Advances in Difference Equations
a SpringerOpen Journal

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Lucas polynomials semi-analytic solution for fractional multi-term initial value problems

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Abstract

Herein, we use the generalized Lucas polynomials to find an approximate numerical solution for fractional initial value problems (FIVPs). The method depends on the operational matrices for fractional differentiation and integration of generalized Lucas polynomials in the Caputo sense. We obtain these solutions using tau and collocation methods. We apply these methods by transforming the FIVP into systems of algebraic equations. The convergence and error analyses are discussed in detail. The applicability and efficiency of the method are tested and verified through numerical examples.

MSC: 65M70; 11B39; 34A08; 41A25

Keywords: Caputo derivative; Fractional differential equations; Spectral methods; Generalized Lucas polynomials

1 Introduction

Ordinary and partial derivatives are special cases of fractional order derivatives. Many scientists are interested in linear and nonlinear fractional differential equations (FDEs). Many phenomena are described using fractional-order differentiation and integration. Their applications appeared in fluid, engineering, mechanics, physics, mathematics, optics, and other fields of science. So the fractional calculus investigates the rules, properties of derivatives, and integrals of noninteger orders. For handling these equations, the researchers apply many numerical methods such as finite difference method [1–3], finite element method [4–6], homotopy analysis method [7, 8], variational iteration method [9–11], a domain decomposition method [12–15], and Haar wavelet method [16, 17].

Recently, the approximate solutions of the fractional differential equations have been evaluated by the spectral methods. These methods help to solve different kinds of differential equations with small error and a small number of unknowns; [18] solutions of fractional differential equations by using Jacobi operational matrix, [19] solutions of third and fifth-order differential equations by using Petrov–Galerkin methods, [20] solutions of fractional differential equations by using shifted Jacobi spectral approximations. The most used spectral methods are the Galerkin, collocation, and tau methods; [21] solutions of time-fractional telegraph equation by using Legendre–Galerkin algorithm, [22] solution for telegraph equation of space fractional order by using Legendre wavelets spectral tau algorithm, [23] solutions of differential problems by using tau method, [24] solutions for the parabolic and elliptic partial equations by the ultra-spherical tau method, [25] solutions for a class of variable-order fractional differential equations by using Jacobi wavelets method. The choice of this method depends on the type of the investigated problem and its initial and boundary conditions. For more applications about numerical and exact solutions of fractional differential models, please see [26–37].

Multi-term fractional IVPs appear in many applications in various disciplines, most of numerical studies use the orthogonal polynomials, only rare studies use nonorthogonal polynomials, this motivates us to use these polynomials as a new basis functions, to test their ability to handle these problems.

In this paper, we solve the fractional ordinary differential equations with initial and boundary conditions applying generalized Lucas polynomials. We obtain the integrated equations and solve them. We use tau and collocation methods to evaluate numerical solutions. We have a system of nonlinear algebraic equations with initial and boundary conditions. Then we solve them by using Mathematica. We compare our numerical results with the Haar wavelet method [38].

There are many techniques in literature to handle multi-term fractional IVPs using orthogonal polynomials, i.e., Legendre, Chebyshev, Jacobi, and others, and there are very few studies on nonorthogonal linearly independent set of polynomials, i.e., Lucas and Fibonacci polynomials. The main advantages of the present technique is that new polynomials can be used as a basis for spectral methods, the generation of these polynomials is easy, and the exponential rate of convergence.

The results in this paper are more efficient and of higher accuracy than the other methods. The sections are organized as follows. In section 2 definitions, properties of fractional calculus, and generalized Lucas polynomials, which are used in the following sections, are introduced. In section 3 derivatives for generalized Lucas polynomials of integer and fractional orders are stated. In section 4 the algorithm of this method is explained. In section 5 we investigate the convergence and error analysis. In section 6 we give some examples and their numerical solutions. In the last section we introduce some conclusions.

2 Preliminaries

In this section, some definitions, properties for fractional calculus [39–41], and the generalized Lucas polynomials [42, 43] are stated. We introduce the important relations for the generalized Lucas polynomials which will be used in the following sections.

