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Oscillatory behavior of a second order nonlinear advanced differential equation with mixed neutral terms

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Abstract

In this paper, we present several new oscillation criteria for a second order nonlinear differential equation with mixed neutral terms of the form

$$(r(t)(z'(t))^{\alpha})' + q(t)x^{\beta}(\sigma(t)) = 0, \quad t \geq t_0,$$

where $z(t) = x(t) + p_1(t)x(\tau(t)) + p_2(t)x(\lambda(t))$ and α, β are ratios of two positive odd integers. Our results improve and complement some well-known results which were published recently in the literature. Two examples are given to illustrate the efficiency of our results.

MSC: 34K06; 34K11

Keywords: Oscillation; Second order; Mixed neutral terms; Advanced differential equation

1 Introduction

In the article, we consider the oscillatory and asymptotic behavior of solutions to a second order nonlinear advanced differential equation with mixed neutral terms of the form

$$\big(r(t)\big(z'(t)\big)^{\alpha}\big)' + q(t)x^{\beta}\big(\sigma(t)\big) = 0, \quad t \geq t_0, \tag{1.1}$$

where $z(t) = x(t) + p_1(t)x(\tau(t)) + p_2(t)x(\lambda(t))$. We assume the following conditions hold throughout this paper.

(H1) α and β are ratios of two positive odd integers;

(H2) $r, \sigma \in C^1([t_0, \infty), (0, \infty))$, $r(t) > 0$, $\sigma(t) \geq t$, $\sigma'(t) \geq 0$, $\lim_{t\to\infty} \sigma(t) = \infty$;

(H3) $\tau, \lambda \in C([t_0, \infty), R)$, $\tau(t) \leq t$, $\lambda(t) \geq t$, $\lim_{t\to\infty} \tau(t) = \lim_{t\to\infty} \lambda(t) = \infty$;

(H4) $p_1, p_2 \in C([t_0, \infty), [0, 1))$, $q \in C([t_0, \infty), [0, \infty))$, $q(t)$ is not identically zero in any interval of $[t_0, \infty)$.

By a solution of Eq. (1.1) we mean a function $x \in C[T_x, \infty)$, $T_x \geq T_0$, which has the property $r(t)(z'(t))^{\alpha} \in C^1([T_x, \infty), R)$ and satisfies (1.1) on $[T_x, \infty)$. In this paper we only consider the nontrivial solution of Eq. (1.1) which satisfies $\sup\{|x(t)| : t \geq T\} > 0$ for all $T \geq T_x$. A solution of (1.1) is called oscillatory if it has arbitrarily large zeros on $[T_x, \infty)$;

Otherwise, it is said to be nonoscillatory. Equation (1.1) is said to be oscillatory if all its solutions oscillate.

Following Trench [21], we shall say that Eq. (1.1) is in canonical form if

$$\int_{t_0}^{\infty} r^{-\frac{1}{\alpha}}(s)\,ds = \infty. \tag{1.2}$$

Conversely, we say that (1.1) is in noncanonical form if

$$\int_{t_0}^{\infty} r^{-\frac{1}{\alpha}}(s)\,ds < \infty. \tag{1.3}$$

Advanced differential equations can find applications in a mass of real world problems where the evolution rate depends on present and future values of the quantity. Therefore, taking into account the impact of potential future actions, an advance could be introduced into the equation, which is available at the present and beneficial in the process of decision making. For instance, we can find numerous applications in mechanical control engineering, economical problems, population dynamics, neural networks and the field of time symmetric electrodynamics; see [14].

The establishment of oscillatory and/or nonoscillatory criteria for differential equations with deviating arguments, which was first studied by Fite [15] in 1921, has always been a very active research field. Several reviews and references of known results can be found in the monographs [3–6]. Up to now, most literature has been devoted to the study of delay differential equations, but few studies have considered the equations with advanced arguments. Therefore, recent studies have attempted to improve the already existing oscillation criteria.

Džurina [12] studied the advanced canonical equation of the form

$$\big(r(t)y'(t)\big)' + q(t)y\big(\sigma(t)\big) = 0$$

and established a new comparison principle by using new monotonic properties of nonoscillatory solutions and iterated exponentiation. Agarwal et al. [7] used an approach that leads to two independent conditions, eliminating increasing and decreasing positive solutions, respectively. Baculíková [9] and Jadlovská [17] investigated the second order linear advanced equation

$$y''(t) + q(t)y\big(\sigma(t)\big) = 0$$

and gave new oscillation results employing some iterative techniques. Recently, Chatzarakis et al. [10] investigated the second order half-linear differential equation with advanced argument

$$\big(r(t)\big(y'(t)\big)^{\alpha}\big)' + q(t)y^{\alpha}\big(\sigma(t)\big) = 0 \tag{1.4}$$

and established new oscillation criteria under the condition (1.3).

Motivated by the above work, we will consider a generalized nonlinear advanced differential equations with mixed neutral terms and establish new sufficient conditions for oscillation of Eq. (1.1) under the condition (1.3). Our results presented in Sect. 2 improve and complement those of Refs. [1, 2, 7–10, 12, 16, 17, 19, 20, 23]. Two examples are addressed to illustrate the efficiency of the main results in Sect. 3 and the conclusions are given in Sect. 4.

2 Main results

In this section, we present some lemmas and our new sufficient conditions for oscillation of Eq. (1.1). For the sake of convenience, we use the following notation:

$$R(t) = \int_{t_0}^{t} r^{-\frac{1}{\alpha}}(s)\,ds, \qquad Q_1(t) = 1 - p_1\bigl(\sigma(t)\bigr) - p_2\bigl(\sigma(t)\bigr)\frac{R(\lambda(\sigma(t)))}{R(\sigma(t))},$$

$$\pi(t) = \int_{t}^{\infty} r^{-\frac{1}{\alpha}}(s)\,ds, \qquad Q_2(t) = 1 - p_1\bigl(\sigma(t)\bigr)\frac{\pi(\tau(\sigma(t)))}{\pi(\sigma(t))} - p_2\bigl(\sigma(t)\bigr),$$

where $t \in [t_0, \infty)$.

In what follows we need only to consider the eventually positive solutions of Eq. (1.1), since if x satisfies Eq. (1.1), then $-x$ is also its solution. Without loss of generality, we only give proofs for the positive solutions. We begin with the following lemmas.

Lemma 2.1 *If $x(t)$ is an eventually positive solution of equation (1.1), then the corresponding function $z(t)$ satisfies one of two cases eventually:*

Case 1. $z(t) > 0$, $r(t)(z'(t))^{\alpha} > 0$ *and* $(r(t)(z'(t))^{\alpha})' \leq 0$;

Case 2. $z(t) > 0$, $r(t)(z'(t))^{\alpha} < 0$ *and* $(r(t)(z'(t))^{\alpha})' \leq 0$.

Proof Suppose that $x(t)$ is an eventually positive solution of equation (1.1). In view of (H3) and (H4), there exists $t_1 \geq t_0$ such that $x(\tau(t)) > 0$, $x(\sigma(t)) > 0$, $x(\lambda(t)) > 0$ for all $t \geq t_1$, then $z(t) = x(t) + p(t)x(\tau(t)) + x(\lambda(t)) \geq x(t) > 0$, for all $t \geq t_1$. From Eq. (1.1) we have

$$\bigl(r(t)\bigl(z'(t)\bigr)^{\alpha}\bigr)' = -q(t)x^{\beta}\bigl(\sigma(t)\bigr) \leq 0, \quad t \geq t_1,$$

which means that $r(t)(z'(t))^{\alpha}$ is nonincreasing for all $t \geq t_1$. Then $r(t)(z'(t))^{\alpha} > 0$ or $r(t)(z'(t))^{\alpha} < 0$, and the proof is complete. □

Lemma 2.2 *If $x(t)$ is a positive solution of equation (1.1) satisfying Case 1 of Lemma 2.1, then*

$$z(t) \geq R(t)r^{\frac{1}{\alpha}}(t)z'(t) \tag{2.1}$$

and $\frac{z(t)}{R(t)}$ is nonincreasing for all $t \geq t_1$. Furthermore,

$$x(t) \geq Q_1(t)z(t) \tag{2.2}$$

on $t \in [t_1, \infty)$.

