(x+y+3t+\frac{5(b-k)^2}{32a}t + \frac{t}{4}(2r^2-1)\mu^2\right)\right]\right\}^{\frac{1}{2}},$$
$$\mu = \pm\frac{(b-k)}{4r}\left(-\frac{5(2r^2-1)+5\epsilon}{6(r^2-1)a}\right)^{\frac{1}{2}}, \tag{73}$$

$$u_{3.4} = v_{3.4} = \left\{\frac{5(b-k)}{8a} + \frac{\epsilon\mu}{2}\sqrt{-\frac{15}{a}}dn\left[\mu\left(x+y+3t+\frac{5(b-k)^2}{32a}t + \frac{t}{4}(2-r^2)\mu^2\right)\right]\right\}^{\frac{1}{2}},$$
$$\mu = \pm\frac{(b-k)}{4}\left(\frac{-5(2-r^2)+5\epsilon r^2}{6(1-r^2)a}\right)^{\frac{1}{2}}, \tag{74}$$

$$u_{3.5} = v_{3.5}$$
$$= \left\{\frac{5(b-k)}{8a}\right.$$
$$\left. + \frac{\epsilon\mu}{2}\sqrt{\frac{15(1-r^2)}{a}}nc\left[\mu\left(x+y+3t+\frac{5(b-k)^2}{32a}t + \frac{t}{4}(2r^2-1)\mu^2\right)\right]\right\}^{\frac{1}{2}},$$
$$\mu = \pm\frac{(b-k)}{4r}\left(\frac{-5(2r^2-1)+5\epsilon}{6(r^2-1)a}\right)^{\frac{1}{2}}, \tag{75}$$

$$u_{3.6} = v_{3.6} = \left\{\frac{5(b-k)}{8a} + \frac{\epsilon\mu}{2}\sqrt{\frac{15}{a}}ns\left[\mu\left(x+y+3t+\frac{5(b-k)^2}{32a}t - \frac{t}{4}(r^2+1)\mu^2\right)\right]\right\}^{\frac{1}{2}},$$
$$\mu = \pm\frac{(b-k)}{4r}\left(\frac{-5(r^2+1)+5\epsilon(r^2-1)}{6a}\right)^{\frac{1}{2}}, \tag{76}$$

$$u_{3.7} = v_{3.7} = \left\{\frac{5(b-k)}{8a} + \frac{\epsilon\mu}{2}\sqrt{\frac{15}{a}}dc\left[\mu\left(x+y+3t+\frac{5(b-k)^2}{32a}t - \frac{t}{4}(r^2+1)\mu^2\right)\right]\right\}^{\frac{1}{2}},$$
$$\mu = \pm\frac{(b-k)}{4r}\left(\frac{-5(r^2+1)+5\epsilon(r^2-1)}{6a}\right)^{\frac{1}{2}}, \tag{77}$$

$$u_{3.8} = v_{3.8} = \left\{ \frac{5(b-k)}{8a} + \frac{\epsilon\mu}{2}\sqrt{\frac{15(r^2-1)}{a}} nd\left[\mu\left(x+y+3t+\frac{5(b-k)^2}{32a}t+\frac{t}{4}(2-r^2)\mu^2\right)\right]\right\}^{\frac{1}{2}},$$

$$\mu = \pm\frac{(b-k)}{4}\left(-\frac{5(2-r^2)+5\epsilon r^2}{6(1-r^2)a}\right)^{\frac{1}{2}}, \tag{78}$$

$$u_{3.9} = v_{3.9} = \left\{ \frac{5(b-k)}{8a} + \frac{\epsilon\mu}{2}\sqrt{\frac{15}{a}} cs\left[\mu\left(x+y+3t+\frac{5(b-k)^2}{32a}t+\frac{t}{4}(2-r^2)\mu^2\right)\right]\right\}^{\frac{1}{2}},$$