2.1 Properties and definitions of fractional calculus

Definition 1 The fractional integral of order β ($\beta \geq 0$) according to Riemann–Liouville is

$$\begin{cases} I^{\beta} g(z) = \frac{1}{\Gamma(\beta)} \int_0^z (z-t)^{\beta-1} g(t)\,dt, & \beta > 0, z > 0, \\ I^0 g(z) = g(z). \end{cases}$$

And I^{β} satisfies the following properties:

$$\begin{cases} I^{\beta} I^{\gamma} = I^{\beta+\gamma}, \\ I^{\beta} I^{\gamma} = I^{\gamma} I^{\beta}, \\ I^{\beta} z^{\upsilon} = \frac{\Gamma(\upsilon+1)}{\Gamma(\upsilon+\beta+1)} z^{\upsilon+\beta}. \end{cases}$$

Definition 2 The fractional derivative of order β according to Caputo

$$D^{\beta}g(z) = I^{m-\beta}D^{m}g(z) = \frac{1}{\Gamma(m-\beta)}\int_{0}^{z}(z-t)^{m-\beta-1}g^{(m)}(t)\,dt, \tag{1}$$

where $m-1<\beta\le m$, and D^{β} satisfies the following properties:

$$\begin{cases} (D^{\beta}I^{\beta}g)(z) = g(z), \\ D^{\beta}z^{\upsilon} = \frac{\Gamma(\upsilon+1)}{\Gamma(\upsilon-\beta+1)}z^{\upsilon-\beta}. \end{cases} \tag{2}$$

For more details about the properties of fractional derivatives, please see [43].

2.2 An overview and relations of generalized Lucas polynomials

Lucas polynomials $L_j(z)$ [43] have the following recurrence relation:

$$L_{j+2}(z) = zL_{j+1}(z) + L_j(z), \tag{3}$$

with the initial values

$$L_0(z) = 2, \qquad L_1(z) = z. \tag{4}$$

Lucas polynomials have Binet's form

$$L_j(z) = \frac{(z+\sqrt{z^2+4})^j + (z-\sqrt{z^2+4})^j}{2^j}, \quad j\ge 0, \tag{5}$$

and also have the power form

$$L_j(z) = j\sum_{i=0}^{\lfloor \frac{j}{2}\rfloor}\frac{1}{j-i}\binom{j-i}{i}z^{j-2i}, \quad j\ge 1, \tag{6}$$

where $\lfloor j\rfloor$ represents the largest integer less than or equal to j. If a and b are nonzero real numbers, the sequence of Lucas polynomials $\{L_j(z)\}_{j\ge 0}$ is generalized by the sequence $\{\varphi_j^{a,b}(z)\}_{j\ge 0}$ generated by the recurrence relation

$$\varphi_j^{a,b}(z) = az\varphi_{j-1}^{a,b}(z) + b\varphi_{j-2}^{a,b}(z), \quad j\ge 2, \tag{7}$$

with the initial values

$$\varphi_0^{a,b}(z) = 2, \qquad \varphi_1^{a,b}(z) = az, \tag{8}$$

so Lucas polynomials $L_j(z)$ are derived from $\varphi_j^{a,b}(z)$ if $a=b=1$. We have the following:

$$\varphi_2^{a,b}(z) = az^2 + 2b, \qquad \varphi_3^{a,b}(z) = a^3z^3 + 3abz, \tag{9}$$

$$\varphi_4^{a,b}(z) = a^4z^4 + 4a^2bz^2 + 2b^2, \qquad \varphi_5^{a,b}(z) = a^5z^5 + 5a^3bz^3 + 5ab^2z, \tag{10}$$

where $\varphi_j^{a,b}(z)$ have the power form

$$\varphi_j^{a,b}(z) = j \sum_{n=0}^{\lfloor \frac{j}{2} \rfloor} \frac{a^{j-2n} b^n \binom{j-n}{n}}{j-n} z^{j-2n}, \quad j \geq 1, \tag{11}$$

and

$$\varphi_j^{a,b}(z) = 2j \sum_{m=0}^{j} \frac{a^m b^{\frac{j-m}{2}} \zeta_{j+k} \binom{j-n}{n}}{j+m} z^m, \quad j \geq 1, \tag{12}$$

where

$$\zeta_\ell = \begin{cases} 1, & \ell \text{ even}, \\ 0, & \ell \text{ odd}. \end{cases} \tag{13}$$

$\varphi_j^{a,b}(z)$ have Binet's form

$$\varphi_j^{a,b}(z) = \frac{(az + \sqrt{a^2z^2 + 4b})^j + (az - \sqrt{a^2z^2 + 4b})^j}{2^j}, \quad j \geq 0. \tag{14}$$

The following relations, used for solving the problems, are very important.

3 Integer and fractional derivatives of generalized Lucas vector

In this section, we state the integer and fractional derivatives of generalized Lucas polynomials in a matrix form.