Proof From Case 1, $z(t) > 0$, $z'(t) > 0$. Combining condition (1.2), we see that

$$z(t) = z(t_1) + \int_{t_1}^{t} z'(s)\,ds \geq \int_{t_1}^{t} \frac{r^{\frac{1}{\alpha}}(s)z'(s)}{r^{\frac{1}{\alpha}}(s)}\,ds \geq R(t)r^{\frac{1}{\alpha}}(t)z'(t)$$

and

$$\left(\frac{z(t)}{R(t)}\right)' = \frac{z'(t)R(t) - R'(t)z(t)}{R^2(t)} = -\frac{z(t) - R(t)r^{\frac{1}{\alpha}}(t)z'(t)}{r^{\frac{1}{\alpha}}(t)R^2(t)} \le 0.$$

Using the monotonicity of $z(t)$ and $\frac{z(t)}{R(t)}$, we have

$$\begin{aligned} x(t) &= z(t) - p_1(t)x(\tau(t)) - p_2(t)x(\lambda(t)) \ge z(t) - p_1(t)z(\tau(t)) - p_2(t)z(\lambda(t)) \\ &\ge \left(1 - p_1(t) - p_2(t)\frac{R(\lambda(t))}{R(t)}\right)z(t) = Q_1(t)z(t). \end{aligned}$$

□

Lemma 2.3 *If $x(t)$ is a positive solution of Eq.* (1.1) *satisfying Case* 2 *of Lemma* 2.1, *then*

$$z(t) \ge -\pi(t)r^{\frac{1}{\alpha}}(t)z'(t), \tag{2.3}$$

and $\frac{z(t)}{\pi(t)}$ is nondecreasing for all $t \ge t_1$. Furthermore,

$$x(t) \ge Q_2(t)z(t) \tag{2.4}$$

on $t \in [t_1, \infty)$.

Proof From Case 2, $z(t) > 0$, $z'(t) < 0$. Using condition (1.3), we have

$$z(l) = z(t) + \int_t^l z'(s)\,ds = z(t) + \int_t^l \frac{r^{\frac{1}{\alpha}}(s)z'(s)}{r^{\frac{1}{\alpha}}(s)}\,ds \le z(t) + r^{\frac{1}{\alpha}}(t)z'(t)\int_t^l r^{-\frac{1}{\alpha}}(s)\,ds.$$

Letting $l \to \infty$, we get

$$0 \le z(t) + \pi(t)r^{\frac{1}{\alpha}}(t)z'(t).$$

Then

$$z(t) \ge -\pi(t)r^{\frac{1}{\alpha}}(t)z'(t),$$

hence

$$\left(\frac{z(t)}{\pi(t)}\right)' = \frac{z'(t)\pi(t) - \pi'(t)z(t)}{\pi^2(t)} = \frac{z(t) + \pi(t)r^{\frac{1}{\alpha}}(t)z'(t)}{r^{\frac{1}{\alpha}}(t)\pi^2(t)} \ge 0.$$

Using the monotonicity of $z(t)$ and $\frac{z(t)}{\pi(t)}$, we have

$$\begin{aligned} x(t) &= z(t) - p_1(t)x(\tau(t)) - p_2(t)x(\lambda(t)) \ge z(t) - p_1(t)z(\tau(t)) - p_2(t)z(\lambda(t)) \\ &\ge \left(1 - p_1(t)\frac{\pi(\tau(t))}{\pi(t)} - p_2(t)\right)z(t) = Q_2(t)z(t). \end{aligned} \tag{2.5}$$

□

Lemma 2.4 *Assume that* (1.3) *holds and*

$$\int_{t_0}^{\infty} q(s)Q_1^{\beta}\bigl(\sigma(s)\bigr)\,ds = \infty. \tag{2.6}$$

Suppose that $x(t)$ *is a positive solution of Eq.* (1.1) *on* $[t_1,\infty)$, *where* $t_1 \in [t_0,\infty)$ *is sufficiently large, then Case* 2 *of Lemma* 2.1 *holds.*

Proof Suppose that $x(t)$ is a positive solution of equation (1.1) on $t \in [t_1,\infty)$. From Lemma 2.1, we have Case 1 and Case 2. If Case 1 holds, then there exists $t_2 \geq t_1$ such that $z'(t) > 0$ on $[t_2,\infty)$. Combining (1.1) and equation (2.2), we get

$$\bigl(r(t)\bigl(z'(t)\bigr)^{\alpha}\bigr)' \leq -q(t)Q_1^{\beta}\bigl(\sigma(t)\bigr)z^{\beta}\bigl(\sigma(t)\bigr), \quad t \geq t_1. \tag{2.7}$$

Define the function w by

$$w(t) := \frac{r(t)(z'(t))^{\alpha}}{z^{\beta}(\sigma(t))} > 0, \quad t \geq t_1.$$

Differentiating the above formula, we have

$$w'(t) \leq -q(t)Q_1^{\beta}\bigl(\sigma(t)\bigr) - \frac{\beta w(t)z'(\sigma(t))\sigma'(t)}{z(\sigma(t))} \leq -q(t)Q_1^{\beta}\bigl(\sigma(t)\bigr). \tag{2.8}$$

Integrating both sides of (2.8) from t_2 to t and using (2.6), we obtain

$$w(t) \leq w(t_2) - \int_{t_2}^{t} q(s)Q_1^{\beta}\bigl(\sigma(s)\bigr)\,ds \to -\infty, \quad \text{as } t \to \infty,$$

which contradicts the fact $w(t) > 0$. Thus, Case 1 is impossible and z satisfies Case 2 for $t \geq t_1$. The proof is complete. □

Theorem 2.5 *Let* $\alpha \geq \beta$. *Assume that* (1.3), (2.6) *and*

$$\int_{t_0}^{\infty} \left(\frac{1}{r(t)}\int_{t_0}^{t} q(s)Q_2^{\beta}\bigl(\sigma(s)\bigr)\,ds\right)^{1/\alpha} dt = \infty, \tag{2.9}$$

hold. Suppose that Eq. (1.1) *has a positive solution* $x(t)$ *on* $[t_1,\infty)$. *Then* $z(t)$ *satisfies Case* 2 *on* $[t_1,\infty)$ *and*

$$\lim_{t\to\infty} x(t) = 0. \tag{2.10}$$

Moreover, there exist positive constants C_1 *and* C_2 *and a real number* $t_* \in [t_1,\infty)$ *such that*

$$C_1Q_2(t)\pi(t) \leq x(t) \leq C_2\exp\left(-\int_{t_0}^{t} \frac{\pi(\sigma(s))(\int_{t_0}^{s} q(u)Q_2^{\beta}(\sigma(u))\,du)^{\frac{1}{\alpha}}}{\pi(s)r^{1/\alpha}(s)}\,ds\right) \tag{2.11}$$

on $t \in [t_*,\infty)$.

Proof Suppose that $x(t)$ is a positive solution of Eq. (1.1) on $[t_1,\infty)$. From Lemma 2.4, we see that $z(t)$ satisfies Case 2 for $t \geq t_1$.