$$\mu = \pm\frac{(b-k)}{4}\left(-\frac{5(2-r^2)+5\epsilon r^2}{6\lambda(1-r^2)}\right)^{\frac{1}{2}}, \tag{79}$$

$$u_{3.10} = v_{3.10} = \left\{ \frac{5(b-k)}{8a} + \frac{\epsilon\mu}{2}\sqrt{\frac{15(1-r^2)}{a}} sc\left[\mu\left(x+y+3t+\frac{5(b-k)^2}{32a}t+\frac{t}{4}(2-r^2)\mu^2\right)\right]\right\}^{\frac{1}{2}},$$

$$\mu = \pm\frac{(b-k)}{4}\left(-\frac{5(2-r^2)+5\epsilon r^2}{6a(1-r^2)}\right)^{\frac{1}{2}}, \tag{80}$$

$$u_{3.11} = v_{3.11} = \left\{ \frac{5(b-k)}{8a} + \frac{\epsilon\mu r}{2}\sqrt{\frac{15(r^2-1)}{a}} sd\left[\mu\left(x+y+3t+\frac{5(b-k)^2}{32a}t+\frac{t}{4}(2r^2-1)\mu^2\right)\right]\right\}^{\frac{1}{2}},$$

$$\mu = \pm\frac{(b-k)}{4r}\left(-\frac{5(2r^2-1)+5\epsilon}{6(r^2-1)a}\right)^{\frac{1}{2}}, \tag{81}$$

$$u_{3.12} = v_{3.12} = \left\{ \frac{5(b-k)}{8a} + \frac{\epsilon\mu}{2}\sqrt{\frac{15}{a}} ds\left[\mu\left(x+y+3t+\frac{5(b-k)^2}{32a}t+\frac{t}{4}(2r^2-1)\mu^2\right)\right]\right\}^{\frac{1}{2}},$$

$$\mu = \pm\frac{(b-k)}{4r}\left(-\frac{5(2r^2-1)+5\epsilon}{6(r^2-1)a}\right)^{\frac{1}{2}}, \tag{82}$$

$$u_{3.13} = v_{3.13} = \left\{ \frac{5(b-k)}{8a} + \frac{\epsilon\mu}{4}\sqrt{-\frac{15}{a}}\left(rcn\left[\mu\left(x+y+3t+\frac{5(b-k)^2}{32a}t+\frac{t}{8}(r^2+1)\mu^2\right)\right] \pm dn\left[\mu\left(x+y+3t+\frac{5(b-k)^2}{32a}t+\frac{t}{8}(r^2+1)\mu^2\right)\right]\right)\right\}^{\frac{1}{2}},$$

$$\mu = \pm\frac{(b-k)}{2(1-r^2)}\left(-\frac{5(r^2+1)+10\epsilon r}{3a}\right)^{\frac{1}{2}}, \tag{83}$$

$$u_{3.14} = v_{3.14} = \left\{ \frac{5(b-k)}{8a} + \frac{\epsilon\mu}{4}\sqrt{\frac{15}{a}}\left(ns\left[\mu\left(x+y+3t+\frac{5(b-k)^2}{32a}t+\frac{t}{8}(1-2r^2)\right)\right]\right.\right.$$

$$\pm cs\left[\mu\left(x+y+3t+\frac{5(b-k)^2}{32a}t+\frac{t}{8}\left(1-2r^2\right)\mu^2\right)\right]\bigg)\bigg\}^{\frac{1}{2}},$$

$$\mu=\pm\frac{(b-k)}{2}\left(-\frac{5(1-2r^2)+10\epsilon\, r\sqrt{r^2-1}}{3a}\right)^{\frac{1}{2}}, \tag{84}$$

$$u_{3.15}=v_{3.15}$$

$$=\left\{\frac{5(b-k)}{8a}+\frac{\epsilon\mu}{4}\sqrt{\frac{15(1-r^2)}{a}}\left(nc\left[\mu\left(x+y+3t+\frac{5(b-k)^2}{32a}t+\frac{t}{8}\left(r^2+1\right)\mu^2\right)\right]\right.\right.$$

