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On implicit impulsive Langevin equation involving mixed order derivatives

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Abstract

In this paper, we consider a nonlocal boundary value problem of nonlinear implicit impulsive Langevin equation involving mixed order derivatives. Sufficient conditions are constructed to discuss the qualitative properties like existence and Ulam's stability of the proposed problem. The main result is verified by an example.

MSC: 26A33; 34A08; 34B27

Keywords: Langevin equation; Caputo derivative; Non–instantaneous impulses; Ulam–Hyers–Rassias stability

1 Introduction

An equation of the form $m\,\frac{d^2z}{dw^2} = \lambda\,\frac{dz}{dw} + \eta(w)$ is called Langevin equation, introduced by Paul Langevin in 1908. Langevin equations have been broadly used to describe stochastic problems in image processing, physics, astronomy, chemistry, defence system, electrical and mechanical engineering. Brownian motion is well described by the Langevin equations when the random oscillation force is supposed to be Gaussian noise. For the removal of noise, mathematicians used fractional order differential equations, also they perform well in reducing the staircase effects as compared to ordinary differential equations. Thus it is very important to learn the idea of fractional Langevin equations; for more details, see [1–4].

Fractional differential equations (FDEs) provide an excellent tool for the description of memory and hereditary properties of different processes and materials. So, contrary to the classical derivative, the fractional derivative is nonlocal. Fractional calculus has played an important role in enhancing the mathematical modeling of several phenomena appearing in engineering and scientific disciplines, such as blood flow systems, control theory, aerodynamics, the nonlinear oscillation of earthquake, polymer rheology, regular variation in thermodynamics, etc. It has been observed that FDEs are more accurate than the integer-order derivatives. Therefore in the last decades, fractional calculus got considerable attention from researchers, see [5–52].

It is well known that the effects of a pulse are inevitable in many processes and phenomena. For example, in the population dynamics systems, there are abrupt changes in population size due to the effects such as diseases, harvesting, and so on. So, researchers have used impulsive differential equations to describe the aforesaid kinds of phenomena.

In recent times, impulsive fractional differential equations are well investigated with different approaches, we recommend the reader to [53–61].

In fields such as numerical analysis, optimization theory, and nonlinear analysis, mostly we deal with the approximate solutions, and hence we need to check how close these solutions are to the actual solutions of the related system or systems. Many approaches can be used for this purpose, but the Ulam–Hyers stability approach is a simple and easy one. The aforesaid stability was first pointed out by Ulam in 1940 [62] and then solved brilliantly by Hyers in 1941 [63]. Afterwards, stability of such form has been known as Ulam–Hyers stability. In 1978, Rassias [64] generalized the Ulam–Hyers approach by considering variables. Thereafter, mathematicians extended the notions for functional, differential, integrals as well as FDEs [65–72].

Recently, many mathematicians have devoted considerable attention to the existence, uniqueness, and different types of Hyers–Ulam stability of the solutions of nonlinear implicit fractional differential equations with Caputo fractional derivative, see [73–75].

Wang *et al.* in [76] studied generalized Ulam–Hyers–Rassias stability of the following fractional differential equation:

$$\begin{cases} {}^{c}\mathcal{D}^{\nu}_{0,w}z(w) = f(w,z(w)), \quad w \in (w_k,s_k], k = 0,1,\dots,m, 0 < \nu < 1, \\ z(w) = g_k(w,z(w)), \quad w \in (s_{k-1},w_k], k = 1,2,\dots,m. \end{cases} \tag{1.1}$$

Zada *et al.* [77] studied the existence and uniqueness of solutions by using Diaz Margolis's fixed point theorem and presented different types of Ulam–Hyers stability for a class of nonlinear implicit fractional differential equations with non-instantaneous integral impulses and nonlinear integral boundary conditions:

$$\begin{cases} {}^{c}\mathcal{D}^{\nu}_{0,w}z(w) = f(w,z(w),{}^{c}\mathcal{D}^{\nu}_{0,w}z(w)), \quad w \in (w_k,s_k], k = 0,1,\dots,m, 0 < \nu < 1, w \in (0,1], \\ z(w) = I^{\nu}_{s_{k-1},w_k}(\xi_k(w,z(w))), \quad w \in (s_{k-1},w_k], k = 0,1,\dots,m, \\ z(0) = \frac{1}{\Gamma\nu}\int_0^T (T-s)^{\nu-1}\eta(s,z(s))\,ds. \end{cases}$$

Zada *et al.* [78] studied the existence and uniqueness of solutions by using Diaz Margolis's fixed point theorem and presented different types of Ulam–Hyers stability for a class of nonlinear implicit sequential fractional differential equations with non-instantaneous integral impulses and multi-point boundary conditions:

$$\begin{cases} {}^{c}\mathcal{D}^{\nu}_{0,w}(\mathcal{D}+\lambda)z(w) = f(w,z(w),{}^{c}\mathcal{D}^{\nu}_{0,w}), \quad w \in (w_k,s_k], k = 0,1,\dots,m, 0 < \nu \le 1, \\ z(w) = g_k(w,z(w)), \quad w \in (s_{k-1},w_k], k = 1,2,\dots,m, \\ z(0) = 0, \qquad z(w_k) = 0, \quad k = 0,1,\dots,m. \end{cases}$$

In this paper, we study the following nonlocal boundary value problem of nonlinear implicit impulsive Langevin equation with mixed derivatives:

$$\begin{cases} {}^{c}\mathcal{D}^{\nu}_{0,w}(\mathcal{D}+\lambda)z(w) = f(w,z(w),{}^{c}\mathcal{D}^{\nu}_{0,w}(\mathcal{D}+\lambda)z(w)), \\ \quad w \in (w_k,s_k], k = 0,1,\dots,m, \\ z(w) = g_k(w,z(w)), \quad w \in (s_{k-1},w_k], k = 1,2,\dots,m, \\ z(0) = z_0, \qquad z(T) = \int_0^{\eta}\frac{1}{\Gamma p}(\eta-s)^{p-1}z(s)\,ds, \quad 0 < \eta < T, \end{cases} \tag{1.2}$$

where ${}^{c}\mathcal{D}^{\nu}_{0,w}$ represents the classical Caputo derivative [8] of order ν with the lower bound zero, $0 = w_0 < s_0 < w_1 < s_1 < \cdots < w_m < s_m = \tau$, τ is the pre-fixed number, and $\lambda \in \mathbb{R} \setminus \{0\}$, $0 < \nu < 1$, $p > 0$, z_0 are constants, $f : [0,\tau] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and $g_k : [s_{k-1}, w_k] \times \mathbb{R} \to \mathbb{R}$ is continuous for all $k = 1, 2, \dots, m$.

In the second section of this paper, we create a uniform framework to originate appropriate formula of solutions for our proposed model. In Sect. 3, we implement the concept of generalized Ulam–Hyers–Rassias stability of Eq. (1.2). Finally, we give an example which supports our main result.

2 Preliminaries

Let $J = [0,\tau]$ and $C(J,\mathbb{R})$ be the space of all continuous functions from J to $\mathbb{R}$ and a piecewise continuous functions space $PC(J,\mathbb{R}) = \{z : f \to \mathbb{R} : z \in ((w_k, w_{k-1})], \mathbb{R}), k = 0, 1, \dots, m,$ and there exist $z(w_k^-)$ and $z(w_k^+), k = 1, 2, \dots, m,$ with $z(w_k^-) = z(w_k^+)\}$.

Consider the linear form of (1.1) as follows:

$$\begin{cases} {}^{c}\mathcal{D}^{\nu}_{0,t}(\mathcal{D} + \lambda)z(w) = f(w), \quad w \in (w_k, s_k], k = 0, 1, \dots, m, 0 < \nu < 1, \\ z(w) = g_k(w), \quad w \in (s_{k-1}, w_k], k = 1, 2, \dots, m, \\ z(0) = z_0, \qquad z(T) = \theta I^p z(\eta) \\ \quad \text{where } I^p z(\eta) = \int_0^\eta \frac{1}{\Gamma p}(\eta - s)^{p-1} z(s)\,ds, 0 < \eta < T. \end{cases} \tag{2.1}$$

We recall some definitions of fractional calculus from [5] as follows.