Since $z(t)$ is nonincreasing and $z(t) > 0$, there exists a constant $c \geq 0$ such that $\lim_{t\to\infty} x(t) = c \geq 0$. We now claim that $c = 0$. If not, assume that $c > 0$, combining (1.1), we have

$$-\bigl(r(t)\bigl(z'(t)\bigr)^{\alpha}\bigr)' = q(t)x^{\beta}\bigl(\sigma(t)\bigr), \quad t \geq t_0.$$

Integrating the above inequality from t_1 to t, we get

$$r(t)\bigl(z'(t)\bigr)^{\alpha} - r(t_1)\bigl(z'(t_1)\bigr)^{\alpha} = -\int_{t_1}^{t} q(s)x^{\beta}\bigl(\sigma(s)\bigr)\,ds, \quad t \geq t_1,$$

which implies that

$$r(t)\bigl(z'(t)\bigr)^{\alpha} \leq -\int_{t_1}^{t} q(s)x^{\beta}\bigl(\sigma(s)\bigr)\,ds, \quad t \geq t_1,$$

then

$$z'(t) \leq -\left(\frac{1}{r(t)}\int_{t_1}^{t} q(s)x^{\beta}\bigl(\sigma(s)\bigr)\,ds\right)^{\frac{1}{\alpha}}, \quad t \geq t_1. \tag{2.12}$$

Integrating (2.12) from t_1 to t, we obtain

$$z(t) - z(t_1) \leq -\int_{t_1}^{t}\left(\frac{1}{r(s)}\int_{t_1}^{s} q(u)x^{\beta}\bigl(\sigma(u)\bigr)\,du\right)^{\frac{1}{\alpha}} ds, \quad t \geq t_1.$$

From Lemma 2.3, we have

$$\begin{aligned} z(t) - z(t_1) &\leq -\int_{t_1}^{t}\left(\frac{1}{r(s)}\int_{t_1}^{s} q(u)Q_2^{\beta}\bigl(\sigma(u)\bigr)z^{\beta}\bigl(\sigma(u)\bigr)\,du\right)^{\frac{1}{\alpha}} ds \\ &\leq -z^{\frac{\beta}{\alpha}}\bigl(\sigma(t)\bigr)\int_{t_1}^{t}\left(\frac{1}{r(s)}\int_{t_1}^{s} q(u)Q_2^{\beta}\bigl(\sigma(u)\bigr)\,du\right)^{\frac{1}{\alpha}} ds \\ &\leq -x^{\frac{\beta}{\alpha}}\bigl(\sigma(t)\bigr)\int_{t_1}^{t}\left(\frac{1}{r(s)}\int_{t_1}^{s} q(u)Q_2^{\beta}\bigl(\sigma(u)\bigr)\,du\right)^{\frac{1}{\alpha}} ds. \end{aligned} \tag{2.13}$$

Letting $t \to \infty$ in the above inequality, we see that $z(t) \to -\infty$ as $t \to \infty$ which is a contradiction. Hence, $c = 0$.

Next, we prove that inequality (2.11) holds. From Lemma 2.3, we see that $\frac{z(t)}{\pi(t)}$ is nondecreasing for all $t \geq t_1$. Thus, there exist $C_1 > 0$ and $t_2 > t_1$ such that

$$z(t) \geq C_1\pi(t), \quad t \geq t_2. \tag{2.14}$$

Using (2.4), we get

$$x(t) \geq Q_2(t)z(t) \geq C_1Q_2(t)\pi(t).$$

Integrating (1.1) from t_2 to t, we have

$$\begin{aligned}
&-r(t)\big(z'(t)\big)^{\alpha} \\
&\quad= -r(t_2)\big(z'(t_2)\big)^{\alpha} + \int_{t_2}^{t} q(s)x^{\beta}\big(\sigma(s)\big)\,ds \\
&\quad\geq -r(t_2)\big(z'(t_2)\big)^{\alpha} + z^{\beta}\big(\sigma(t)\big)\int_{t_2}^{t} q(s)Q_2^{\beta}\big(\sigma(s)\big)\,ds \\
&\quad\geq -r(t_2)\big(z'(t_2)\big)^{\alpha} + z^{\beta}\big(\sigma(t)\big)\int_{t_0}^{t} q(s)Q_2^{\beta}\big(\sigma(s)\big)\,ds - z^{\beta}\big(\sigma(t)\big)\int_{t_0}^{t_2} q(s)Q_2^{\beta}\big(\sigma(s)\big)\,ds.
\end{aligned}$$

In view of (2.10), there exists $t_3 > t_2$ such that

$$-r(t_3)\big(z'(t_3)\big)^{\alpha} - z^{\beta}\big(\sigma(t)\big)\int_{t_0}^{t_3} q(s)Q_2^{\beta}\big(\sigma(s)\big)\,ds > 0, \quad t \geq t_3.$$

Therefore,

$$-r(t)\big(z'(t)\big)^{\alpha} \geq z^{\beta}\big(\sigma(t)\big)\int_{t_0}^{t} q(s)Q_2^{\beta}\big(\sigma(s)\big)\,ds, \quad t \geq t_3.$$

Using Lemma 2.3 in the above inequality, we find

$$\begin{aligned}
-r(t)\big(z'(t)\big)^{\alpha} &\geq \frac{z^{\beta-\alpha}(\sigma(t))z^{\alpha}(\sigma(t))}{\pi^{\alpha}(\sigma(t))}\pi^{\alpha}\big(\sigma(t)\big)\int_{t_0}^{t} q(s)Q_2^{\beta}\big(\sigma(s)\big)\,ds \\
&\geq m_1\frac{z^{\alpha}(t)}{\pi^{\alpha}(t)}\pi^{\alpha}\big(\sigma(t)\big)\int_{t_0}^{t} q(s)Q_2^{\beta}\big(\sigma(s)\big)\,ds,
\end{aligned}$$

where $m_1 > 0$ is a constant and $z^{\beta-\alpha}(\sigma(t)) \geq m_1$ for $t \geq t_3$, which implies that

$$\frac{z'(t)}{z(t)} \leq -\frac{m_1\pi(\sigma(t))}{\pi(t)r^{1/\alpha}(t)}\left(\int_{t_0}^{t} q(s)Q_2^{\beta}\big(\sigma(s)\big)\,ds\right)^{\frac{1}{\alpha}}. \tag{2.15}$$

Integrating (2.15) from t_3 to t, we get

$$\begin{aligned}
x(t) \leq z(t) \leq z(t_3)\exp&\left(-\int_{t_3}^{t}\frac{m_1\pi(\sigma(u))}{\pi(u)r^{1/\alpha}(u)}\left(\int_{t_0}^{u} q(s)Q_2^{\beta}\big(\sigma(s)\big)\,ds\right)^{\frac{1}{\alpha}}du\right) \\
= C_2\exp&\left(-\int_{t_0}^{t}\frac{m_1\pi(\sigma(u))}{\pi(u)r^{1/\alpha}(u)}\left(\int_{t_0}^{u} q(s)Q_2^{\beta}\big(\sigma(s)\big)\,ds\right)^{\frac{1}{\alpha}}du\right),
\end{aligned}$$

where

$$C_2 := z(t_3)\exp\left(-\int_{t_0}^{t_3}\frac{m_1\pi(\sigma(u))}{\pi(u)r^{1/\alpha}(u)}\left(\int_{t_0}^{u} q(s)Q_2^{\beta}\big(\sigma(s)\big)\,ds\right)^{\frac{1}{\alpha}}du\right) > 0.$$

The proof is complete. □

Theorem 2.6 *Assume that* (1.3) *and* (2.6) *hold. If*

$$\int_{t_0}^{\infty}\left(\frac{1}{r(t)}\int_{t_0}^{t} q(s)Q_2^{\beta}\big(\sigma(s)\big)\pi^{\beta}\big(\sigma(s)\big)\,ds\right)^{\frac{1}{\alpha}}dt=\infty, \tag{2.16}$$

then Eq. (1.1) *is oscillatory.*

Proof Suppose that $x(t)$ is a positive solution of equation (1.1) on $[t_1,\infty)$. From Lemma 2.4, we see that z satisfies Case 2 for $t\geq t_1$.