$$\left.\left.\pm dc\left[\mu\left(x+y+3t+\frac{5(b-k)^2}{32a}t+\frac{t}{8}\left(r^2+1\right)\mu^2\right)\right]\right)\right\}^{\frac{1}{2}},$$

$$\mu=\pm\frac{(b-k)}{2(1-r^2)}\left(-\frac{5(r^2+1)+10\epsilon\, r}{3a}\right)^{\frac{1}{2}}, \tag{85}$$

$$u_{3.16}=v_{3.16}$$

$$=\left\{\frac{5(b-k)}{8a}+\frac{\epsilon\mu}{4}\sqrt{\frac{15}{a}}\left(ns\left[\mu\left(x+y+3t+\frac{5(b-k)^2}{32a}t+\frac{t}{8}\left(r^2-2\right)\mu^2\right)\right]\right.\right.$$

$$\left.\left.\pm ds\left[\mu\left(x+y+3t+\frac{5(b-k)^2}{32a}t+\frac{t}{8}\left(r^2-2\right)\mu^2\right)\right]\right)\right\}^{\frac{1}{2}},$$

$$\mu=\pm\frac{(b-k)}{2r^2}\left(-\frac{5(r^2-2)+10\epsilon\sqrt{1-r^2}}{3a}\right)^{\frac{1}{2}}, \tag{86}$$

$$u_{3.17}=v_{3.17}$$

$$=\left\{\frac{5(b-k)}{8a}+\frac{\epsilon\mu r}{4}\sqrt{\frac{15}{a}}\left(sn\left[\mu\left(x+y+3t+\frac{5(b-k)^2}{32a}t+\frac{t}{8}\left(r^2-2\right)\mu^2\right)\right]\right.\right.$$

$$\left.\left.\pm icn\left[\mu\left(x+y+3t+\frac{5(b-k)^2}{32a}t+\frac{t}{8}\left(r^2-2\right)\mu^2\right)\right]\right)\right\}^{\frac{1}{2}},$$

$$\mu=\pm\frac{(b-k)}{2r^2}\left(-\frac{5(r^2-2)+10\epsilon\sqrt{1-r^2}}{3a}\right)^{\frac{1}{2}}, \tag{87}$$

$$u_{3.18}=v_{3.18}$$

$$=\left\{\frac{5(b-k)}{8a}+\frac{\epsilon\mu r}{4}\sqrt{\frac{15}{a}}\right.$$

$$\times\left(sn\left[\mu\left(x+y+3t+\frac{5(b-k)^2}{32a}t+\frac{t}{8}\left(r^2-2\right)\mu^2\right)\right]\right)$$

$$\times\left(\sqrt{1-r^2}sn\left[\mu\left(x+y+3t+\frac{5(b-k)^2}{32a}t+\frac{t}{8}\left(r^2-2\right)\mu^2\right)\right]\right.$$

$$\left.\left.\pm cn\left[\mu\left(x+y+3t+\frac{5(b-k)^2}{32a}t+\frac{t}{8}\left(r^2-2\right)\mu^2\right)\right]\right)^{-1}\right\}^{\frac{1}{2}},$$

$$\mu=\pm\frac{(b-k)}{2r^2}\left(\frac{-5(r^2-2)+10\epsilon\sqrt{1-r^2}}{3a}\right)^{\frac{1}{2}}, \tag{88}$$

