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Sixteen practically solvable systems of difference equations

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Abstract
Closed-form formulas for general solutions to sixteen hyperbolic-cotangent-type systems of difference equations of interest are obtained, showing their practical solvability and completely solving a solvability problem for some concrete values of delays.

MSC: 39A45

Keywords: System of difference equations; General solution; Closed-form formula; Product-type system

1 Introduction

1.1 A little on solvability

Solvability of recurrence relations was started to be studied long time ago [1–3]. That the relations are, in fact, difference equations has been also known for a long time [4].

Solvability of linear difference equations with constant coefficients, that is, of the equation

$$x_n = a_1 x_{n-1} + a_2 x_{n-2} + \cdots + a_m x_{n-m}, \quad n \ge l, \tag{1}$$

was known to De Moivre [2] (see also [5]). The study was continued by other known scientists (see, e.g., [4, 6]). For some presentations of classical solvability results consult, e.g., [7–14].

In the last two decades the topic re-attracted some interest, although it seems that a considerable part of the results in the topic are not quite original (see, e.g., some of the comments in [15–19]). Nevertheless, the recent investigation opened a door for further studies of related equations and systems (see, e.g., [20–23] and the references therein).

On the other hand, Papaschinopoulos, Schinas, and their collaborators devoted a part of their investigations to some symmetric-type systems, that is, to those of the form

$$u_n = g(u_{n-s}, v_{n-t}), \qquad v_n = g(v_{n-s}, u_{n-t}),$$

as well as to some related ones obtained from the systems by modifying the function g in various ways (see, e.g., [24–32]). Regarding the solvability of systems, they paid more attention to their invariants (see, e.g., [25–27, 29, 31, 32]). Their study on systems of difference equations motivated some other authors to do some research on the related ones (see, e.g., [15, 16, 19, 22, 23, 33–37] and the related references therein). Besides the papers on nonlinear two-dimensional systems of difference equations, there are also some on three-dimensional ones, as well as on systems of higher orders (see, e.g., [16, 29]).

It should be pointed out that since the beginning of the study of solvability of difference equations, many problems have been motivated by concrete applications, which to some extent lasts up to the present time (see, e.g., [3, 4, 6, 8, 9, 13, 38–47]). It is interesting to note that among the first ones were the problems from combinatorics (see, e.g., [2–4, 6]). But recall also that from the ancient times many recurrent relations have been connected to various population models, as it was the case with the Fibonacci model for the growth of a rabbit population (see, e.g., [47]). The main idea in solvability theory is to obtain some formulas for the general solution to a concrete difference equation or a class of difference equations, and based on them to get some information on the long-term behavior of their solutions.

1.2 Results related to the ones presented here

Before we continue, recall some notations. Let $\mathbb{N} = \{1, 2, 3, \dots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{Z} = -\mathbb{N} \cup \mathbb{N}_0$, $\mathbb{R}$ be the set of reals, and $\mathbb{C}$ be that of complex numbers. The notation $s = \overline{l_1, l_2}$, where $l_1, l_2 \in \mathbb{Z}$, $l_1 \le l_2$, is the same as $\{s \in \mathbb{Z} : l_1 \le s \le l_2\}$.

The equation

$$z_{n+1} = \frac{z_{n-k} z_{n-l} + \alpha}{z_{n-k} + z_{n-l}}, \quad n \in \mathbb{N}_0, \tag{2}$$

where $k, l \in \mathbb{N}_0$, $\alpha \in \mathbb{R}$ (or $\mathbb{C}$), is one of those which attracted some attention, especially, during the last two decades. Positive solutions to some concrete cases of the equation have been studied by several authors. But it was shown in [48] that the results on the global stability of such solutions are essentially known, since they follow from a result in [49]. For some multi-dimensional extensions of the result in [49], see [50] and [51]. It is easy to see that in (2) we may assume $\alpha = 1$. In this case the right-hand side of (2) has the form of known hyperbolic-cotangent sum formula. As usual, such difference equations are candidates for solvable ones, the fact known for a long time (the observation can be found in the old book [7], and it would not be surprising to find it even in some much earlier sources). A natural way for studying solvability of equation (2) can be found in [52].

The corresponding close-to-symmetric system in the case $k = 0$, $l = 1$, that is, the system

$$x_{n+1} = \frac{u_n v_{n-1} + a}{u_n + v_{n-1}}, \qquad y_{n+1} = \frac{w_n s_{n-1} + a}{w_n + s_{n-1}}, \quad n \in \mathbb{N}_0, \tag{3}$$

where

$$a, u_j, w_j, v_{j'}, s_{j'} \in \mathbb{C}, \quad j = 0, j' = -1, 0,$$

whereas u_n, v_n, w_n, s_n are x_n or y_n, has been studied recently in [53] and [54]. Systems of this form were studied for the first time in [22].

Quite recently in [55] we have given another solution to the solvability problem for system (3), which is related to the method for solving equation (2) presented in [56]. Maybe more important is that we have also shown therein theoretical solvability of the following generalization of system (3):

$$x_{n+1} = \frac{u_{n-k}v_{n-l} + a}{u_{n-k} + v_{n-l}}, \qquad y_{n+1} = \frac{w_{n-k}s_{n-l} + a}{w_{n-k} + s_{n-l}}, \quad n \in \mathbb{N}_0, \tag{4}$$

where $k, l \in \mathbb{N}_0$,

$$a, u_{-j}, w_{-j}, v_{-j'}, s_{-j'} \in \mathbb{C}, \quad j = \overline{0,k}, j' = \overline{0,l},$$

and each of the sequences u_n, v_n, w_n, s_n is equal to x_n or y_n.

We would like also to say that general solutions to the systems investigated in [53–55] were presented in terms of the Fibonacci sequence, that is, in terms of the sequence defined by

$$a_{n+2} = a_{n+1} + a_n$$

satisfying the initial conditions $a_1 = a_2 = 1$ (see, e.g., [39, 43, 47, 57] for some information on the sequence). Such kind of representations seems to have been quite popular among some authors in the last ten years or so. For some explanations in this direction, see [15]. There we have shown that to some difference equations and systems of difference equations can be naturally associated some linear difference equations with constant coefficients such that some of their solutions can be used in representations of the general solutions to the equations and systems. For example, in [15] it naturally appears the sequence

$$a_{n+2} = \alpha a_{n+1} + \beta a_n$$

such that $a_0 = 0$ and $a_1 = 1$. For some related results and their applications in representations of general solutions to some classes of difference equations and systems, see also [16–19].

1.3 What is done in this paper and how

Here, we continue our study in [55] by finding general solutions to the systems of difference equations in (4) in the case $k = 1$ and $l = 2$, complementing the results in [55]. We show that, in the case, the systems are *practically solvable*. This means that for each of the sixteen systems in (3) there is a finite number of closed-form formulas representing its general solution (for some more explanations related to the notion, as well as for some examples, see [55]). This is done by considerable use of some methods and ideas on product-type difference equations and systems, which can be found, e.g., in the following recent papers: [33–37] (see also the related references therein).

Here we also show that for each of the systems in (3) there are a naturally associated homogeneous linear difference equation with constant coefficients and a specially chosen solution to the linear equation by which the general solution to the system can be represented. Unlike the closed-form formulas presented in [55], this time the associated linear difference equations are not connected to the Fibonacci sequence, at all. The closed-form formulas obtained here show the usual diversity of representations that we have noticed in some previous studies (see, e.g., [15, 16, 19, 33–37]). However, the connectedness of the systems studied here and [55] is through the general homogeneous linear difference equation with constant coefficients.

How is dealt with the systems in (4) in the case $a = 0$ is well known. Namely, in this case the systems are easily solved by some simple changes of variables, which transform them to linear solvable ones (for more details, see [55]). Hence, we omit here studying the case and leave it to the interested reader as a simple exercise (see also [55]).

Let us also mention that the majority of solvable difference equations are solved by using the idea, that is, by using suitable changes of variables which transform them to known solvable ones. Many results of this type, out of the books and papers quoted above, can be found, e.g., in [4, 7–9, 15–19, 21–23, 33–37, 52–55, 58].

2 Auxiliary results

Here we present some auxiliary results, which are used in the proofs of the main ones.

The following Lagrange result is well known and can be proved in several ways (see, e.g., [37, 39, 59]).

Lemma 1 *Let x_k, $k = \overline{1,m}$, be the roots of the polynomial*

$$q_m(x) = b_m x^m + \cdots + b_1 x + b_0,$$

where $b_m \neq 0$. Assume that the roots are distinct.

Then

$$\sum_{k=1}^{m} \frac{x_k^j}{q_m'(x_k)} = 0,$$

when $0 \le j \le m-2$, and

$$\sum_{k=1}^{m} \frac{x_k^{m-1}}{q_m'(x_k)} = \frac{1}{b_m}.$$

Special cases of the following interesting, quite useful, and applicable lemma, which should be folklore, have been used in some of our recent papers (see, e.g., [34, 35, 37]). Here we present it in full generality and will frequently use in the rest of the paper.

Lemma 2 *Consider equation (1), where $l \in \mathbb{Z}$, $a_j \in \mathbb{C}$, $j = \overline{1,m}$, $a_m \neq 0$. Let t_k, $k = \overline{1,m}$, be the roots of the characteristic polynomial*

$$p_m(t) = t^m - a_1 t^{m-1} - a_2 t^{m-2} - \cdots - a_m,$$

associated with equation (1). Assume that the roots are distinct.

Then the solution to equation (1) such that

$$x_{j-m} = 0, \quad j = \overline{l, l+m-2}, \quad \textit{and} \quad x_{l-1} = 1 \tag{5}$$

is given by

$$x_n = \sum_{k=1}^{m} \frac{t_k^{n+m-l}}{p'_m(t_k)} \tag{6}$$

for $n \geq l - m$.

Proof By a well-known result from theory of homogeneous linear difference equations with constant coefficients, a general solution to equation (1), in the case when the roots t_k, $k = \overline{1,m}$, of the characteristic polynomial p_m are distinct, has the following form:

$$x_n = c_1 t_1^n + c_2 t_2^n + \cdots + c_m t_m^n, \quad n \geq l - m, \tag{7}$$

where c_k, $k = \overline{1,m}$, are arbitrary constants (see, e.g., [7–9, 11–14]).

Since the sequence x_n defined in (6) has the form in (7), with

$$c_k = \frac{t_k^{m-l}}{p'_m(t_k)}, \quad k = \overline{1,m},$$

it is obviously a solution to equation (1).

On the other hand, by Lemma 1 we see that this solution satisfies the initial conditions given in (5), from which together with the well-known fact that each solution to equation (1) is uniquely defined by m consecutive terms, the lemma follows. □

Remark 1 We would also like to say that we will frequently use here the fact that, when $a_m \neq 0$, every solution to equation (1) is naturally prolonged on the whole domain $\mathbb{Z}$ by using the following obvious consequence of recurrence relation (1):

$$x_{n-m} = \frac{x_n - a_1 x_{n-1} - a_2 x_{n-2} - \cdots - a_{m-1} x_{n-m+1}}{a_m}. \tag{8}$$

Since each solution to equation (1) is uniquely defined by m consecutive members, it follows that the formulas obtained on the domain $n \geq l$ also hold on $\mathbb{Z}$. Specially, formula (6) presents the solution to equation (1) with initial conditions (5) not only on the set $n \geq l$, but also on the whole $\mathbb{Z}$.

Remark 2 Recall that

$$p'_m(t_k) = \prod_{j=1, j\neq k}^{m} (t_k - t_j)$$

for each $k \in \{1, 2, \ldots, m\}$.

From this and due to the observation in Remark 1, it follows that formula (6) can be also written in the following form:

$$x_n = \sum_{k=1}^{m} \frac{t_k^{n+m-l}}{\prod_{j=1, j\neq k}^{m} (t_k - t_j)}$$

for every $n \in \mathbb{Z}$.

3 Main results

Before we present and give proofs of our main results, we first transform the systems studied here to some simpler forms by using some changes of variables, which are naturally imposed.

In order to do this, first note that from the equations in (4) with $k = 1$ and $l = 2$, after some simple calculations, we have that the following two pairs of relations hold:

$$x_{n+1} \pm \sqrt{a} = \frac{(u_{n-1} \pm \sqrt{a})(v_{n-2} \pm \sqrt{a})}{u_{n-1} + v_{n-2}} \quad \text{and} \quad y_{n+1} \pm \sqrt{a} = \frac{(w_{n-1} \pm \sqrt{a})(s_{n-2} \pm \sqrt{a})}{w_{n-1} + s_{n-2}}$$

for $n \in \mathbb{N}_0$, from which it follows that

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \frac{u_{n-1}+\sqrt{a}}{u_{n-1}-\sqrt{a}} \cdot \frac{v_{n-2}+\sqrt{a}}{v_{n-2}-\sqrt{a}}, \qquad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \frac{w_{n-1}+\sqrt{a}}{w_{n-1}-\sqrt{a}} \cdot \frac{s_{n-2}+\sqrt{a}}{s_{n-2}-\sqrt{a}} \tag{9}$$

for $n \in \mathbb{N}_0$.