From (1.1), (2.4) and (2.14), we obtain

$$\begin{aligned}-\big(r(t)\big(z'(t)\big)^{\alpha}\big)' &= q(t)x^{\beta}\big(\sigma(t)\big)\\ &\geq q(t)Q_2^{\beta}\big(\sigma(t)\big)z^{\beta}\big(\sigma(t)\big)\\ &\geq C_1^{\beta}q(t)Q_2^{\beta}\big(\sigma(t)\big)\pi^{\beta}\big(\sigma(t)\big),\quad t\geq t_2\geq t_1.\end{aligned} \tag{2.17}$$

Integrating (2.17) from t_2 to t, we have

$$-r(t)\big(z'(t)\big)^{\alpha}\geq C_1^{\beta}\int_{t_2}^{t} q(s)Q_2^{\beta}\big(\sigma(s)\big)\pi^{\beta}\big(\sigma(s)\big)\,ds, \tag{2.18}$$

that is,

$$-z'(t)\geq\frac{C_1^{\beta/\alpha}}{r^{1/\alpha}(t)}\left(\int_{t_2}^{t} q(s)Q_2^{\beta}\big(\sigma(s)\big)\pi^{\beta}\big(\sigma(s)\big)\,ds\right)^{\frac{1}{\alpha}}.$$

Integrating the above inequality from t_2 to t, we get

$$z(t)\leq z(t_2)-\int_{t_2}^{t}\frac{C_1^{\beta/\alpha}}{r^{1/\alpha}(u)}\left(\int_{t_2}^{u} q(s)Q_2^{\beta}\big(\sigma(s)\big)\pi^{\beta}\big(\sigma(s)\big)\,ds\right)^{\frac{1}{\alpha}}du\to-\infty,$$

which contradicts the condition (2.16). The proof is complete. □

Theorem 2.7 *Let $\alpha\leq\beta$. Assume that* (1.3), (2.6) *and*

$$\int_{t_1}^{t} q(s)Q_2^{\beta}\big(\sigma(s)\big)\pi^{\beta}\big(\sigma(s)\big)\,ds=\infty \tag{2.19}$$

hold. If

$$\limsup_{t\to\infty}\pi^{\beta}\big(\sigma(t)\big)\int_{t_1}^{t} q(s)Q_2^{\beta}\big(\sigma(s)\big)\,ds>1\quad\text{when }\alpha=\beta \tag{2.20}$$

and

$$\limsup_{t\to\infty}\pi^{\beta}\big(\sigma(t)\big)\int_{t_1}^{t} q(s)Q_2^{\beta}\big(\sigma(s)\big)\,ds>0\quad\text{when }\alpha<\beta, \tag{2.21}$$

then Eq. (1.1) *is oscillatory.*

Proof Suppose that $x(t)$ is a positive solution of equation (1.1) on $[t_1,\infty)$. From Lemma 2.4, we see that $z(t)$ satisfies Case 2 for $t \ge t_1$. Combining (2.18) and (2.19), we have

$$\lim_{t\to\infty}\left(-r(t)\left(z'(t)\right)^{\alpha}\right) = \infty. \tag{2.22}$$

Integrating (1.1) from t_1 to t and using (2.4) and the fact that $z(t)$ is nonincreasing, we get

$$\begin{aligned} -r(t)\left(z'(t)\right)^{\alpha} &= -r(t_1)\left(z'(t_1)\right)^{\alpha} + \int_{t_1}^{t} q(s)x^{\beta}\left(\sigma(s)\right)ds \\ &\ge z^{\beta}\left(\sigma(t)\right)\int_{t_1}^{t} q(s)Q_2^{\beta}\left(\sigma(s)\right)ds. \end{aligned} \tag{2.23}$$

Noting (2.3) and $\sigma(t) \ge t$, we obtain

$$\begin{aligned} W(t) := -r(t)\left(z'(t)\right)^{\alpha} &\ge -r^{\frac{\beta}{\alpha}}\left(\sigma(t)\right)\left(z'\left(\sigma(t)\right)\right)^{\beta}\pi^{\beta}\left(\sigma(t)\right)\int_{t_1}^{t} q(s)Q_2^{\beta}\left(\sigma(s)\right)ds \\ &= \left(-r\left(\sigma(t)\right)\left(z'\left(\sigma(t)\right)\right)^{\alpha}\right)^{\frac{\beta}{\alpha}}\pi^{\beta}\left(\sigma(t)\right)\int_{t_1}^{t} q(s)Q_2^{\beta}\left(\sigma(s)\right)ds \\ &\ge \left(-r(t)\left(z'(t)\right)^{\alpha}\right)^{\frac{\beta}{\alpha}}\pi^{\beta}\left(\sigma(t)\right)\int_{t_1}^{t} q(s)Q_2^{\beta}\left(\sigma(s)\right)ds \\ &= W^{\frac{\beta}{\alpha}}(t)\pi^{\beta}\left(\sigma(t)\right)\int_{t_1}^{t} q(s)Q_2^{\beta}\left(\sigma(s)\right)ds. \end{aligned}$$

Hence,

$$W^{1-\frac{\beta}{\alpha}}(t) \ge \pi^{\beta}\left(\sigma(t)\right)\int_{t_1}^{t} q(s)Q_2^{\beta}\left(\sigma(s)\right)ds.$$

Taking lim sup of both sides of the above inequality as $t \to \infty$, we arrive at a contradiction to (2.20) when $\alpha = \beta$ and (2.21) when $\alpha < \beta$. The proof is complete. □

By attaching a condition, the dependence on the initial constant t_1 can be easily eliminated.

Corollary 2.8 *Let $\alpha \le \beta$. Assume that* (1.3), (2.6), (2.9) *and* (2.19) *hold. If*

$$\limsup_{t\to\infty}\pi^{\beta}\left(\sigma(t)\right)\int_{t_0}^{t} q(s)Q_2^{\beta}\left(\sigma(s)\right)ds > 1 \quad \text{when } \alpha = \beta$$

and

$$\limsup_{t\to\infty}\pi^{\beta}\left(\sigma(t)\right)\int_{t_0}^{t} q(s)Q_2^{\beta}\left(\sigma(s)\right)ds > 0 \quad \text{when } \alpha < \beta,$$

then Eq. (1.1) *is oscillatory.*

Proof As in the proof of Theorem 2.7, we conclude that (2.23) holds. In view of (2.10), then there exists $t_2 > t_1$ such that

$$-r(t)\left(z'(t)\right)^{\alpha} - z^{\beta}\left(\sigma(t)\right)\int_{t_0}^{t} q(s)Q_2^{\beta}\left(\sigma(s)\right)ds > 0.$$

It is clear that

$$-r(t)\big(z'(t)\big)^{\alpha} \geq -r(t_1)\big(z'(t_1)\big)^{\alpha} + \int_{t_0}^{t} q(s)x^{\beta}\big(\sigma(s)\big)\,ds - \int_{t_0}^{t_1} q(s)x^{\beta}\big(\sigma(s)\big)\,ds$$
$$\geq z^{\beta}\big(\sigma(t)\big)\int_{t_0}^{t} q(s)Q_2^{\beta}\big(\sigma(s)\big)\,ds, \quad t > t_1. \tag{2.24}$$

The rest of the proof is similar to that of Theorem 2.7 and hence we omit it. □

In order to prove a main theorem of this paper, we review an auxiliary result obtained by Wu et al. [22, Lemma 2.3].

Lemma 2.9 *Let $\varphi(u) = Au - B(u-C)^{(\alpha+1)/\alpha}$ where $\alpha > 0$ is a quotient of two odd positive integers, A and $C \in \mathbb{R}$, and $B > 0$. Then $\varphi(u)$ attains its maximum value on $u^* = C + (\frac{A\alpha}{B(\alpha+1)})^{\alpha}$, and*

$$\max_{u\in\mathbb{R}} \varphi(u) = \varphi\big(u^*\big) = AC + \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{A^{\alpha+1}}{B^{\alpha}}. \tag{2.25}$$

The proof of the above lemma is simple and can be obtained directly by the change of the variable. We omit it.

Theorem 2.10 *Let $\alpha > \beta$. Assume that (1.3) and (2.6) hold. If there exists a function $\rho \in C^1([t_0,\infty),(0,\infty))$ such that*

$$\limsup_{t\to\infty}\left(\frac{\pi^{\alpha}(t)}{\rho(t)}\int_{t_2}^{t} \rho(s)q(s)Q_2^{\beta}\big(\sigma(s)\big)\left(\frac{\pi(\sigma(s))}{\pi(s)}\right)^{\beta} - \frac{\alpha^{\alpha}C_4^{\alpha}(\rho'(s))^{\alpha+1}r(s)}{(\alpha+1)^{\alpha+1}\beta^{\alpha}\rho^{\alpha}(s)}\,ds\right) > 1, \tag{2.26}$$

for any positive constants $C_4 > 0$ and $t_2 \geq t_0$, then Eq. (1.1) is oscillatory.