3.1 Integer derivatives for generalized Lucas matrix

Suppose that the function $W(z)$ can be expanded in terms of generalized Lucas polynomials

$$W(z) = \sum_{i=0}^{\infty} e_i \varphi_i^{a,b}(z). \tag{15}$$

If we approximate this function as

$$W(z) \approx W_N(z) = \sum_{i=0}^{N} e_i \varphi_i^{a,b}(z) = E^T \Phi(z), \tag{16}$$

where

$$E^T = [e_0, e_1, \ldots, e_N], \tag{17}$$

$$\Phi(z) = \left[\varphi_0^{a,b}(z), \varphi_1^{a,b}(z), \ldots, \varphi_N^{a,b}(z)\right]^T. \tag{18}$$

If the first derivative of $\frac{d\Phi(z)}{dz}$ is written as

$$\frac{d\Phi(z)}{dz} = H^{(1)} \Phi(z), \tag{19}$$

where $H^{(1)} = (H^{(1)}_{nm})$ is $(N+1) \times (N+1)$ matrix of derivatives.

$$H^{(1)}_{nm} = \begin{cases} (-1)^{\frac{n-m+1}{2}} nab^{\frac{n-m-1}{2}} \xi_m, & n > m, (n+m) \text{ odd}, \\ 0, & \text{otherwise}, \end{cases} \tag{20}$$

and

$$\xi_k = \begin{cases} \frac{1}{2}, & k = 0, \\ 1, & \text{otherwise}. \end{cases} \tag{21}$$

From (15) we can write $\frac{d^i \Phi(z)}{dz^i}$, $i \geq 1$,

$$\frac{d^i \Phi(z)}{dz^i} = H^{(i)} \Phi(z) = \left(H^{(1)}\right)^i \Phi(z). \tag{22}$$

3.2 Fractional derivatives for generalized Lucas matrix

We state in this section the fractional derivative of generalized Lucas matrix, which is the general case for integer derivative.

Theorem 1 *The fractional derivatives of generalized Lucas vector, which is defined in* (14), *have the form* [42]

$$D^\beta \Phi(z) = \frac{d^\beta \Phi(z)}{dz^\beta} = z^{-\beta} H^{(\beta)} \Phi(z), \quad \beta > 0, \tag{23}$$

where $H^{(\beta)} = (H^\beta_{nm})$ *is* $(N+1) \times (N+1)$ *lower triangular matrix of the form*

$$H^{(\beta)} = \begin{bmatrix} 0 & 0 & \cdots & 0, \\ \vdots & \vdots & \vdots & \vdots \\ \gamma_\beta(\lceil\beta\rceil, 0) & \gamma_\beta(\lceil\beta\rceil, 1) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \gamma_\beta(N, 0) & \gamma_\beta(N, 1) & \cdots & \gamma_\beta(N, N) \end{bmatrix}, \tag{24}$$

where $\lceil\beta\rceil$ *represents the smallest integer greater than or equal to* β. *And* H^β_{nm} *and* $\gamma_\beta(n,m)$ *have the elements in the form*

$$H^\beta_{nm} = \begin{cases} \gamma_\beta(n,m), & n \geq \lceil\beta\rceil, m \geq n, \\ 0, & \textit{otherwise}, \end{cases} \tag{25}$$

$$\gamma_\beta(n,m) = \sum_{i=\lceil\beta\rceil}^{n} \frac{(-1)^{\frac{i-m}{2}} ni! \zeta_{n+i} \zeta_{m+i} \xi_m b^{\frac{n-m}{2}} (\frac{n+i}{2} - 1)!}{(\frac{n-i}{2})!(\frac{i-m}{2})!(\frac{m+i}{2})! \Gamma(1+i-\beta)}. \tag{26}$$

4 The algorithm of the method

In this section, we explain the method for solving the boundary FDE with constant coefficients by using generalized Lucas polynomials

$$D^\beta W(z) + D^\alpha W(z) + W''(z) + W(z) = g(z), \tag{27}$$

with the boundary conditions

$$W^{(i)}(0) = a_i, \quad i = 0, 1, 2, \ldots, \tag{28}$$

where $0 < \alpha \leq 1$, $1 < \beta \leq 2$. Suppose that equation (27) has the approximating solution

$$W(z) \approx W_N(z) = E^T \Phi(z). \tag{29}$$

By using Theorem 1, we have

$$D^\beta W(z) \approx z^{-\beta} E^T H^{(\beta)} \Phi(z), \tag{30}$$

and now the residual of equation (27) has the form

$$R(z) = z^{-\beta} E^T H^{(\beta)} \Phi(z) + z^{-\alpha} E^T H^{(\alpha)} \Phi(z) + E^T H^{(2)} \Phi(z) + E^T \Phi(z) - g(z). \tag{31}$$