Definition 2.1 The fractional integral of order ν from 0 to w for the function f is defined by

$$I^{\nu}_{0,w} f(w) = \frac{1}{\Gamma(\nu)} \int_0^w f(s)(w-s)^{\nu-1}\,ds, \quad w > 0, \ \nu > 0,$$

where $\Gamma(\cdot)$ is the gamma function.

Definition 2.2 The Riemann–Liouville fractional derivative of fractional order ν from 0 to w for a function f can be written as

$${}^{L}\mathcal{D}^{\nu}_{0,w} f(w) = \frac{1}{\Gamma(n-\nu)} \frac{d^n}{dt^n} \int_0^w \frac{f(s)}{(w-s)^{\nu+1-n}}\,ds, \quad w > 0, \ n-1 < \nu < n,$$

where $\Gamma(\cdot)$ is the gamma function.

Definition 2.3 The Caputo derivative of fractional order ν from 0 to w for a function f can be defined as

$${}^{c}\mathcal{D}^{\nu}_{0,w} f(w) = \frac{1}{\Gamma(n-\nu)} \int_0^w (w-s)^{n-\nu-1} f^n(s)\,ds, \quad \text{where } n = [\nu] + 1.$$

Definition 2.4 The general form of the classical Caputo derivative of order ν of a function f can be given as

$${}^{c}\mathcal{D}^{\nu}_{0,w} = {}^{L}\mathcal{D}^{\nu}_{0,w}\left(f(w) - \sum_{k=0}^{n-1} \frac{w^k}{k!} f^{(k)}(0)\right), \quad w > 0, \ n-1 < \nu < n.$$

Remark 2.5

(i) If $f(\cdot) \in C^m([0,\infty),\mathbb{R})$, then

$$
{}^{L}\mathcal{D}^{\nu}_{0,w}f(w) = \frac{1}{\Gamma(m-\nu)}\int_0^w \frac{f^m(s)}{(w-s)^{\nu+1-m}}\,ds = I^{m-\nu}_{0,w}f^{(m)}(w),
$$
$$
w > 0, m-1 < \nu < m.
$$

(ii) In Definition 2.4, the integrable function f can be discontinuous function. This fact can support us in considering impulsive fractional problems in the sequel.

Lemma 2.6 ([5]) *The fractional differential equation* ${}^{c}D^{\nu}f(w) = 0$ *with* $\nu > 0$, *involving Caputo differential operator* ${}^{c}D^{\nu}$, *has a solution in the following form*:

$$
f(w) = c_0 + c_1 w + c_2 w^2 + \cdots + c_{m-1}w^{n-1},
$$

where $c_k \in \mathbb{R}$, $k = 0,1,\ldots,m-1$, *and* $m = [\nu + 1]$.

Lemma 2.7 ([5]) *For arbitrary* $\nu > 0$, *we have*

$$
I^{\nu}\big({}^{c}\mathcal{D}^{\nu}f(w)\big) = c_0 + c_1 w + c_2 w^2 + \cdots + c_{m-1}w^{n-1},
$$

where $c_k \in \mathbb{R}$, $k = 0,1,\ldots,m-1$, *and* $m = [\nu + 1]$.

Lemma 2.8 ([6]) *Let* $\nu > 0$ *and* $\beta > 0, f \in L^1([a,b])$.
Then

$$
I^{\nu}I^{\beta}f(w) = I^{\nu+\beta}f(w), \qquad {}^{c}\mathcal{D}^{\nu}_{0,w}\big({}^{c}\mathcal{D}^{\beta}_{0,w}f(w)\big) = {}^{c}\mathcal{D}^{\nu+\beta}_{0,w}f(w), \quad \text{and}
$$
$$
I^{\nu}\mathcal{D}^{\nu}_{0,w}f(w) = f(w), \quad w \in [a,b].
$$

Lemma 2.9 *The function* $z \in PC(J,\mathbb{R})$ *is a solution of* (2.1) *if and only if*

$$
z(w) = \begin{cases}
\begin{cases}
\int_0^w e^{-\lambda(w-s)}I^{\nu}f(s)\,ds + \frac{A_{11}}{\Gamma(p+1)}\int_0^{\eta}(\eta-s)^p e^{-\lambda(\eta-s)}I^{\nu}f(s)\,ds \\
\quad - A_{11}\int_0^T e^{-\lambda(T-s)}I^{\nu}f(s)\,ds \\
\quad + (A_{11}(\eta^p E_{(1,p+1)}(aw) - e^{\lambda T}) + e^{\lambda T})z_0, \quad w \in (0,s_0],
\end{cases} \\
\begin{cases} g_k(w), \quad w \in (s_{k-1},w_k], k = 1,2,\ldots,m, \end{cases} \\
\begin{cases}
\int_0^w e^{-\lambda(w-s)}I^{\nu}f(s)\,ds + \frac{M}{\Gamma(p+1)}\int_0^{\eta}(\eta-s)^p e^{-\lambda(\eta-s)}I^{\nu}f(s)\,ds \\
\quad - M\int_0^T e^{-\lambda(T-s)}I^{\nu}f(s)\,ds \\
\quad + N\int_0^{w_k} e^{-\lambda(w_k-s)}I^{\nu}f(s)\,ds + Ng_k(w_k), \quad w \in (w_k,s_k], k = 0,1,\ldots,m,
\end{cases}
\end{cases}
$$

where

$$
A_{11} = \frac{\lambda\Gamma(p+1)}{(1-e^{-\lambda T})\Gamma(p+1) - \eta^p + \Gamma(p+1)\eta^p E_{(1,p+1)}(aw)},
$$
$$
B_k = \frac{\lambda\Gamma(p+1)(\eta^p E_{(1,p+1)}(aw) - e^{-\lambda T})}{\delta_k},
$$

$$A_k = \frac{\delta_k \lambda \Gamma(p+1) - \Gamma(p+1)(1-e^{\lambda w_k})(\lambda \Gamma(p+1)(\eta^p E_{(1,p+1)}(aw) - e^{-\lambda T}))}{\delta_k((1-e^{-\lambda T})\Gamma(p+1) - \eta^p + \Gamma(p+1)\eta^p E_{(1,p+1)}(aw))},$$

$$M_k = \frac{A_k(1-e^{-\lambda w})}{\lambda} - \frac{\Gamma(p+1)e^{-\lambda w}(1-e^{-\lambda w_k})}{\delta_k},$$

$$N_k = \frac{B_k(1-e^{-\lambda w})}{\lambda} - \frac{(1-e^{-\lambda T})\Gamma(p+1) - \eta^p + \Gamma(p+1)\eta^p E_{(1,p+1)}(aw)}{\delta_k e^{\lambda w}},$$

$$\delta_k = 2\Gamma(p+1)\big(e^{-\lambda w_k} - e^{-\lambda(w_k+T)} + \eta^p E_{(1,p+1)}(aw)e^{-\lambda w_k}\big) - \eta^p e^{-\lambda w_k} - \Gamma(p+1)E_{(1,p+1)}(aw).$$

Proof Let z be a solution of problem (2.1), we have the following cases.

Case 1: For $w \in [0, s_0]$, we consider

$$^{c}\mathcal{D}^{\nu}_{0,w}(\mathcal{D}+\lambda)z(w) = f(w), \qquad z(0) = z_0, \quad \text{and} \quad z(T) = I^p z(\eta),$$

where $\mathcal{D}$ denotes an ordinary differential operator. In light of Lemma 2.7 and an ordinary integration, we get

$$z(w) = \int_0^w e^{-\lambda(w-s)} I^\nu f(s)\,ds + c_0\left(\frac{1-e^{-\lambda w}}{\lambda}\right) + c_1 e^{-\lambda w}. \tag{2.2}$$

Using boundary conditions, we get

$$z(w) = \int_0^w e^{-\lambda(w-s)} I^\nu f(s)\,ds + \frac{A_{11}}{\Gamma(p+1)}\int_0^\eta (\eta-s)^p e^{-\lambda(\eta-s)} I^\nu f(s)\,ds - A_{11}\int_0^T e^{-\lambda(T-s)} I^\nu f(s)\,ds + \big(A_{11}\big(\eta^p E_{(1,p+1)}(aw) - e^{\lambda T}\big) + e^{\lambda T}\big)z_0.$$

For $w \in (s_0, w_1]$, $z(w) = g_1(w)$.