Proof Suppose that $x(t)$ is a positive solution of equation (1.1) on $[t_1,\infty)$. From Lemma 2.4, we see that $z(t)$ satisfies Case 2 for $t \geq t_1$. Define the generalized Riccati substitution $w(t)$ by

$$w(t) := \rho(t)\left(\frac{r(t)(z'(t))^{\alpha}}{z^{\beta}(t)} + \frac{1}{\pi^{\alpha}(t)}\right), \quad t \geq t_1. \tag{2.27}$$

By virtue of (2.3), we have $w(t) \geq 0$ for $t \geq t_1$. Differentiating on both sides of (2.27), we obtain

$$w'(t) = \frac{\rho'(t)}{\rho(t)}w(t) + \rho(t)\frac{(r(t)(z'(t))^{\alpha})'}{z^{\beta}(t)} - \frac{\beta\rho(t)r(t)(z'(t))^{\alpha}z'(t)}{z^{\beta+1}(t)} + \frac{\alpha\rho(t)}{r^{1/\alpha}(t)\pi^{\alpha}(t)}$$
$$\leq \frac{\rho'(t)}{\rho(t)}w(t) + \rho(t)\frac{(r(t)(z'(t))^{\alpha})'}{z^{\beta}(t)} - \frac{\beta(w(t) - \frac{\rho(t)}{\pi^{\alpha}(t)})^{\frac{\alpha+1}{\alpha}}}{C_4(\rho(t)(r(t))^{1/\alpha}} + \frac{\alpha\rho(t)}{r^{1/\alpha}(t)\pi^{\alpha}(t)}, \tag{2.28}$$

where $C_4 > 0$ is a constant and such that $z^{1-\frac{\beta}{\alpha}}(t) \leq C_4$ for $t \geq t_1$.

Combining (1.1), $\sigma(t) \geq t$ and Lemma 2.3, we have

$$\bigl(r(t)\bigl(z'(t)\bigr)^{\alpha}\bigr)' \leq -q(t)Q_2^{\beta}\bigl(\sigma(t)\bigr)z^{\beta}\bigl(\sigma(t)\bigr) \leq -q(t)Q_2^{\beta}\bigl(\sigma(t)\bigr)\left(\frac{\pi(\sigma(t))}{\pi(t)}\right)^{\beta} z^{\beta}\bigl(\sigma(t)\bigr) \tag{2.29}$$

for $t > t_2$, where $t_2 \in [t_1, \infty)$ is large enough. Substituting (2.29) into (2.28), it follows that

$$\begin{aligned} w'(t) &\leq -\rho(t)q(t)Q_2^{\beta}\bigl(\sigma(t)\bigr)\left(\frac{\pi(\sigma(t))}{\pi(t)}\right)^{\beta} + \frac{\rho'(t)}{\rho(t)}w(t) \\ &\quad - \frac{\beta}{C_4(\rho(t)(r(t))^{1/\alpha}}\left(w(t) - \frac{\rho(t)}{\pi^{\alpha}(t)}\right)^{\frac{\alpha+1}{\alpha}} + \frac{\alpha\rho(t)}{r^{1/\alpha}(t)\pi^{\alpha}(t)}. \end{aligned} \tag{2.30}$$

Using (2.25) with

$$A := \frac{\rho'(t)}{\rho(t)}, \qquad B := \frac{\beta}{C_4(\rho(t)(r(t))^{1/\alpha}}, \qquad C := \frac{\rho(t)}{\pi^{\alpha}(t)},$$

we obtain

$$\begin{aligned} w'(t) &\leq -\rho(t)q(t)Q_2^{\beta}\bigl(\sigma(t)\bigr)\left(\frac{\pi(\sigma(t))}{\pi(t)}\right)^{\beta} + \frac{\rho'(t)}{\pi^{\alpha}(t)} \\ &\quad + \frac{\alpha^{\alpha}C_4^{\alpha}(\rho'(t))^{\alpha+1}r(t)}{(\alpha+1)^{\alpha+1}\beta^{\alpha}\rho^{\alpha}(t)} + \frac{\alpha\rho(t)}{r^{1/\alpha}(t)\pi^{\alpha}(t)} \\ &\leq -\rho(t)q(t)Q_2^{\beta}\bigl(\sigma(t)\bigr)\left(\frac{\pi(\sigma(t))}{\pi(t)}\right)^{\beta} + \left(\frac{\rho(t)}{\pi^{\alpha}(t)}\right)'. \end{aligned} \tag{2.31}$$

Integrating (2.31) from t_2 to t, we have

$$\begin{aligned} &\int_{t_2}^{t} \rho(s)q(s)Q_2^{\beta}\bigl(\sigma(s)\bigr)\left(\frac{\pi(\sigma(s))}{\pi(s)}\right)^{\beta} - \frac{\alpha^{\alpha}C_4^{\alpha}(\rho'(s))^{\alpha+1}r(s)}{(\alpha+1)^{\alpha+1}\beta^{\alpha}\rho^{\alpha}(s)}\,ds - \frac{\rho(t)}{\pi^{\alpha}(t)} + \frac{\rho(t_2)}{\pi^{\alpha}(t_2)} \\ &\quad \leq w(t_2) - w(t). \end{aligned}$$

In view of (2.27), we see that

$$\begin{aligned} &\int_{t_2}^{t} \rho(s)q(s)Q_2^{\beta}\bigl(\sigma(s)\bigr)\left(\frac{\pi(\sigma(s))}{\pi(s)}\right)^{\beta} - \frac{\alpha^{\alpha}C_4^{\alpha}(\rho'(s))^{\alpha+1}r(s)}{(\alpha+1)^{\alpha+1}\beta^{\alpha}\rho^{\alpha}(s)}\,ds \\ &\quad \leq \rho(t_2)\frac{r(t_2)(z'(t_2))^{\alpha}}{z^{\beta}(t_2)} - \rho(t)\frac{r(t)(z'(t))^{\alpha}}{z^{\beta}(t)}. \end{aligned} \tag{2.32}$$

On the other hand, from (2.3), we have

$$-\frac{\rho(t)}{\pi^{\alpha}(t)} \leq \rho(t)\frac{r(t)(z'(t))^{\alpha}}{z^{\beta}(t)} \leq 0. \tag{2.33}$$

Substituting (2.33) into (2.32), we obtain

$$\int_{t_2}^{t} q(s)Q_2^{\beta}\bigl(\sigma(s)\bigr)\left(\frac{\pi(\sigma(s))}{\pi(s)}\right)^{\beta} - \frac{\alpha^{\alpha}C_4^{\alpha}(\rho'(s))^{\alpha+1}r(s)}{(\alpha+1)^{\alpha+1}\beta^{\alpha}\rho^{\alpha}(s)}\,ds \leq \frac{\rho(t)}{\pi^{\alpha}(t)}. \tag{2.34}$$

Multiplying both sides of (2.34) by $\frac{\pi^{\alpha}(t)}{\rho(t)}$ and taking lim sup on both sides of the resulting inequality as $t \to \infty$, we obtain a contradiction to (2.26). The proof is complete. □

Since $\rho(t)$ can be taken appropriately, Theorem 2.10 is more flexible in studying the oscillation of (1.1). When $\rho(t) = \pi^{\alpha}(t)$, $\rho(t) = \pi^{\beta}(t)$, $\rho(t) = 1$, respectively, the following results are obtained.

Corollary 2.11 *Assume that $\alpha > \beta$, (1.3) and (2.6) hold. If*

$$\limsup_{t\to\infty}\left(\int_{t_2}^{t} q(s)Q_2^{\beta}\big(\sigma(s)\big)\pi^{\beta}\big(\sigma(s)\big)\pi^{\alpha-\beta}(s) - \frac{C_4^{\alpha}\alpha^{2\alpha+1}}{\beta^{\alpha}(\alpha+1)^{\alpha+1}r^{1/\alpha}(s)\pi^{\alpha}(s)}\,ds\right) > 1,$$

for any positive constants $C_4 > 0$ and $t_2 \ge t_0$, then Eq. (1.1) is oscillatory.