$$
\begin{aligned}
u_{3.19} &= v_{3.19} \\
&= \left\{ \frac{5(b-k)}{8a} + \frac{\epsilon\mu}{4}\sqrt{\frac{15}{a}} \left(rsn\left[\mu\left(x+y+3t+\frac{5(b-k)^2}{32a}t+\frac{t}{8}\left(1-2r^2\right)\mu^2 \right) \right] \right.\right. \\
&\quad \left.\left. \pm idn\left[\mu\left(x+y+3t+\frac{5(b-k)^2}{32a}t+\frac{t}{8}\left(1-2r^2\right)\mu^2 \right) \right] \right) \right\}^{\frac{1}{2}}, \\
\mu &= \pm\frac{(b-k)}{2}\left(-\frac{5(1-2r^2)+10\epsilon\, r\sqrt{r^2-1}}{3a} \right)^{\frac{1}{2}},
\end{aligned} \tag{89}
$$

$$
\begin{aligned}
u_{3.20} = v_{3.20} &= \left\{ \frac{5(b-k)}{8a} + \frac{\epsilon\mu}{4}\sqrt{\frac{15}{a}} \frac{sn[\mu(x+y+3t+\frac{5(b-k)^2}{32a}t+\frac{t}{8}(1-2r^2)\mu^2)]}{1\pm cn[\mu(x+y+3t+\frac{5(b-k)^2}{32a}t+\frac{t}{8}(1-2r^2)\mu^2)]} \right\}^{\frac{1}{2}}, \\
\mu &= \pm\frac{(b-k)}{2}\left(-\frac{5(1-2r^2)+10\epsilon\, r\sqrt{r^2-1}}{3a} \right)^{\frac{1}{2}},
\end{aligned} \tag{90}
$$

$$
\begin{aligned}
u_{3.21} = v_{3.21} &= \left\{ \frac{5(b-k)}{8a} + \frac{\epsilon\mu r}{4}\sqrt{\frac{15}{a}} \frac{sn[\mu(x+y+3t+\frac{5(b-k)^2}{32a}t+\frac{t}{8}(r^2-2)\mu^2]}{1\pm dn[\mu(x+y+3t+\frac{5(b-k)^2}{32a}t+\frac{t}{8}(r^2-2)\mu^2]} \right\}^{\frac{1}{2}}, \\
\mu &= \pm\frac{(b-k)}{2r}\left(-\frac{5(r^2-2)+5\epsilon\sqrt{r^4-5r^2+4}}{3a} \right)^{\frac{1}{2}},
\end{aligned} \tag{91}
$$

$$
\begin{aligned}
u_{3.22} &= v_{3.22} \\
&= \left\{ \frac{5(b-k)}{8a} + \frac{\epsilon\mu}{4}\sqrt{\frac{15(r^2-1)}{a}} \frac{dn[\mu(x+y+3t+\frac{5(b-k)^2}{32a}t+\frac{t}{8}(r^2+1)\mu^2)]}{1\pm rsn[\mu(x+y+3t+\frac{5(b-k)^2}{32a}t+\frac{t}{8}(r^2+1)\mu^2)]} \right\}^{\frac{1}{2}}, \\
\mu &= \pm\frac{\beta}{2(1-r^2)}\left(-\frac{5(r^2+1)+10\epsilon r}{3a} \right)^{\frac{1}{2}},
\end{aligned} \tag{92}
$$

$$
\begin{aligned}
u_{3.23} &= v_{3.23} \\
&= \left\{ \frac{5(b-k)}{8a} + \frac{\epsilon\mu}{4}\sqrt{-\frac{15(r^2-1)}{a}} \right. \\
&\quad \left. \times \frac{cn[\mu(x+y+3t+\frac{5(b-k)^2}{32a}t+\frac{t}{8}(r^2+1)\mu^2)]}{1\pm sn[\mu(x+y+3t+\frac{5(b-k)^2}{32a}t+\frac{t}{8}(r^2+1)\mu^2)]} \right\}^{\frac{1}{2}}, \\
\mu &= \pm\frac{(b-k)}{2(1-r^2)}\left(-\frac{5(r^2+1)+10\epsilon r}{3a} \right)^{\frac{1}{2}},
\end{aligned} \tag{93}
$$

