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On the asymptotic analysis of bounded solutions to nonlinear differential equations of second order

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Abstract

In this paper, we consider two different models of nonlinear ordinary differential equations (ODEs) of second order. We construct two new Lyapunov functions to investigate boundedness of solutions of those nonlinear ODEs of second order. By using the Lyapunov direct or second method and inequality techniques, we prove two new theorems on the boundedness solutions of those ODEs of second order as $t \to \infty$. When we compare the conditions of the theorems of this paper with those of Meng in (J. Syst. Sci. Math. Sci. 15(1):50–57, 1995) and Sun and Meng in (Ann. Differ. Equ. 18(1):58–64 2002), we can see that our theorems have less restrictive conditions than those in (Meng in J. Syst. Sci. Math. Sci. 15(1):50–57, 1995) and Sun and Meng in (Ann. Differ. Equ. 18(1):58–64 2002) because of the two new suitable Lyapunov functions. Next, in spite of the use of the Lyapunov second method here and in (Meng in J. Syst. Sci. Math. Sci. 15(1):50–57, 1995; Sun and Meng in Ann. Differ. Equ. 18(1):58–64 2002), the proofs of the results of this paper are proceeded in a very different way from that used in the literature for the qualitative analysis of ODEs of second order. Two examples are given to show the applicability of our results. At the end, we can conclude that the results of this paper generalize and improve the results of Meng in (J. Syst. Sci. Math. Sci. 15(1):50–57, 1995), Sun and Meng in (Ann. Differ. Equ. 18(1):58–64 2002), and some other that can be found in the literature, and they have less restrictive conditions than those in these references.

MSC: 34C11; 34D05; 34D20

Keywords: Nonlinear ODEs; Second order; Boundedness; The Lyapunov second method

1 Introduction

It is well known that linear and nonlinear ODEs of second order can arise during many applications in various scientific areas such as physics, biology, chemistry, biophysics, mechanics, medicine, aerodynamics, economy, atomic energy, control theory, information theory, population dynamics, electrodynamics of complex media, and so on. Therefore, qualitative behaviors of solutions to ODEs of second order, stability, boundedness, convergence, instability, integrability, globally existence of solutions, etc., have been extensively investigated in the literature by this time. For a comprehensive treatment of these qualitative properties of solutions of ODEs of second order and some applications, we refer the readers to [1–45, 47–54] and the references cited therein. Besides, the boundedness of solutions of nonlinear ODEs of second order are of particular interest in applications (see [1–3, 6, 8, 16–18, 23–46, 48–54]).

During the investigations of the qualitative properties of solutions to ODEs of second order, fixed point method, perturbation theory, variations of parameters formulas, the Lyapunov second method, and so on have been used to get information without solving equation(s) under study. It is worth mentioning that, to the best of our knowledge, probably, up to now in the related literature, the Lyapunov second method has been the most effective tool to study these qualitative properties of solutions of nonlinear ODEs of higher order without solving them. This method requires the construction of a suitable function or functional, which gives meaningful result(s) for the problem under study. However, the construction of the Lyapunov functions for nonlinear ODEs of higher order still remains an open problem in the literature for now.

As for the motivation of the results of this paper, the boundedness and square integrability, etc. of solutions of the following ODEs of second order:

$$x'' + \big(q_1(t) + q_2(t)\big)x = 0, \tag{1}$$

$$\frac{d}{dt}\big(r(t)x'\big) + q(t)x = 0, \tag{2}$$

$$x'' + p(t)x' + \big(q_1(t) + q_2(t)\big)x = 0, \tag{3}$$

$$x'' + p(t)x' + \big(q_1(t) + q_2(t)\big)x = f(t), \tag{4}$$

and

$$\frac{d}{dt}\big(r(t)x'\big) + p(t)x' + \big(q_1(t) + q_2(t)\big)x = f(t,x) \tag{5}$$

have been discussed by Meng [15], Sun and Meng [23], and some others cited in the references [15] and [23]. The proofs of the results of Meng [15], Sun and Meng [23], and the others were done by using the Lyapunov second method and some well-known inequalities. In [15] and [23], the authors obtained some interesting results on the boundedness and square integrability of solutions of these ODEs of second order, (1)–(5). Here, in particular, we should insist that Meng [15] considered ODE (4). He obtained certain sufficient conditions under which all the solutions of ODE (4) are bounded and square integrable. He proved three theorems on boundedness and square integrability of solutions of ODE (4). Each of the theorems in [15] is a slight different variant of the other ones.

In this paper, motivated by the works of Meng [15] and Sun and Meng [23], firstly, we deal with the following nonlinear ODE of second order:

$$x'' + p(t)g\big(x'\big) + q_1(t)h(x) + q_2(t)x = f\big(t,x,x'\big), \tag{6}$$

where $x \in \Re$, $\Re = (-\infty,\infty)$, $t \in \Re^+$, $\Re^+ = [0,\infty)$. Throughout this paper, it is assumed that $p, q_2 \in C(\Re^+,\Re)$, $q_1 \in C^1(\Re^+,\Re)$, $g \in C^1(\Re,\Re)$, $h \in C^1(\Re,\Re)$, and $f \in C(\Re^+ \times \Re^2,\Re)$, $g(0) = 0$ and $h(0) = 0$. By these assumptions, the existence of the solutions of ODE (6) is guaranteed. We also assume that the functions g, h, and f satisfy the Lipschitz condition in the unknown function x and its derivative x'. Hence, the uniqueness of solutions of ODE (6) is guaranteed.

Let us compare ODEs (1)–(5) with ODE (6). It is clear that ODE (6) is a general form of ODEs (1), (3). Next, when $p(t) = 0$ or $g(x') = 0$, $h(x) = x$, $f(t,x,x') = 0$, and $q_1(t) + q_2(t) = q(t)$, then ODE (6) is reduced to ODE (2) provided that $r(t) = 1$.

Similarly, when $g(x') = x'$, $h(x) = x$ and $f(t,x,x') = 0$ depends only on t, then ODE (6) is reduced to ODE (4). In addition, when $g(x') = x'$, $h(x) = x$ and $f(t,x,x')$ depends only on t and x, then ODE (6) is reduced to ODE (5) provided that $r(t) = 1$.

Secondly, we consider the following nonlinear ODE of second order:

$$x'' + \phi\big(t,x,x'\big) + q_1(t)x + q_2(t)\theta(x) = q\big(t,x,x'\big), \tag{7}$$

where $x \in \Re$, $t \in \Re^+$, $q_1 \in C(\Re^+,\Re)$, $q_2 \in C^1(\Re^+,\Re)$, $\theta \in C^1(\Re,\Re)$, $\phi \in C^1(\Re^+ \times \Re^2,\Re)$ and $q \in C(\Re^+ \times \Re^2,\Re)$, $\theta(0) = 0$, and $\phi(t,x,0) = 0$. By these assumptions, the existence of the solutions of ODE (7) is guaranteed. We also assume that the functions ϕ, θ, and q satisfy the Lipschitz condition in the unknown function x and its derivative x'. Hence, the uniqueness of solutions of ODE (7) is guaranteed.