Case 2: For $w \in (w_1, s_1]$, we consider

$$^{c}\mathcal{D}^{\nu}_{0,w}(\mathcal{D}+\lambda)z(w) = f(w) \quad \text{with } z(w_1) = g_1(w_1).$$

Since $z(w_1) = g_1(w_1)$, then Eq. (2.2) is of the following type:

$$g_1(w_1) = \int_0^{w_1} e^{-\lambda(w_1-s)} I^\nu f(s)\,ds + c_0\left(\frac{1-e^{-\lambda w_1}}{\lambda}\right) + c_1 e^{-\lambda w_1}. \tag{2.3}$$

Using boundary conditions, we get

$$z(w) = \int_0^w e^{-\lambda(w-s)} I^\nu f(s)\,ds + \frac{M_1}{\Gamma(p+1)}\int_0^\eta (\eta-s)^p e^{-\lambda(\eta-s)} I^\nu f(s)\,ds - M_1\int_0^T e^{-\lambda(T-s)} I^\nu f(s)\,ds + N_1\int_0^{w_1} e^{-\lambda(w_1-s)} I^\nu f(s)\,ds + N_1 g_1(w_1),$$

where

$$A_{11} = \frac{\lambda\Gamma(p+1)}{(1-e^{-\lambda T})\Gamma(p+1) - \eta^p + \Gamma(p+1)\eta^p E_{(1,p+1)}(aw)},$$

$$
\begin{aligned}
B_{22} &= \frac{\lambda\Gamma(p+1)(\eta^p E_{(1,p+1)}(aw) - e^{-\lambda T})}{\delta_1}, \\
A_{22} &= \frac{\delta_1\lambda\Gamma(p+1) - \Gamma(p+1)(1-e^{\lambda w_1})(\lambda\Gamma(p+1)(\eta^p E_{(1,p+1)}(aw) - e^{-\lambda T}))}{\delta_1((1-e^{-\lambda T})\Gamma(p+1) - \eta^p + \Gamma(p+1)\eta^p E_{(1,p+1)}(aw))}, \\
M_1 &= \frac{A_{22}(1-e^{-\lambda w})}{\lambda} - \frac{\Gamma(p+1)e^{-\lambda w}(1-e^{-\lambda w_1})}{\delta_1}, \\
N_1 &= \frac{B_{22}(1-e^{-\lambda w})}{\lambda} - \frac{(1-e^{-\lambda T})\Gamma(p+1) - \eta^p + \Gamma(p+1)\eta^p E_{(1,p+1)}(aw)}{\delta_1 e^{\lambda w}}, \\
\delta_1 &= 2\Gamma(p+1)\big(e^{-\lambda w_1} - e^{-\lambda(w_1+T)} + \eta^p E_{(1,p+1)}(aw)e^{-\lambda w_1}\big) \\
&\quad - \eta^p e^{-\lambda w_1} - \Gamma(p+1)E_{(1,p+1)}(aw).
\end{aligned}
$$

Generally speaking, for $w \in (s_{k-1}, w_k]$, $z(w_k) = g_k(w_k)$.

Case 3: For $w \in (w_k, s_k]$, we consider

$$
\mathcal{D}^{\nu}_{0,w}(\mathcal{D}+\lambda)z(w) = f(w) \quad \text{with } z(w_k) = g_k(w_k) \quad \text{and} \quad u(T) = I^p z(\eta).
$$

By repeating again the same process, we have

$$
\begin{aligned}
z(w) &= \int_0^w e^{-\lambda(w-s)} I^\nu f(s)\,ds + \frac{M_k}{\Gamma(p+1)}\int_0^\eta (\eta-s)^p e^{-\lambda(\eta-s)} I^\nu f(s)\,ds \\
&\quad - M_k\int_0^T e^{-\lambda(T-s)} I^\nu f(s)\,ds + N_k\int_0^{w_k} e^{-\lambda(w_k-s)} I^\nu f(s)\,ds + N_k g_k(w_k),
\end{aligned}
$$

where

$$
\begin{aligned}
A_{11} &= \frac{\lambda\Gamma(p+1)}{(1-e^{-\lambda T})\Gamma(p+1) - \eta^p + \Gamma(p+1)\eta^p E_{(1,p+1)}(aw)}, \\
B_k &= \frac{\lambda\Gamma(p+1)(\eta^p E_{(1,p+1)}(aw) - e^{-\lambda T})}{\delta_k}, \\
A_k &= \frac{\delta_k\lambda\Gamma(p+1) - \Gamma(p+1)(1-e^{\lambda w_k})(\lambda\Gamma(p+1)(\eta^p E_{(1,p+1)}(aw) - e^{-\lambda T}))}{\delta_k((1-e^{-\lambda T})\Gamma(p+1) - \eta^p + \Gamma(p+1)\eta^p E_{(1,p+1)}(aw))}, \\
M_k &= \frac{A_k(1-e^{-\lambda w})}{\lambda} - \frac{\Gamma(p+1)e^{-\lambda w}(1-e^{-\lambda w_k})}{\delta_k}, \\
N_k &= \frac{B_k(1-e^{-\lambda w})}{\lambda} - \frac{(1-e^{-\lambda T})\Gamma(p+1) - \eta^p + \Gamma(p+1)\eta^p E_{(1,p+1)}(aw)}{\delta_k e^{\lambda w}}, \\
\delta_k &= 2\Gamma(p+1)\big(e^{-\lambda w_k} - e^{-\lambda(w_k+T)} + \eta^p E_{(1,p+1)}(aw)e^{-\lambda w_k}\big) \\
&\quad - \eta^p e^{-\lambda w_k} - \Gamma(p+1)E_{(1,p+1)}(aw). \qquad \square
\end{aligned}
$$

3 Stability result

By the ideas of stability in [1, 65, 79], we can generate a generalized Ulam–Hyers–Rassias stability concept for Eq. (1.2).

Let $\epsilon, \psi \geq 0$ and $\varphi \in PC(J, \mathbb{R}_+)$ be nondecreasing, consider

$$\begin{cases} |{}^c\mathcal{D}^\nu_{0,w}(\mathcal{D}+\lambda)z(w) - f(w, z(w), {}^c\mathcal{D}^\nu_{0,w}(\mathcal{D}+\lambda)z(w))| \leq \varphi(w), \\ \quad w \in (w_k, s_k], k = 0, 1, \dots, m, 0 < \nu < 1, \\ |z(w) - Ng_k(w, z(w))| \leq \psi, \quad w \in (s_{k-1}, w_k], k = 1, 2, \dots, m. \end{cases} \tag{3.1}$$

Remark 3.1 A function $z \in PC(J, \mathbb{R})$ is a solution of inequality (3.1) if and only if there are $G \in PC(J, \mathbb{R})$ and a sequence G_k, $k = 1, 2, \dots, m$ (which depends on z) such that

(i) $|G(w)| \leq \varphi(w)$, $w \in J$, and $|G_k| \leq \psi$, $k = 1, 2, \dots, m$;
(ii) ${}^c\mathcal{D}^\nu_{0,w}(\mathcal{D}+\lambda)z(w) = f(w, z(w), {}^c\mathcal{D}^\nu_{0,w}(\mathcal{D}+\lambda)z(w)) + G(w)$, $w \in (w_k, s_k]$, $k = 1, 2, \dots, m$;
(iii) $z(w) = N_k g_k(w, z(w)) + G_k$, $w \in (s_{k-1}, w_k]$, $k = 1, 2, \dots, m$.