Then we have

$$z^\beta R(z) = E^T H^{(\beta)} \Phi(z) + z^{\beta-\alpha} E^T H^{(\beta)} \Phi(z) + z^\beta E^T H^{(2)} \Phi(z) + z^\beta E^T \Phi(z) - z^\beta g(z). \tag{32}$$

By using the tau method, we obtain the system of equations

$$\int_0^1 z^\beta R(z) \varphi_i^{a,b}(z)\, dz = 0, \quad i = 0, 1, \ldots. \tag{33}$$

With the boundary conditions (28), we have

$$E^T H^{(i)} \Phi(0) = a_i, \quad i = 0, 1, 2, \ldots. \tag{34}$$

Equations (33)–(34) give a linear system of equations in coefficients e_i, $i = 0, 1, \ldots, N$. These coefficients can be efficiently solved by Gaussian elimination.

5 Investigation of convergence and error analysis

In this section, we explain the convergence and error analysis of generalized Lucas expansion. The following lemmas are satisfied.

Lemma 1 *For all $t \in [0, 1]$, the following inequality holds for generalized Lucas polynomials*:

$$\varphi_i^{a,b} \leq 2\bigl(a + \sqrt{a^2 + b}\bigr)^i. \tag{35}$$

Proof See Abd-Elhameed and Youssri (2017) [43]. □

Lemma 2

$$\varphi_i^{a,b} = 2i \sum_{k=0}^{i} \lambda_{i,k} t^k, \tag{36}$$

where

$$\lambda_{i,k} = \frac{a^k b^{\frac{i-k}{2}} \zeta_{i+k} \binom{\frac{i+k}{2}}{\frac{i-k}{2}}}{i+k}. \tag{37}$$

Proof See Abd-Elhameed and Youssri (2017) [43]. □

Theorem 2 *If $W(z)$ is defined on $[0,1]$ and $|W^{(i)}(0)| \leq L^i$, $i \geq 0$, where L is a positive constant and if $W(z)$ has the expansion*

$$W(z) = \sum_{i=0}^{\infty} e_i \varphi_i^{a,b}(z). \tag{38}$$

Then one has

$$|e_i| \leq \frac{|a|^{-i} L^i \cosh(2|a|^{-1} b^{\frac{1}{2}} L)}{i!}. \tag{39}$$

Proof See Abd-Elhameed and Youssri (2017) [43]. □

If $\varepsilon_N = \max |W(z) - W_N(z)|$, then we have the following truncation error.

Theorem 3 *We have the following truncation error estimate*:

$$\varepsilon_N < \frac{2e^{L(1+\sqrt{1+a^{-2}b})} \cosh(2L(1+\sqrt{1+a^{-2}b}))(1+\sqrt{1+a^{-2}b})^{N+1}}{(N+1)!} \tag{40}$$

or

$$\varepsilon_N < \frac{2e^{L\rho} \cosh(2L\rho)\rho^{N+1}}{(N+1)!}, \tag{41}$$

where $\rho = 1 + \sqrt{1 + a^{-2}b}$.

Proof See Abd-Elhameed and Youssri (2017) [43]. □

Lemma 3 *The derivatives of $\varphi_i^{(\alpha)}$, $\varphi_i^{(\beta)}$, and φ_i'' are denoted by the following estimates*:

(i) $|\varphi_i^{(\alpha)}| \leq 2i^3$,

(ii) $|\varphi_i^{(\beta)}| \leq 2i^3$,

(iii) $|\varphi_i''| \leq 2i^3$.