Let us compare ODEs (1)–(5) with ODE (7). It is clear that ODE (7) is a general form of ODEs (1), (3), and (4). Next, ODE (7) can be reduced to ODEs (2) and (5) provided that $r(t) = 1$. These cases can be examined in detail as before. However, we omit the details of the comparison for the sake of brevity.

Finally, when we look at ODEs (1)–(5), we see that ODEs (1)–(4) are linear and ODE (5) has a slightly modified nonlinear form. Next, ODEs (6) and (7) include and improve ODEs (1)–(4) from the linear cases to the more general nonlinear forms. In addition, ODE (5) has a simple nonlinear function; however, each of ODEs (6) and (7) has three nonlinear functions.

This paper investigates the problem of asymptotic boundedness of solutions of ODEs (6) and (7) as $t \to \infty$. Here, instead of simple linear and nonlinear ODEs of second order, we discuss the mentioned problem for two general nonlinear ODEs of second order compared to those in [15, 23] and the one that can be found in the literature. This fact is the first contribution of this paper to the topic and the existing literature.

Next, in [15] and [23], the authors benefited from the Lyapunov second method. However, the style and arrangements through the proofs of the results of Meng [15] and Sun and Meng [23] are very different than the classical ones that can be found in the literature. Here, the authors of this paper would not like to discuss the differences for the sake of brevity. The next and second contribution of this paper is that we can obtain the results of Meng [15] and Sun and Meng [23] under less restrictive conditions in the particular cases.

Later, Meng [15] and Sun and Meng [23] gave examples to show the applicability of the theoretical results obtained in [15] and [23], respectively. However, in [15] and [23], the graphs of the orbits of the solutions of the given examples were not drawn. In this paper, two examples with the graphs of paths of their solutions are given to show and illustrate that the established conditions can be applicable clearly. This is the third contribution of this paper to the topic and the existing literature.

Finally, to best of our knowledge from the literature, no paper was devoted to the asymptotic analysis of bounded solutions of ODEs (6) and (7). By this work, we attempt to fill that gap for the relevant literature. Next, it should be noted that the construction of suitable Lyapunov functions for ODE(s) under study enables us to get more restrictive conditions for the qualitative behaviors of solutions. This can be seen when we compare the Lyapunov functions and the results of this paper with those of Meng [15], Sun and Meng [23], and those that can be found in the literature. These are the main contributions of this paper to the relevant topics and literature.

Next, let $y(t) = \frac{dx}{dt} = x'(t)$. Hence, ODEs (6) and (7) can be transformed into the following systems of ODEs, respectively:

$$\begin{aligned} x' &= y, \\ y' &= -p(t)g(y) - q_1(t)h(x) - q_2(t)x + f(t,x,y) \end{aligned} \tag{8}$$

and

$$\begin{aligned} x' &= y, \\ y' &= -\phi(t,x,y) - q_1(t)x - q_2(t)\theta(x) + q(t,x,y). \end{aligned} \tag{9}$$

Let

$$g_1(y) = \begin{cases} y^{-1}g(y), & y \neq 0, \\ g_y'(0), & y = 0, \end{cases}$$

$$\theta_1(x) = \begin{cases} x^{-1}\theta(x), & x \neq 0, \\ \theta_x'(0), & x = 0, \end{cases}$$

$$h_1(x) = \begin{cases} x^{-1}h(x), & x \neq 0, \\ h_x'(0), & x = 0 \end{cases}$$

and

$$\phi_1(t,x,y) = \begin{cases} x^{-1}\phi(t,x,y), & y \neq 0, \\ \phi_y'(t,x,0), & y = 0. \end{cases}$$

2 Asymptotic analysis

Firstly, we prove a theorem on the boundedness of solutions of ODE (6) as $t \to \infty$ by using the second method of Lyapunov. Before stating the theorem, we give the following hypotheses.

Hypotheses A

$(H1)$ *Let g_0 and h_0 be positive constants such that the following conditions hold*:

$$g(0) = 0, \qquad y^{-1}g(y) \geq g_0 \geq 1, \quad \forall y \neq 0 \text{ as } y \in \Re$$

and

$$h(0) = 0, \qquad x^{-1}h(x) \geq h_0 \geq 1 > 0, \quad \forall x \neq 0 \text{ as } x \in \Re.$$

(*H*2) *Let* $\alpha_1(t)$, $Q(t)$, *and* $\lambda(t)$ *be continuous functions such that the following conditions hold*:

$$\begin{aligned}
&p(t) > 0, \qquad q_1 \in C^1[a,\infty), \qquad q_1(t) > 0, \quad \forall t \in \Re^+, \\
&\bigl|f(t,x,y)\bigr| \le \bigl|\alpha_1(t)\bigr|, \quad \forall t \in \Re^+, \forall x, y \in \Re, \\
&Q(t) = \frac{1}{2}\bigl[q_1'(t) + 2p(t)q_1(t)\bigr] > 0, \quad \forall \in t \in \Re^+, \\
&\int_a^\infty \frac{q_2^2(s)}{\lambda^2(s)Q(s)}\,ds < \infty, \qquad \int_a^\infty \frac{\alpha_1^2(s)}{Q(s)}\,ds < \infty,
\end{aligned}$$

and

$$\lambda^2(t) \ge 1, \quad \forall t \in [a,\infty).$$

We now give the first theorem of this paper.

Theorem 1 *We suppose that hypotheses* (*H*1) *and* (*H*2) *hold. Then, any solution of ODE* (6) *satisfies*

$$\bigl|x(t)\bigr| \le O(1), \qquad \left|\frac{dx}{dt}\right| \le O\bigl(\sqrt{q_1(t)}\bigr) \quad \textit{as } t \to \infty.$$

Proof To proceed with the proof, we benefit from the Lyapunov second method. Hence, we define a Lyapunov function $W(t) = W(x,y)$ by

$$W(x,y) = 2\int_0^x h(\xi)\,d\xi + \frac{1}{q_1(t)}y^2. \tag{10}$$

By using the hypotheses of (*H*1) and (*H*2), we obtain

$$W(x,y) = 0 \quad \text{if and only if} \quad x = 0 \quad \text{and} \quad y = 0.$$

Next, if we use hypothesis (*H*1), we have

$$\begin{aligned}
W(x,y) &= 2\int_0^x \frac{h(\xi)}{\xi}\xi\,d\xi + \frac{1}{q_1(t)}y^2 \ge 2\int_0^x h_0\xi\,d\xi + \frac{1}{q_1(t)}y^2 \\
&\ge 2\int_0^x \xi\,d\xi + \frac{1}{q_1(t)}y^2 \ge x^2 + \frac{1}{q_1(t)}y^2 \ge 0.
\end{aligned}$$