Definition 3.2 Equation (1.2) is generalized Ulam–Hyers–Rassias stable with respect to (φ, ψ) if there exists $c_{f,\nu,g_k,\varphi} > 0$ such that, for each solution $z \in PC(J, \mathbb{R})$ of inequality (3.1), there is a solution $z_0 \in PC(J, \mathbb{R})$ of Eq. (1.2) with

$$\left|z(w) - z_0(w)\right| \leq c_{f,\nu,g_k,\varphi}\left(\varphi(w) + \psi\right), \quad w \in J.$$

Remark 3.3 If $z \in PC(J, \mathbb{R})$ is a solution of inequality (3.1), then z is a solution of the following integral inequality:

$$\begin{cases} \begin{cases} |z(w) - \int_0^w e^{-\lambda(w-s)} I^\nu f(s, z(s), {}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)z(s))\,ds \\ \qquad - \frac{A_{11}}{\Gamma(p+1)} \int_0^\eta (\eta - s)^p e^{-\lambda(\eta-s)} I^\nu f(s, z(s), {}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)z(s))\,ds \\ \qquad + A_{11} \int_0^T e^{-\lambda(T-s)} I^\nu f(s, z(s), {}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)z(s))\,ds \\ \qquad - (A_{11}(\eta^p E_{(1,p+1)}(aw) - e^{\lambda T}) + e^{\lambda T})z_0|, \\ \quad \leq \int_0^w e^{-\lambda(w-s)} I^\nu \varphi(s)\,ds - \frac{A_{11}}{\Gamma(p+1)} \int_0^\eta (\eta - s)^p e^{-\lambda(\eta-s)} I^\nu \varphi(s)\,ds \\ \qquad + A_{11} \int_0^T e^{-\lambda(T-s)} I^\nu \varphi(s)\,ds, \quad w \in (0, s_0]; \end{cases} \\ \begin{cases} |z(w) - N_k g_k(w, z(w))| \leq \psi, \quad w \in (s_{k-1}, w_k], k = 1, 2, \dots, m; \end{cases} \\ \begin{cases} |z(w) - \int_0^w e^{-\lambda(w-s)} I^\nu f(s, z(s), {}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)z(s))\,ds \\ \qquad - \frac{M_k}{\Gamma(p+1)} \int_0^\eta (\eta - s)^p e^{-\lambda(\eta-s)} I^\nu f(s, z(s), {}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)z(s))\,ds \\ \qquad + M_k \int_0^T e^{-\lambda(T-s)} I^\nu f(s, z(s), {}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)z(s))\,ds \\ \qquad - N_k \int_0^{w_k} e^{-\lambda(w_k-s)} I^\nu f(s, z(s), {}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)z(s))\,ds - N_k g_k(w_k, z(w_k))| \\ \quad \leq \int_0^w e^{-\lambda(w-s)} I^\nu \varphi(s)\,ds + \frac{M_k}{\Gamma(p+1)} \int_0^\eta (\eta - s)^p e^{-\lambda(\eta-s)} I^\nu \varphi(s)\,ds \\ \qquad + M_k \int_0^T e^{-\lambda(T-s)} I^\nu \varphi(s)\,ds + N_k \int_0^{w_k} e^{-\lambda(w_k-s)} I^\nu \varphi(s)\,ds + \psi, \\ \quad w \in (w_k, s_k], k = 0, 1, \dots, m. \end{cases} \end{cases} \tag{3.2}$$

In fact, by Remark 3.1, we get

$$\begin{cases} {}^c\mathcal{D}^\nu_{0,w}(\mathcal{D}+\lambda)z(w) = f(w, z(w), {}^c\mathcal{D}^\nu_{0,w}(\mathcal{D}+\lambda)z(w)) + G(w), \\ \quad w \in (w_k, s_k], k = 0, 1, \dots, m, 0 < \nu < 1, \\ z(w) = N_k g_k(w, z(w)) + G_k, \quad w \in (s_{k-1}, w_k], k = 1, 2, \dots, m. \end{cases} \tag{3.3}$$

Clearly, the solution of Eq. (3.3) is given by

$$z(w)=\begin{cases}\begin{cases}\int_0^w e^{-\lambda(w-s)}I^\nu(f(s,z(s),{}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)z(s))+G(s))\,ds\\ \quad+\frac{A_{11}}{\Gamma(p+1)}\int_0^\eta(\eta-s)^p e^{-\lambda(\eta-s)}I^\nu(f(s,z(s),{}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)z(s))+G(s))\,ds\\ \quad-A_{11}\int_0^T e^{-\lambda(T-s)}I^\nu(f(s,z(s),{}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)z(s))+G(s))\,ds\\ \quad+(A_{11}(\eta^p E_{(1,p+1)}(aw)-e^{\lambda T})+e^{\lambda T})z_0,\quad w\in(0,s_0],\end{cases}\\ \begin{cases}N_k g_k(w,z(w)),\quad w\in(s_{k-1},w_k],k=1,2,\ldots,m,\end{cases}\\ \begin{cases}\int_0^w e^{-\lambda(w-s)}I^\nu(f(s,z(s),{}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)z(s))+G(s))\,ds\\ \quad+\frac{M_k}{\Gamma(p+1)}\int_0^\eta(\eta-s)^p e^{-\lambda(\eta-s)}I^\nu(f(s,z(s),{}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)z(s))+G(s))\,ds\\ \quad-M_k\int_0^T e^{-\lambda(T-s)}I^\nu(f(s,z(s),{}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)z(s))+G(s))\,ds\\ \quad+N_k\int_0^{w_k} e^{-\lambda(w_k-s)}I^\nu(f(s,z(s),{}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)z(s))+G(s))\,ds\\ \quad+N_k g_k(w_k,z(w_k))+G_k,\quad w\in(w_k,s_k],k=0,1,\ldots,m.\end{cases}\end{cases}$$

For $w\in(w_k,s_k]$, $k=0,1,\ldots,m$, we get

$$\begin{aligned}&\biggl|z(w)-\int_0^w e^{-\lambda(w-s)}I^\nu f\bigl(s,z(s),{}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)z(s)\bigr)\,ds\\&\quad-\frac{M_k}{\Gamma(p+1)}\int_0^\eta(\eta-s)^p e^{-\lambda(\eta-s)}I^\nu f\bigl(s,z(s),{}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)z(s)\bigr)\,ds-N_k g_k\bigl(w_k,z(w_k)\bigr)\\&\quad+M_k\int_0^T e^{-\lambda(T-s)}I^\nu f\bigl(s,z(s),{}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)z(s)\bigr)\,ds\\&\quad-N_k\int_0^{w_k} e^{-\lambda(w_k-s)}I^\nu f\bigl(s,z(s),{}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)z(s)\bigr)\,ds\biggr|\\&\le\biggl|\int_0^w e^{-\lambda(w-s)}I^\nu G(s)\,ds\biggr|+\biggl|\frac{M_k}{\Gamma(p+1)}\int_0^\eta(\eta-s)^p e^{-\lambda(\eta-s)}I^\nu G(s)\,ds\biggr|\\&\quad+\biggl|M_k\int_0^T e^{-\lambda(T-s)}I^\nu G(s)\,ds\biggr|+\biggl|N_k\int_0^{w_k} e^{-\lambda(w_k-s)}I^\nu G(s)\,ds\biggr|+|G_k|\\&\le\int_0^w e^{-\lambda(w-s)}I^\nu\varphi(s)\,ds+\frac{M_k}{\Gamma(p+1)}\int_0^\eta(\eta-s)^p e^{-\lambda(\eta-s)}I^\nu\varphi(s)\,ds\\&\quad+M_k\int_0^T e^{-\lambda(T-s)}I^\nu\varphi(s)\,ds+N_k\int_0^{w_k} e^{-\lambda(w_k-s)}I^\nu\varphi(s)\,ds+\psi.\end{aligned}$$

Proceeding the above, we derive that

$$\bigl|z(w)-N_k g_k\bigl(w,z(w)\bigr)\bigr|\le|G_k|\le\psi,\quad w\in(s_{k-1},w_k],k=1,2,\ldots,m,$$

and

$$\begin{aligned}&\biggl|z(w)-\int_0^w e^{-\lambda(w-s)}I^\nu f\bigl(s,z(s),{}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)z(s)\bigr)\,ds\\&\quad-\frac{A_{11}}{\Gamma(p+1)}\int_0^\eta(\eta-s)^p e^{-\lambda(\eta-s)}I^\nu f\bigl(s,z(s),{}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)z(s)\bigr)\,ds\\&\quad+A_{11}\int_0^T e^{-\lambda(T-s)}I^\nu f\bigl(s,z(s),{}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)z(s)\bigr)\,ds\end{aligned}$$

$$
\begin{aligned}
&\quad - \left(A_{11}\left(\eta^p E_{(1,p+1)}(aw) - e^{\lambda T}\right) + e^{\lambda T}\right)z_0 \bigg| \\
&\le \left|\int_0^w e^{-\lambda(w-s)} I^\nu G(s)\,ds\right| + \left|\frac{A_{11}}{\Gamma(p+1)}\int_0^\eta (\eta-s)^p e^{-\lambda(\eta-s)} I^\nu G(s)\,ds\right| \\
&\quad + \left|A_{11}\int_0^T e^{-\lambda(T-s)} I^\nu G(s)\,ds\right| \\
&\le \int_0^w e^{-\lambda(w-s)} I^\nu \varphi(s)\,ds + \frac{A_{11}}{\Gamma(p+1)}\int_0^\eta (\eta-s)^p e^{-\lambda(\eta-s)} I^\nu \varphi(s)\,ds \\
&\quad + A_{11}\int_0^T e^{-\lambda(T-s)} I^\nu \varphi(s)\,ds, \quad w \in (0, s_0].
\end{aligned}
$$

4 Main results

This section is started with the following definition.