Corollary 2.12 *Assume that $\alpha > \beta$, (1.3) and (2.6) hold. If*

$$\limsup_{t\to\infty}\left(\pi^{\alpha-\beta}(t)\int_{t_2}^{t} q(s)Q_2^{\beta}\big(\sigma(s)\big)\pi^{\beta}\big(\sigma(s)\big) - \frac{C_4^{\alpha}\alpha^{\alpha}\beta}{(\alpha+1)^{\alpha+1}r^{1/\alpha}(s)\pi^{\alpha-\beta+1}(s)}\,ds\right) > 1,$$

for any positive constants $C_4 > 0$ and $t_2 \ge t_0$, then Eq. (1.1) is oscillatory.

Corollary 2.13 *Assume that $\alpha \ge \beta$, (1.3) and (2.6) hold. If*

$$\limsup_{t\to\infty}\left(\pi^{\alpha}(t)\int_{t_2}^{t} q(s)Q_2^{\beta}\big(\sigma(s)\big)\left(\frac{\pi(\sigma(s))}{\pi(s)}\right)^{\beta} ds\right) > 1,$$

for any $t_2 \ge t_0$, then Eq. (1.1) is oscillatory.

Remark When $\alpha = \beta$, we can choose $C_4 = 1$ in Theorem 2.10, Corollary 2.11, Corollary 2.12, respectively.

Lemma 2.14 *Let $\alpha \le \beta$. Assume that (1.3) and (2.6) hold. Suppose that equation (1.1) has a positive solution $x(t)$ on $[t_1, \infty)$ and that γ and δ are constants satisfying*

$$0 \le \gamma + \delta < 1, \tag{2.35}$$

$$0 \le \gamma \le Lq(t)Q_2^{\beta}\big(\sigma(t)\big)\pi^{\beta}\big(\sigma(t)\big)\pi(t)r^{\frac{1}{\alpha}}(t), \tag{2.36}$$

where $L > 0$ is a constant and such that $(r(t)(z'(t))^{\alpha})^{\frac{\beta-\alpha}{\alpha}} \ge L$ for $t \ge t_1$, and

$$0 \le \delta \le m_1\pi\big(\sigma(t)\big)\left(\int_{t_1}^{t} q(s)Q_1^{\beta}\big(\sigma(s)\big)\,ds\right)^{\frac{1}{\alpha}}, \tag{2.37}$$

where $m_1 > 0$ is a constant and such that $z^{\beta-\alpha}(\sigma(t)) \ge m_1$. Then there exists $t_* \in [t_1, \infty)$ such that

$$\frac{z}{\pi^{1-\gamma}}$$

is nondecreasing and

$$\frac{z}{\pi^{\delta}}$$

is nonincreasing on $[t_*, \infty)$.

Proof From Lemma 2.4, we see that $z(t)$ satisfies Case 2. Using (1.1), (2.3), (2.4) and (2.36), we obtain

$$\begin{aligned}
&\bigl(-r(t)\bigl(z'(t)\bigr)^{\alpha}\pi^{\gamma}(t)\bigr)' \\
&\quad= -\bigl(r(t)\bigl(z'(t)\bigr)^{\alpha}\bigr)'\pi^{\gamma}(t) + \gamma r(t)\bigl(z'(t)\bigr)^{\alpha}\frac{\pi^{\gamma-1}(t)}{r^{1/\alpha}(t)} \\
&\quad= q(t)x^{\beta}\bigl(\sigma(t)\bigr)\pi^{\gamma}(t) + \gamma r(t)\bigl(z'(t)\bigr)^{\alpha}\frac{\pi^{\gamma-1}(t)}{r^{1/\alpha}(t)} \\
&\quad\geq -q(t)Q_2^{\beta}\bigl(\sigma(t)\bigr)\pi^{\gamma}(t)r^{\beta/\alpha}\bigl(\sigma(t)\bigr)\bigl(z'\bigl(\sigma(t)\bigr)\bigr)^{\beta}\pi^{\beta}\bigl(\sigma(t)\bigr) \\
&\qquad+ \gamma r(t)\bigl(z'(t)\bigr)^{\alpha}\frac{\pi^{\gamma-1}(t)}{r^{1/\alpha}(t)} \\
&\quad\geq -q(t)Q_2^{\beta}\bigl(\sigma(t)\bigr)\pi^{\gamma}(t)r(t)\bigl(z'(t)\bigr)^{\alpha}\bigl(r\bigl(\sigma(t)\bigr)\bigl(z'\bigl(\sigma(t)\bigr)\bigr)^{\alpha}\bigr)^{\frac{\beta-\alpha}{\alpha}}\pi^{\beta}\bigl(\sigma(t)\bigr) \\
&\qquad+ \gamma r(t)\bigl(z'(t)\bigr)^{\alpha}\frac{\pi^{\gamma-1}(t)}{r^{1/\alpha}(t)} \\
&\quad\geq -q(t)Q_2^{\beta}\bigl(\sigma(t)\bigr)\pi^{\gamma}(t)r(t)\bigl(z'(t)\bigr)^{\alpha}L\pi^{\beta}\bigl(\sigma(t)\bigr) \\
&\qquad+ \gamma r(t)\bigl(z'(t)\bigr)^{\alpha}\frac{\pi^{\gamma-1}(t)}{r^{1/\alpha}(t)} \\
&\quad= -r(t)\bigl(z'(t)\bigr)^{\alpha}\pi^{\gamma}(t)\biggl[Lq(t)Q_2^{\beta}\bigl(\sigma(t)\bigr)\pi^{\beta}\bigl(\sigma(t)\bigr) - \frac{\gamma}{\pi(t)r^{1/\alpha}(t)}\biggr] \geq 0,
\end{aligned} \tag{2.38}$$

where $L > 0$ is a constant and such that $(r(t)(z'(t))^{\alpha})^{\frac{\beta-\alpha}{\alpha}} \geq L$ for $t \geq t_1$. Thus, $-r(t)(z'(t))^{\alpha} \times \pi^{\gamma}(t)$ is nondecreasing eventually, that is, there exists a $t_2 \in [t_1, \infty)$ such that $-r(t)(z'(t))^{\alpha} \times \pi^{\gamma}(t)$ is nondecreasing for $t \geq t_2$. So, we have

$$\begin{aligned}
z(t) &\geq -\int_t^{\infty}\frac{r^{1/\alpha}(s)\pi^{\gamma}(s)}{r^{1/\alpha}(s)\pi^{\gamma}(s)}z'(s)\,ds \\
&\geq -r^{1/\alpha}(t)z'(t)\pi^{\gamma}(t)\int_t^{\infty}\frac{1}{r^{1/\alpha}(s)\pi^{\gamma}(s)}\,ds.
\end{aligned} \tag{2.39}$$

In view of

$$\int_t^{\infty}\frac{1}{r^{1/\alpha}(s)\pi^{\gamma}(s)}\,ds = \frac{\pi^{1-\gamma}(t)}{1-\gamma}, \tag{2.40}$$

we get

$$z(t) \geq -r^{1/\alpha}(t)z'(t)\frac{\pi(t)}{1-\gamma}. \tag{2.41}$$

Hence,

$$\left(\frac{z(t)}{\pi^{1-\gamma}(t)}\right)' = \frac{r^{1/\alpha}(t)z'(t)\frac{\pi(t)}{1-\gamma} + z(t)}{(1-\gamma)r^{1/\alpha}(t)\pi^{2-\gamma}(t)} \geq 0,$$

that is, $\frac{z(t)}{\pi^{1-\gamma}(t)}$ is nondecreasing.