If we differentiate the Lyapunov function W in (10) along the solutions of system of ODEs (8) and use hypotheses (*H*1) and (*H*2), we derive

$$\begin{aligned}
\frac{d}{dt}W &= -\frac{q_1'(t)}{q_1^2(t)}y^2 - \frac{2p(t)}{q_1(t)}g(y)y - \frac{2q_2(t)}{q_1(t)}xy + \frac{2}{q_1(t)}yf(t,x,y) \\
&\le -\frac{q_1'(t)}{q_1^2(t)}y^2 - \frac{2p(t)}{q_1(t)}y^2 - \frac{2q_2(t)}{q_1(t)}xy + \frac{2}{q_1(t)}yf(t,x,y) \\
&= -\frac{2}{q_1^2(t)}\left[\frac{q_1'(t)}{2} + p(t)q_1(t)\right]y^2 - \frac{2q_2(t)}{q_1(t)}xy + \frac{2}{q_1(t)}yf(t,x,y).
\end{aligned}$$

Let

$$Q(t)=\frac{1}{2}\bigl[q_1'(t)+2p(t)q_1(t)\bigr].$$

Then we have

$$\frac{dW}{dt}\le -\frac{2q_2(t)}{q_1(t)}xy-\frac{2Q(t)}{q_1^2(t)}y^2+\frac{2}{q_1(t)}yf(t,x,y).$$

Let $a>0$, $b,x\in\Re$. We consider the inequality

$$-ax^2+bx\le -\frac{a}{2}x^2+\frac{b^2}{2a}. \tag{11}$$

If we apply this inequality, that is, inequality (11), to the terms

$$-\frac{2Q(t)}{q_1^2(t)}y^2+\frac{2}{q_1(t)}yf(t,x,y),$$

then we can derive

$$-\frac{2Q(t)}{q_1^2(t)}y^2+\frac{2}{q_1(t)}yf(t,x,y)\le -\frac{Q(t)}{q_1^2(t)}y^2+\frac{f^2(t,x,y)}{Q(t)}.$$

Hence, we have

$$\begin{aligned}\frac{dW}{dt}&\le -\frac{2q_2(t)}{q_1(t)}xy-\frac{Q(t)}{q_1^2(t)}y^2+\frac{f^2(t,x,y)}{Q(t)}\\&\le -\frac{2q_2(t)}{q_1(t)}xy-\frac{Q(t)}{q_1^2(t)}y^2+\frac{\alpha_1^2(t)}{Q(t)}\end{aligned} \tag{12}$$

by using hypothesis $(H2)$.

Let

$$E(t)=-\frac{2q_2(t)}{q_1(t)}xy-\frac{Q(t)}{q_1^2(t)}y^2.$$

We arrange this function as follows:

$$E(t)=-\frac{Q(t)}{q_1^2(t)}\left[\lambda(t)y+\frac{q_1(t)q_2(t)}{\lambda(t)Q(t)}x\right]^2+\frac{q_2^2(t)}{\lambda^2(t)Q(t)}x^2+\frac{Q(t)}{q_1^2(t)}\bigl(\lambda^2(t)-1\bigr)y^2.$$

Since the first term of $E(t)$ is negative, it is clear that

$$E(t)\le \frac{q_2^2(t)}{\lambda^2(t)Q(t)}x^2+\frac{Q(t)}{q_1^2(t)}\bigl(\lambda^2(t)-1\bigr)y^2. \tag{13}$$

When we consider together inequalities (12) and (13), it follows that

$$\frac{dW}{dt}\le \frac{q_2^2(t)}{\lambda^2(t)Q(t)}x^2+\frac{Q(t)}{q_1^2(t)}\bigl(\lambda^2(t)-1\bigr)y^2+\frac{\alpha_1^2(t)}{Q(t)}. \tag{14}$$

Next, we assume that

$$\frac{q_2^2(t)}{\lambda^2(t)Q(t)} = \frac{Q(t)}{q_1(t)}\bigl(\lambda^2(t) - 1\bigr).$$

Hence, from the last inequality, we can derive that

$$\lambda^2(t) = \frac{Q^2(t) + \sqrt{Q^4(t) + 4q_1(t)q_2^2(t)Q^2(t)}}{2Q^2(t)}, \quad t \in [a, \infty).$$

From this equality, we can see that

$$\lambda^2(t) \geq 1, \quad t \in [a, \infty).$$

Thus, we obtain

$$E(t) \leq \frac{q_2^2(t)}{\lambda^2(t)Q(t)}\left[x^2 + \frac{1}{q_1(t)}y^2\right]. \tag{15}$$

We observe from inequalities (14) and (15) that

$$\frac{dW}{dt} \leq \frac{q_2^2(t)}{\lambda^2(t)Q(t)}\left[x^2 + \frac{1}{q_1(t)}y^2\right] + \frac{\alpha_1^2(t)}{Q(t)}.$$

We also have

$$x^2 + \frac{1}{q_1(t)}y^2 \leq W(t).$$

Clearly, by the last inequality, we can derive

$$\frac{dW}{dt} - \frac{q_2^2(t)}{\lambda^2(t)Q(t)}W(t) \leq \frac{\alpha^2(t)}{Q(t)}.$$

Multiplying this inequality by $\exp(-\int_a^t \frac{q_2^2(s)}{\lambda^2(s)Q(s)}\,ds)$, we obtain

$$\frac{d}{dt}\left[W(t)\exp\left(-\int_a^t \frac{q_2^2(s)}{\lambda^2(s)Q(s)}\,ds\right)\right] \leq \frac{\alpha^2(t)}{Q(t)}\exp\left(-\int_a^t \frac{q_2^2(s)}{\lambda^2(s)Q(s)}\,ds\right).$$

Integrating this inequality from a to t, we get

$$W(t)\exp\left(-\int_a^t \frac{q_2^2(s)}{\lambda^2(s)Q(s)}\,ds\right) - W(a) \leq \int_a^t \left[\frac{\alpha^2(s)}{Q(s)}\exp\left(-\int_a^t \frac{q_2^2(\tau)}{\lambda^2(\tau)Q(\tau)}\,d\tau\right)\right]ds.$$

Hence, it follows that

$$\begin{aligned} W(t) &\leq W(a)\exp\left(\int_a^t \frac{q_2^2(s)}{\lambda^2(s)Q(s)}\,ds\right) + \int_a^t \left[\frac{\alpha^2(s)}{Q(s)}\exp\left(\int_s^t \frac{q_2^2(\tau)}{\lambda^2(\tau)Q(\tau)}\,d\tau\right)\right]ds \\ &\leq W(a)\exp\left(\int_a^\infty \frac{q_2^2(s)}{\lambda^2(s)Q(s)}\,ds\right) + \int_a^\infty \left[\frac{\alpha^2(s)}{Q(s)}\exp\left(\int_s^\infty \frac{q_2^2(\tau)}{\lambda^2(\tau)Q(\tau)}\,d\tau\right)\right]ds. \end{aligned}$$