Definition 4.1 For a nonempty set V, a function $d : V \times V \to [0,\infty]$ is called a generalized metric on V if and only if d satisfies:

- ◇ $d(\upsilon_1,\upsilon_2) = 0$ if and only if $\upsilon_1 = \upsilon_2$;
- ◇ $d(\upsilon_1,\upsilon_2) = d(\upsilon_2,\upsilon_1)$ for all $\upsilon_1,\upsilon_2 \in V$;
- ◇ $d(\upsilon_1,\upsilon_3) \le d(\upsilon_1,\upsilon_2) + d(\upsilon_2,\upsilon_3)$ for all $\upsilon_1,\upsilon_2,\upsilon_3 \in V$.

Lemma 4.2 (see [80] (Generalized Diaz–Margolis's fixed point theorem)) *Let (V,d) be a generalized complete metric space. Assume that $T : V \to V$ is a strictly contractive operator with the Lipschitz constant $L < 1$. If there exists $k \ge 0$ such that $d(T^{k+1}v, T^k v) < \infty$ for some v in V, then the following statements are true*:

- (B_1) *The sequence $\{T^n v\}$ converges to a fixed point v^* of T;*
- (B_2) *The unique fixed point of T is $v^* \in V^* = \{u \in V \text{ such that } d(T^k v, u) < \infty\}$;*
- (B_3) *$u \in V^*$, then $d(u, v^*) \le \frac{1}{1-L} d(Tu, u)$.*

We can introduce some assumptions as follows:

- (H_1) $f \in C(J \times \mathbb{R}, \mathbb{R})$.
- (H_2) *There exists a positive constant Ł$_f$ such that*

$$
\begin{aligned}
&\left|f(w,u,m) - f(w,v,n)\right| \le Ł_{f_1}|u-v| + Ł_{f_2}|m-n| \\
&\quad \text{for each } w \in J \text{ and all } u,v,m,n \in \mathbb{R}.
\end{aligned}
$$

- (H_3) *$g_k \in C((s_{k-1}, w_k] \times \mathbb{R}, \mathbb{R})$ and there are positive constants Ł$_{gk}$, $k = 1,2,\dots,m$, such that*

$$
\left|g_k(w,v) - g_k(w,v)\right| \le Ł_{gk}|u-v| \quad \text{for each } w \in (s_{k-1}, w_k] \text{ and all } u,v \in \mathbb{R}.
$$

- (H_4) *Let $\varphi \in C(J, \mathbb{R}_+)$ be a nondecreasing function, there exist $C_\varphi, C_\gamma > 0$ such that*

$$
\int_0^w I^\nu\left(\varphi(s)\right)ds \le C_\varphi \varphi(w) \quad \text{for each } w \in J, \tag{4.1}
$$

$$
\int_0^w (\mathcal{D} + \lambda)\left(\varphi(s)\right)ds \le C_\gamma \varphi(w) \quad \text{for each } w \in J. \tag{4.2}
$$

Theorem 4.3 *Suppose that (H_1)–(H_2) are satisfied and also a function $z \in PC(J,\mathbb{R})$ satisfies (3.1). Then there exists a unique solution z_0 of Eq. (1.2) such that*

$$z_0(w) = \begin{cases} \begin{cases} \int_0^w e^{-\lambda(w-s)} I^\nu f(s, z_0(s), {}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)z_0(s))\,ds \\ \quad + \frac{A_{11}}{\Gamma(p+1)} \int_0^\eta (\eta-s)^p e^{-\lambda(\eta-s)} I^\nu f(s, z_0(s), {}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)z_0(s))\,ds \\ \quad - A_{11} \int_0^T e^{-\lambda(T-s)} I^\nu f(s, z_0(s), {}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)z_0(s))\,ds \\ \quad + (A_{11}(\eta^p E_{(1,p+1)}(aw) - e^{\lambda T}) + e^{\lambda T}) z_0, \quad w \in (0, s_0], \end{cases} \\ \begin{cases} g_k(w, z_0(w)), \quad w \in (s_{k-1}, w_k], k = 1,2,\dots,m, \end{cases} \\ \begin{cases} \int_0^w e^{-\lambda(w-s)} I^\nu f(s, z_0(s), {}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)z_0(s))\,ds \\ \quad + \frac{M_k}{\Gamma(p+1)} \int_0^\eta (\eta-s)^p e^{-\lambda(\eta-s)} I^\nu f(s, z_0(s), {}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)z_0(s))\,ds \\ \quad - M_k \int_0^T e^{-\lambda(T-s)} I^\nu f(s, z_0(s), {}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)z_0(s))\,ds \\ \quad + N_k \int_0^{w_k} e^{-\lambda(w_k-s)} I^\nu f(s, z_0(s), {}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)z_0(s))\,ds \\ \quad + N_k g_k(w_k, z_0(w_k)), \quad w \in (w_k, s_k], \end{cases} \end{cases} \tag{4.3}$$

and

$$\begin{aligned} \left|z(w) - z_0(w)\right| \le{}& \left\{\left(\left(\frac{1-e^{-\lambda w}}{\lambda}\right) + \frac{M_k}{\Gamma(p+1)}\frac{\eta^{p+1}}{p+1}\left(\frac{1-e^{-\lambda\eta}}{\lambda}\right) + M_k\left(\frac{1-e^{-\lambda T}}{\lambda}\right)\right.\right. \\ &\left.\left. + N_k\left(\frac{1-e^{-\lambda w_k}}{\lambda}\right)\right)(Ł_{f_1}C_\varphi + Ł_{f_2}C_\gamma) + N_k Ł_{gk}\right\}\left(\frac{\varphi(w)+\psi}{1-Ł}\right) \end{aligned} \tag{4.4}$$

for all $w \in J$ if $0 < \nu < 1$, and

$$Ł = \max\{Ł_1, Ł_2\} < 1, \tag{4.5}$$

where

$$\begin{aligned} Ł_1 ={}& \max\left\{\left(\left(\frac{1-e^{-\lambda w}}{\lambda}\right) + \frac{M_k}{\Gamma(p+1)}\frac{\eta^{p+1}}{p+1}\left(\frac{1-e^{-\lambda\eta}}{\lambda}\right) + M_k\left(\frac{1-e^{-\lambda T}}{\lambda}\right)\right.\right. \\ &\left.\left. + N_k\left(\frac{1-e^{-\lambda w_k}}{\lambda}\right)\right)(Ł_{f_1}C_\varphi + Ł_{f_2}C_\gamma) + N_k Ł_{gk} \text{ such that } k = 1,2,\dots,m\right\}, \\ Ł_2 ={}& \max\left\{Ł_{f_1}\left(\frac{1-e^{-\lambda w}}{\lambda}\right)\left(\frac{w^\nu}{\Gamma(\nu+1)}\right) + Ł_{f_2}\lambda\left(\frac{1-e^{-\lambda w}}{\lambda}\right)\right. \\ & + Ł_{f_2}\lambda\frac{M_k}{\Gamma(p+1)}\left(\frac{\eta^{p+1}}{p+1}\right)\left(\frac{1-e^{-\lambda\eta}}{\lambda}\right) \\ & + Ł_{f_1}\frac{M_k}{\Gamma(p+1)}\left(\frac{\eta^{p+1}}{p+1}\right)\left(\frac{1-e^{-\lambda\eta}}{\lambda}\right)\left(\frac{\eta^\nu}{\Gamma(\nu+1)}\right) + Ł_{f_1}M_k\left(\frac{1-e^{-\lambda T}}{\lambda}\right)\left(\frac{T^\nu}{\Gamma(\nu+1)}\right) \\ & + Ł_{f_2}\lambda M_k\left(\frac{1-e^{-\lambda T}}{\lambda}\right) + Ł_{f_1} i N_k\left(\frac{1-e^{-\lambda w_k}}{\lambda}\right)\left(\frac{w_k^\nu}{\Gamma(\nu+1)}\right) + Ł_{f_2}\lambda N_k\left(\frac{1-e^{-\lambda w_k}}{\lambda}\right) \\ &\left. + N_k Ł_{gk}, \text{ such that } k = 0,1,\dots,m\right\}. \end{aligned}$$