Next, we prove that $\frac{z}{\pi^\delta}$ is nonincreasing. Proceeding as in the proof of Theorem 2.5, we obtain (2.15), that is,

$$z(t) \leq -\frac{1}{m_1} r^{1/\alpha}(t)z'(t)\frac{\pi(t)}{\pi(\sigma(t))}\left(\int_{t_0}^t q(s)Q_2^\beta\big(\sigma(s)\big)\,ds\right)^{-\frac{1}{\alpha}}, \tag{2.42}$$

where $m_1 > 0$ is a constant and such that $z^{\beta-\alpha}(\sigma(t)) \geq m_1$. On the other hand, we see

$$\left(\frac{z(t)}{\pi^\delta(t)}\right)' = \frac{z'(t)}{\pi^\delta(t)} + \frac{\delta z(t)}{\pi^{\delta+1}(t)r^{1/\alpha}(t)}.$$

Using the inequality (2.42), we obtain

$$\begin{aligned}\left(\frac{z(t)}{\pi^\delta(t)}\right)' &\leq \frac{z'(t)}{\pi^\delta(t)} - \frac{\delta z'(t)}{m_1\pi^\delta(t)\pi(\sigma(t))}\left(\int_{t_0}^t q(s)Q_2^\beta\big(\sigma(s)\big)\,ds\right)^{-\frac{1}{\alpha}} \\ &= \frac{z'(t)}{\pi^\delta(t)}\left(1 - \frac{\delta}{m_1\pi(\sigma(t))}\left(\int_{t_0}^t q(s)Q_2^\beta\big(\sigma(s)\big)\,ds\right)^{-\frac{1}{\alpha}}\right) \leq 0. \end{aligned} \tag{2.43}$$

Thus, $\frac{z(t)}{\pi^\delta(t)}$ is nonincreasing. The proof is complete. □

Theorem 2.15 *Let $\alpha \leq \beta$. Assume that (1.3) and (2.6) hold. Suppose that γ and δ are constants satisfying (2.35)–(2.37). Also, there exists a constant $M > 0$ such that $\int_{t_0}^t q(t)\,ds \leq M$ for all $t \geq t_0$. If*

$$\limsup_{t\to\infty} \pi^\gamma(t)\pi^{1-\gamma-\delta}\big(\sigma(t)\big)\int_{t_1}^t \pi^{\delta\beta}\big(\sigma(s)\big)q(s)Q_2^\beta\big(\sigma(s)\big)\,ds > (1-\gamma)^\beta \quad \text{when } \alpha = \beta \tag{2.44}$$

and

$$\limsup_{t\to\infty} \pi^\gamma(t)\pi^{1-\gamma-\delta}\big(\sigma(t)\big)\int_{t_1}^t \pi^{\delta\beta}\big(\sigma(s)\big)q(s)Q_2^\beta\big(\sigma(s)\big)\,ds = \infty \quad \text{when } \alpha < \beta, \tag{2.45}$$

for any $t_1 \geq t_0$, then Eq. (1.1) is oscillatory.

Proof Suppose that $x(t)$ is a positive solution of Eq. (1.1) on $[t_1, \infty)$. From Lemma 2.4, we see that z satisfies Case 2 for $t \geq t_1$.

Integrating from t_1 to t and combining Lemma 2.14, we have

$$\begin{aligned}W(t) &:= -r(t)\bigl(z'(t)\bigr)^{\alpha} = -r(t_1)\bigl(z'(t_1)\bigr)^{\alpha} + \int_{t_1}^{t} q(s)x^{\beta}\bigl(\sigma(s)\bigr)\,ds \\ &\geq \int_{t_1}^{t}\left(\frac{z(\sigma(s))}{\pi^{\delta}(\sigma(s))}\right)^{\beta}\pi^{\delta\beta}\bigl(\sigma(s)\bigr)q(s)Q_2^{\beta}\bigl(\sigma(s)\bigr)\,ds \\ &\geq \left(\frac{z(\sigma(t))}{\pi^{\delta}(\sigma(t))}\right)^{\beta}\int_{t_1}^{t}\pi^{\delta\beta}\bigl(\sigma(s)\bigr)q(s)Q_2^{\beta}\bigl(\sigma(s)\bigr)\,ds \\ &\geq \left(\frac{z(\sigma(t))\pi^{1-\gamma}(\sigma(t))}{\pi^{\delta}(\sigma(t))\pi^{1-\gamma}(\sigma(t))}\right)^{\beta}\int_{t_1}^{t}\pi^{\delta\beta}\bigl(\sigma(s)\bigr)q(s)Q_2^{\beta}\bigl(\sigma(s)\bigr)\,ds \\ &\geq \left(\frac{z(t)\pi^{1-\gamma-\delta}(\sigma(t))}{\pi^{1-\gamma}(t)}\right)^{\beta}\int_{t_1}^{t}\pi^{\delta\beta}\bigl(\sigma(s)\bigr)q(s)Q_2^{\beta}\bigl(\sigma(s)\bigr)\,ds. \end{aligned}\tag{2.46}$$

Clearly, one can see that the function $W(t)$ is bounded due to Eq. (1.1) and condition $\int_{t_0}^{t} q(t)\,ds \leq M$. Using (2.41) in the above inequality, we obtain

$$W(t) \geq W^{\frac{\beta}{\alpha}}(t)\left(\frac{\pi^{\gamma}(t)\pi^{1-\gamma-\delta}(\sigma(t))}{1-\gamma}\right)^{\beta}\int_{t_1}^{t}\pi^{\delta\beta}\bigl(\sigma(s)\bigr)q(s)Q_2^{\beta}\bigl(\sigma(s)\bigr)\,ds,$$

that is,

$$W^{1-\frac{\beta}{\alpha}}(t)(t) \geq \left(\frac{\pi^{\gamma}(t)\pi^{1-\gamma-\delta}(\sigma(t))}{1-\gamma}\right)^{\beta}\int_{t_1}^{t}\pi^{\delta\beta}\bigl(\sigma(s)\bigr)q(s)Q_2^{\beta}\bigl(\sigma(s)\bigr)\,ds.$$

Taking lim sup on both sides of this inequality, we arrive at a contradiction to (2.44) when $\alpha = \beta$ and (2.45) when $\alpha < \beta$. The proof is complete. □

3 Examples

In this section, we present two examples to illustrate our main results.

Example 3.1 Consider the following second order differential equation:

$$\left[t^4\left(x(t) + \frac{1}{9}x\left(\frac{t}{2}\right)\right)'\right]' + t^6x^{\frac{5}{3}}(3t) = 0, \quad t \geq 1. \tag{3.1}$$

Clearly,

$$\begin{aligned} &r(t) = t^4, \qquad p_1(t) = \frac{1}{9}, \qquad p_2(t) = 0, \qquad q(t) = t^6, \\ &\tau(t) = \frac{t}{2}, \qquad \sigma(t) = 3t, \qquad \alpha = 1, \qquad \beta = \frac{5}{3}. \end{aligned}$$

By $\pi(t) = \int_t^{\infty} r^{-\frac{1}{\alpha}}(s)\,ds$, we have $\pi(t) = \frac{1}{3}t^{-3}$, and condition (1.3) holds. Notice that $\frac{\pi(\tau(\sigma(t)))}{\pi(\sigma(t))} = \frac{\frac{1}{3}(\frac{3t}{2})^{-3}}{\frac{1}{3}(3t)^{-3}} = 8$, and

$$Q_1^{\beta}(t) = \left(1 - p_1\bigl(\sigma(t)\bigr) - p_2\bigl(\sigma(t)\bigr)\frac{R(\lambda(\sigma(t)))}{R(\sigma(t))}\right)^{\beta} = \left(\frac{8}{9}\right)^{\frac{5}{3}},$$

$$Q_2^\beta(t) = \left(1 - p_1\bigl(\sigma(t)\bigr)\frac{\pi(\tau(\sigma(t)))}{\pi(\sigma(t))} - p_2\bigl(\sigma(t)\bigr)\right)^\beta = \left(\frac{1}{9}\right)^{\frac{5}{3}}.$$

Letting $t_1 = t_0 = 1$, we have

$$\limsup_{t\to\infty} \int_1^t s^6 \left(\frac{8}{9}\right)^{\frac{5}{3}} ds = \infty,$$

and condition (2.6) is satisfied. To verify conditions (2.19) and (2.21), we find

$$\int_{t_1}^t q(s)Q_2^\beta\bigl(\sigma(s)\bigr)\pi^\beta\bigl(\sigma(s)\bigr)\,ds = \int_1^t s^6\left(\frac{1}{9}\right)^{\frac{5}{3}}\left(\frac{1}{3}\right)^{\frac{5}{3}}(3s)^{-5}\,ds = \infty$$

and

$$\limsup_{t\to\infty} \pi^\beta\bigl(\sigma(t)\bigr)\int_{t_1}^t q(s)Q_2^\beta\bigl(\sigma(s)\bigr)\,ds = \limsup_{t\to\infty}\left(\frac{1}{3}\right)^{\frac{5}{3}}(3t)^{-5}\int_1^t s^6\left(\frac{1}{9}\right)^{\frac{5}{3}} ds > 0,$$

which show that (2.19) and (2.21) hold. Hence, by Theorem 2.7, Eq. (3.1) is oscillatory.