By hypothesis $(H2)$, we know that

$$\int_a^\infty \frac{\alpha^2(s)}{Q(s)}\,ds < \infty, \qquad \int_a^\infty \frac{q_2^2(s)}{\lambda^2(s)Q(s)}\,ds < \infty \quad \text{and} \quad \int_s^\infty \frac{q_2^2(\tau)}{\lambda^2(\tau)Q(\tau)}\,d\tau < \infty.$$

Hence, we can assume that

$$\begin{aligned} &W(a)\exp\left(\int_a^\infty \frac{q_2^2(s)}{\lambda^2(s)Q(s)}\,ds\right) \\ &\quad + \int_a^\infty \left[\frac{\alpha^2(s)}{Q(s)}\exp\left(\int_s^\infty \frac{q_2^2(\tau)}{\lambda^2(\tau)Q(\tau)}\,d\tau\right)\right]ds = C_0, \quad C_0 > 0, C_0 \in \Re. \end{aligned}$$

Now, it is observed that

$$W(t) \le C_0$$

and

$$x^2 + \frac{1}{q_1(t)}y^2 \le W(t) \le C_0.$$

Therefore, we can conclude that

$$\left|x(t)\right| \le \sqrt{C_0}, \qquad \left|y(t)\right| \le \sqrt{C_0 q_1(t)}.$$

Thus, we have

$$\left|x(t)\right| \le O(1), \qquad \left|y(t)\right| \le O\left(\sqrt{q_1(t)}\right) \quad \text{as } t \to \infty.$$

These two inequalities complete the proof of Theorem 1. □

Example 1 In a particular case of ODE (6), we consider the following nonlinear ODE of second order:

$$\begin{aligned} &x'' + \left(\frac{1}{t^{10}} - \frac{6}{t}\right)\left(2x' + x'\exp\left(-\left(x'\right)^2\right) + t^{12}\left(3 + \sin x\right)x - t^\alpha x \right. \\ &\quad = \frac{1}{\sqrt{1 + t^2 + x^2 + (x')^2}}, \quad t \ge 2, \alpha < 5, \alpha \in \Re. \end{aligned} \tag{16}$$

Let $y = \frac{dx}{dt}$. Then ODE (16) can be stated as the following system of ODEs:

$$\begin{aligned} \frac{dx}{dt} &= y, \\ \frac{dy}{dt} &= -\left(\frac{1}{t^{10}} - \frac{6}{t}\right)\left(2y + y\exp\left(-y^2\right)\right) - t^{12}(3 + \sin x)x \\ &\quad + t^\alpha x + \frac{1}{\sqrt{1 + t^2 + x^2 + y^2}}, \quad t \ge 2, \alpha < 5, \alpha \in \Re. \end{aligned} \tag{17}$$

Let us compare system (17) with system (8). Then the existence of the following relations can be derived respectively:

$$
\begin{aligned}
&g(y) = 2y + y\exp\bigl(-y^2\bigr), \quad g(0) = 0, \\
&y^{-1}g(y) = 2 + \exp\bigl(-y^2\bigr) \geq 2 = g_0 > 1, \quad \forall y \neq 0 \text{ as } y \in \Re, \\
&h(x) = (3 + \sin x)x, \quad h(0) = 0, \\
&x^{-1}h(x) = 3 + \sin x \geq 2 = h_0 > 1, \quad \forall x \neq 0 \text{ as } x \in \Re, \\
&p(t) = \frac{1}{t^{10}} - \frac{6}{t} > 0, \quad t \geq 2, \forall t \in \Re^+, \\
&q_1(t) = t^{12} > 0, \quad t \geq 2, \qquad q_2(t) = -t^{\alpha}, \qquad \alpha < 5, \quad \alpha \in \Re, \text{ say } \alpha = 4, \\
&f(t,x,y) = \frac{1}{\sqrt{1 + t^2 + x^2 + (x')^2}}, \\
&\bigl|f(t,x,y)\bigr| \leq \frac{1}{\sqrt{1+t^2}} = \alpha(t), \\
&Q(t) = \frac{1}{2}\bigl[q_1'(t) + 2p(t)q_1(t)\bigr] = t^2 > 0, \quad t \geq 2 = a, \\
&\lambda^2(t) = \frac{Q^2(t) + \sqrt{Q^4(t) + 4q_1(t)q_2^2(t)Q^2(t)}}{2Q^2(t)} \\
& = \frac{t^4 + \sqrt{t^8 + 4t^{16+2\alpha}}}{2t^4} \\
& = \frac{1 + \sqrt{1 + 4t^{8+2\alpha}}}{2} > 1, \\
&\lambda^2(t) > 1, \quad t \in [2, \infty), \\
&\int_a^{\infty} \frac{\alpha^2(s)}{Q(s)}\,ds = \int_2^{\infty} \frac{1}{s^2(s^2+1)}\,ds = \frac{1}{2} + \operatorname{arctg} 2 - \frac{\pi}{2} < \infty,
\end{aligned}
$$

and

$$
\int_a^{\infty} \frac{q_2^2(s)}{\lambda^2(s)Q(s)}\,ds = 2\int_2^{\infty} \frac{t^{2\alpha}}{t^2 + \sqrt{t^4 + 4t^{12+2\alpha}}}\,ds \leq \int_2^{\infty} \frac{t^{2\alpha}}{t^{6+\alpha}}\,ds = \int_2^{\infty} \frac{1}{t^{6-\alpha}}\,ds < \infty.
$$

It is now notable that the above discussion shows that all the hypotheses of Theorem 1, $(H1)$ and $(H2)$, can be applicable. Therefore, all the solutions of system (17) satisfy

$$
\bigl|x(t)\bigr| \leq O(1), \qquad \bigl|y(t)\bigr| \leq O\bigl(t^6\bigr) \quad \text{as } t \to \infty.
$$

Then we can say the same for all the solutions of ODE (16) and their first derivatives.

We can observe the behaviors of the paths of ODE (16) by Figs. 1–4. Example 1 has been solved with MATLAB-Simulink and the following graphs have been obtained.

Next, we prove the boundedness properties of solution of ODE (7) as $t \to \infty$ using also the Lyapunov direct or second method. Below are the hypotheses required for the results.