Bearing in mind that u_n, v_n, w_n, s_n can be x_n or y_n, from (9) sixteen different systems are obtained. They are as follows:

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \frac{x_{n-1}+\sqrt{a}}{x_{n-1}-\sqrt{a}} \cdot \frac{x_{n-2}+\sqrt{a}}{x_{n-2}-\sqrt{a}}, \qquad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \frac{x_{n-1}+\sqrt{a}}{x_{n-1}-\sqrt{a}} \cdot \frac{x_{n-2}+\sqrt{a}}{x_{n-2}-\sqrt{a}}, \tag{10}$$

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \frac{x_{n-1}+\sqrt{a}}{x_{n-1}-\sqrt{a}} \cdot \frac{x_{n-2}+\sqrt{a}}{x_{n-2}-\sqrt{a}}, \qquad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \frac{y_{n-1}+\sqrt{a}}{y_{n-1}-\sqrt{a}} \cdot \frac{x_{n-2}+\sqrt{a}}{x_{n-2}-\sqrt{a}}, \tag{11}$$

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \frac{x_{n-1}+\sqrt{a}}{x_{n-1}-\sqrt{a}} \cdot \frac{x_{n-2}+\sqrt{a}}{x_{n-2}-\sqrt{a}}, \qquad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \frac{x_{n-1}+\sqrt{a}}{x_{n-1}-\sqrt{a}} \cdot \frac{y_{n-2}+\sqrt{a}}{y_{n-2}-\sqrt{a}}, \tag{12}$$

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \frac{x_{n-1}+\sqrt{a}}{x_{n-1}-\sqrt{a}} \cdot \frac{x_{n-2}+\sqrt{a}}{x_{n-2}-\sqrt{a}}, \qquad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \frac{y_{n-1}+\sqrt{a}}{y_{n-1}-\sqrt{a}} \cdot \frac{y_{n-2}+\sqrt{a}}{y_{n-2}-\sqrt{a}}, \tag{13}$$

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \frac{y_{n-1}+\sqrt{a}}{y_{n-1}-\sqrt{a}} \cdot \frac{x_{n-2}+\sqrt{a}}{x_{n-2}-\sqrt{a}}, \qquad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \frac{x_{n-1}+\sqrt{a}}{x_{n-1}-\sqrt{a}} \cdot \frac{x_{n-2}+\sqrt{a}}{x_{n-2}-\sqrt{a}}, \tag{14}$$

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \frac{y_{n-1}+\sqrt{a}}{y_{n-1}-\sqrt{a}} \cdot \frac{x_{n-2}+\sqrt{a}}{x_{n-2}-\sqrt{a}}, \qquad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \frac{y_{n-1}+\sqrt{a}}{y_{n-1}-\sqrt{a}} \cdot \frac{x_{n-2}+\sqrt{a}}{x_{n-2}-\sqrt{a}}, \tag{15}$$

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \frac{y_{n-1}+\sqrt{a}}{y_{n-1}-\sqrt{a}} \cdot \frac{x_{n-2}+\sqrt{a}}{x_{n-2}-\sqrt{a}}, \qquad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \frac{x_{n-1}+\sqrt{a}}{x_{n-1}-\sqrt{a}} \cdot \frac{y_{n-2}+\sqrt{a}}{y_{n-2}-\sqrt{a}}, \tag{16}$$

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \frac{y_{n-1}+\sqrt{a}}{y_{n-1}-\sqrt{a}} \cdot \frac{x_{n-2}+\sqrt{a}}{x_{n-2}-\sqrt{a}}, \qquad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \frac{y_{n-1}+\sqrt{a}}{y_{n-1}-\sqrt{a}} \cdot \frac{y_{n-2}+\sqrt{a}}{y_{n-2}-\sqrt{a}}, \tag{17}$$

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \frac{x_{n-1}+\sqrt{a}}{x_{n-1}-\sqrt{a}} \cdot \frac{y_{n-2}+\sqrt{a}}{y_{n-2}-\sqrt{a}}, \qquad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \frac{x_{n-1}+\sqrt{a}}{x_{n-1}-\sqrt{a}} \cdot \frac{x_{n-2}+\sqrt{a}}{x_{n-2}-\sqrt{a}}, \tag{18}$$

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \frac{x_{n-1}+\sqrt{a}}{x_{n-1}-\sqrt{a}} \cdot \frac{y_{n-2}+\sqrt{a}}{y_{n-2}-\sqrt{a}}, \qquad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \frac{y_{n-1}+\sqrt{a}}{y_{n-1}-\sqrt{a}} \cdot \frac{x_{n-2}+\sqrt{a}}{x_{n-2}-\sqrt{a}}, \tag{19}$$

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \frac{x_{n-1}+\sqrt{a}}{x_{n-1}-\sqrt{a}} \cdot \frac{y_{n-2}+\sqrt{a}}{y_{n-2}-\sqrt{a}}, \qquad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \frac{x_{n-1}+\sqrt{a}}{x_{n-1}-\sqrt{a}} \cdot \frac{y_{n-2}+\sqrt{a}}{y_{n-2}-\sqrt{a}}, \tag{20}$$

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \frac{x_{n-1}+\sqrt{a}}{x_{n-1}-\sqrt{a}} \cdot \frac{y_{n-2}+\sqrt{a}}{y_{n-2}-\sqrt{a}}, \qquad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \frac{y_{n-1}+\sqrt{a}}{y_{n-1}-\sqrt{a}} \cdot \frac{y_{n-2}+\sqrt{a}}{y_{n-2}-\sqrt{a}}, \tag{21}$$

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \frac{y_{n-1}+\sqrt{a}}{y_{n-1}-\sqrt{a}} \cdot \frac{y_{n-2}+\sqrt{a}}{y_{n-2}-\sqrt{a}}, \qquad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \frac{x_{n-1}+\sqrt{a}}{x_{n-1}-\sqrt{a}} \cdot \frac{x_{n-2}+\sqrt{a}}{x_{n-2}-\sqrt{a}}, \tag{22}$$

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}}=\frac{y_{n-1}+\sqrt{a}}{y_{n-1}-\sqrt{a}}\cdot\frac{y_{n-2}+\sqrt{a}}{y_{n-2}-\sqrt{a}},\qquad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}}=\frac{y_{n-1}+\sqrt{a}}{y_{n-1}-\sqrt{a}}\cdot\frac{x_{n-2}+\sqrt{a}}{x_{n-2}-\sqrt{a}}, \tag{23}$$

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}}=\frac{y_{n-1}+\sqrt{a}}{y_{n-1}-\sqrt{a}}\cdot\frac{y_{n-2}+\sqrt{a}}{y_{n-2}-\sqrt{a}},\qquad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}}=\frac{x_{n-1}+\sqrt{a}}{x_{n-1}-\sqrt{a}}\cdot\frac{y_{n-2}+\sqrt{a}}{y_{n-2}-\sqrt{a}}, \tag{24}$$

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}}=\frac{y_{n-1}+\sqrt{a}}{y_{n-1}-\sqrt{a}}\cdot\frac{y_{n-2}+\sqrt{a}}{y_{n-2}-\sqrt{a}},\qquad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}}=\frac{y_{n-1}+\sqrt{a}}{y_{n-1}-\sqrt{a}}\cdot\frac{y_{n-2}+\sqrt{a}}{y_{n-2}-\sqrt{a}} \tag{25}$$

for $n\in\mathbb{N}_0$.

Let

$$\gamma_n=\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\quad\text{and}\quad \delta_n=\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}} \tag{26}$$

for $n\ge -2$.

Then (10)–(25) respectively become

$$\gamma_{n+1}=\gamma_{n-1}\gamma_{n-2},\qquad \delta_{n+1}=\gamma_{n-1}\gamma_{n-2}, \tag{27}$$

$$\gamma_{n+1}=\gamma_{n-1}\gamma_{n-2},\qquad \delta_{n+1}=\delta_{n-1}\gamma_{n-2}, \tag{28}$$

$$\gamma_{n+1}=\gamma_{n-1}\gamma_{n-2},\qquad \delta_{n+1}=\gamma_{n-1}\delta_{n-2}, \tag{29}$$

$$\gamma_{n+1}=\gamma_{n-1}\gamma_{n-2},\qquad \delta_{n+1}=\delta_{n-1}\delta_{n-2}, \tag{30}$$

$$\gamma_{n+1}=\delta_{n-1}\gamma_{n-2},\qquad \delta_{n+1}=\gamma_{n-1}\gamma_{n-2}, \tag{31}$$

$$\gamma_{n+1}=\delta_{n-1}\gamma_{n-2},\qquad \delta_{n+1}=\delta_{n-1}\gamma_{n-2}, \tag{32}$$

$$\gamma_{n+1}=\delta_{n-1}\gamma_{n-2},\qquad \delta_{n+1}=\gamma_{n-1}\delta_{n-2}, \tag{33}$$

$$\gamma_{n+1}=\delta_{n-1}\gamma_{n-2},\qquad \delta_{n+1}=\delta_{n-1}\delta_{n-2}, \tag{34}$$

$$\gamma_{n+1}=\gamma_{n-1}\delta_{n-2},\qquad \delta_{n+1}=\gamma_{n-1}\gamma_{n-2}, \tag{35}$$

$$\gamma_{n+1}=\gamma_{n-1}\delta_{n-2},\qquad \delta_{n+1}=\delta_{n-1}\gamma_{n-2}, \tag{36}$$

$$\gamma_{n+1}=\gamma_{n-1}\delta_{n-2},\qquad \delta_{n+1}=\gamma_{n-1}\delta_{n-2}, \tag{37}$$

$$\gamma_{n+1}=\gamma_{n-1}\delta_{n-2},\qquad \delta_{n+1}=\delta_{n-1}\delta_{n-2}, \tag{38}$$

$$\gamma_{n+1}=\delta_{n-1}\delta_{n-2},\qquad \delta_{n+1}=\gamma_{n-1}\gamma_{n-2}, \tag{39}$$

$$\gamma_{n+1}=\delta_{n-1}\delta_{n-2},\qquad \delta_{n+1}=\delta_{n-1}\gamma_{n-2}, \tag{40}$$

$$\gamma_{n+1}=\delta_{n-1}\delta_{n-2},\qquad \delta_{n+1}=\gamma_{n-1}\delta_{n-2}, \tag{41}$$

$$\gamma_{n+1}=\delta_{n-1}\delta_{n-2},\qquad \delta_{n+1}=\delta_{n-1}\delta_{n-2}, \tag{42}$$

$n\in\mathbb{N}_0$.

Now we are going to study the solvability of systems (27)–(42). As we have already mentioned, we employ here some methods which have been used to product-type systems and can be found, e.g., in [34–36].

3.1 Solution to system (27)

Clearly, in this case we have

$$\gamma_n=\delta_n,\quad n\in\mathbb{N}. \tag{43}$$

Let

$$a_1 = 1, \qquad b_1 = 1, \qquad c_1 = 0. \tag{44}$$

Then we have

$$y_n = y_{n-2}y_{n-3} = y_{n-2}^{a_1}y_{n-3}^{b_1}y_{n-4}^{c_1} \tag{45}$$

for $n \in \mathbb{N}$.

Iterating (45) we have

$$y_n = (y_{n-4}y_{n-5})^{a_1}y_{n-3}^{b_1}y_{n-4}^{c_1} = y_{n-3}^{b_1}y_{n-4}^{a_1+c_1}y_{n-5}^{a_1} = y_{n-3}^{a_2}y_{n-4}^{b_2}y_{n-5}^{c_2}$$

for $n \ge 3$, where

$$a_2 := b_1, \qquad b_2 := a_1 + c_1, \qquad c_2 := a_1.$$

Suppose that

$$y_n = y_{n-k-1}^{a_k}y_{n-k-2}^{b_k}y_{n-k-3}^{c_k}, \tag{46}$$

$$a_k = b_{k-1}, \qquad b_k = a_{k-1} + c_{k-1}, \qquad c_k = a_{k-1}, \tag{47}$$

for $k \in \mathbb{N} \setminus \{1\}$ and every $n \ge k+1$.