Proof By applying the differential operators to the right-hand side of equation (29) and noting that $t < 1$, and finally by induction on i, we get the desired results. □

Theorem 4 *If $W(z) = \sum_{i=0}^{\infty} e_i \varphi_i^{a,b}(z)$ is the exact solution of equation (21) satisfies the hypotheses of Eq. (6) and $W(z)$ is approximated by $W_N(z) = \sum_{i=0}^{N} e_i \varphi_i^{a,b}(z)$, then we have the following global error estimate*:

$$\epsilon_N = \left| W_N'' + W_N^{(\alpha)} + W_N^{(\beta)} + W_N - g \right| < \frac{\Omega N^{\xi}}{2^N}, \tag{42}$$

where Ω is a generic constant and

$$\xi = 3A + 1, \quad A = \frac{L}{|a|}. \tag{43}$$

Proof Now the global error estimate

$$\epsilon_N = \left|W''_N + W_N^{(\alpha)} + W_N^{(\beta)} + W_N - g\right|. \tag{44}$$

□

From equation (21) we have

$$\epsilon_N = \left|W''_N - W'' + W_N^{(\alpha)} - W^{(\alpha)} + W_N^{(\beta)} - W^{(\beta)} + W_N - W\right|. \tag{45}$$

By the triangle inequality

$$\epsilon_N \le \left|W''_N - W''\right| + \left|W_N^{(\alpha)} - W^{(\alpha)}\right| + \left|W_N^{(\beta)} - W^{(\beta)}\right| + |W_N - W|. \tag{46}$$

And hence

$$\epsilon_N \le \sum_{i=N+1}^{\infty} |e_i|\left|\varphi''_i\right| + \sum_{i=N+1}^{\infty} |e_i|\left|\varphi_i^{(\alpha)}\right| + \sum_{i=N+1}^{\infty} |e_i|\left|\varphi_i^{(\beta)}\right| + \frac{2e^{L\rho\cosh(2L\rho)}\rho^{N+1}}{(N+1)!}, \tag{47}$$

where $A = \frac{L}{|a|}$, $B = \frac{2\sqrt{b}L}{|a|}$, by Theorem 2, Lemma 3, and Theorem 3, and application of the series comparison test, we have

$$\sum_{i=N+1}^{\infty} \left(\frac{A^i\cosh(B)}{i!}\right)\left(\left|\varphi''_i\right| + \left|\varphi_i^{(\alpha)}\right| + \left|\varphi_i^{(\beta)}\right|\right) + \frac{2e^{L\rho\cosh(2L\rho)}\rho^{N+1}}{(N+1)!} \tag{48}$$

$$< \sum_{i=N+1}^{\infty} \frac{6A^i\cosh(B)i^3}{i!} + \frac{2e^{L\rho\cosh(2L\rho)}\rho^{N+1}}{(N+1)!}. \tag{49}$$

Therefore

$$\epsilon_N < 6\cosh(B)\frac{N^{3A+1}}{2^N} + \frac{2e^{L\rho\cosh(2L\rho)}\rho^{N+1}}{(N+1)!} \tag{50}$$

$$= \Omega_1\frac{N^{3A+1}}{2^N} + \Omega_2\frac{\rho^{N+1}}{(N+1)!} < \Omega\frac{N^{3A+1}}{2^N}, \tag{51}$$

where $\Omega = \max(\Omega_1, \Omega_2)$.

6 Numerical examples

In this section, we solve some examples on equations (21), (22) using the generalized Lucas polynomials.

Example 1 Consider the following fractional-order initial value problem [38]:

$$D^\beta W(z) + \frac{3}{57}W(z) = z + \frac{3z^{\beta+1}}{57\Gamma(\beta+2)}, \quad 0 < z < 1, 1 < \beta \le 2, \tag{52}$$

with the boundary conditions

$$W(0) = 0, \qquad W(1) = \frac{1}{\Gamma(\beta+2)}. \tag{53}$$

The exact solution of equation (52) is $W(z) = \frac{z^{\beta+1}}{\Gamma(\beta+1)}$. The residual of this equation:

$$z^{\beta}R(z) = E^{T}H^{(\beta)}\Phi(z) + \frac{3}{57}z^{\beta}E^{T}\Phi(z) - z^{\beta+1} - \frac{3z^{2\beta+1}}{57\Gamma(\beta+2)}. \tag{54}$$

For $N = 3$, we have $H^{(\beta)}$ in the form

$$H^{(\beta)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\Gamma(2-\beta)} & 0 & 0 \\ \frac{-2b}{\Gamma(3-\beta)} & 0 & \frac{2}{\Gamma(3-\beta)} & 0, \\ 0 & \frac{3b(\beta-5)\beta}{\Gamma(4-\beta)} & 0 & \frac{6}{\Gamma(4-\beta)} \end{bmatrix}. \tag{55}$$