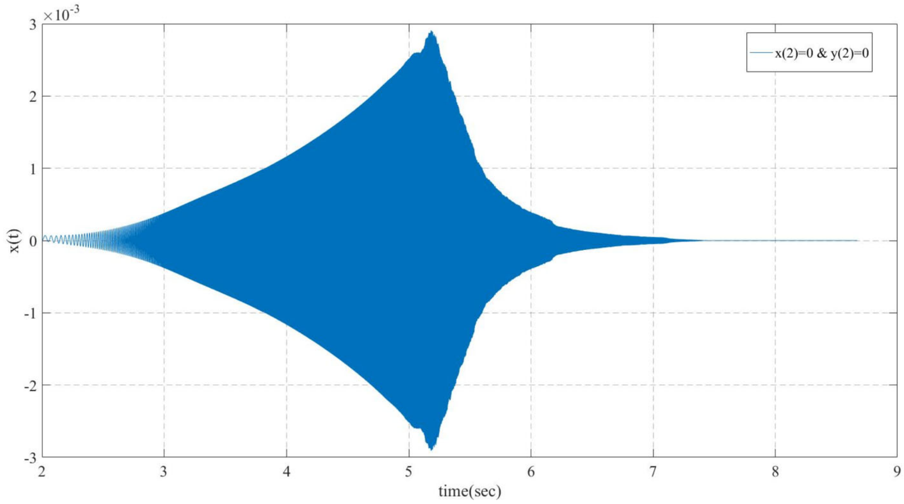

Figure 1 Trajectory of $x(t)$ for Example 1 when $x(2) = 0$ and $y(2) = 0$

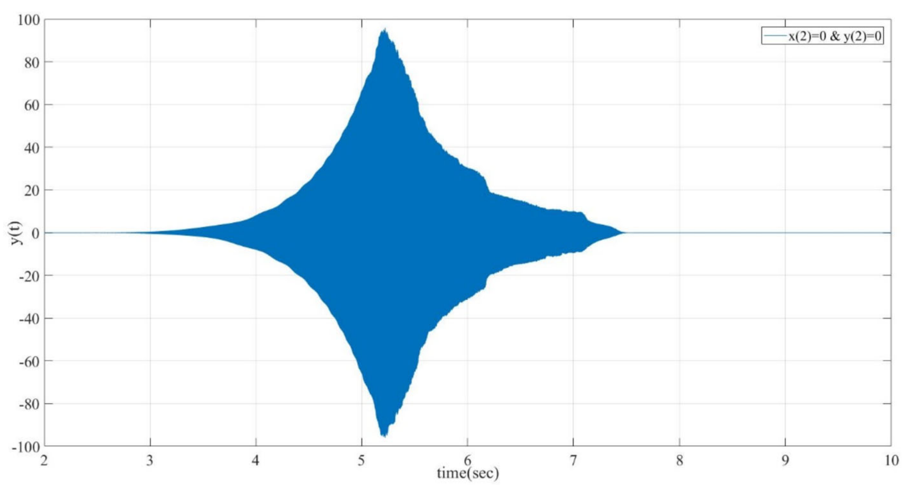

Figure 2 Trajectory of $y(t)$ for Example 1 when $x(2) = 0$ and $y(2) = 0$

Hypotheses B

$(H3)$ *Let θ_0 and ϕ_0 be positive constants such that the following conditions hold*:

$$\theta(0) = 0, \qquad x^{-1}\theta(x) \geq \theta_0 \geq 1 \quad \text{for all } x \neq 0 \text{ as } x \in \Re,$$

and

$$\phi(t,x,0) = 0, \qquad y^{-1}\phi(t,x,y) \geq \phi_0 \geq 1 \quad \text{for all } t \in \Re^+, y \neq 0, \text{ as } x,y \in \Re.$$

$(H4)$ *Let $\beta(t)$, $Q_1(t)$, and $\mu(t)$ be functions such that the following conditions hold*:

$$\left|q(t,x,y)\right| \leq \left|\beta(t)\right| \quad \text{for } \forall t \in \Re^+, \forall x,y \in \Re,$$

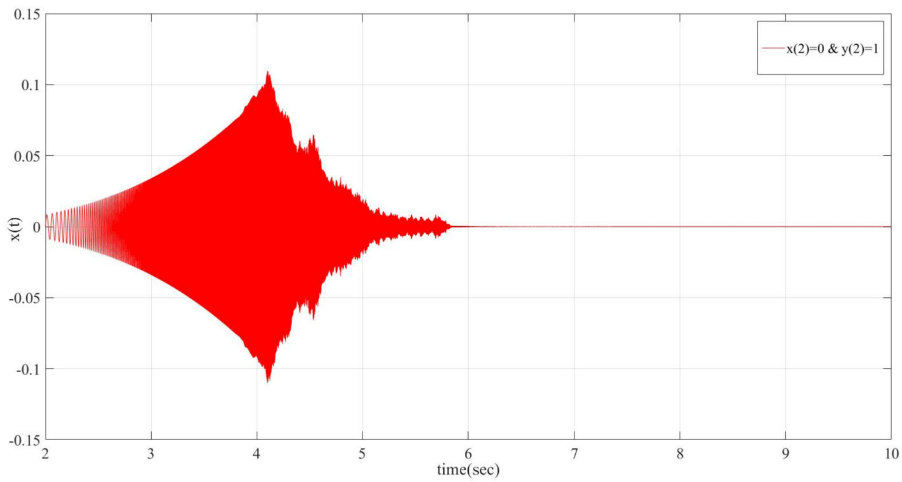

Figure 3 Trajectory of $x(t)$ for Example 1 when $x(2) = 0$ and $y(2) = 1$

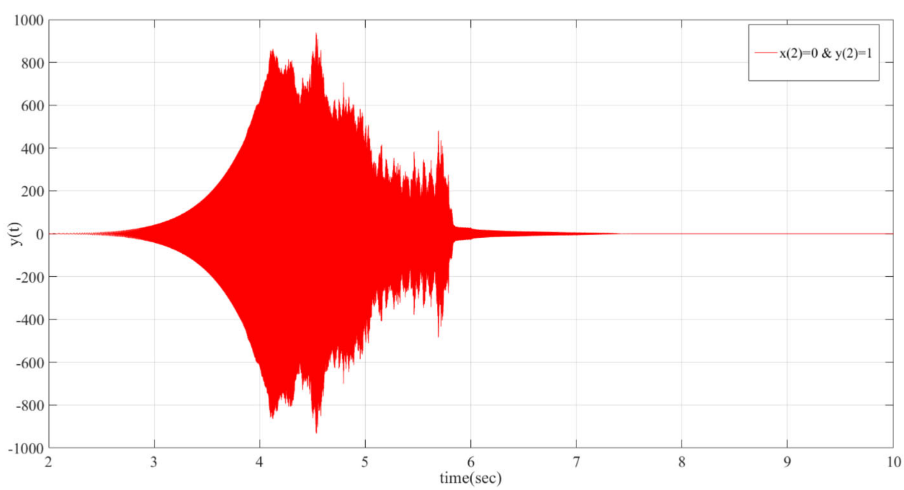

Figure 4 Trajectory of $y(t)$ for Example 1 when $x(2) = 0$ and $y(2) = 1$

$$q_2(t) > 0, \quad \forall t \in \Re^+, \qquad Q_1(t) = \frac{q_2'(t)}{2} + q_2(t),$$
$$\int_a^\infty \frac{\beta^2(s)}{Q_1(s)}\, ds < \infty, \qquad \int_a^\infty \frac{q_1^2(s)}{\mu^2(s)Q_1(s)}\, ds < \infty,$$

and

$$\mu^2(t) \geq 1, \quad \forall t \in [a, \infty).$$

The next result of this paper is given below.