From (45)–(47), we have

$$\begin{aligned} y_n &= (y_{n-k-3}y_{n-k-4})^{a_k}y_{n-k-2}^{b_k}y_{n-k-3}^{c_k} \\ &= y_{n-k-2}^{b_k}y_{n-k-3}^{a_k+c_k}y_{n-k-4}^{a_k} \\ &= y_{n-k-2}^{a_{k+1}}y_{n-k-3}^{b_{k+1}}y_{n-k-4}^{c_{k+1}}, \end{aligned}$$

where

$$a_{k+1} := b_k, \qquad b_{k+1} := a_k + c_k, \qquad c_{k+1} := a_k.$$

The inductive argument shows that (46) and (47) hold for every $k \ge 2$ and $n \ge k+1$.

Note that from (47) we have

$$a_n = a_{n-2} + a_{n-3}, \quad n \ge 4 \tag{48}$$

(in fact, (48) holds for every $n \in \mathbb{Z}$; see Remark 1).

Besides, by using (47), we have

$$a_0 = 0, \qquad a_{-1} = 1, \qquad a_{-2} = a_{-3} = 0, \qquad a_{-4} = 1 \tag{49}$$

(see, e.g., the corresponding calculations in [33] and [37]).

Letting $k = n-1$ in (46) and using (47), we get

$$\begin{aligned}\gamma_n &= \gamma_0^{a_{n-1}}\gamma_{-1}^{b_{n-1}}\gamma_{-2}^{c_{n-1}}\\ &= \gamma_0^{a_{n-1}}\gamma_{-1}^{a_n}\gamma_{-2}^{a_{n-2}}\end{aligned} \tag{50}$$

for $n \in \mathbb{N}$ (moreover, (50) holds for $n \geq -2$, which is easily checked by a simple calculation and the use of (49)).

From (43) and (50), it follows that

$$\delta_n = \gamma_0^{a_{n-1}}\gamma_{-1}^{a_n}\gamma_{-2}^{a_{n-2}} \tag{51}$$

for $n \in \mathbb{N}$.

The characteristic equation associated with equation (48) is

$$P_3(\lambda) = \lambda^3 - \lambda - 1 = 0. \tag{52}$$

Let λ_j, $j = \overline{1,3}$, be the roots of polynomial P_3. Since the discriminant $\Delta = 23$ of the polynomial is different from zero, it follows that the roots are distinct (recall that if a polynomial of the third order has the form $Q_3(\lambda) = \lambda^3 + p\lambda + q$, then the discriminant is given by $\Delta = 4p^3 + 27q^2$). Moreover, since the discriminant is positive, one of the roots is real, whereas the two other ones are complex conjugate. They can be calculated by using known methods (see, e.g., [59]). Since this is a routine thing and we have done similar calculations a few times recently [34, 36], we leave it to the reader.

By Lemma 2, we have that the solution to equation (48) such that

$$a_{-3} = a_{-2} = 0 \quad \text{and} \quad a_{-1} = 1$$

is given by

$$a_n = \sum_{j=1}^{3} \frac{\lambda_j^{n+3}}{P_3'(\lambda_j)}, \quad n \in \mathbb{Z}. \tag{53}$$

From the above consideration along with the relations in (26), we see that the following corollary holds.

Corollary 1 *Consider system (10) with $a \neq 0$. Then its general solution is*

$$x_n = \sqrt{a}\frac{(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}})^{a_{n-1}}(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}})^{a_n}(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}})^{a_{n-2}}+1}{(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}})^{a_{n-1}}(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}})^{a_n}(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}})^{a_{n-2}}-1}, \quad n \geq -2,$$

$$y_n = \sqrt{a}\frac{(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}})^{a_{n-1}}(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}})^{a_n}(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}})^{a_{n-2}}+1}{(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}})^{a_{n-1}}(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}})^{a_n}(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}})^{a_{n-2}}-1}, \quad n \in \mathbb{N},$$

where a_n is given by (53).

3.2 Solution to system (28)

From the previous case we see that (50) holds. Besides, we also have

$$\delta_n = \delta_{n-2}\gamma_{n-3}, \quad n \in \mathbb{N}. \tag{54}$$

From (50), (54), and some calculation, we have

$$\begin{aligned}\delta_{2n} &= \delta_0 \prod_{j=0}^{n-1} \gamma_{2j-1} \\ &= \delta_0 \prod_{j=0}^{n-1} \gamma_0^{a_{2j-2}} \gamma_{-1}^{a_{2j-1}} \gamma_{-2}^{a_{2j-3}} \\ &= \delta_0 \gamma_0^{\sum_{j=0}^{n-1} a_{2j-2}} \gamma_{-1}^{\sum_{j=0}^{n-1} a_{2j-1}} \gamma_{-2}^{\sum_{j=0}^{n-1} a_{2j-3}}\end{aligned} \tag{55}$$

for $n \in \mathbb{N}_0$ and

$$\begin{aligned}\delta_{2n+1} &= \delta_{-1} \prod_{j=0}^{n} \gamma_{2j-2} \\ &= \delta_{-1} \prod_{j=0}^{n} \gamma_0^{a_{2j-3}} \gamma_{-1}^{a_{2j-2}} \gamma_{-2}^{a_{2j-4}} \\ &= \delta_{-1} \gamma_0^{\sum_{j=0}^{n} a_{2j-3}} \gamma_{-1}^{\sum_{j=0}^{n} a_{2j-2}} \gamma_{-2}^{\sum_{j=0}^{n} a_{2j-4}}\end{aligned} \tag{56}$$

for $n \geq -1$.

Further, by using (48) and (49), we have

$$\sum_{j=0}^{n-1} a_{2j-2} = \sum_{j=0}^{n-1} (a_{2j+1} - a_{2j-1}) = a_{2n-1} - 1, \tag{57}$$

$$\sum_{j=0}^{n-1} a_{2j-1} = \sum_{j=0}^{n-1} (a_{2j+2} - a_{2j}) = a_{2n}, \tag{58}$$

$$\sum_{j=0}^{n-1} a_{2j-3} = \sum_{j=0}^{n-1} (a_{2j} - a_{2j-2}) = a_{2n-2}, \tag{59}$$

$$\sum_{j=0}^{n-1} a_{2j-4} = \sum_{j=0}^{n-1} (a_{2j-1} - a_{2j-3}) = a_{2n-3} \tag{60}$$

for $n \in \mathbb{N}_0$.

Employing (57)–(60) in (55) and (56), we obtain

$$\delta_{2n} = \delta_0 \gamma_0^{a_{2n-1}-1} \gamma_{-1}^{a_{2n}} \gamma_{-2}^{a_{2n-2}} \tag{61}$$

for $n \in \mathbb{N}_0$ and

$$\delta_{2n+1} = \delta_{-1} \gamma_0^{a_{2n}} \gamma_{-1}^{a_{2n+1}-1} \gamma_{-2}^{a_{2n-1}} \tag{62}$$

for $n \geq -1$.

Hence, (50), (61), and (62) present the general solution to the system of difference equations (28). This consideration along with (26) shows that the following corollary holds.

Corollary 2 *Consider system* (11) *with* $a \neq 0$. *Then its general solution is*

$$x_n = \sqrt{a}\frac{(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}})^{a_{n-1}}(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}})^{a_n}(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}})^{a_{n-2}}+1}{(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}})^{a_{n-1}}(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}})^{a_n}(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}})^{a_{n-2}}-1}, \quad n \geq -2,$$

$$y_{2n} = \sqrt{a}\frac{(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}})(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}})^{a_{2n-1}-1}(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}})^{a_{2n}}(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}})^{a_{2n-2}}+1}{(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}})(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}})^{a_{2n-1}-1}(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}})^{a_{2n}}(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}})^{a_{2n-2}}-1}, \quad n \in \mathbb{N}_0,$$

$$y_{2n+1} = \sqrt{a}\frac{(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}})(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}})^{a_{2n}}(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}})^{a_{2n+1}-1}(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}})^{a_{2n-1}}+1}{(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}})(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}})^{a_{2n}}(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}})^{a_{2n+1}-1}(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}})^{a_{2n-1}}-1}, \quad n \geq -1,$$

where a_n *is given by* (53).

3.3 Solution to system (29)

Note that (50) holds, and we have

$$\delta_n = \gamma_{n-2}\delta_{n-3} \tag{63}$$

for $n \in \mathbb{N}$, that is,

$$\delta_{3n+i} = \gamma_{3n-2+i}\delta_{3(n-1)+i} \tag{64}$$

for $n \in \mathbb{N}$, $i = -2, -1, 0$.

We have

$$\begin{aligned}\delta_{3n} &= \delta_0 \prod_{j=1}^{n} \gamma_{3j-2} \\ &= \delta_0 \prod_{j=1}^{n} \gamma_0^{a_{3j-3}} \gamma_{-1}^{a_{3j-2}} \gamma_{-2}^{a_{3j-4}} \\ &= \delta_0 \gamma_0^{\sum_{j=1}^{n} a_{3j-3}} \gamma_{-1}^{\sum_{j=1}^{n} a_{3j-2}} \gamma_{-2}^{\sum_{j=1}^{n} a_{3j-4}}\end{aligned} \tag{65}$$

for $n \in \mathbb{N}_0$,

$$\begin{aligned}\delta_{3n+1} &= \delta_{-2} \prod_{j=0}^{n} \gamma_{3j-1} \\ &= \delta_{-2} \prod_{j=0}^{n} \gamma_0^{a_{3j-2}} \gamma_{-1}^{a_{3j-1}} \gamma_{-2}^{a_{3j-3}} \\ &= \delta_{-2} \gamma_0^{\sum_{j=0}^{n} a_{3j-2}} \gamma_{-1}^{\sum_{j=0}^{n} a_{3j-1}} \gamma_{-2}^{\sum_{j=0}^{n} a_{3j-3}}\end{aligned} \tag{66}$$

for $n \ge -1$, and

$$\begin{aligned}\delta_{3n+2} &= \delta_{-1}\prod_{j=0}^{n}\gamma_{3j}\\ &= \delta_{-1}\prod_{j=0}^{n}\gamma_0^{a_{3j-1}}\gamma_{-1}^{a_{3j}}\gamma_{-2}^{a_{3j-2}}\\ &= \delta_{-1}\gamma_0^{\sum_{j=0}^{n}a_{3j-1}}\gamma_{-1}^{\sum_{j=0}^{n}a_{3j}}\gamma_{-2}^{\sum_{j=0}^{n}a_{3j-2}} \qquad (67)\end{aligned}$$

for $n \ge -1$.

Further, by using (48) and (49), we have

$$\sum_{j=0}^{n}a_{3j-3} = \sum_{j=0}^{n}(a_{3j-1}-a_{3j-4}) = a_{3n-1}-1, \qquad (68)$$

$$\sum_{j=0}^{n}a_{3j-2} = \sum_{j=0}^{n}(a_{3j}-a_{3j-3}) = a_{3n}, \qquad (69)$$

$$\sum_{j=0}^{n}a_{3j-1} = \sum_{j=0}^{n}(a_{3j+1}-a_{3j-2}) = a_{3n+1}, \qquad (70)$$

$$\sum_{j=0}^{n}a_{3j} = \sum_{j=0}^{n}(a_{3j+2}-a_{3j-1}) = a_{3n+2}-1, \qquad (71)$$

$$\sum_{j=1}^{n}a_{3j-3} = \sum_{j=1}^{n}(a_{3j-1}-a_{3j-4}) = a_{3n-1}-1, \qquad (72)$$

$$\sum_{j=1}^{n}a_{3j-2} = \sum_{j=1}^{n}(a_{3j}-a_{3j-3}) = a_{3n}, \qquad (73)$$

$$\sum_{j=1}^{n}a_{3j-4} = \sum_{j=1}^{n}(a_{3j-2}-a_{3j-5}) = a_{3n-2}. \qquad (74)$$

Employing (68)–(74) in (65)–(67), we obtain

$$\delta_{3n} = \delta_0\gamma_0^{a_{3n-1}-1}\gamma_{-1}^{a_{3n}}\gamma_{-2}^{a_{3n-2}} \qquad (75)$$

for $n \in \mathbb{N}_0$,

$$\delta_{3n+1} = \delta_{-2}\gamma_0^{a_{3n}}\gamma_{-1}^{a_{3n+1}}\gamma_{-2}^{a_{3n-1}-1} \qquad (76)$$

for $n \ge -1$, and

$$\delta_{3n+2} = \delta_{-1}\gamma_0^{a_{3n+1}}\gamma_{-1}^{a_{3n+2}-1}\gamma_{-2}^{a_{3n}} \qquad (77)$$

for $n \ge -1$.

Formulas (50), (75)–(77) present the general solution to system (29). This consideration along with (26) shows that the following corollary holds.