We apply the generalized Lucas tau method and obtain the following equations:

$$\begin{aligned}
&\frac{2}{65{,}835\sqrt{\pi}}\bigl(-294 - 47{,}025\sqrt{\pi} + 2772\sqrt{\pi}e_0 + 990\sqrt{\pi}ae_1 + 87{,}780a^2e_2 + 770\sqrt{\pi}a^2e_2 \\
&\quad + 2772\sqrt{\pi}be_2 + 131{,}670a^3e_3 + 630\sqrt{\pi}a^3e_3 + 2970\sqrt{\pi}abe_3\bigr) = 0, \\
&\frac{a}{855{,}855\sqrt{\pi}}\bigl(-10{,}010 - 475{,}475\sqrt{\pi} + 25{,}740\sqrt{\pi}e_0 + 10{,}010\sqrt{\pi}ae_1 + 855{,}855a^2e_2 \\
&\quad + 8190\sqrt{\pi}a^2e_2 + 25{,}740\sqrt{\pi}be_2 + 1{,}369{,}368a^3e_3 + 6930\sqrt{\pi}a^3e_3 \\
&\quad + 30{,}030\sqrt{\pi}abe_3\bigr) = 0,
\end{aligned}$$

with the boundary conditions

$$\begin{aligned}
&e_0 + be_2 = 0, \\
&2e_0 + ae_1 + \frac{1}{4}\Bigl(\bigl(a - \sqrt{a^2+4b}\bigr)^2 + \bigl(a + \sqrt{a^2+4b}\bigr)^2\Bigr)e_2 \\
&\quad + \frac{1}{8}\Bigl(\bigl(a - \sqrt{a^2+4b}\bigr)^3 + \bigl(a + \sqrt{a^2+4b}\bigr)^3\Bigr)e_3 = \frac{4}{3\sqrt{\pi}}.
\end{aligned}$$

We solve these equations by Mathematica, we obtain

$$\begin{aligned}
e_0 &= -\frac{33b(-222{,}794 + 3{,}871{,}765\sqrt{\pi} + 57{,}000\pi)}{10a^3\sqrt{\pi}(107{,}324{,}217 - 341{,}088\sqrt{\pi} + 320\pi)}, \\
e_1 &= \frac{1}{6a^3(107{,}324{,}217 - 341{,}088\sqrt{\pi} + 320\pi)} \\
&\quad \times \bigl[715b\sqrt{\pi}(-27{,}588 - 393{,}483\sqrt{\pi} + 11{,}400\pi) + \\
&\quad + a^2\bigl(1{,}430{,}989{,}560 + 3{,}581{,}424\sqrt{\pi} - 468{,}479{,}005\pi - 1{,}045{,}000\pi^{\frac{3}{2}}\bigr)\bigr],
\end{aligned} \tag{56}$$

Table 1 Maximum absolute errors E of Example 1

z	$m=32$ $\beta=1.4$	$N=16$ $\beta=1.4$	$m=32$ $\beta=1.6$	$N=16$ $\beta=1.6$	$m=32$ $\beta=1.8$	$N=16$ $\beta=1.8$	$m=32$ $\beta=2$	$N=16$ $\beta=2$
0.1	$1.4\cdot10^{-6}$	$1.9\cdot10^{-7}$	$1.5\cdot10^{-7}$	$6.9\cdot10^{-8}$	$1.4\cdot10^{-7}$	$1.6\cdot10^{-8}$	$3.5\cdot10^{-8}$	$5.5\cdot10^{-18}$
0.2	$6.9\cdot10^{-9}$	$7.2.0\cdot10^{-7}$	$3.8\cdot10^{-8}$	$7.3\cdot10^{-8}$	$8.9\cdot10^{-7}$	$1.7\cdot10^{-8}$	$6.5\cdot10^{-8}$	$2.6\cdot10^{-19}$
0.3	$3.3\cdot10^{-8}$	$6.02\cdot10^{-7}$	$5.7\cdot10^{-7}$	$6.0\cdot10^{-8}$	$1.1\cdot10^{-7}$	$1.3\cdot10^{-8}$	$8.7\cdot10^{-8}$	$6.0\cdot10^{-18}$
0.4	$3.1\cdot10^{-7}$	$9.6\cdot10^{-8}$	$3.8\cdot10^{-7}$	$3.4\cdot10^{-8}$	$1.8\cdot10^{-7}$	$7.7\cdot10^{-9}$	$9.7\cdot10^{-8}$	$2.0\cdot10^{-18}$
0.5	$7.8\cdot10^{-7}$	$8.1\cdot10^{-10}$	$3.6\cdot10^{-7}$	$2.8\cdot10^{-10}$	$2.6\cdot10^{-7}$	$6.6\cdot10^{-11}$	$8.9\cdot10^{-8}$	$1.1\cdot10^{-17}$
0.6	$1.5\cdot10^{-6}$	$9.6\cdot10^{-8}$	$2.3\cdot10^{-7}$	$2.8\cdot10^{-8}$	$6.0\cdot10^{-7}$	$5.6\cdot10^{-9}$	$5.9\cdot10^{-8}$	$4.1\cdot10^{-18}$
0.7	$4.8\cdot10^{-7}$	$1.1\cdot10^{-7}$	$3.6\cdot10^{-8}$	$3.3\cdot10^{-8}$	$1.0\cdot10^{-6}$	$6.7\cdot10^{-9}$	$4.8\cdot10^{-9}$	$5.5\cdot10^{-18}$
0.8	$7.9\cdot10^{-7}$	$1.1\cdot10^{-7}$	$2.2\cdot10^{-7}$	$2.9\cdot10^{-8}$	$1.8\cdot10^{-7}$	$5.3\cdot10^{-9}$	$8.0\cdot10^{-8}$	$6.1\cdot10^{-17}$
0.9	$1.1\cdot10^{-6}$	$6.5\cdot10^{-8}$	$5.5\cdot10^{-7}$	$1.5\cdot10^{-8}$	$3.5\cdot10^{-7}$	$2.6\cdot10^{-9}$	$1.9\cdot10^{-7}$	$2.2\cdot10^{-17}$