Theorem 2 *Suppose that hypotheses* $(H3)$ *and* $(H4)$ *hold. Then any solution of ODE* (7) *satisfies*

$$|x(t)| \le O(1), \qquad \left|\frac{dx}{dt}\right| \le O\bigl(\sqrt{q_2(t)}\bigr) \quad \text{as } t \to \infty.$$

Proof We define a Lyapunov function $W_1(t) = W_1(x, y)$ by

$$W_1(x,y) = 2\int_0^x \theta(\xi)\,d\xi + \frac{1}{q_2(t)}y^2. \tag{18}$$

By the hypotheses of Theorem 2, it follows that

$$W_1(x,y) = 0 \quad \text{if and only if} \quad x = 0 \quad \text{and} \quad y = 0.$$

In addition, by hypothesis $(H3)$, we get

$$\begin{aligned} W_1(x,y) &= 2\int_0^x \frac{\theta(\xi)}{\xi}\xi\,d\xi + \frac{1}{q_2(t)}y^2 \\ &\ge 2\int_0^x \xi\,d\xi + \frac{1}{q_2(t)}y^2 \\ &= x^2 + \frac{1}{q_2(t)}y^2 \ge 0. \end{aligned}$$

Next, if we calculate the time derivative of the function $W_1(x, y)$ given by (18) along solutions of system (9), we have

$$\begin{aligned} \frac{d}{dt}W_1 &= -\frac{q_2'(t)}{q_2^2(t)}y^2 - \frac{2}{q_2(t)}\phi(t,x,y)y - 2\frac{q_1(t)}{q_2(t)}xy + \frac{2}{q_2(t)}yq(t,x,y) \\ &= -\frac{q_2'(t)}{q_2^2(t)}y^2 - \frac{2}{q_2(t)}\frac{\phi(t,x,y)}{y}y^2 - 2\frac{q_1(t)}{q_2(t)}xy + \frac{2}{q_2(t)}yq(t,x,y). \end{aligned}$$

Using hypothesis $(H3)$, we obtain

$$\begin{aligned} \frac{d}{dt}W_1 &\le -\frac{q_2'(t)}{q_2^2(t)}y^2 - \frac{2}{q_2(t)}y^2 - 2\frac{q_1(t)}{q_2(t)}xy + \frac{2}{q_2(t)}yq(t,x,y) \\ &= -\frac{2}{q_2^2(t)}\left[\frac{q_2'(t)}{2} + q_2(t)\right]y^2 - \frac{2q_1(t)}{q_2(t)}xy + \frac{2}{q_2(t)}yq(t,x,y). \end{aligned}$$

Let

$$Q_1(t) = \frac{q_2'(t)}{2} + q_2(t).$$

This representation yields that

$$\frac{dW_1}{dt} \le -\frac{2q_1(t)}{q_2(t)}xy - \frac{2Q_1(t)}{q_2^2(t)}y^2 + \frac{2}{q_2(t)}yq(t,x,y).$$

If we consider inequality (11) and the following terms:

$$-\frac{2Q_1(t)}{q_2^2(t)}y^2 + \frac{2}{q_2(t)}yq(t,x,y),$$

then we have

$$-\frac{2Q_1(t)}{q_2^2(t)}y^2 + \frac{2}{q_2(t)}yq(t,x,y) \le -\frac{Q_1(t)}{q_2^2(t)}y^2 + \frac{q^2(t,x,y)}{Q_1(t)}.$$

Hence, we observe that

$$\begin{aligned}\frac{dW_1}{dt} &\le -\frac{2q_1(t)}{q_2(t)}xy - \frac{Q_1(t)}{q_2^2(t)}y^2 + \frac{q^2(t,x,y)}{Q_1(t)} \\ &\le -\frac{2q_1(t)}{q_2(t)}xy - \frac{Q_1(t)}{q_2^2(t)}y^2 + \frac{\beta^2(t)}{Q_1(t)}\end{aligned} \tag{19}$$

by hypothesis $(H4)$.

Let

$$E_1(t) = -\frac{2q_1(t)}{q_2(t)}xy - \frac{Q_1(t)}{q_2^2(t)}y^2.$$

We can arrange this function as follows:

$$E_1(t) = -\frac{Q_1(t)}{q_2^2(t)}\left[\mu(t)y + \frac{q_1(t)q_2(t)}{\mu(t)Q_1(t)}x\right]^2 + \frac{q_1^2(t)}{\mu^2(t)Q_1(t)}x^2 + \frac{Q_1(t)}{q_2^2(t)}\bigl(\mu^2(t)-1\bigr)y^2.$$

Clearly, from this expression, it follows that

$$E_1(t) \le \frac{q_1^2(t)}{\mu^2(t)Q_1(t)}x^2 + \frac{Q_1(t)}{q_2^2(t)}\bigl(\mu^2(t)-1\bigr)y^2. \tag{20}$$

Using inequalities (19) and (20), we obtain

$$\frac{dW_1}{dt} \le \frac{q_1^2(t)}{\mu^2(t)Q_1(t)}x^2 + \frac{Q_1(t)}{q_2^2(t)}\bigl(\mu^2(t)-1\bigr)y^2 + \frac{\beta^2(t)}{Q_1(t)}. \tag{21}$$

Now, we assume that

$$\frac{q_1^2(t)}{\mu^2(t)Q_1(t)} = \frac{Q_1(t)}{q_2(t)}\bigl(\mu^2(t)-1\bigr).$$

Hence, we can derive that

$$\mu^2(t) = \frac{Q_1^2(t) + \sqrt{Q_1^4(t) + 4q_2(t)q_1^2(t)Q_1^2(t)}}{2Q_1^2(t)}, \quad t \in [a,\infty).$$

From this equality, we have

$$\mu^2(t) \ge 1, \quad t \in [a,\infty).$$

Then we obtain

$$E_1(t) \le \frac{q_1^2(t)}{\mu^2(t)Q_1(t)}\left[x^2 + \frac{1}{q_2(t)}y^2\right]. \tag{22}$$