Corollary 3 *Consider system* (12) *with $a \neq 0$. Then its general solution is*

$$x_n = \sqrt{a}\frac{(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}})^{a_{n-1}}(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}})^{a_n}(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}})^{a_{n-2}}+1}{(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}})^{a_{n-1}}(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}})^{a_n}(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}})^{a_{n-2}}-1}, \quad n \geq -2,$$

$$y_{3n} = \sqrt{a}\frac{(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}})(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}})^{a_{3n-1}-1}(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}})^{a_{3n}}(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}})^{a_{3n-2}}+1}{(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}})(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}})^{a_{3n-1}-1}(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}})^{a_{3n}}(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}})^{a_{3n-2}}-1}, \quad n \in \mathbb{N}_0,$$

$$y_{3n+1} = \sqrt{a}\frac{(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}})(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}})^{a_{3n}}(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}})^{a_{3n+1}}(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}})^{a_{3n-1}-1}+1}{(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}})(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}})^{a_{3n}}(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}})^{a_{3n+1}}(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}})^{a_{3n-1}-1}-1}, \quad n \geq -1,$$

$$y_{3n+2} = \sqrt{a}\frac{(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}})(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}})^{a_{3n+1}}(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}})^{a_{3n+2}-1}(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}})^{a_{3n}}+1}{(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}})(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}})^{a_{3n+1}}(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}})^{a_{3n+2}-1}(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}})^{a_{3n}}-1}, \quad n \geq -1,$$

where sequence a_n is given by (53).

3.4 Solution to system (30)

Obviously, formula (50) holds, and we have

$$\delta_n = \delta_0^{a_{n-1}}\delta_{-1}^{a_n}\delta_{-2}^{a_{n-2}} \tag{78}$$

for $n \geq -2$.

Therefore, formulas (50) and (78) present the general solution to the system of difference equations (30). This consideration along with (26) shows that the following corollary holds.

Corollary 4 *Consider system* (13) *with $a \neq 0$. Then its general solution is*

$$x_n = \sqrt{a}\frac{(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}})^{a_{n-1}}(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}})^{a_n}(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}})^{a_{n-2}}+1}{(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}})^{a_{n-1}}(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}})^{a_n}(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}})^{a_{n-2}}-1},$$

$$y_n = \sqrt{a}\frac{(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}})^{a_{n-1}}(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}})^{a_n}(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}})^{a_{n-2}}+1}{(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}})^{a_{n-1}}(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}})^{a_n}(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}})^{a_{n-2}}-1}$$

for $n \geq -2$, where a_n is given by (53).

3.5 Solution to system (31)

From the equations in (31) we have

$$\gamma_{n+1} = \gamma_{n-2}\gamma_{n-3}\gamma_{n-4}, \quad n \geq 2. \tag{79}$$

Let

$$a_1 = b_1 = c_1 = 1, \qquad d_1 = e_1 = 0. \tag{80}$$

Then

$$\gamma_{n+1} = \gamma_{n-2}^{a_1}\gamma_{n-3}^{b_1}\gamma_{n-4}^{c_1}\gamma_{n-5}^{d_1}\gamma_{n-6}^{e_1}, \quad n \geq 2. \tag{81}$$

Employing (79) in (81), we have

$$\begin{aligned} y_{n+1} &= y_{n-2}^{a_1} y_{n-3}^{b_1} y_{n-4}^{c_1} y_{n-5}^{d_1} y_{n-6}^{e_1} \\ &= (y_{n-5} y_{n-6} y_{n-7})^{a_1} y_{n-3}^{b_1} y_{n-4}^{c_1} y_{n-5}^{d_1} y_{n-6}^{e_1} \\ &= y_{n-3}^{b_1} y_{n-4}^{c_1} y_{n-5}^{a_1+d_1} y_{n-6}^{a_1+e_1} y_{n-7}^{a_1} \\ &= y_{n-3}^{a_2} y_{n-4}^{b_2} y_{n-5}^{c_2} y_{n-6}^{d_2} y_{n-7}^{e_2} \end{aligned}$$

for $n \geq 5$, where

$$a_2 := b_1, \qquad b_2 := c_1, \qquad c_2 := a_1 + d_1, \qquad d_2 := a_1 + e_1, \qquad e_2 := a_1.$$

Suppose that

$$y_{n+1} = y_{n-k-1}^{a_k} y_{n-k-2}^{b_k} y_{n-k-3}^{c_k} y_{n-k-4}^{d_k} y_{n-k-5}^{e_k} \tag{82}$$

and

$$a_k = b_{k-1}, \qquad b_k = c_{k-1}, \qquad c_k = a_{k-1} + d_{k-1}, \qquad d_k = a_{k-1} + e_{k-1}, \qquad e_k = a_{k-1} \tag{83}$$

for $k \in \mathbb{N} \setminus \{1\}$ and $n \geq k + 3$.

By using (79) in (82), we get

$$\begin{aligned} y_{n+1} &= y_{n-k-1}^{a_k} y_{n-k-2}^{b_k} y_{n-k-3}^{c_k} y_{n-k-4}^{d_k} y_{n-k-5}^{e_k} \\ &= (y_{n-k-4} y_{n-k-5} y_{n-k-6})^{a_k} y_{n-k-2}^{b_k} y_{n-k-3}^{c_k} y_{n-k-4}^{d_k} y_{n-k-5}^{e_k} \\ &= y_{n-k-2}^{b_k} y_{n-k-3}^{c_k} y_{n-k-4}^{a_k+d_k} y_{n-k-5}^{a_k+e_k} y_{n-k-6}^{a_k} \\ &= y_{n-k-2}^{a_{k+1}} y_{n-k-3}^{b_{k+1}} y_{n-k-4}^{c_{k+1}} y_{n-k-5}^{d_{k+1}} y_{n-k-6}^{e_{k+1}}, \end{aligned}$$

where

$$a_{k+1} := b_k, \qquad b_{k+1} := c_k, \qquad c_{k+1} := a_k + d_k, \qquad d_{k+1} := a_k + e_k, \qquad e_{k+1} := a_k$$

for $k \geq 2$ and every $n \geq k + 4$. The inductive argument shows that (82) and (83) hold for $2 \leq k \leq n - 3$.

From (83) it follows that

$$a_n = a_{n-3} + a_{n-4} + a_{n-5}, \qquad n \geq 6, \tag{84}$$

(in fact (84) holds for every $n \in \mathbb{Z}$, see Remark 1) and

$$\begin{aligned} &a_0 = 0, \qquad a_{-1} = 0, \qquad a_{-2} = 1, \qquad a_{-j} = 0, \quad j = \overline{3,6}, \\ &a_{-7} = 1, \qquad a_{-8} = -1. \end{aligned} \tag{85}$$

If we take $k = n - 4$ in equation (82), which is obtained when n is replaced by $n - 1$, we obtain

$$\begin{aligned}
\gamma_n &= \gamma_2^{a_{n-4}}\gamma_1^{b_{n-4}}\gamma_0^{c_{n-4}}\gamma_{-1}^{d_{n-4}}\gamma_{-2}^{e_{n-4}} \\
&= (\delta_0\gamma_{-1})^{a_{n-4}}(\delta_{-1}\gamma_{-2})^{b_{n-4}}\gamma_0^{c_{n-4}}\gamma_{-1}^{d_{n-4}}\gamma_{-2}^{e_{n-4}} \\
&= \delta_0^{a_{n-4}}\delta_{-1}^{b_{n-4}}\gamma_0^{c_{n-4}}\gamma_{-1}^{a_{n-4}+d_{n-4}}\gamma_{-2}^{b_{n-4}+e_{n-4}} \\
&= \delta_0^{a_{n-4}}\delta_{-1}^{a_{n-3}}\gamma_0^{a_{n-2}}\gamma_{-1}^{a_{n-1}}\gamma_{-2}^{a_{n-3}+a_{n-5}} \qquad (86)
\end{aligned}$$

for $n \geq -2$.

From the second equation in (31) and (86), we have

$$\begin{aligned}
\delta_n &= \gamma_{n-2}\gamma_{n-3} \\
&= \delta_0^{a_{n-6}+a_{n-7}}\delta_{-1}^{a_{n-5}+a_{n-6}}\gamma_0^{a_{n-4}+a_{n-5}}\gamma_{-1}^{a_{n-3}+a_{n-4}}\gamma_{-2}^{a_{n-3}+a_{n-5}} \qquad (87)
\end{aligned}$$

for $n \geq -1$.

The characteristic polynomial associated with equation (84) is

$$P_5(\lambda) = \lambda^5 - \lambda^2 - \lambda - 1.$$

Since

$$P_5(\lambda) = \lambda^5 + \lambda^3 - \lambda^3 - \lambda^2 - \lambda - 1 = (\lambda^2 + 1)(\lambda^3 - \lambda - 1),$$

we have that three roots of P_5 coincide with the roots λ_j, $j = \overline{1,3}$, of polynomial (52), whereas $\lambda_{4,5} = \pm i$.

By Lemma 2, the solution to equation (84) such that $a_{-k} = 0$, $k = \overline{3,6}$, and $a_{-2} = 1$ is

$$a_n = \sum_{j=1}^{5} \frac{\lambda_j^{n+6}}{P_5'(\lambda_j)}, \quad n \in \mathbb{Z}. \qquad (88)$$

Formulas (86) and (87) present the general solution to system (31). This consideration along with (26) shows that the following corollary holds.

Corollary 5 *Consider system* (14) *with* $a \neq 0$. *Then its general solution is*

$$x_n = \sqrt{a}\,\frac{(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}})^{a_{n-4}}(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}})^{a_{n-3}}(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}})^{a_{n-2}}(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}})^{a_{n-1}}(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}})^{a_{n-3}+a_{n-5}} + 1}{(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}})^{a_{n-4}}(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}})^{a_{n-3}}(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}})^{a_{n-2}}(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}})^{a_{n-1}}(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}})^{a_{n-3}+a_{n-5}} - 1}$$

for $n \geq -2$ *and*

$$y_n = \sqrt{a}\,\frac{(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}})^{a_{n-6}+a_{n-7}}(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}})^{a_{n-5}+a_{n-6}}(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}})^{a_{n-4}+a_{n-5}}(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}})^{a_{n-3}+a_{n-4}}(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}})^{a_{n-3}+a_{n-5}} + 1}{(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}})^{a_{n-6}+a_{n-7}}(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}})^{a_{n-5}+a_{n-6}}(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}})^{a_{n-4}+a_{n-5}}(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}})^{a_{n-3}+a_{n-4}}(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}})^{a_{n-3}+a_{n-5}} - 1}$$

for $n \geq -1$, *where* a_n *is given by* (88).

3.6 Solution to system (32)

From (32) we have

$$\gamma_n = \delta_n, \quad n \in \mathbb{N}.$$

So, we have

$$\gamma_{n+1} = \gamma_{n-1}\gamma_{n-2}, \quad n \geq 2. \tag{89}$$

Hence, by using (50) it follows that

$$\begin{aligned}
\gamma_n &= \gamma_2^{a_{n-3}}\gamma_1^{a_{n-2}}\gamma_0^{a_{n-4}} \\
&= (\delta_0\gamma_{-1})^{a_{n-3}}(\delta_{-1}\gamma_{-2})^{a_{n-2}}\gamma_0^{a_{n-4}} \\
&= \delta_0^{a_{n-3}}\delta_{-1}^{a_{n-2}}\gamma_0^{a_{n-4}}\gamma_{-1}^{a_{n-3}}\gamma_{-2}^{a_{n-2}}
\end{aligned} \tag{90}$$

for $n \in \mathbb{N}_0$, where a_n is the solution to equation (48) such that $a_{-3} = a_{-2} = 0$ and $a_{-1} = 1$, and consequently

$$\delta_n = \delta_0^{a_{n-3}}\delta_{-1}^{a_{n-2}}\gamma_0^{a_{n-4}}\gamma_{-1}^{a_{n-3}}\gamma_{-2}^{a_{n-2}} \tag{91}$$

for $n \in \mathbb{N}$.

Formulas (90) and (91) are the closed-form formulas for the general solution to system (32). This consideration along with (26) shows that the following corollary holds.

Corollary 6 *Consider system (15) with $a \neq 0$. Then its general solution is*

$$x_n = \sqrt{a}\frac{(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}})^{a_{n-3}}(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}})^{a_{n-2}}(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}})^{a_{n-4}}(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}})^{a_{n-3}}(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}})^{a_{n-2}}+1}{(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}})^{a_{n-3}}(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}})^{a_{n-2}}(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}})^{a_{n-4}}(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}})^{a_{n-3}}(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}})^{a_{n-2}}-1}$$

for $n \in \mathbb{N}_0$, and

$$y_n = \sqrt{a}\frac{(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}})^{a_{n-3}}(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}})^{a_{n-2}}(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}})^{a_{n-4}}(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}})^{a_{n-3}}(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}})^{a_{n-2}}+1}{(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}})^{a_{n-3}}(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}})^{a_{n-2}}(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}})^{a_{n-4}}(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}})^{a_{n-3}}(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}})^{a_{n-2}}-1}$$

for $n \in \mathbb{N}$, where a_n is given by (53).