$$
\begin{aligned}
u_{3.24} &= v_{3.24} \\
&= \left\{ \frac{5(b-k)}{8a} + \frac{\epsilon\mu(r^2-1)}{4}\sqrt{\frac{15}{a}} \right. \\
&\quad \times \left(sn\left[\mu\left(x+y+3t+\frac{5(b-k)^2}{32a}t+\frac{t}{8}\left(r^2+1\right)\mu^2 \right) \right] \right) \\
&\quad \times \left(dn\left[\mu\left(x+y+3t+\frac{5(b-k)^2}{32a}t+\frac{t}{8}\left(r^2+1\right)\mu^2 \right) \right] \right. \\
&\quad \left.\left. \pm cn\left[\mu\left(x+y+3t+\frac{5(b-k)^2}{32a}t+\frac{t}{8}\left(r^2+1\right)\mu^2 \right) \right] \right)^{-1} \right\}^{\frac{1}{2}},
\end{aligned}
$$

$$\mu = \pm\frac{(b-k)}{2(1-r^2)}\left(-\frac{5(r^2+1)+10\epsilon r}{3a}\right)^{\frac{1}{2}}, \tag{94}$$

$$\begin{aligned} u_{3.25} &= v_{3.25} \\ &= \left\{\frac{5(b-k)}{8a}+\frac{\epsilon\mu r^2}{4}\sqrt{\frac{15}{a}}\frac{cn[\mu(x+y+3t+\frac{5(b-k)^2}{32a}t+\frac{t}{8}(r^2-2)\mu^2])]}{\sqrt{1-r^2}\pm dn[\mu(x+y+3t+\frac{5(b-k)^2}{32a}t+\frac{t}{8}(r^2-2)\mu^2])]}\right\}^{\frac{1}{2}}, \end{aligned}$$

$$\mu = \pm\frac{(b-k)}{2r^2}\left(\frac{-5(r^2-2)+10\epsilon\sqrt{1-r^2}}{3a}\right)^{\frac{1}{2}}. \tag{95}$$

Of course, these are Jacobi elliptic function solutions.

3.4 Solutions in general case

From the properties of the Jacobi elliptic functions, we know that when $r \to 1$, for these solutions listed from (19) to (40), some of them will become zero or constants, the others will degenerate soliton-like solutions, the degenerated solitons for Eq. (2) are expressed by

$$\begin{aligned} u_{4.1} &= v_{4.1} \\ &= \left\{\frac{(b-k)(2n+1)}{2a(n+2)}-\epsilon\frac{(b-k)(2n+1)}{2a(n+2)}\right. \\ &\quad \left.\times\tanh\left[\frac{(b-k)n}{n+2}\sqrt{\frac{2n+1}{4a(n+1)}}\left(x+y+3t+\frac{(b-k)^2(2n+1)t}{a(n+2)^2(n+1)}\right)\right]\right\}^{\frac{1}{n}}, \end{aligned} \tag{96}$$

$$\begin{aligned} u_{4.2} &= v_{4.2} \\ &= \left\{\frac{(b-k)(2n+1)}{2a(n+2)}-\epsilon\frac{\sqrt{2}(b-k)(2n+1)}{2a(n+2)}\right. \\ &\quad \left.\times\operatorname{sech}\left[\frac{(b-k)n}{n+2}\sqrt{-\frac{2n+1}{2a(n+1)}}\left(x+y+3t+\frac{(b-k)^2(2n+1)t}{a(n+2)^2(n+1)}\right)\right]\right\}^{\frac{1}{n}}, \end{aligned} \tag{97}$$

$$\begin{aligned} u_{4.3} &= v_{4.3} \\ &= \left\{\frac{(b-k)(2n+1)}{2a(n+2)}-\epsilon\frac{(b-k)(2n+1)}{2a(n+2)}\right. \\ &\quad \left.\times\coth\left[\frac{(b-k)n}{n+2}\sqrt{\frac{2n+1}{4a(n+1)}}\left(x+y+3t+\frac{(b-k)^2(2n+1)t}{a(n+2)^2(n+1)}\right)\right]\right\}^{\frac{1}{n}}, \end{aligned} \tag{98}$$