Inequalities (21) and (22) together yield that

$$\frac{dW_1}{dt} \le \frac{q_1^2(t)}{\mu^2(t)Q_1(t)}\left[x^2 + \frac{1}{q_2(t)}y^2\right] + \frac{\beta^2(t)}{Q_1(t)}.$$

Since

$$x^2 + \frac{1}{q_2(t)}y^2 \le W_1(t),$$

then

$$\frac{dW_1}{dt} - \frac{q_1^2(t)}{\mu^2(t)Q_1(t)}W_1(t) \le \frac{\beta^2(t)}{Q_1(t)}.$$

Multiplying this inequality by $\exp(-\int_a^t \frac{q_1^2(s)}{\mu^2(s)Q_1(s)}\,ds)$, we obtain

$$\frac{d}{dt}\left[W_1(t)\exp\left(-\int_a^t \frac{q_1^2(s)}{\mu^2(s)Q_1(s)}\,ds\right)\right] \le \frac{\beta^2(t)}{Q_1(t)}\exp\left(-\int_a^t \frac{q_1^2(s)}{\mu^2(s)Q_1(s)}\,ds\right).$$

Integrating this inequality from a to t, we get

$$W_1(t)\exp\left(-\int_a^t \frac{q_1^2(s)}{\mu^2(s)Q_1(s)}\,ds\right) - W_1(a) \le \int_a^t \left[\frac{\beta^2(s)}{Q_1(s)}\exp\left(-\int_a^t \frac{q_1^2(\tau)}{\mu^2(\tau)Q_1(\tau)}\,d\tau\right)\right]ds.$$

Hence, it follows that

$$\begin{aligned} W_1(t) &\le W_1(a)\exp\left(\int_a^t \frac{q_1^2(s)}{\mu^2(s)Q_1(s)}\,ds\right) + \int_a^t \left[\frac{\beta^2(s)}{Q_1(s)}\exp\left(\int_s^t \frac{q_1^2(\tau)}{\mu^2(\tau)Q_1(\tau)}\,d\tau\right)\right]ds \\ &\le W_1(a)\exp\left(\int_a^\infty \frac{q_1^2(s)}{\mu^2(s)Q_1(s)}\,ds\right) + \int_a^\infty \left[\frac{\beta^2(s)}{Q_1(s)}\exp\left(\int_s^\infty \frac{q_1^2(\tau)}{\mu^2(\tau)Q_1(\tau)}\,d\tau\right)\right]ds. \end{aligned}$$

By hypothesis $(H4)$, we know

$$\int_a^\infty \frac{\beta^2(s)}{Q_1(s)}\,ds < \infty, \qquad \int_a^\infty \frac{q_1^2(s)}{\mu^2(s)Q_1(s)}\,ds < \infty \quad \text{and} \quad \int_s^\infty \frac{q_1^2(\tau)}{\mu^2(\tau)Q_1(\tau)}\,d\tau < \infty.$$

Hence, we can assume that

$$\begin{aligned} &W_1(a)\exp\left(\int_a^\infty \frac{q_1^2(s)}{\mu^2(s)Q_1(s)}\,ds\right) \\ &\quad + \int_a^\infty \left[\frac{\beta^2(s)}{Q_1(s)}\exp\left(\int_s^\infty \frac{q_1^2(\tau)}{\mu^2(\tau)Q_1(\tau)}\,d\tau\right)\right]ds = M_0, \quad M_0 > 0, M_0 \in \Re. \end{aligned}$$

Now, it is observed that

$$W_1(t) \le M_0$$

and

$$x^2 + \frac{1}{q_2(t)} y^2 \le W_1(t) \le M_0.$$

Therefore, we can conclude that

$$|x(t)| \le \sqrt{M_0}, \qquad |y(t)| \le \sqrt{M_0 q_2(t)}.$$

That is, we have

$$|x(t)| \le O(1), \qquad |y(t)| \le O\bigl(\sqrt{q_2(t)}\bigr) \quad \text{as } t \to \infty.$$

These two estimates finish the proof of Theorem 2. □

Example 2 As a particular case of ODE (7), we consider the nonlinear ODE of second order

$$\begin{aligned} &x'' + \bigl(3x' + x' \exp\bigl(-t^2 - x^2\bigr)\bigr) + \bigl(6\exp(2t)\bigr)x + \exp(8t)(2x + \sin x) \\ &\quad = \frac{\sin t}{\exp(t)(1 + \exp(x^2))}. \end{aligned} \tag{23}$$

Let $y = \frac{dx}{dt}$. Then ODE (23) can be expressed as the following system:

$$\begin{aligned} \frac{dx}{dt} &= y, \\ \frac{dy}{dt} &= -(3y + y\exp\bigl(-t^2 - x^2\bigr) - \bigl(6\exp(2t)\bigr)x \\ &\quad - \exp(8t)(2x + \sin x) + \frac{\sin t}{\exp(t)(1 + \exp(x^2))}. \end{aligned} \tag{24}$$

Let us compare system (24) with system (9). It can be observed that the following relations hold respectively:

$$\begin{aligned} &\theta(x) = 2x + \sin x, \quad \theta(0) = 0, \\ &\frac{\theta(x)}{x} = 2 + \frac{\sin x}{x} \ge 1 = \theta_0, \quad \forall x \neq 0, x \in \Re, \\ &\phi(t,x,y) = 3y + y\exp\bigl(-t^2 - x^2\bigr), \quad \phi(t,x,0) = 0, \\ &\frac{\phi(t,x,y)}{y} = 3 + \exp\bigl(-t^2 - x^2\bigr) \ge 3 = \phi_0 > 1, \quad \forall t \in \Re^+, \forall y \neq 0 \text{ as } x, y \in \Re, \\ &q_1(t) = 6\exp(2t), \qquad q_2(t) = \exp(8t), \quad \forall t \in \Re^+, \\ &q(t,x,y) = \frac{\sin t}{\exp(t)(1 + \exp(x^2))}, \qquad \bigl|q(t,x,y)\bigr| \le \frac{1}{\exp(t)} = \beta(t), \\ &Q_1(t) = \frac{q_2'(t)}{2} + 2q_2(t) = 6\exp(8t) > 0, \quad \forall t \in \Re^+, \end{aligned}$$