3.7 Solution to system (33)

Combining the equations in (33), we obtain

$$\gamma_{n+3} = \gamma_n^2\gamma_{n-1}\gamma_{n-3}^{-1}, \quad n \in \mathbb{N}. \tag{92}$$

Let

$$a_1 := 2, \qquad b_1 := 1, \qquad c_1 := 0, \qquad d_1 := -1, \qquad e_1 := 0, \qquad f_1 := 0.$$

Then

$$\begin{aligned}\gamma_n &= \gamma_{n-3}^{a_1}\gamma_{n-4}^{b_1}\gamma_{n-5}^{c_1}\gamma_{n-6}^{d_1}\gamma_{n-7}^{e_1}\gamma_{n-8}^{f_1}\\ &= \left(\gamma_{n-6}^{2}\gamma_{n-7}\gamma_{n-9}^{-1}\right)^{a_1}\gamma_{n-4}^{b_1}\gamma_{n-5}^{c_1}\gamma_{n-6}^{d_1}\gamma_{n-7}^{e_1}\gamma_{n-8}^{f_1}\\ &= \gamma_{n-4}^{b_1}\gamma_{n-5}^{c_1}\gamma_{n-6}^{2a_1+d_1}\gamma_{n-7}^{a_1+e_1}\gamma_{n-8}^{f_1}\gamma_{n-9}^{-a_1}\\ &= \gamma_{n-4}^{a_2}\gamma_{n-5}^{b_2}\gamma_{n-6}^{c_2}\gamma_{n-7}^{d_2}\gamma_{n-8}^{e_2}\gamma_{n-9}^{f_2}\end{aligned}$$

for $n \ge 7$, where

$$a_2 := b_1, \qquad b_2 := c_1, \qquad c_2 := 2a_1 + d_1, \qquad d_2 := a_1 + e_1, \qquad e_2 := f_1, \qquad f_2 := -a_1.$$

Suppose that

$$\gamma_n = \gamma_{n-k-2}^{a_k}\gamma_{n-k-3}^{b_k}\gamma_{n-k-4}^{c_k}\gamma_{n-k-5}^{d_k}\gamma_{n-k-6}^{e_k}\gamma_{n-k-7}^{f_k} \tag{93}$$

for $k \in \mathbb{N} \setminus \{1\}$ and every $n \ge k+5$, and

$$\begin{aligned}&a_k = b_{k-1}, \qquad b_k = c_{k-1}, \qquad c_k = 2a_{k-1} + d_{k-1},\\ &d_k = a_{k-1} + e_{k-1}, \qquad e_k = f_{k-1}, \qquad f_k = -a_{k-1}.\end{aligned} \tag{94}$$

Then

$$\begin{aligned}\gamma_n &= \gamma_{n-k-2}^{a_k}\gamma_{n-k-3}^{b_k}\gamma_{n-k-4}^{c_k}\gamma_{n-k-5}^{d_k}\gamma_{n-k-6}^{e_k}\gamma_{n-k-7}^{f_k}\\ &= \left(\gamma_{n-k-5}^{2}\gamma_{n-k-6}\gamma_{n-k-8}^{-1}\right)^{a_k}\gamma_{n-k-3}^{b_k}\gamma_{n-k-4}^{c_k}\gamma_{n-k-5}^{d_k}\gamma_{n-k-6}^{e_k}\gamma_{n-k-7}^{f_k}\\ &= \gamma_{n-k-3}^{b_k}\gamma_{n-k-4}^{c_k}\gamma_{n-k-5}^{2a_k+d_k}\gamma_{n-k-6}^{a_k+e_k}\gamma_{n-k-7}^{f_k}\gamma_{n-k-8}^{-a_k}\\ &= \gamma_{n-k-3}^{a_{k+1}}\gamma_{n-k-4}^{b_{k+1}}\gamma_{n-k-5}^{c_{k+1}}\gamma_{n-k-6}^{d_{k+1}}\gamma_{n-k-7}^{e_{k+1}}\gamma_{n-k-8}^{f_{k+1}}\end{aligned}$$

for $n \ge k+3$, where

$$\begin{aligned}&a_{k+1} = b_k, \qquad b_{k+1} = c_k, \qquad c_{k+1} = 2a_k + d_k,\\ &d_{k+1} = a_k + e_k, \qquad e_{k+1} = f_k, \qquad f_{k+1} = -a_k.\end{aligned}$$

The inductive argument implies that (93) and (94) hold for every $k, n \in \mathbb{N}$ such that $2 \le k \le n-5$.

From (94) it follows that

$$a_n = 2a_{n-3} + a_{n-4} - a_{n-6}, \quad n \ge 7 \tag{95}$$

(in fact (95) holds for every $n \in \mathbb{Z}$, see Remark 1) and

$$a_0 = 0, \qquad a_{-1} = 0, \qquad a_{-2} = 1, \qquad a_{-j} = 0, \quad j = \overline{3,7}, \qquad a_{-8} = -1. \tag{96}$$

If in (93) we take $k = n - 5$, we get

$$\begin{aligned}
\gamma_n &= \gamma_3^{a_{n-5}}\gamma_2^{b_{n-5}}\gamma_1^{c_{n-5}}\gamma_0^{d_{n-5}}\gamma_{-1}^{e_{n-5}}\gamma_{-2}^{f_{n-5}}\\
&= (\gamma_0\gamma_{-1}\delta_{-2})^{a_{n-5}}(\delta_0\gamma_{-1})^{b_{n-5}}(\delta_{-1}\gamma_{-2})^{c_{n-5}}\gamma_0^{d_{n-5}}\gamma_{-1}^{e_{n-5}}\gamma_{-2}^{f_{n-5}}\\
&= \gamma_0^{a_{n-5}+d_{n-5}}\gamma_{-1}^{a_{n-5}+b_{n-5}+e_{n-5}}\gamma_{-2}^{c_{n-5}+f_{n-5}}\delta_0^{b_{n-5}}\delta_{-1}^{c_{n-5}}\delta_{-2}^{a_{n-5}}\\
&= \gamma_0^{a_{n-2}-a_{n-5}}\gamma_{-1}^{a_{n-1}-a_{n-4}}\gamma_{-2}^{a_{n-3}-a_{n-6}}\delta_0^{a_{n-4}}\delta_{-1}^{a_{n-3}}\delta_{-2}^{a_{n-5}}
\end{aligned}\tag{97}$$

for $n \ge -2$.

From the first equation in (33) and (97) it follows that

$$\begin{aligned}
\delta_n &= \gamma_{n+2}/\gamma_{n-1}\\
&= \gamma_0^{a_n-2a_{n-3}+a_{n-6}}\gamma_{-1}^{a_{n+1}-2a_{n-2}+a_{n-5}}\gamma_{-2}^{a_{n-1}-2a_{n-4}+a_{n-7}}\\
&\quad\times \delta_0^{a_{n-2}-a_{n-5}}\delta_{-1}^{a_{n-1}-a_{n-4}}\delta_{-2}^{a_{n-3}-a_{n-6}}\\
&= \gamma_0^{a_{n-4}}\gamma_{-1}^{a_{n-3}}\gamma_{-2}^{a_{n-5}}\delta_0^{a_{n-2}-a_{n-5}}\delta_{-1}^{a_{n-1}-a_{n-4}}\delta_{-2}^{a_{n-3}-a_{n-6}}
\end{aligned}\tag{98}$$

for $n \ge -2$.

The characteristic polynomial associated with equation (95) is

$$P_6(t) = t^6 - 2t^3 - t^2 + 1 = \left(t^3 - t - 1\right)\left(t^3 + t - 1\right).$$

Let t_j, $j = \overline{1,6}$, be its roots. Then $t_j = \lambda_j$, $j = \overline{1,3}$ (the roots of polynomial (52)), whereas the other three roots of P_6 are the roots of the polynomial $t^3 + t - 1$, which are routinely found (see, e.g., [34, 36, 59]).

So, by Lemma 2, the solution to equation (95) such that $a_{-k} = 0$, $k = \overline{3,7}$, and $a_{-2} = 1$ is

$$a_n = \sum_{j=1}^{6}\frac{t_j^{n+7}}{P_6'(t_j)}, \quad n \in \mathbb{Z}.\tag{99}$$

Formulas (97) and (98) present the general solution to system (33). This consideration along with (26) shows that the following corollary holds.

Corollary 7 *Consider system* (16) *with* $a \neq 0$. *Then its general solution is*

$$x_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{b_{n-2}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{b_{n-1}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{b_{n-3}}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{a_{n-4}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{a_{n-3}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{a_{n-5}}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{b_{n-2}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{b_{n-1}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{b_{n-3}}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{a_{n-4}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{a_{n-3}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{a_{n-5}}-1},$$

$$y_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{a_{n-4}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{a_{n-3}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{a_{n-5}}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{b_{n-2}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{b_{n-1}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{b_{n-3}}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{a_{n-4}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{a_{n-3}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{a_{n-5}}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{b_{n-2}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{b_{n-1}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{b_{n-3}}-1}$$

for $n \ge -2$, *where the sequence* a_n *is given by* (99) *and* $b_n = a_n - a_{n-3}$.

3.8 Solution to system (34)

This system is obtained from system (29) by interchanging letters ζ and η.

Thus, its general solution is

$$\gamma_{3n} = \gamma_0 \delta_0^{a_{3n-1}-1} \delta_{-1}^{a_{3n}} \delta_{-2}^{a_{3n-2}} \tag{100}$$

for $n \in \mathbb{N}_0$,

$$\gamma_{3n+1} = \gamma_{-2} \delta_0^{a_{3n}} \delta_{-1}^{a_{3n+1}} \delta_{-2}^{a_{3n-1}-1} \tag{101}$$

for $n \ge -1$,

$$\gamma_{3n+2} = \gamma_{-1} \delta_0^{a_{3n+1}} \delta_{-1}^{a_{3n+2}-1} \delta_{-2}^{a_{3n}} \tag{102}$$

for $n \ge -1$, and

$$\delta_n = \delta_0^{a_{n-1}} \delta_{-1}^{a_n} \delta_{-2}^{a_{n-2}} \tag{103}$$

for $n \ge -2$.

Formulas (100)–(103) present the general solution to the system of difference equations (34). This consideration along with (26) shows that the following corollary holds.

Corollary 8 *Consider system* (17) *with* $a \neq 0$. *Then its general solution is*

$$\begin{aligned}
x_{3n} &= \sqrt{a}\frac{(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}})(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}})^{a_{3n-1}-1}(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}})^{a_{3n}}(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}})^{a_{3n-2}}+1}{(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}})(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}})^{a_{3n-1}-1}(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}})^{a_{3n}}(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}})^{a_{3n-2}}-1}, \quad n \in \mathbb{N}_0,\\
x_{3n+1} &= \sqrt{a}\frac{(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}})(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}})^{a_{3n}}(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}})^{a_{3n+1}}(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}})^{a_{3n-1}-1}+1}{(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}})(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}})^{a_{3n}}(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}})^{a_{3n+1}}(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}})^{a_{3n-1}-1}-1}, \quad n \ge -1,\\
x_{3n+2} &= \sqrt{a}\frac{(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}})(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}})^{a_{3n+1}}(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}})^{a_{3n+2}-1}(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}})^{a_{3n}}+1}{(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}})(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}})^{a_{3n+1}}(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}})^{a_{3n+2}-1}(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}})^{a_{3n}}-1}, \quad n \ge -1,\\
y_n &= \sqrt{a}\frac{(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}})^{a_{n-1}}(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}})^{a_n}(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}})^{a_{n-2}}+1}{(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}})^{a_{n-1}}(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}})^{a_n}(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}})^{a_{n-2}}-1}, \quad n \ge -2,
\end{aligned}$$

where a_n *is given by* (53).

3.9 Solution to system (35)

Combining the equations in (35), we have that the following recurrence relation holds:

$$\gamma_n = \gamma_{n-2}\gamma_{n-5}\gamma_{n-6} \tag{104}$$

for $n \ge 4$.