Proof Consider the space of piecewise continuous functions

$$V = \{\mu : J \to \mathbb{R} \text{ such that } \mu \in PC(J,\mathbb{R})\},$$

endowed with the generalized metric on V, defined by

$$\begin{aligned} d(\mu,\upsilon) = \inf\bigl\{&C_1+C_2 \in [0,+\infty] \text{ such that} \\ &\bigl|\mu(w)-\upsilon(w)\bigr| \le (C_1+C_2)\bigl(\varphi(w)+\psi\bigr) \text{ for all } w\in J\bigr\}, \end{aligned} \tag{4.6}$$

where

$$C_1 \in \bigl\{C\in[0,\infty] \text{ such that } \bigl|\mu(w)-\upsilon(w)\bigr| \le C\varphi(w) \text{ for all } w\in(w_k,s_k], k=0,1,\ldots,m\bigr\}$$

and

$$C_2 \in \bigl\{C\in[0,\infty] \text{ such that } \bigl|\mu(w)-\upsilon(w)\bigr| \le C\psi \text{ for all } w\in(s_{k-1},w_k], k=1,2,\ldots,m\bigr\}.$$

It is easy to verify that (V,d) is a complete generalized metric space [80].

Define an operator $\Lambda : V \to V$ by

$$(\Lambda z)(w) = \begin{cases} \begin{cases} \int_0^w e^{-\lambda(w-s)} I^\nu f(s,z(s),{}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)z(s))\,ds \\ \quad + \frac{A_{11}}{\Gamma(p+1)}\int_0^\eta (\eta-s)^p e^{-\lambda(\eta-s)} I^\nu f(s,z(s),{}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)z(s))\,ds \\ \quad - A_{11}\int_0^T e^{-\lambda(T-s)} I^\nu f(s,z(s),{}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)z(s))\,ds \\ \quad + (A_{11}(\eta^p E_{(1,p+1)}(aw) - e^{\lambda T}) + e^{\lambda T})z_0, \quad w\in(0,s_0], \end{cases} \\ \begin{cases} g_k(w,z(w)), \quad w\in(s_{k-1},w_k], k=1,2,\ldots,m, \end{cases} \\ \begin{cases} \int_0^w e^{-\lambda(w-s)} I^\nu f(s,z(s),{}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)z(s))\,ds \\ \quad + \frac{M_k}{\Gamma(p+1)}\int_0^\eta (\eta-s)^p e^{-\lambda(\eta-s)} I^\nu f(s,z(s))\,ds \\ \quad - M_k\int_0^T e^{-\lambda(T-s)} I^\nu f(s,z(s),{}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)z(s))\,ds \\ \quad + N_k\int_0^{w_k} e^{-\lambda(t_k-s)} I^\nu f(s,z(s),{}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)z(s))\,ds \\ \quad + N_k g_k(w_k,z(w_k)), \quad w\in(w_k,s_k], k=0,1,\ldots,m, \end{cases} \end{cases} \tag{4.7}$$

for all z belonging to V and $w\in J$. Obviously, according to (H_1), Λ is a well-defined operator.

Next we shall verify that Λ is strictly contractive on V. Note that according to definition of (V,d), for any $\mu,\upsilon\in V$, it is possible to find $C_1,C_2\in[0,\infty]$ such that

$$\bigl|\mu(w)-\upsilon(w)\bigr| \le \begin{cases} C_1\varphi(w), & w\in(w_k,s_k], k=0,\ldots,m, \\ C_2\psi, & w\in(s_{k-1},w_k], k=1,\ldots,m. \end{cases} \tag{4.8}$$

From the definition of Λ in Eq. (4.7), (H_2), (H_3), and (4.8), we obtain the following.

Case 1: For $w\in[0,s_0]$,

$$\begin{aligned} &\bigl|(\Lambda\mu)(w)-(\Lambda\upsilon)(w)\bigr| \\ &\quad = \biggl|\int_0^w e^{-\lambda(w-s)} I^\nu f\bigl(s,\mu(s),{}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)\mu(s)\bigr)\,ds \\ &\qquad - \int_0^w e^{-\lambda(w-s)} I^\nu f\bigl(s,\upsilon(s),{}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)\upsilon(s)\bigr)\,ds \end{aligned}$$

$$
\begin{aligned}
&+\frac{A_{11}}{\Gamma(p+1)}\int_0^\eta(\eta-s)^p e^{-\lambda(\eta-s)}I^\nu f\bigl(s,\mu(s),{}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)\mu(s)\bigr)\,ds\\
&-A_{11}\int_0^T e^{-\lambda(T-s)}I^\nu f\bigl(s,\mu(s),{}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)\mu(s)\bigr)\,ds\\
&+\bigl(A_{11}\bigl(\eta^p E_{(1,p+1)}(aw)-e^{\lambda T}\bigr)+e^{\lambda T}\bigr)z_0\\
&-\frac{A_{11}}{\Gamma(p+1)}\int_0^\eta(\eta-s)^p e^{-\lambda(\eta-s)}I^\nu f\bigl(s,\upsilon(s),{}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)\upsilon(s)\bigr)\,ds\\
&+A_{11}\int_0^T e^{-\lambda(T-s)}I^\nu f\bigl(s,\upsilon(s),{}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)\upsilon(s)\bigr)\,ds\\
&-\bigl(A_{11}\bigl(\eta^p E_{(1,p+1)}(aw)-e^{\lambda T}\bigr)+e^{\lambda T}\bigr)z_0\Big|\\
\le{}&\Big|\int_0^w e^{-\lambda(w-s)}I^\nu f\bigl(s,\mu(s),{}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)\mu(s)\bigr)\,ds\\
&-\int_0^w e^{-\lambda(w-s)}I^\nu f\bigl(s,\upsilon(s),{}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)\upsilon(s)\bigr)\,ds\Big|\\
&+\Big|\frac{A_{11}}{\Gamma(p+1)}\int_0^\eta(\eta-s)^p e^{-\lambda(\eta-s)}I^\nu f\bigl(s,\mu(s),{}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)\mu(s)\bigr)\,ds\\
&-\frac{A_{11}}{\Gamma(p+1)}\int_0^\eta(\eta-s)^p e^{-\lambda(\eta-s)}I^\nu f\bigl(s,\upsilon(s),{}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)\upsilon(s)\bigr)\,ds\Big|\\
&+\Big|A_{11}\int_0^T e^{-\lambda(T-s)}I^\nu f\bigl(s,\upsilon(s),{}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)\upsilon(s)\bigr)\,ds\\
&-A_{11}\int_0^T e^{-\lambda(T-s)}I^\nu f\bigl(s,\mu(s),{}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)\mu(s)\bigr)\,ds\Big|\\
\le{}&\int_0^w e^{-\lambda(w-s)}I^\nu\big|f\bigl(s,\mu(s),{}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)\mu(s)\bigr)\,ds-f\bigl(s,\upsilon(s),{}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)\upsilon(s)\bigr)\,ds\big|\\
&+A_{11}\int_0^T e^{-\lambda(T-s)}I^\nu\big|f\bigl(s,\upsilon(s),{}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)\upsilon(s)\bigr)\,ds-f\bigl(s,\mu(s),{}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)\mu(s)\bigr)\,ds\big|\\
&+\frac{A_{11}}{\Gamma(p+1)}\int_0^\eta(\eta-s)^p e^{-\lambda(\eta-s)}I^\nu\big|f\bigl(s,\mu(s),{}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)\mu(s)\bigr)\,ds\\
&-f\bigl(s,\upsilon(s),{}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)\upsilon(s)\bigr)\,ds\big|\\
\le{}&Ł_{f_1}\int_0^w e^{-\lambda(w-s)}I^\nu\big|\mu(s)-\upsilon(s)\big|\,ds\\
&+Ł_{f_2}\int_0^w e^{-\lambda(w-s)}I^\nu\big|{}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)\mu(s)-{}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)\upsilon(s)\big|\,ds\\
&+Ł_{f_1}A_{11}\int_0^T e^{-\lambda(T-s)}I^\nu\big|\upsilon(s)-\mu(s)\big|\,ds\\
&+Ł_{f_1}\frac{A_{11}}{\Gamma(p+1)}\int_0^\eta(\eta-s)^p e^{-\lambda(\eta-s)}I^\nu\big|\mu(s)-\upsilon(s)\big|\,ds\\
&+Ł_{f_2}A_{11}\int_0^T e^{-\lambda(T-s)}I^\nu\big|{}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)\upsilon(s)-{}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)\mu(s)\big|\,ds
\end{aligned}
$$