Table 2 Comparison between different errors E of Example 2

z	[38]	[44]	$a=1, b=2, N=10$	Exact
0.1	0.05934820	0.05934300	$2.9074\cdot10^{-15}$	0.05934303
0.2	0.11014318	0.11013418	$2.75252\cdot10^{-14}$	0.11013421
0.3	0.15103441	0.15102438	$2.28456\cdot10^{-14}$	0.15102441
0.4	0.18048329	0.18047531	$1.94317\cdot10^{-14}$	0.18047535
0.5	0.19673826	0.19673463	$5.55112\cdot10^{-18}$	0.19673467
0.6	0.19780653	0.19780792	$1.97675\cdot10^{-14}$	0.19780797
0.7	0.18142196	0.18142718	$2.36283\cdot10^{-14}$	0.18142725
0.8	0.14500893	0.14501532	$2.90212\cdot10^{-14}$	0.14501540
0.9	0.08564186	0.08564623	$3.13083\cdot10^{-15}$	0.08564632

$$
\begin{aligned}
e_2 &= \frac{33(-222{,}794 + 3{,}871{,}765\sqrt{\pi} + 57{,}000\pi)}{5a^2(107{,}324{,}217 - 341{,}088\sqrt{\pi} + 320\pi)}, \\
e_3 &= -\frac{143(-27{,}588 - 393{,}483\sqrt{\pi} + 11{,}400\pi)}{6a^3(107{,}324{,}217 - 341{,}088\sqrt{\pi} + 320\pi)}.
\end{aligned}
\tag{57}
$$

In Table 1, we compare our results for the case $a = b = 1$ and $N = 16$ with the results of [38] for $m = 32$ for different values of β. In Table 2, we compare between different solutions of Example 2.

Example 2 Consider the following fractional-order initial value problem [38]:

$$
D^{\beta} W(z) - D^{\alpha} W(z) = e^{z-1} + 1, \quad 0 < \beta \le 1, 1 < \alpha \le 2, \tag{58}
$$

with the boundary conditions

$$
W(0) = 0, \qquad W(1) = 0. \tag{59}
$$

The exact solution of equation (38) is $W(z) = z(1 - e^{z-1})$. The residual of this equation is

$$
z^{\beta} R(z) = E^T H^{(\beta)} \Phi(z) - z^{\beta-\alpha} E^T H^{(\alpha)} \Phi(z) - z^{\beta} e^{z-1} - z^{\beta}. \tag{60}
$$