Let

$$a_1 := 1, \qquad b_1 := 0, \qquad c_1 := 0, \qquad d_1 := 1, \qquad e_1 := 1, \qquad f_1 := 0. \tag{105}$$

Then equation (104) can be written as follows:

$$y_n = y_{n-2}^{a_1} y_{n-3}^{b_1} y_{n-4}^{c_1} y_{n-5}^{d_1} y_{n-6}^{e_1} y_{n-7}^{f_1}, \quad n \geq 4. \tag{106}$$

Employing (104) in (106), we have

$$\begin{aligned} y_n &= y_{n-2}^{a_1} y_{n-3}^{b_1} y_{n-4}^{c_1} y_{n-5}^{d_1} y_{n-6}^{e_1} y_{n-7}^{f_1} \\ &= (y_{n-4} y_{n-7} y_{n-8})^{a_1} y_{n-3}^{b_1} y_{n-4}^{c_1} y_{n-5}^{d_1} y_{n-6}^{e_1} y_{n-7}^{f_1} \\ &= y_{n-3}^{b_1} y_{n-4}^{a_1+c_1} y_{n-5}^{d_1} y_{n-6}^{e_1} y_{n-7}^{a_1+f_1} y_{n-8}^{a_1} \\ &= y_{n-3}^{a_2} y_{n-4}^{b_2} y_{n-5}^{c_2} y_{n-6}^{d_2} y_{n-7}^{e_2} y_{n-8}^{f_2} \end{aligned} \tag{107}$$

for $n \geq 6$, where

$$\begin{aligned} &a_2 := b_1, \qquad b_2 := a_1 + c_1, \qquad c_2 := d_1, \\ &d_2 := e_1, \qquad e_2 := a_1 + f_1, \qquad f_2 := a_1. \end{aligned} \tag{108}$$

Similarly as in the case of equation (92), it is proved that

$$y_n = y_{n-k-1}^{a_k} y_{n-k-2}^{b_k} y_{n-k-3}^{c_k} y_{n-k-4}^{d_k} y_{n-k-5}^{e_k} y_{n-k-6}^{f_k} \tag{109}$$

for $k \in \mathbb{N} \setminus \{1\}$ and $n \geq k + 4$, and that

$$\begin{aligned} &a_k = b_{k-1}, \qquad b_k = a_{k-1} + c_{k-1}, \qquad c_k = d_{k-1}, \\ &d_k = e_{k-1}, \qquad e_k = a_{k-1} + f_{k-1}, \qquad f_k = a_{k-1}. \end{aligned} \tag{110}$$

From (110), we have

$$a_n = a_{n-2} + a_{n-5} + a_{n-6}, \quad n \geq 7 \tag{111}$$

(in fact (111) holds for every $n \in \mathbb{Z}$, see Remark 1) and

$$\begin{aligned} &a_0 = 0, \qquad a_{-1} = 1, \qquad a_{-j} = 0, \quad j = \overline{2,6}, \\ &a_{-7} = 1, \qquad a_{-8} = -1, \qquad a_{-9} = 1. \end{aligned} \tag{112}$$

Taking $k = n - 4$ in (109), we obtain

$$\begin{aligned} y_n &= y_3^{a_{n-4}} y_2^{b_{n-4}} y_1^{c_{n-4}} y_0^{d_{n-4}} y_{-1}^{e_{n-4}} y_{-2}^{f_{n-4}} \\ &= (y_{-1}\delta_0\delta_{-2})^{a_{n-4}} (y_0\delta_{-1})^{b_{n-4}} (y_{-1}\delta_{-2})^{c_{n-4}} y_0^{d_{n-4}} y_{-1}^{e_{n-4}} y_{-2}^{f_{n-4}} \\ &= y_0^{b_{n-4}+d_{n-4}} y_{-1}^{a_{n-4}+c_{n-4}+e_{n-4}} y_{-2}^{f_{n-4}} \delta_0^{a_{n-4}} \delta_{-1}^{b_{n-4}} \delta_{-2}^{a_{n-4}+c_{n-4}} \\ &= y_0^{a_{n-1}} y_{-1}^{a_n} y_{-2}^{a_{n-5}} \delta_0^{a_{n-4}} \delta_{-1}^{a_{n-3}} \delta_{-2}^{a_{n-2}} \end{aligned} \tag{113}$$

for $n \geq -2$.

Employing (113) in the second equation in (35), it follows that

$$\begin{aligned}\delta_n &= \gamma_{n-2}\gamma_{n-3}\\ &= \gamma_0^{a_{n-3}+a_{n-4}}\gamma_{-1}^{a_{n-2}+a_{n-3}}\gamma_{-2}^{a_{n-7}+a_{n-8}}\delta_0^{a_{n-6}+a_{n-7}}\delta_{-1}^{a_{n-5}+a_{n-6}}\delta_{-2}^{a_{n-4}+a_{n-5}}\end{aligned} \tag{114}$$

for $n \geq -2$.

The roots of the characteristic polynomial

$$\widetilde{P}_6(t) = t^6 - t^4 - t - 1 = \left(t^3 - t - 1\right)\left(t^3 + 1\right)$$

associated with equation (111) are

$$t_1 = \lambda_1, \qquad t_2 = \lambda_2, \qquad t_3 = \lambda_3, \qquad t_4 = -1, \qquad t_{5,6} = e^{\pm i\frac{\pi}{3}}, \tag{115}$$

where λ_j, $j = \overline{1,3}$, are the roots of polynomial (52).

By Lemma 2, we see that the solution to equation (111) such that $a_{-k} = 0$, $k = \overline{2,6}$, and $a_{-1} = 1$ is

$$a_n = \sum_{j=1}^{6} \frac{t_j^{n+6}}{\widetilde{P}_6'(t_j)}, \quad n \in \mathbb{Z}. \tag{116}$$

This consideration along with (26) shows that the following corollary holds.

Corollary 9 *Consider system (18) with $a \neq 0$. Then its general solution is*

$$x_n = \sqrt{a}\frac{(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}})^{a_{n-1}}(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}})^{a_n}(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}})^{a_{n-5}}(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}})^{a_{n-4}}(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}})^{a_{n-3}}(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}})^{a_{n-2}}+1}{(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}})^{a_{n-1}}(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}})^{a_n}(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}})^{a_{n-5}}(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}})^{a_{n-4}}(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}})^{a_{n-3}}(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}})^{a_{n-2}}-1},$$

$$y_n = \sqrt{a}\frac{(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}})^{b_{n-3}}(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}})^{b_{n-2}}(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}})^{b_{n-7}}(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}})^{b_{n-6}}(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}})^{b_{n-5}}(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}})^{b_{n-4}}+1}{(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}})^{b_{n-3}}(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}})^{b_{n-2}}(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}})^{b_{n-7}}(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}})^{b_{n-6}}(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}})^{b_{n-5}}(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}})^{b_{n-4}}-1}$$

for $n \geq -2$, where the sequence a_n is given by (116) and

$$b_n = a_n + a_{n-1}.$$

3.10 Solution to system (36)

Combining the equations in (36), it follows that the following relation holds:

$$\gamma_n = \gamma_{n-2}^2\gamma_{n-4}^{-1}\gamma_{n-6} \tag{117}$$

for $n \geq 4$.

Let

$$a_1 = 2, \qquad b_1 = -1, \qquad c_1 = 1. \tag{118}$$

Since equation (117) is with interlacing indices of order two (see, e.g., [17, 23]), it can be written as follows:

$$\gamma_{2n+i}=\gamma_{2(n-1)+i}^{a_1}\gamma_{2(n-2)+i}^{b_1}\gamma_{2(n-3)+i}^{c_1} \tag{119}$$

for $n\ge 2$ and $i=0,1$.

By using (117) in (119) we have

$$\begin{aligned}\gamma_{2n+i}&=\left(\gamma_{2(n-2)+i}^{2}\gamma_{2(n-3)+i}^{-1}\gamma_{2(n-4)+i}\right)^{a_1}\gamma_{2(n-2)+i}^{b_1}\gamma_{2(n-3)+i}^{c_1}\\&=\gamma_{2(n-2)+i}^{2a_1+b_1}\gamma_{2(n-3)+i}^{-a_1+c_1}\gamma_{2(n-4)+i}^{a_1}\\&=\gamma_{2(n-2)+i}^{a_2}\gamma_{2(n-3)+i}^{b_2}\gamma_{2(n-4)+i}^{c_2}\end{aligned}$$

for $n\ge 3$, $i=0,1$, where

$$a_2:=2a_1+b_1,\qquad b_2:=-a_1+c_1,\qquad c_2:=a_1.$$

As in the case of equation (45), it is proved that

$$\gamma_{2n+i}=\gamma_{2(n-k)+i}^{a_k}\gamma_{2(n-k-1)+i}^{b_k}\gamma_{2(n-k-2)+i}^{c_k} \tag{120}$$

for $i=0,1$, and

$$a_k=2a_{k-1}+b_{k-1},\qquad b_k=-a_{k-1}+c_{k-1},\qquad c_k=a_{k-1} \tag{121}$$

for $k\in\mathbb{N}\setminus\{1\}$ and every $n\ge k+1$.

From (121) it follows that

$$a_n=2a_{n-1}-a_{n-2}+a_{n-3},\quad n\ge 4 \tag{122}$$

(in fact recurrent relation (122) holds for every $n\in\mathbb{Z}$, Remark 1), and it is easily shown that

$$a_0=1,\qquad a_{-1}=0,\qquad a_{-2}=0,\qquad a_{-3}=1,\qquad a_{-4}=1. \tag{123}$$

For $k=n-1$, from (120), we have

$$\begin{aligned}\gamma_{2n}&=\gamma_2^{a_{n-1}}\gamma_0^{b_{n-1}}\gamma_{-2}^{c_{n-1}}\\&=(\gamma_0\delta_{-1})^{a_{n-1}}\gamma_0^{b_{n-1}}\gamma_{-2}^{c_{n-1}}\\&=\gamma_0^{a_{n-1}+b_{n-1}}\gamma_{-2}^{c_{n-1}}\delta_{-1}^{a_{n-1}}\\&=\gamma_0^{a_n-a_{n-1}}\gamma_{-2}^{a_{n-2}}\delta_{-1}^{a_{n-1}}\end{aligned} \tag{124}$$

for $n\ge -1$, and

$$\begin{aligned}\gamma_{2n+1}&=\gamma_3^{a_{n-1}}\gamma_1^{b_{n-1}}\gamma_{-1}^{c_{n-1}}\\&=(\gamma_{-1}\delta_0\delta_{-2})^{a_{n-1}}(\gamma_{-1}\delta_{-2})^{b_{n-1}}\gamma_{-1}^{c_{n-1}}\end{aligned}$$

$$
\begin{aligned}
&= \gamma_{-1}^{a_{n-1}+b_{n-1}+c_{n-1}} \delta_0^{a_{n-1}} \delta_{-2}^{a_{n-1}+b_{n-1}} \\
&= \gamma_{-1}^{a_{n+1}-a_n} \delta_0^{a_{n-1}} \delta_{-2}^{a_n-a_{n-1}}
\end{aligned}
\tag{125}
$$

for $n \ge -1$.

From (36), (124), and (125), it follows that

$$
\begin{aligned}
\delta_{2n+1} &= \gamma_{2n+4}/\gamma_{2n+2} \\
&= \gamma_0^{a_{n+2}-2a_{n+1}+a_n} \gamma_{-2}^{a_n-a_{n-1}} \delta_{-1}^{a_{n+1}-a_n} \\
&= \gamma_0^{a_{n-1}} \gamma_{-2}^{a_n-a_{n-1}} \delta_{-1}^{a_{n+1}-a_n}
\end{aligned}
\tag{126}
$$

for $n \ge -1$, and

$$
\begin{aligned}
\delta_{2n} &= \gamma_{2n+3}/\gamma_{2n+1} \\
&= \gamma_{-1}^{a_{n+2}-2a_{n+1}+a_n} \delta_0^{a_n-a_{n-1}} \delta_{-2}^{a_{n+1}-2a_n+a_{n-1}} \\
&= \gamma_{-1}^{a_{n-1}} \delta_0^{a_n-a_{n-1}} \delta_{-2}^{a_{n-2}}
\end{aligned}
\tag{127}
$$

for $n \ge -1$ ((126) and (127) are also obtained from (124) and (125) due to the symmetry of system (36)).

Now note that the characteristic polynomial associated with difference equation (122) is given by

$$
\widetilde{P}_3(t) = t^3 - 2t^2 + t - 1.
$$

Let $\widetilde{t}_j$, $j = \overline{1,3}$, be the roots of polynomial $\widetilde{P}_3$ (they are also found in a routine way [34, 36, 59], which we leave to the interested reader).

By using Lemma 2, we see that the solution to equation (122) satisfying the following conditions

$$
a_{-2} = a_{-1} = 0 \quad \text{and} \quad a_0 = 1
$$

is

$$
a_n = \sum_{j=1}^{3} \frac{\widetilde{t}_j^{\,n+2}}{\widetilde{P}_3'(\widetilde{t}_j)}, \quad n \in \mathbb{Z}.
\tag{128}
$$

The above consideration along with the changes of variables in (26) shows that the following corollary holds.