$$\begin{aligned} u_{4.4} &= v_{4.4} \\ &= \left\{\frac{(b-k)(2n+1)}{2a(n+2)}-i\epsilon\frac{\sqrt{2}(b-k)(2n+1)}{2a(n+2)}\right. \\ &\quad \left.\times csch\left[\frac{(b-k)n}{n+2}\sqrt{-\frac{2n+1}{2a(n+1)}}\left(x+y+3t+\frac{(b-k)^2(2+1)t}{a(n+2)^2(n+1)}\right)\right]\right\}^{\frac{1}{n}}, \end{aligned} \tag{99}$$

$$\begin{aligned} u_{4.5} &= v_{4.5} \\ &= \left\{\frac{(b-k)(2n+1)}{2a(n+2)}-\epsilon\frac{(b-k)(2n+1)}{2a(n+2)}\right. \\ &\quad \left.\times\frac{\cosh(\frac{(b-k)n}{n+2}\sqrt{\frac{2n+1}{a(p+1)}}(x+y+3t+\frac{(b-k)^2(2n+1)t}{a(n+2)^2(n+1)}))\pm 1}{\sinh(\frac{(b-k)n}{n+2}\sqrt{\frac{2n+1}{a(n+1)}}(x+y+3t+\frac{(b-k)^2(2n+1)t}{a(n+2)^2(n+1)}))}\right\}^{\frac{1}{n}}, \end{aligned} \tag{100}$$

$$\begin{aligned} u_{4.6} &= v_{4.6} \\ &= \left\{ \frac{(b-k)(2n+1)}{2a(p+2)} - \epsilon \frac{(b-k)(2n+1)}{2a(n+2)} \right. \\ &\quad \left. \times \frac{\sinh(\frac{(b-k)n}{n+2}\sqrt{\frac{2n+1}{a(n+1)}}[x+y+3t+\frac{(b-k)^2(2n+1)t}{a(n+2)^2(n+1)}]) \pm i}{\cosh(\frac{(b-k)n}{n+2}\sqrt{\frac{2n+1}{a(n+1)}}[x+y+3t+\frac{(b-k)^2(2n+1)t}{a(n+2)^2(n+1)}])} \right\}^{\frac{1}{n}}, \end{aligned} \tag{101}$$

$$\begin{aligned} u_{4.7} &= v_{4.7} \\ &= \left\{ \frac{(b-k)(2n+1)}{2a(n+2)} - \epsilon \frac{(b-k)(2n+1)}{2a(n+2)} \right. \\ &\quad \left. \times \frac{\sinh(\frac{(b-k)n}{n+2}\sqrt{\frac{2n+1}{a(n+1)}}(x+y+3t+\frac{(b-k)^2(2n+1)t}{a(n+2)^2(n+1)}))}{1 \pm \cosh(\frac{(b-k)n}{n+2}\sqrt{\frac{2n+1}{a(n+1)}}[x+y+3t+\frac{(b-k)^2(2p+1)t}{a(n+2)^2(n+1)}])} \right\}^{\frac{1}{n}}, \end{aligned} \tag{102}$$

$$\begin{aligned} u_{4.8} &= v_{4.8} \\ &= \left\{ \frac{(b-k)(2n+1)}{2a(n+2)} - \epsilon \frac{\sqrt{2}(b-k)(2n+1)}{2a(n+2)} \right. \\ &\quad \left. \times \sec\left[\frac{(b-k)p}{p+2}\sqrt{\frac{2n+1}{2a(n+1)}}\left(x+y+3t+\frac{(b-k)^2(2n+1)t}{a(n+2)^2(n+1)}\right)\right]\right\}^{\frac{1}{n}}, \end{aligned} \tag{103}$$