Example 3.2 Consider the following second order differential equation:

$$\left[t^4\left(x(t) + \frac{1}{9}x\left(\frac{t}{2}\right)\right)'\right]' + t^6 x^{\frac{1}{3}}(3t) = 0, \quad t \ge 1. \tag{3.2}$$

It is easy to find that

$$r(t) = t^4, \qquad p_1(t) = \frac{1}{9}, \qquad p_2(t) = 0, \qquad q(t) = t^6,$$
$$\tau(t) = \frac{t}{2}, \qquad \sigma(t) = 3t, \qquad \alpha = 1, \qquad \beta = \frac{1}{3}.$$

From $\pi(t) = \int_t^\infty r^{-\frac{1}{\alpha}}(s)\,ds$, we have $\pi(t) = \frac{1}{3}t^{-3}$ and condition (1.3) holds. In view of $\frac{\pi(\tau(\sigma(t)))}{\pi(\sigma(t))} = \frac{\frac{1}{3}(\frac{3t}{2})^{-3}}{\frac{1}{3}(3t)^{-3}} = 8$, we obtain

$$Q_1^\beta(t) = \left(1 - p_1\bigl(\sigma(t)\bigr) - p_2\bigl(\sigma(t)\bigr)\frac{R(\lambda(\sigma(t)))}{R(\sigma(t))}\right)^\beta = \left(\frac{8}{9}\right)^{\frac{1}{3}},$$
$$Q_2^\beta(t) = \left(1 - p_1\bigl(\sigma(t)\bigr)\frac{\pi(\tau(\sigma(t)))}{\pi(\sigma(t))} - p_2\bigl(\sigma(t)\bigr)\right)^\beta = \left(\frac{1}{9}\right)^{\frac{1}{3}}.$$

Letting $t_1 = t_0 = 1$, we see that

$$\limsup_{t\to\infty} \int_1^t s^6 \left(\frac{8}{9}\right)^{\frac{1}{3}} ds = \infty$$

and condition (2.6) is satisfied. Setting $\rho(t) = 1$, we have

$$\begin{aligned}
&\limsup_{t\to\infty}\left(\pi^{\alpha}(t)\int_{t_2}^{t} q(s)Q_2^{\beta}\bigl(\sigma(s)\bigr)\left(\frac{\pi(\sigma(s))}{\pi(s)}\right)^{\beta} ds\right) \\
&\quad = \limsup_{t\to\infty}\frac{1}{3}t^{-3}\int_{1}^{t} s^{6}\left(\frac{1}{9}\right)^{\frac{1}{3}}\frac{1}{3}\,ds > 1.
\end{aligned}$$

Now, all conditions of Corollary 2.13 hold. Hence, Eq. (3.2) is oscillatory.

4 Conclusions

In this paper, we have obtained several new oscillation criteria for a second order nonlinear advanced differential equation with mixed neutral terms. Our results improve and complement some well-known results which were published recently in the literature. Two examples are given to illustrate the efficiency of our results. We believe that the proof method and the obtained results may be generalized to the differential equations, such as those in [11, 13, 18].

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References

1. Agarwal, R.P., Bohner, M., Li, T., Zhang, C.: Oscillation of second-order Emden–Fowler neutral delay differential equations. Ann. Mat. Pura Appl. (4) **193**, 1861–1875 (2014)
2. Agarwal, R.P., Bohner, M., Li, T., Zhang, C.: Even-order half-linear advanced differential equations: improved criteria in oscillatory and asymptotic properties. Appl. Math. Comput. **266**, 481–490 (2015)
3. Agarwal, R.P., Bohner, M., Li, W.: Nonoscillation and Oscillation: Theory for Functional Differential Equations. Monographs and Textbooks in Pure and Applied Mathematics, vol. 267. Dekker, New York (2004)
4. Agarwal, R.P., Grace, S.R., O'Regan, D.: Oscillation Theory for Second Order Linear, Half-Linear, Superlinear and Sublinear Dynamic Equations. Kluwer Academic, Dordrecht (2002)
5. Agarwal, R.P., Grace, S.R., O'Regan, D.: Oscillation Theory for Second Order Dynamic Equations. Series in Mathematical Analysis and Applications. Taylor & Francis, London (2003)
6. Agarwal, R.P., Grace, S.R., O'Regan, D.: Oscillation Theory for Difference and Functional Differential Equations. Springer, Berlin (2013)
7. Agarwal, R.P., Zhang, C., Li, T.: New Kamenev-type oscillation criteria for second-order nonlinear advanced dynamic equations. Appl. Math. Comput. **225**, 822–828 (2013)
8. Agarwal, R.P., Zhang, C., Li, T.: Some remarks on oscillation of second order neutral differential equations. Appl. Math. Comput. **274**, 178–181 (2016)
9. Baculíková, B.: Oscillatory behavior of the second order functional differential equations. Appl. Math. Lett. **72**, 35–41 (2017)
10. Chatzarakis, G.E., Džurina, J., Jadlovská, I.: New oscillation criteria for second-order half-linear advanced differential equations. Appl. Math. Comput. **347**, 404–416 (2019)

11. Chatzarakis, G.E., Grace, S.R., Jadlovská, I., Li, T., Tunç, E.: Oscillation criteria for third-order Emden–Fowler differential equations with unbounded neutral coefficients. Complexity **2019**, Article ID 5691758 (2019)
12. Džurina, J.: A comparison theorem for linear delay differential equations. Arch. Math. **135**, 531–540 (2003)
13. Džurina, J., Grace, S.R., Jadlovská, I., Li, T.: Oscillation criteria for second-order Emden–Fowler delay differential equations with a sublinear neutral term. Math. Nachr. https://doi.org/10.1002/mana.201800196
14. Elsgolts, L.E., Norkin, S.B.: Introduction to the Theory and Application of Differential Equations with Deviating Arguments. Academic Press, New York (1973)
15. Fite, W.B.: Properties of the solutions of certain functional-differential equations. Trans. Am. Math. Soc. **22**, 311–319 (1921)
16. Grace, S.R., Džurina, J., Jadlovská, I., Li, T.: An improved approach for studying oscillation of second-order neutral delay differential equations. J. Inequal. Appl. **2018**, 193 (2018)
17. Jadlovská, I.: Iterative oscillation results for second-order differential equations with advanced argument. Electron. J. Differ. Equ. **2017**, 162 (2017)
18. Li, T., Rogovchenko, Y.V.: Asymptotic behavior of higher-order quasilinear neutral differential equations. Abstr. Appl. Anal. **2014**, Article ID 395368 (2014)
19. Li, T., Rogovchenko, Y.V.: Oscillation of second-order neutral differential equations. Math. Nachr. **288**, 1150–1162 (2015)
20. Li, T., Rogovchenko, Y.V.: Oscillation criteria for second-order superlinear Emden–Fowler neutral differential equations. Monatshefte Math. **184**, 489–500 (2017)
21. Trench, W.F.: Canonical forms and principal systems for general disconjugate equations. Trans. Am. Math. Soc. **189**, 319–327 (1973)
22. Wu, H., Erbe, L., Peterson, A.: Oscillation of solution to second-order half-linear delay dynamic equations on time scales. Electron. J. Differ. Equ. **2016**, 71 (2016)
23. Zhang, C., Agarwal, R.P., Bohner, M., Li, T.: Oscillation of second-order nonlinear neutral dynamic equations with noncanonical operators. Bull. Malays. Math. Sci. Soc. **38**, 761–778 (2015)