Example 3 Consider the following fractional-order initial value problem [38]:

$$
\begin{aligned}
&D^{\beta} W(z) + \frac{e^{-3\pi}}{\sqrt{\pi}} W(z) \\
&\quad = \frac{e^{-3\pi}}{40\sqrt{\pi}} \bigl(z^2\bigl(40z^2 - 74z + 33\bigr) + 4e^{3\pi}\sqrt{z}\bigl(128z^2 - 148z + 33\bigr)\bigr), \quad 1 \le \beta < 2,
\end{aligned}
\tag{61}
$$

with the boundary conditions

$$W(0) = 0, \qquad W(1) = -\frac{1}{40}. \tag{62}$$

The exact solution of equation (41) is $W(z) = z^2(z^3 - \frac{37}{20}z + \frac{33}{40})$. The residual of this equation is

$$\begin{aligned} z^\beta R(z) &= E^T H^{(\beta)} \Phi(z) + \frac{e^{-3\pi}}{\sqrt{\pi}} z^\beta E^T \Phi(z) \\ &\quad - \left(\frac{e^{-3\pi}}{40\sqrt{\pi}} \left(z^2 \left(40z^2 - 74z + 33 \right) + 4e^{3\pi} \sqrt{z} \left(128z^2 - 148z + 33 \right) \right) \right) z^\beta. \end{aligned}$$

We apply our algorithm for the case $a = 1$, $b = 2$, $N = 5$, $\beta = \frac{3}{2}$, which yields

$$\begin{aligned} &e_1 = 511/10, \qquad e_3 = -237/20, \qquad e_5 = 1, \\ &e_0 = -33/2, \qquad e_2 = 233/4, \qquad e_4 = 0, \end{aligned}$$

and consequently,

$$W(z) = -(33/2)2 + (511/10)z + (233/4)\left(4 + z^2\right) - (237/20)\left(6z + z^3\right) + 20z + 10z^3 + z^5,$$

which is the exact solution.

Example 4 Consider the following fractional-order initial value problem [38]:

$$W''(z) + \frac{8}{17} D^\beta W(z) + \frac{13}{51} W(z) = \frac{z^{\frac{-1}{2}}}{89{,}250\sqrt{\pi}} \left(48p(z) + 7\sqrt{z} q(z) \right), \quad 1 \leq \beta < 2, \tag{63}$$

where

$$p(z) = 16{,}000z^4 - 32{,}480z^3 + 21{,}280z^2 - 4746z + 189, \tag{64}$$

$$q(z) = 3250z^5 - 9425z^4 + 264{,}880z^3 - 44 \tag{65}$$

with the boundary conditions

$$W(0) = 0, \qquad W(1) = 0. \tag{66}$$

The exact solution of equation (45) is $W(z) = z^5 - \frac{29z^4}{10} + \frac{76z^3}{25} - \frac{339z^2}{250} + \frac{27z}{125}$. The residual of this equation is

$$\begin{aligned} z^\beta R(z) &= z^\beta E^T H^{(2)} \Phi(z) + \frac{8}{17} E^T H^{(\beta)} \Phi(z) + \frac{13}{51} z^\beta E^T \Phi(z) \\ &\quad - \left(\frac{z^{\frac{-1}{2}+\beta}}{89{,}250\sqrt{\pi}} \left(48p(z) + 7\sqrt{z} q(z) \right) \right). \end{aligned} \tag{67}$$

We apply our algorithm for the case $a = 2$, $b = 1$, $N = 5$, $\beta = \frac{3}{2}$, which yields

$$\begin{aligned} &e_1 = -1439/2000, \qquad e_3 = 179/800, \qquad e_5 = 1/32, \\ &e_0 = -819/4000, \qquad e_2 = 193/500, \qquad e_4 = -29/160, \end{aligned}$$

and consequently,

$$\begin{aligned} W(z) = {}& -(819/4000)2 - (1439/2000)2z + (193/500)\bigl(2 + 4z^2\bigr) + (179/800)\bigl(6z + 8z^3\bigr) \\ & - (29/160)\bigl(2 + 16z^2 + 16z^4\bigr) + (1/32)\bigl(10z + 40z^3 + 32z^5\bigr), \end{aligned}$$

which is the exact solution.

7 Conclusion

Herein, a generalized Lucas polynomial sequence approach based on the operational matrix of fractional derivatives Lucas polynomials to spectrally solve fractional multi-term initial value problem was successfully applied to handle these equations. Four examples to a system of linear algebraic equations were solved by Mathematica software showing the exponential rate of convergence of the method. This method can be modified in the future work to solve different types of ordinary and partial FDEs with nonhomogeneous conditions and with variable coefficients.

Acknowledgements
The authors are very grateful to the anonymous referees for careful reviewing and crucial comments, which enabled us to improve the manuscript.

Funding
This research received no specific grant from any funding agency in the public, commercial, or not-for-profit sectors.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to this work. All authors read and approved the final manuscript.

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Publisher's Note

Received: 20 July 2019 Accepted: 1 November 2019 Published online: 14 November 2019

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