$$
\begin{aligned}
&+ Ł_{f_2}\frac{A_{11}}{\Gamma(p+1)}\int_0^\eta(\eta-s)^p e^{-\lambda(\eta-s)}I^\nu\big|{}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)\mu(s)-{}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)\upsilon(s)\big|\,ds\\
&= Ł_{f_1}\int_0^w e^{-\lambda(w-s)}I^\nu\big|\mu(s)-\upsilon(s)\big|\,ds + Ł_{f_2}\int_0^w e^{-\lambda(w-s)}I^{\nu c}\mathcal{D}^\nu(\mathcal{D}+\lambda)\big|\mu(s)-\upsilon(s)\big|\,ds\\
&+ Ł_{f_1}A_{11}\int_0^T e^{-\lambda(T-s)}I^\nu\big|\upsilon(s)-\mu(s)\big|\,ds\\
&+ Ł_{f_2}A_{11}\int_0^T e^{-\lambda(T-s)}I^{\nu c}\mathcal{D}^\nu(\mathcal{D}+\lambda)\big|\upsilon(s)-\mu(s)\big|\,ds\\
&+ Ł_{f_1}\frac{A_{11}}{\Gamma(p+1)}\int_0^\eta(\eta-s)^p e^{-\lambda(\eta-s)}I^\nu\big|\mu(s)-\upsilon(s)\big|\,ds\\
&+ Ł_{f_2}\frac{A_{11}}{\Gamma(p+1)}\int_0^\eta(\eta-s)^p e^{-\lambda(\eta-s)}I^{\nu c}\mathcal{D}^\nu(\mathcal{D}+\lambda)\big|\mu(s)-\upsilon(s)\big|\,ds\\
&\le Ł_{f_1}C_1\int_0^w e^{-\lambda(w-s)}I^\nu\big|\varphi(s)\big|\,ds + Ł_{f_2}C_1\int_0^w e^{-\lambda(w-s)}I^{\nu c}\mathcal{D}^\nu(\mathcal{D}+\lambda)\big|\varphi(s)\big|\,ds\\
&+ Ł_{f_1}C_1A_{11}\int_0^T e^{-\lambda(T-s)}I^\nu\big|\varphi(s)\big|\,ds + Ł_{f_2}C_1A_{11}\int_0^T e^{-\lambda(T-s)}I^{\nu c}\mathcal{D}^\nu(\mathcal{D}+\lambda)\big|\varphi(s)\big|\,ds\\
&+ \frac{A_{11}C_1Ł_{f_1}}{\Gamma(p+1)}\int_0^\eta(\eta-s)^p e^{-\lambda(\eta-s)}I^\nu\big|\varphi(s)\big|\,ds\\
&+ \frac{A_{11}C_1Ł_{f_2}}{\Gamma(p+1)}\int_0^\eta(\eta-s)^p e^{-\lambda(\eta-s)}I^{\nu c}\mathcal{D}^\nu(\mathcal{D}+\lambda)\big|\varphi(s)\big|\,ds\\
&= Ł_{f_1}C_1\int_0^w e^{-\lambda(w-s)}I^\nu\big|\varphi(s)\big|\,ds + Ł_{f_2}C_1\int_0^w e^{-\lambda(w-s)}(\mathcal{D}+\lambda)\big|\varphi(s)\big|\,ds\\
&+ Ł_{f_1}C_1A_{11}\int_0^T e^{-\lambda(T-s)}I^\nu\big|\varphi(s)\big|\,ds + Ł_{f_2}C_1A_{11}\int_0^T e^{-\lambda(T-s)}(\mathcal{D}+\lambda)\big|\varphi(s)\big|\,ds\\
&+ \frac{A_{11}C_1Ł_{f_1}}{\Gamma(p+1)}\int_0^\eta(\eta-s)^p e^{-\lambda(\eta-s)}I^\nu\big|\varphi(s)\big|\,ds\\
&+ \frac{A_{11}C_1Ł_{f_2}}{\Gamma(p+1)}\int_0^\eta(\eta-s)^p e^{-\lambda(\eta-s)}(\mathcal{D}+\lambda)\big|\varphi(s)\big|\,ds\\
&\le Ł_{f_1}C_1\left(\int_0^w e^{-\lambda(w-s)}\,ds\right)\left(\int_0^w I^\nu\big(\varphi(s)\big)\,ds\right)\\
&+ Ł_{f_2}C_1\left(\int_0^w e^{-\lambda(w-s)}\,ds\right)\left(\int_0^w(\mathcal{D}+\lambda)\big(\varphi(s)\big)\,ds\right)\\
&+ Ł_{f_1}C_1A_{11}\left(\int_0^T e^{-\lambda(T-s)}\,ds\right)\left(\int_0^T I^\nu\big(\varphi(s)\big)\,ds\right)\\
&+ Ł_{f_2}C_1A_{11}\left(\int_0^T e^{-\lambda(T-s)}\,ds\right)\left(\int_0^T(\mathcal{D}+\lambda)\big(\varphi(s)\big)\,ds\right)\\
&+ Ł_{f_1}C_1\frac{A_{11}}{\Gamma(p+1)}\left(\int_0^\eta(\eta-s)^p\,ds\right)\left(\int_0^\eta e^{-\lambda(\eta-s)}\,ds\right)\left(\int_0^\eta I^\nu\big(\varphi(s)\big)\,ds\right)\\
&+ Ł_{f_2}C_1\frac{A_{11}}{\Gamma(p+1)}\left(\int_0^\eta(\eta-s)^p\,ds\right)\left(\int_0^\eta e^{-\lambda(\eta-s)}\,ds\right)\left(\int_0^\eta(\mathcal{D}+\lambda)\big(\varphi(s)\big)\,ds\right)\\
&\le Ł_{f_1}C_1\left(\frac{1-e^{-\lambda w}}{\lambda}\right)C_\varphi\varphi(w) + Ł_{f_2}C_1\left(\frac{1-e^{-\lambda w}}{\lambda}\right)C_\gamma\varphi(w)
\end{aligned}
$$

$$
\begin{aligned}
&+ Ł_{f_1} C_1 A_{11}\left(\frac{1-e^{-\lambda T}}{\lambda}\right) C_\varphi \varphi(w) + Ł_{f_2} C_1 A_{11}\left(\frac{1-e^{-\lambda T}}{\lambda}\right) C_\gamma \varphi(w) \\
&+ Ł_{f_1} C_1 \frac{A_{11}}{\Gamma(p+1)} \frac{\eta^{p+1}}{p+1}\left(\frac{1-e^{-\lambda \eta}}{\lambda}\right) C_\varphi \varphi(w) \\
&+ Ł_{f_2} C_1 \frac{A_{11}}{\Gamma(p+1)} \frac{\eta^{p+1}}{p+1}\left(\frac{1-e^{-\lambda \eta}}{\lambda}\right) C_\gamma \varphi(w) \\
={}& C_1\left(\frac{1-e^{-\lambda w}}{\lambda}\right)\varphi(w)(Ł_{f_1} C_\varphi + Ł_{f_2} C_\gamma) + C_1 A_{11}\left(\frac{1-e^{-\lambda T}}{\lambda}\right)\varphi(w)(Ł_{f_1} C_\varphi + Ł_{f_2} C_\gamma) \\
&+ C_1 \frac{A_{11}}{\Gamma(p+1)} \frac{\eta^{p+1}}{p+1}\left(\frac{1-e^{-\lambda \eta}}{\lambda}\right)\varphi(w)(Ł_{f_1} C_\varphi + Ł_{f_2} C_\gamma) \\
={}& \left(\left(\frac{1-e^{-\lambda w}}{\lambda}\right) + A_{11}\left(\frac{1-e^{-\lambda T}}{\lambda}\right) + \frac{A_{11}}{\Gamma(p+1)} \frac{\eta^{p+1}}{p+1}\left(\frac{1-e^{-\lambda \eta}}{\lambda}\right)\right) \\
&\times C_1 \varphi(w)(Ł_{f_1} C_\varphi + Ł_{f_2} C_\gamma).
\end{aligned}
$$