Corollary 10 *Consider system (19) with $a \neq 0$. Then its general solution is*

$$
\begin{aligned}
x_{2n} &= \sqrt{a}\,\frac{\big(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\big)^{a_n-a_{n-1}}\big(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\big)^{a_{n-2}}\big(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\big)^{a_{n-1}}+1}{\big(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\big)^{a_n-a_{n-1}}\big(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\big)^{a_{n-2}}\big(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\big)^{a_{n-1}}-1}, \\
x_{2n+1} &= \sqrt{a}\,\frac{\big(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\big)^{a_{n+1}-a_n}\big(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\big)^{a_{n-1}}\big(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\big)^{a_n-a_{n-1}}+1}{\big(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\big)^{a_{n+1}-a_n}\big(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\big)^{a_{n-1}}\big(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\big)^{a_n-a_{n-1}}-1},
\end{aligned}
$$

$$y_{2n}=\sqrt{a}\frac{(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}})^{a_{n-1}}(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}})^{a_n-a_{n-1}}(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}})^{a_{n-2}}+1}{(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}})^{a_{n-1}}(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}})^{a_n-a_{n-1}}(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}})^{a_{n-2}}-1}$$

$$y_{2n+1}=\sqrt{a}\frac{(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}})^{a_{n-1}}(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}})^{a_n-a_{n-1}}(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}})^{a_{n+1}-a_n}+1}{(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}})^{a_{n-1}}(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}})^{a_n-a_{n-1}}(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}})^{a_{n+1}-a_n}-1}$$

for $n\ge -1$, where the sequence a_n is given by (128).

Remark 3 Equation (117) can be also solved directly, that is, without reducing it to two non-interlaced difference equations. Namely, let

$$\widetilde{a}_1:=2,\qquad \widetilde{b}_1:=0,\qquad \widetilde{c}_1:=-1,\qquad \widetilde{d}_1:=0,\qquad \widetilde{e}_1:=1,\qquad \widetilde{f}_1:=0,$$

then equation (117) can be written as follows:

$$\gamma_n=\gamma_{n-2}^{\widetilde{a}_1}\gamma_{n-3}^{\widetilde{b}_1}\gamma_{n-4}^{\widetilde{c}_1}\gamma_{n-5}^{\widetilde{d}_1}\gamma_{n-6}^{\widetilde{e}_1}\gamma_{n-7}^{\widetilde{f}_1} \tag{129}$$

for $n\ge 4$.

By using (117) in (129) we have

$$\begin{aligned}\gamma_n&=\gamma_{n-2}^{\widetilde{a}_1}\gamma_{n-3}^{\widetilde{b}_1}\gamma_{n-4}^{\widetilde{c}_1}\gamma_{n-5}^{\widetilde{d}_1}\gamma_{n-6}^{\widetilde{e}_1}\gamma_{n-7}^{\widetilde{f}_1},\\&=\left(\gamma_{n-4}^2\gamma_{n-6}^{-1}\gamma_{n-8}\right)^{\widetilde{a}_1}\gamma_{n-3}^{\widetilde{b}_1}\gamma_{n-4}^{\widetilde{c}_1}\gamma_{n-5}^{\widetilde{d}_1}\gamma_{n-6}^{\widetilde{e}_1}\gamma_{n-7}^{\widetilde{f}_1}\\&=\gamma_{n-3}^{\widetilde{b}_1}\gamma_{n-4}^{2\widetilde{a}_1+\widetilde{c}_1}\gamma_{n-5}^{\widetilde{d}_1}\gamma_{n-6}^{-\widetilde{a}_1+\widetilde{e}_1}\gamma_{n-7}^{\widetilde{f}_1}\gamma_{n-8}^{\widetilde{a}_1}\\&=\gamma_{n-3}^{\widetilde{a}_2}\gamma_{n-4}^{\widetilde{b}_2}\gamma_{n-5}^{\widetilde{c}_2}\gamma_{n-6}^{\widetilde{d}_2}\gamma_{n-7}^{\widetilde{e}_2}\gamma_{n-8}^{\widetilde{f}_2}\end{aligned}$$

for $n\ge 6$, where

$$\widetilde{a}_2:=\widetilde{b}_1,\qquad \widetilde{b}_2:=2\widetilde{a}_1+\widetilde{c}_1,\qquad \widetilde{c}_2:=\widetilde{d}_1,\qquad \widetilde{d}_2:=-\widetilde{a}_1+\widetilde{e}_1,\qquad \widetilde{e}_2:=\widetilde{f}_1,\qquad \widetilde{f}_2:=\widetilde{a}_1.$$

Similar to equation (104) it is proved that

$$\gamma_n=\gamma_{n-k-1}^{\widetilde{a}_k}\gamma_{n-k-2}^{\widetilde{b}_k}\gamma_{n-k-3}^{\widetilde{c}_k}\gamma_{n-k-4}^{\widetilde{d}_k}\gamma_{n-k-5}^{\widetilde{e}_k}\gamma_{n-k-6}^{\widetilde{f}_k} \tag{130}$$

and

$$\begin{aligned}&\widetilde{a}_k=\widetilde{b}_{k-1},\qquad \widetilde{b}_k=2\widetilde{a}_{k-1}+\widetilde{c}_{k-1},\qquad \widetilde{c}_k=\widetilde{d}_{k-1},\\&\widetilde{d}_k=-\widetilde{a}_{k-1}+\widetilde{e}_{k-1},\qquad \widetilde{e}_k=\widetilde{f}_{k-1},\qquad \widetilde{f}_k=\widetilde{a}_{k-1}\end{aligned} \tag{131}$$

for $2\le k\le n-4$.

From (131) it is obtained that

$$\widetilde{a}_n=2\widetilde{a}_{n-2}-\widetilde{a}_{n-4}+\widetilde{a}_{n-6},\quad n\ge 7 \tag{132}$$

(in fact, recurrent relation (132) holds for every $n\in\mathbb{Z}$, see Remark 1), and we have that

$$\widetilde{a}_{-j}=0,\quad j=\overline{2,6},\qquad \widetilde{a}_{-1}=1,\qquad \widetilde{a}_0=0.$$

Now note that the characteristic polynomial associated with difference equation (132) is

$$Q_6(t) = t^6 - 2t^4 + t^2 - 1.$$

Since

$$Q_6(t) = \big(t(t^2-1)\big)^2 - 1 = \big(t^3 - t - 1\big)\big(t^3 - t + 1\big),$$

we see that Q_6 is solvable by radicals.

Let $\widehat{t}_j$, $j = \overline{1,6}$, be the roots of polynomial Q_6. Then, clearly, $\widehat{t}_j = \lambda_j$, $j = \overline{1,3}$, where λ_j, $j = \overline{1,3}$, are the roots of polynomial (52), whereas the other three roots of Q_6 are the roots of the polynomial

$$\widetilde{Q}_3(t) := t^3 - t + 1,$$

which are routinely found (see, e.g., [34, 36, 59]).

So, by Lemma 2, we see that the solution to equation (132) such that

$$\widetilde{a}_{-k} = 0, \quad k = \overline{2,6}, \quad \text{and} \quad \widetilde{a}_{-1} = 1$$

is

$$\widetilde{a}_n = \sum_{j=1}^{6} \frac{\widehat{t}_j^{\,n+6}}{Q_6'(\widehat{t}_j)}, \quad n \in \mathbb{Z}. \tag{133}$$

From (26), (36), (130) with $k = n - 4$, the recurrent relations in (131), and (133), another set of closed-form formulas for general solution to system (19) is obtained.

3.11 Solution to system (37)

The form of system (37) shows that

$$\gamma_n = \delta_n, \quad n \in \mathbb{N}.$$

Hence,

$$\gamma_n = \gamma_{n-2}\gamma_{n-3}$$

for $n \geq 4$.

By using (50) we have

$$\begin{aligned}
\gamma_n &= \gamma_3^{a_{n-4}}\gamma_2^{a_{n-3}}\gamma_1^{a_{n-5}} \\
&= (\gamma_{-1}\delta_0\delta_{-2})^{a_{n-4}}(\gamma_0\delta_{-1})^{a_{n-3}}(\gamma_{-1}\delta_{-2})^{a_{n-5}} \\
&= \gamma_0^{a_{n-3}}\gamma_{-1}^{a_{n-4}+a_{n-5}}\delta_0^{a_{n-4}}\delta_{-1}^{a_{n-3}}\delta_{-2}^{a_{n-4}+a_{n-5}} \\
&= \gamma_0^{a_{n-3}}\gamma_{-1}^{a_{n-2}}\delta_0^{a_{n-4}}\delta_{-1}^{a_{n-3}}\delta_{-2}^{a_{n-2}}
\end{aligned} \tag{134}$$

for $n \in \mathbb{N}$, where a_n is the solution to equation (48) such that $a_{-3} = a_{-2} = 0$ and $a_{-1} = 1$, and consequently

$$\delta_n = \gamma_0^{a_{n-3}} \gamma_{-1}^{a_{n-2}} \delta_0^{a_{n-4}} \delta_{-1}^{a_{n-3}} \delta_{-2}^{a_{n-2}} \tag{135}$$

for $n \in \mathbb{N}$.

It is easy to see that formula (135) holds also for $n = 0$.

Formulas (134) and (135) present the general solution to system (37). This consideration along with (26) shows that the following corollary holds.

Corollary 11 *Consider system* (20) *with* $a \neq 0$. *Then its general solution is*

$$x_n = \sqrt{a}\frac{(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}})^{a_{n-3}}(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}})^{a_{n-2}}(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}})^{a_{n-4}}(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}})^{a_{n-3}}(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}})^{a_{n-2}} + 1}{(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}})^{a_{n-3}}(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}})^{a_{n-2}}(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}})^{a_{n-4}}(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}})^{a_{n-3}}(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}})^{a_{n-2}} - 1}$$

for $n \in \mathbb{N}$ *and*

$$y_n = \sqrt{a}\frac{(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}})^{a_{n-3}}(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}})^{a_{n-2}}(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}})^{a_{n-4}}(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}})^{a_{n-3}}(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}})^{a_{n-2}} + 1}{(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}})^{a_{n-3}}(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}})^{a_{n-2}}(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}})^{a_{n-4}}(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}})^{a_{n-3}}(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}})^{a_{n-2}} - 1}$$

for $n \in \mathbb{N}_0$, *where* a_n *is given by* (53).

3.12 Solution to system (38)

This system is got from (28) by interchanging letters ζ and η. Hence, its general solution is

$$\gamma_{2n} = \gamma_0 \delta_0^{a_{2n-1}-1} \delta_{-1}^{a_{2n}} \delta_{-2}^{a_{2n-2}} \tag{136}$$

for $n \in \mathbb{N}_0$ and

$$\gamma_{2n+1} = \gamma_{-1} \delta_0^{a_{2n}} \delta_{-1}^{a_{2n+1}-1} \delta_{-2}^{a_{2n-1}} \tag{137}$$

for $n \geq -1$, and

$$\delta_n = \delta_0^{a_{n-1}} \delta_{-1}^{a_n} \delta_{-2}^{a_{n-2}} \tag{138}$$

for $n \geq -2$.

Formulas (136)–(138) present the general solution to system (38). This consideration along with (26) shows that the following corollary holds.

Corollary 12 *Consider system* (21) *with* $a \neq 0$. *Then its general solution is*

$$x_{2n} = \sqrt{a}\frac{(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}})(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}})^{a_{2n-1}-1}(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}})^{a_{2n}}(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}})^{a_{2n-2}} + 1}{(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}})(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}})^{a_{2n-1}-1}(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}})^{a_{2n}}(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}})^{a_{2n-2}} - 1}, \quad n \in \mathbb{N}_0,$$

$$x_{2n+1} = \sqrt{a}\frac{(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}})(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}})^{a_{2n}}(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}})^{a_{2n+1}-1}(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}})^{a_{2n-1}} + 1}{(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}})(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}})^{a_{2n}}(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}})^{a_{2n+1}-1}(\frac{y_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}})^{a_{2n-1}} - 1}, \quad n \geq -1,$$

$$y_n = \sqrt{a}\frac{(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}})^{a_{n-1}}(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}})^{a_n}(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}})^{a_{n-2}}+1}{(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}})^{a_{n-1}}(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}})^{a_n}(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}})^{a_{n-2}}-1}, \quad n \geq -2,$$

where a_n is given by (53).

3.13 Solution to system (39)

Combining the equations in (39), we obtain

$$\gamma_n = \gamma_{n-4}\gamma_{n-5}^2\gamma_{n-6} \tag{139}$$

for $n \geq 4$.

Let

$$a_1 := 1, \qquad b_1 := 2, \qquad c_1 := 1, \qquad d_1 := 0, \qquad e_1 := 0, \qquad f_1 := 0. \tag{140}$$

Then equation (140) can be written as follows:

$$\gamma_n = \gamma_{n-4}^{a_1}\gamma_{n-5}^{b_1}\gamma_{n-6}^{c_1}\gamma_{n-7}^{d_1}\gamma_{n-8}^{e_1}\gamma_{n-9}^{f_1} \tag{141}$$

for $n \geq 4$.