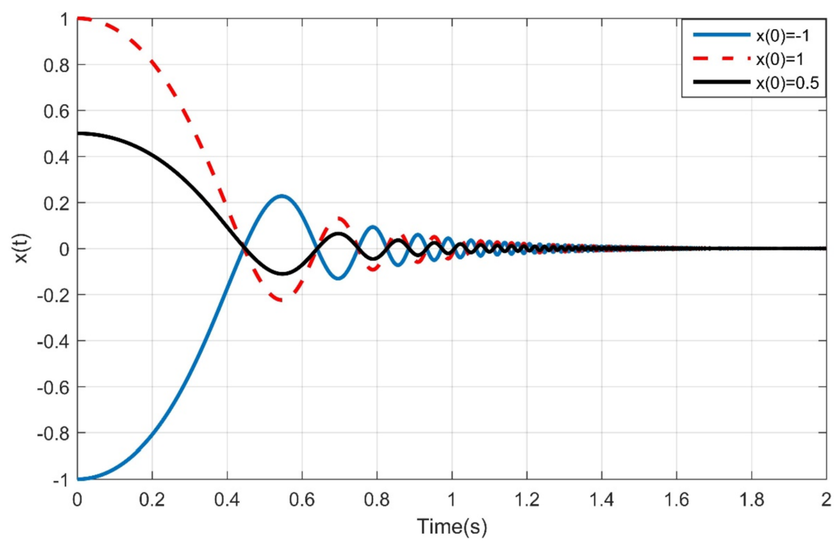

Figure 5 Trajectories of $x(t)$ for Example 2 when $x(0) = -1$, $x(0) = 1$ and $x(0) = 0.5$

$$\begin{aligned}
\mu^2(t) &= \frac{Q_1^2(t) + \sqrt{Q_1^4(t) + 4q_2(t)q_1^2(t)Q_1^2(t)}}{2Q_1^2(t)} \\
&= \frac{36\exp(16t) + \sqrt{(36)^2\exp(32t) + 4(36)^2\exp(28t)}}{72\exp(16t)} \\
&= \frac{1 + \sqrt{1 + 4\exp(-8t)}}{2} > 1, \quad \mu^2(t) \geq 1, \forall t \in \Re^+, \\
\int_a^\infty \frac{\beta^2(s)}{Q_1(s)}\,ds &= \frac{1}{6}\int_0^\infty \frac{1}{\exp(10s)}\,ds = \frac{1}{60} < \infty,
\end{aligned}$$

and

$$\int_a^\infty \frac{q_1^2(s)}{\mu^2(s)Q_1(s)}\,ds \leq 6\int_0^\infty \frac{1}{\exp(2s)}\,ds = 3 < \infty.$$

It is now notable that the discussion given shows that all the hypotheses of Theorem 2, ($H3$) and ($H4$), can be applied and hold. Thus, all the solutions of system (24) satisfy

$$|x(t)| \leq O(1), \qquad |y(t)| \leq O\big(\sqrt{6}\exp(t)\big) \quad \text{as } t \to \infty.$$

Therefore, all the solutions of ODE (23) and their first derivatives satisfy the inequalities

$$|x(t)| \leq O(1), \qquad |x'(t)| \leq O\big(\sqrt{6}\exp(t)\big) \quad \text{as } t \to \infty.$$

The results above as $t \to \infty$ can be seen as shown in Figs. 5 and 6. Example 2 has been solved using MATLAB-Simulink and the following graphs have been obtained.

3 Discussion

Let us compare the results of this paper, Theorem 1 and Theorem 2, with those of Meng [15] and Sun and Meng [23], respectively.

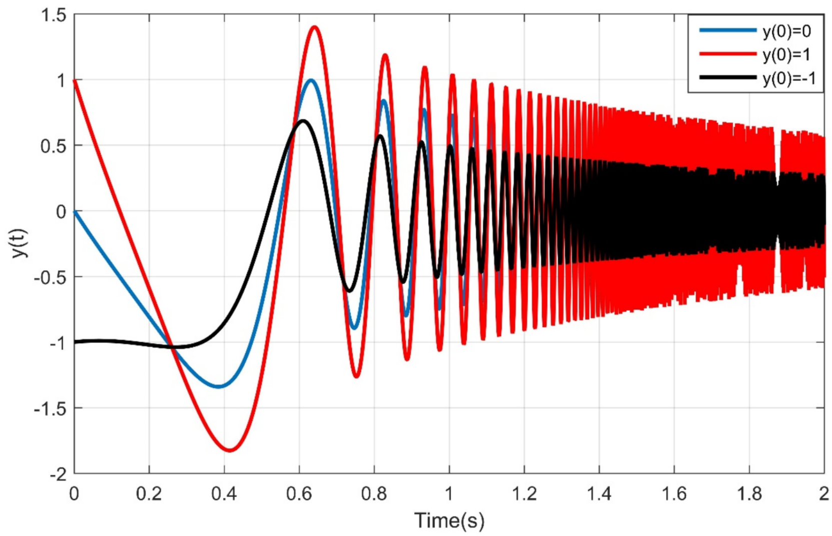

Figure 6 Trajectories of $y(t)$ for Example 2 when $y(0) = 0$, $y(0) = 1$ and $y(0) = -1$

(1^0) We observe that our equations ODEs (6) and (7) generalize and improve ODEs (1)–(4) from the linear ODEs to the nonlinear ODEs. Next, ODEs (6) and (7) also generalize and improve ODE (5) for the case as $r(t) = 1$.

(2^0) The Lyapunov function

$$V(t) = F(t)\left[x^2(t) + \frac{1}{q_1(t)}\left(\frac{dx}{dt}\right)^2\right]$$

was employed by Meng [15]. This Lyapunov function includes the function F but this function is not involved in ODE (4). However, in this paper, as a basic tool, we constructed two new Lyapunov functions as follows:

$$W(x,y) = 2\int_0^x h(\xi)\,d\xi + \frac{1}{q_1(t)}y^2$$

and

$$W_1(x,y) = 2\int_0^x \theta(\xi)\,d\xi + \frac{1}{q_2(t)}y^2.$$

These Lyapunov functions depend only on the functions appearing in ODEs (6) and (7), respectively, but they do not depend on the function F. Hence, we obtain the results of Meng [15] and Sun and Meng [23] under less restrictive conditions by using these two Lyapunov functions. In the proofs of Theorem 1 and Theorem 2, we did not need the conditions

$$F(t) > 0, \qquad \frac{d}{dt}F(t) \geq 0.$$

In addition, the functions $Q(t)$, $\lambda(t)$ and $Q_1(t)$, $\mu(t)$ in Theorem 1 and Theorem 2, respectively, have very simple forms compared to the functions $Q(t)$, $\lambda(t)$ used in

Meng [15, Theorem 1]. Moreover, the integrability conditions in Theorem 1 and Theorem 2 have much simpler forms than those used in Meng [15, Theorem 1].

(3^0) Let $r(t) = 1$ in ODE (5). We can observe that the conditions of Theorem 1 and Theorem 2 are less restrictive than those of Sun and Meng [23, Theorem 2]. Hence, we obtain the results of Sun and Meng [23] under weaker conditions.

Thus, the results of this paper improve and include the results of [23].

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The authors declare that there is no conflict of interests.

Authors' contributions
All authors read and approved the final manuscript.

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