$$\begin{aligned} u_{4.9} &= v_{4.9} \\ &= \left\{ \frac{(b-k)(2n+1)}{2a(n+2)} - \epsilon \frac{\sqrt{2}(b-k)(2n+1)}{2a(n+2)} \right. \\ &\quad \left. \times \csc\left[\frac{(b-k)n}{n+2}\sqrt{\frac{2n+1}{2a(n+1)}}\left(x+y+3t+\frac{(b-k)^2(2n+1)t}{a(n+2)^2(n+1)}\right)\right]\right\}^{\frac{1}{n}}, \end{aligned} \tag{104}$$

$$\begin{aligned} u_{4.10} &= v_{4.10} \\ &= \left\{ \frac{(b-k)(2n+1)}{2a(n+2)} - i\epsilon \frac{(b-k)(2n+1)}{2a(n+2)} \right. \\ &\quad \left. \times \cot\left[\frac{(b-k)n}{n+2}\sqrt{-\frac{2n+1}{4a(n+1)}}\left(x+y+3t+\frac{(b-k)^2(2n+1)t}{a(n+2)^2(n+1)}\right)\right]\right\}^{\frac{1}{n}}, \end{aligned} \tag{105}$$

$$\begin{aligned} u_{4.11} &= v_{4.11} \\ &= \left\{ \frac{(b-k)(2n+1)}{2a(n+2)} - i\epsilon \frac{(b-k)(2n+1)}{2a(n+2)} \right. \\ &\quad \left. \times \tan\left[\frac{(b-k)n}{n+2}\sqrt{-\frac{2n+1}{4a(n+1)}}\left(x+y+3t+\frac{(b-k)^2(2n+1)t}{a(n+2)^2(n+1)}\right)\right]\right\}^{\frac{1}{p}}, \end{aligned} \tag{106}$$

$$\begin{aligned} u_{4.12} &= v_{4.12} \\ &= \left\{ \frac{(b-k)(2n+1)}{2a(n+2)} - i\epsilon \frac{(b-k)(2n+1)}{2a(n+2)} \right. \\ &\quad \left. \times \frac{1 \pm \cos(\frac{(b-k)n}{n+2}\sqrt{-\frac{2n+1}{a(n+1)}}(x+y+3t+\frac{(b-k)^2(2n+1)t}{a(n+2)^2(n+1)}))}{\sin(\frac{(b-k)n}{n+2}\sqrt{-\frac{2n+1}{a(n+1)}}(x+y+3t+\frac{(b-k)^2(2p+1)t}{a(n+2)^2(n+1)}))}]\right\}^{\frac{1}{n}}, \end{aligned} \tag{107}$$

$$\begin{aligned} u_{4.13} &= v_{4.13} \\ &= \left\{ \frac{(b-k)(2n+1)}{2a(n+2)} - i\epsilon \frac{(b-k)(2n+1)}{2a(n+2)} \right. \end{aligned}$$

$$\times \frac{1 \pm \sin(\frac{(b-k)n}{n+2}\sqrt{-\frac{2n+1}{a(n+1)}}[x+y+3t+\frac{(b-k)^2(2n+1)t}{a(n+2)^2(n+1)}])}{\cos(\frac{(b-k)n}{n+2}\sqrt{-\frac{2n+1}{a(n+1)}}[x+y+3t+\frac{(b-k)^2(2n+1)t}{a(n+2)^2(n+1)}])}]\Bigg\}^{\frac{1}{n}}, \tag{108}$$