By using (139) in (141) we have

$$\begin{aligned} \gamma_n &= \left(\gamma_{n-8}\gamma_{n-9}^2\gamma_{n-10}\right)^{a_1}\gamma_{n-5}^{b_1}\gamma_{n-6}^{c_1}\gamma_{n-7}^{d_1}\gamma_{n-8}^{e_1}\gamma_{n-9}^{f_1} \\ &= \gamma_{n-5}^{b_1}\gamma_{n-6}^{c_1}\gamma_{n-7}^{d_1}\gamma_{n-8}^{a_1+e_1}\gamma_{n-9}^{2a_1+f_1}\gamma_{n-10}^{a_1} \\ &= \gamma_{n-5}^{a_2}\gamma_{n-6}^{b_2}\gamma_{n-7}^{c_2}\gamma_{n-8}^{d_2}\gamma_{n-9}^{e_2}\gamma_{n-10}^{f_2} \end{aligned} \tag{142}$$

for $n \geq 8$, where

$$\begin{aligned} &a_2 := b_1, \qquad b_2 := c_1, \qquad c_2 := d_1, \\ &d_2 := a_1 + e_1, \qquad e_2 := 2a_1 + f_1, \qquad f_2 := a_1. \end{aligned} \tag{143}$$

Similar to equation (79), we get

$$\gamma_n = \gamma_{n-k-3}^{a_k}\gamma_{n-k-4}^{b_k}\gamma_{n-k-5}^{c_k}\gamma_{n-k-6}^{d_k}\gamma_{n-k-7}^{e_k}\gamma_{n-k-8}^{f_k} \tag{144}$$

and

$$\begin{aligned} &a_k = b_{k-1}, \qquad b_k = c_{k-1}, \qquad c_k = d_{k-1}, \\ &d_k = a_{k-1} + e_{k-1}, \qquad e_k = 2a_{k-1} + f_{k-1}, \qquad f_k = a_{k-1} \end{aligned} \tag{145}$$

for $k \in \mathbb{N} \setminus \{1\}$ and every $n \geq k + 6$.

From (145) we obtain

$$a_n = a_{n-4} + 2a_{n-5} + a_{n-6} \tag{146}$$

and

$$a_0 = 0, \qquad a_{-1} = 0, \qquad a_{-2} = 0, \qquad a_{-3} = 1,$$
$$a_{-j} = 0, \quad j = \overline{4,8}, \qquad a_{-9} = 1. \tag{147}$$

Taking $k = n - 6$ in (144), we have

$$\begin{aligned}\gamma_n &= \gamma_3^{a_{n-6}}\gamma_2^{b_{n-6}}\gamma_1^{c_{n-6}}\gamma_0^{d_{n-6}}\gamma_{-1}^{e_{n-6}}\gamma_{-2}^{f_{n-6}}\\ &= (\delta_0\gamma_{-1}\gamma_{-2})^{a_{n-6}}(\delta_0\delta_{-1})^{a_{n-5}}(\delta_{-1}\delta_{-2})^{a_{n-4}}\gamma_0^{a_{n-3}}\gamma_{-1}^{a_{n-2}-a_{n-6}}\gamma_{-2}^{a_{n-7}}\\ &= \gamma_0^{a_{n-3}}\gamma_{-1}^{a_{n-2}}\gamma_{-2}^{a_{n-6}+a_{n-7}}\delta_0^{a_{n-5}+a_{n-6}}\delta_{-1}^{a_{n-4}+a_{n-5}}\delta_{-2}^{a_{n-4}}\end{aligned} \tag{148}$$

for $n \ge -2$.

From (39) and (148), we have

$$\begin{aligned}\delta_n &= \gamma_{n-2}\gamma_{n-3}\\ &= \gamma_0^{a_{n-5}+a_{n-6}}\gamma_{-1}^{a_{n-4}+a_{n-5}}\gamma_{-2}^{a_{n-8}+2a_{n-9}+a_{n-10}}\\ &\quad\times \delta_0^{a_{n-7}+2a_{n-8}+a_{n-9}}\delta_{-1}^{a_{n-6}+2a_{n-7}+a_{n-8}}\delta_{-2}^{a_{n-6}+a_{n-7}}\\ &= \gamma_0^{a_{n-5}+a_{n-6}}\gamma_{-1}^{a_{n-4}+a_{n-5}}\gamma_{-2}^{a_{n-4}}\delta_0^{a_{n-3}}\delta_{-1}^{a_{n-2}}\delta_{-2}^{a_{n-6}+a_{n-7}}\end{aligned} \tag{149}$$

for $n \ge -2$.

The characteristic polynomial associated with equation (146) is

$$\widehat{P}_6(t) = t^6 - t^2 - 2t - 1 = \left(t^3 - t - 1\right)\left(t^3 + t + 1\right).$$

If t_j, $j = \overline{1,6}$, are the roots of the polynomial, then clearly $\widehat{t_j} = \lambda_j$, $j = \overline{1,3}$, where λ_j, $j = \overline{1,3}$, are the roots of polynomial (52), whereas the other three roots of $\widehat{P}_6$ are the roots of the polynomial $t^3 + t + 1$, which are routinely found.

In view of Lemma 2 we see that the solution to (146) such that $a_{-k} = 0$, $k = \overline{4,8}$, and $a_{-3} = 1$ is

$$a_n = \sum_{j=1}^{6}\frac{t_j^{n+8}}{\widehat{P}_6'(t_j)}, \quad n \in \mathbb{Z}. \tag{150}$$

Formulas (148) and (149) present the general solution to system (39). This consideration along with (26) shows that the following corollary holds.

Corollary 13 *Consider system* (22) *with $a \ne 0$. Then its general solution is*

$$x_n = \sqrt{a}\frac{(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}})^{a_{n-3}}(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}})^{a_{n-2}}(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}})^{b_{n-6}}(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}})^{b_{n-5}}(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}})^{b_{n-4}}(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}})^{a_{n-4}}+1}{(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}})^{a_{n-3}}(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}})^{a_{n-2}}(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}})^{b_{n-6}}(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}})^{b_{n-5}}(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}})^{b_{n-4}}(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}})^{a_{n-4}}-1},$$

$$y_n = \sqrt{a}\frac{(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}})^{b_{n-5}}(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}})^{b_{n-4}}(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}})^{a_{n-4}}(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}})^{a_{n-3}}(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}})^{a_{n-2}}(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}})^{b_{n-6}}+1}{(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}})^{b_{n-5}}(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}})^{b_{n-4}}(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}})^{a_{n-4}}(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}})^{a_{n-3}}(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}})^{a_{n-2}}(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}})^{b_{n-6}}-1}$$

for $n \ge -2$, where the sequence a_n is given by (150) *and $b_n = a_n + a_{n-1}$.*

3.14 Solution to system (40)

System (40) is obtained from (35) by interchanging letters ζ and η. Hence, its general solution is

$$\delta_n = \delta_0^{a_{n-1}} \delta_{-1}^{a_n} \delta_{-2}^{a_{n-5}} \gamma_0^{a_{n-4}} \gamma_{-1}^{a_{n-3}} \gamma_{-2}^{a_{n-2}} \tag{151}$$

for $n \ge -2$ and

$$\gamma_n = \delta_0^{a_{n-3}+a_{n-4}} \delta_{-1}^{a_{n-2}+a_{n-3}} \delta_{-2}^{a_{n-7}+a_{n-8}} \gamma_0^{a_{n-6}+a_{n-7}} \gamma_{-1}^{a_{n-5}+a_{n-6}} \gamma_{-2}^{a_{n-4}+a_{n-5}} \tag{152}$$

for $n \ge -2$.

Formulas (151) and (152) present the general solution to system (40). This consideration along with (26) shows that the following corollary holds.

Corollary 14 *Consider system* (23) *with* $a \ne 0$. *Then its general solution is*

$$x_n = \sqrt{a}\frac{(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}})^{b_{n-3}}(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}})^{b_{n-2}}(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}})^{b_{n-7}}(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}})^{b_{n-6}}(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}})^{b_{n-5}}(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}})^{b_{n-4}}+1}{(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}})^{b_{n-3}}(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}})^{b_{n-2}}(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}})^{b_{n-7}}(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}})^{b_{n-6}}(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}})^{b_{n-5}}(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}})^{b_{n-4}}-1},$$

$$y_n = \sqrt{a}\frac{(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}})^{a_{n-1}}(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}})^{a_n}(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}})^{a_{n-5}}(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}})^{a_{n-4}}(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}})^{a_{n-3}}(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}})^{a_{n-2}}+1}{(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}})^{a_{n-1}}(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}})^{a_n}(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}})^{a_{n-5}}(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}})^{a_{n-4}}(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}})^{a_{n-3}}(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}})^{a_{n-2}}-1}$$

for $n \ge -2$, *where the sequence* a_n *is given by* (116) *and* $b_n = a_n + a_{n-1}$.

3.15 Solution to system (41)

System (41) is obtained from (31) by interchanging letters ζ and η. Hence, its general solution is

$$\gamma_n = \gamma_0^{a_{n-6}+a_{n-7}} \gamma_{-1}^{a_{n-5}+a_{n-6}} \delta_0^{a_{n-4}+a_{n-5}} \delta_{-1}^{a_{n-3}+a_{n-4}} \delta_{-2}^{a_{n-3}+a_{n-5}} \tag{153}$$

for $n \ge -1$ and

$$\delta_n = \gamma_0^{a_{n-4}} \gamma_{-1}^{a_{n-3}} \delta_0^{a_{n-2}} \delta_{-1}^{a_{n-1}} \delta_{-2}^{a_{n-3}+a_{n-5}} \tag{154}$$

for $n \ge -2$.

Formulas (153) and (154) present the general solution to system (41). This consideration along with (26) shows that the following corollary holds.

Corollary 15 *Consider system* (24) *with* $a \ne 0$. *Then its general solution is*

$$x_n = \sqrt{a}\frac{(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}})^{a_{n-6}+a_{n-7}}(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}})^{a_{n-5}+a_{n-6}}(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}})^{a_{n-4}+a_{n-5}}(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}})^{a_{n-3}+a_{n-4}}(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}})^{a_{n-3}+a_{n-5}}+1}{(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}})^{a_{n-6}+a_{n-7}}(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}})^{a_{n-5}+a_{n-6}}(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}})^{a_{n-4}+a_{n-5}}(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}})^{a_{n-3}+a_{n-4}}(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}})^{a_{n-3}+a_{n-5}}-1}$$

for $n \ge -1$ *and*

$$y_n = \sqrt{a}\frac{(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}})^{a_{n-4}}(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}})^{a_{n-3}}(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}})^{a_{n-2}}(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}})^{a_{n-1}}(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}})^{a_{n-3}+a_{n-5}}+1}{(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}})^{a_{n-4}}(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}})^{a_{n-3}}(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}})^{a_{n-2}}(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}})^{a_{n-1}}(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}})^{a_{n-3}+a_{n-5}}-1}$$

for $n \ge -2$, *where* a_n *is given by* (88).

3.16 Solution to system (42)

This system of difference equations is obtained from system (27) by interchanging letters ζ and η only. Hence, its solution is given by

$$\gamma_n = \delta_0^{a_{n-1}}\delta_{-1}^{a_n}\delta_{-2}^{a_{n-2}} \tag{155}$$

for $n \in \mathbb{N}$ and

$$\delta_n = \delta_0^{a_{n-1}}\delta_{-1}^{a_n}\delta_{-2}^{a_{n-2}} \tag{156}$$

for $n \geq -2$.

Formulas (155) and (156) present the general solution to system (42). This consideration along with (26) shows that the following corollary holds.

Corollary 16 *Consider system* (25) *with* $a \neq 0$. *Then its general solution is*

$$x_n = \sqrt{a}\frac{(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}})^{a_{n-1}}(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}})^{a_n}(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}})^{a_{n-2}}+1}{(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}})^{a_{n-1}}(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}})^{a_n}(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}})^{a_{n-2}}-1}, \quad n \in \mathbb{N},$$

$$y_n = \sqrt{a}\frac{(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}})^{a_{n-1}}(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}})^{a_n}(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}})^{a_{n-2}}+1}{(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}})^{a_{n-1}}(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}})^{a_n}(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}})^{a_{n-2}}-1}, \quad n \geq -2,$$

where a_n *is given by* (53).

Remark 4 Corollaries 1–16 show that all systems (10)–(25) are practically solvable. This is obviously equivalent with the practical solvability of system (4) when $k = 1$ and $l = 2$, which is one of the results that we wanted to present in this paper.

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