$$\begin{aligned} u_{4.14} &= v_{4.14} \\ &= \left\{\frac{(b-k)(2n+1)}{2a(n+2)} - i\epsilon\frac{(b-k)(2n+1)}{2a(n+2)}\right. \\ &\quad \left.\times \frac{\sin(\frac{(b-k)n}{n+2}\sqrt{-\frac{2n+1}{a(n+1)}}(x+y+3t+\frac{(b-k)^2(2n+1)t}{a(n+2)^2(n+1)}))}{1 \pm \cos(\frac{(b-k)n}{n+2}\sqrt{-\frac{2n+1}{a(n+1)}}[x+y+3t+\frac{(b-k)^2(2n+1)t}{a(n+2)^2(n+1)}])}\right\}^{\frac{1}{n}}, \end{aligned} \tag{109}$$

$$\begin{aligned} u_{4.15} &= v_{4.15} \\ &= \left\{\frac{(b-k)(2n+1)}{2a(n+2)} - i\epsilon\frac{(b-k)(2n+1)}{2a(n+2)}\right. \\ &\quad \left.\times \frac{\cos(\frac{(b-k)n}{n+2}\sqrt{-\frac{2n+1}{a(n+1)}}[x+y+3t+\frac{(b-k)^2(2n+1)t}{a(p+2)^2(n+1)}])}{1 \pm \sin(\frac{(b-k)n}{n+2}\sqrt{-\frac{2n+1}{2a(n+1)}}(x+y+3t+\frac{(b-k)^2(2n+1)t}{a(n+2)^2(n+1)}))}\right\}^{\frac{1}{n}}. \end{aligned} \tag{110}$$

In the solutions from $u_{1.1}, v_{1.1}$ to $u_{3.25}, v_{3.25}$, if $r \to 0$ or $r \to 1$, these solutions can also be found from $u_{4.1}, v_{4.1}$ to $u_{4.15}, v_{4.15}$ while $n = 1, 2$ and $\frac{1}{2}$ respectively. Specially, when $r \to 1$, $u_{2.17}$ and $u_{3.17}$ can be expressed by

$$\begin{aligned} u_{2.17.1} = v_{2.17} &= \left\{\frac{2(b-k)}{5a} \pm \frac{2(b-k)}{5a}\left(\tanh\left[\frac{(b-k)}{5}\sqrt{\frac{4}{3a}}(x+y-ct)\right]\right.\right. \\ &\quad \left.\left.\pm i\,\mathrm{sech}\left[\frac{(b-k)}{5}\sqrt{\frac{4}{3a}}\left(x+y+3t+\frac{16(b-k)^2}{75a}\right)\right]\right)\right\}^2, \end{aligned} \tag{111}$$

$$\begin{aligned} u_{3.17.1} = v_{3.17} &= \left\{\frac{5(b-k)}{8a} + \frac{5\epsilon(b-k)}{8a} + \left(\tanh\left[\frac{b-k}{2}\sqrt{\frac{5}{3a}}\left(x+y+3t+\frac{5(b-k)^2}{48a}t\right)\right]\right.\right. \\ &\quad \left.\left.\pm i\,\mathrm{sech}\left[\frac{b-k}{2}\sqrt{\frac{5}{3a}}\left(x+y+3t+\frac{5(b-k)^2}{48a}t\right)\right]\right)\right\}^{\frac{1}{2}}, \end{aligned} \tag{112}$$

which are the same as $u_{4.6}, v_{4.6}$ in the case of $n = \frac{1}{2}, 2$, respectively.

4 Conclusion

The auxiliary differential equation approach has been successfully employed to seek the exact solutions of the modified $(2+1)$-dimensional Konopelchenko–Dubrovsky equations (2) which possess high order nonlinear terms. After handling the mKD equations, it shows some new solutions compared with these obtained in [22, 25] by this approach. On the one hand, for special exponent n such as 1, 2, and 1/2, we derive many Jacobi function solutions, some of which can degenerate triangular function solutions and solitons. On the other hand, the paper presents abundant new exact traveling wave solutions for general exponent n.

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Competing interests
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Authors' contributions
All authors contributed equally and significantly in writing this paper. All authors have read and approved the final paper.

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