Case 2: For $w \in (s_{k-1}, w_k]$, we have

$$
\left|(\Lambda\mu)(w) - (\Lambda\upsilon)(w)\right| = \left|g_k\big(w,\mu(w)\big) - g_k\big(w,\upsilon(w)\big)\right| \le Ł_{gk}\left|\mu(w) - \upsilon(w)\right| \le Ł_{gk} C_2 \psi.
$$

Case 3: For $w \in (w_k, s_k]$ and $s \in (w_k, s_k]$,

$$
\begin{aligned}
&\left|(\Lambda\mu)(w) - (\Lambda\upsilon)(w)\right| \\
={}& \left| \int_0^w e^{-\lambda(w-s)} I^\nu f\big(s,\mu(s),{}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)\mu(s)\big)\,ds \right. \\
&- \int_0^w e^{-\lambda(w-s)} I^\nu f\big(s,\upsilon(s),{}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)\upsilon(s)\big)\,ds \\
&+ \frac{M_k}{\Gamma(p+1)} \int_0^\eta (\eta-s)^p e^{-\lambda(\eta-s)} I^\nu f\big(s,\mu(s),{}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)\mu(s)\big)\,ds \\
&- M_k \int_0^T e^{-\lambda(T-s)} I^\nu f\big(s,\mu(s),{}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)\mu(s)\big)\,ds \\
&+ N_k \int_0^{w_k} e^{-\lambda(w_k-s)} I^\nu f\big(s,\mu(s),{}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)\mu(s)\big)\,ds \\
&- \frac{M_k}{\Gamma(p+1)} \int_0^\eta (\eta-s)^p e^{-\lambda(\eta-s)} I^\nu f\big(s,\upsilon(s),{}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)\upsilon(s)\big)\,ds \\
&+ M_k \int_0^T e^{-\lambda(T-s)} I^\nu f\big(s,\upsilon(s),{}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)\upsilon(s)\big)\,ds \\
&- N_k \int_0^{w_k} e^{-\lambda(w_k-s)} I^\nu f\big(s,\upsilon(s),{}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)\upsilon(s)\big)\,ds \\
&\left. + N_k g_k\big(w_k,\mu(w_k)\big) - N_k g_k\big(w_k,\nu(w_k)\big) \right| \\
\le{}& \int_0^w e^{-\lambda(w-s)} I^\nu \big| f\big(s,\mu(s),{}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)\mu(s)\big) \\
&- f\big(s,\upsilon(s),{}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)\upsilon(s)\big)\big|\,ds + N_k Ł_{gk} C_2 \psi \\
&+ \frac{M_k}{\Gamma(p+1)} \int_0^\eta (\eta-s)^p e^{-\lambda(\eta-s)} I^\nu \big| f\big(s,\mu(s),{}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)\mu(s)\big)
\end{aligned}
$$

$$
\begin{aligned}
&-f\big(s,\upsilon(s),{}^{c}\mathcal{D}^{\nu}(\mathcal{D}+\lambda)\upsilon(s)\big)\big|\,ds\\
&+M_k\int_0^T e^{-\lambda(T-s)}I^{\nu}\big|f\big(s,\upsilon(s),{}^{c}\mathcal{D}^{\nu}(\mathcal{D}+\lambda)\upsilon(s)\big)-f\big(s,\mu(s),{}^{c}\mathcal{D}^{\nu}(\mathcal{D}+\lambda)\mu(s)\big)\big|\,ds\\
&+N_k\int_0^{w_k} e^{-\lambda(w_k-s)}I^{\nu}\big|f\big(s,\mu(s),{}^{c}\mathcal{D}^{\nu}(\mathcal{D}+\lambda)\mu(s)\big)-f\big(s,\upsilon(s),{}^{c}\mathcal{D}^{\nu}(\mathcal{D}+\lambda)\upsilon(s)\big)\big|\,ds\\
\le{}&Ł_{f_1}\int_0^w e^{-\lambda(w-s)}I^{\nu}\big|\mu(s)-\upsilon(s)\big|\,ds\\
&+Ł_{f_2}\int_0^w e^{-\lambda(w-s)}I^{\nu}\big|{}^{c}\mathcal{D}^{\nu}(\mathcal{D}+\lambda)\mu(s)-{}^{c}\mathcal{D}^{\nu}(\mathcal{D}+\lambda)\upsilon(s)\big|\,ds\\
&+Ł_{f_1}\frac{M_k}{\Gamma(p+1)}\int_0^{\eta}(\eta-s)^p e^{-\lambda(\eta-s)}I^{\nu}\big|\mu(s)-\upsilon(s)\big|\,ds\\
&+Ł_{f_1}M_k\int_0^T e^{-\lambda(T-s)}I^{\nu}\big|\upsilon(s)-\mu(s)\big|\,ds\\
&+Ł_{f_2}\frac{M_k}{\Gamma(p+1)}\int_0^{\eta}(\eta-s)^p e^{-\lambda(\eta-s)}I^{\nu}\big|{}^{c}\mathcal{D}^{\nu}(\mathcal{D}+\lambda)\mu(s)-{}^{c}\mathcal{D}^{\nu}(\mathcal{D}+\lambda)\upsilon(s)\big|\,ds\\
&+Ł_{f_2}M_k\int_0^T e^{-\lambda(T-s)}I^{\nu}\big|{}^{c}\mathcal{D}^{\nu}(\mathcal{D}+\lambda)\upsilon(s)-{}^{c}\mathcal{D}^{\nu}(\mathcal{D}+\lambda)\mu(s)\big|\,ds\\
&+Ł_{f_2}N_k\int_0^{w_k} e^{-\lambda(w_k-s)}I^{\nu}\big|{}^{c}\mathcal{D}^{\nu}(\mathcal{D}+\lambda)\mu(s)-{}^{c}\mathcal{D}^{\nu}(\mathcal{D}+\lambda)\upsilon(s)\big|\,ds\\
&+Ł_{f_1}N_k\int_0^{w_k} e^{-\lambda(w_k-s)}I^{\nu}\big|\mu(s)-\upsilon(s)\big|\,ds+N_kŁ_{gk}C_2\psi\\
={}&Ł_{f_1}\int_0^w e^{-\lambda(w-s)}I^{\nu}\big|\mu(s)-\upsilon(s)\big|\,ds+Ł_{f_2}\int_0^w e^{-\lambda(w-s)}I^{\nu c}\mathcal{D}^{\nu}(\mathcal{D}+\lambda)\big|\mu(s)-\upsilon(s)\big|\,ds\\
&+Ł_{f_1}\frac{M_k}{\Gamma(p+1)}\int_0^{\eta}(\eta-s)^p e^{-\lambda(\eta-s)}I^{\nu}\big|\mu(s)-\upsilon(s)\big|\,ds\\
&+Ł_{f_1}N_k\int_0^{w_k} e^{-\lambda(w_k-s)}I^{\nu}\big|\mu(s)-\upsilon(s)\big|\,ds\\
&+Ł_{f_2}\frac{M_k}{\Gamma(p+1)}\int_0^{\eta}(\eta-s)^p e^{-\lambda(\eta-s)}I^{\nu c}\mathcal{D}^{\nu}(\mathcal{D}+\lambda)\big|\mu(s)-\upsilon(s)\big|\,ds\\
&+Ł_{f_1}M_k\int_0^T e^{-\lambda(T-s)}I^{\nu}\big|\upsilon(s)-\mu(s)\big|\,ds\\
&+Ł_{f_2}M_k\int_0^T e^{-\lambda(T-s)}I^{\nu c}\mathcal{D}^{\nu}(\mathcal{D}+\lambda)\big|\upsilon(s)-\mu(s)\big|\,ds\\
&+Ł_{f_2}N_k\int_0^{w_k} e^{-\lambda(w_k-s)}I^{\nu c}\mathcal{D}^{\nu}(\mathcal{D}+\lambda)\big|\mu(s)-\upsilon(s)\big|\,ds+N_kŁ_{gk}C_2\psi\\
\le{}&Ł_{f_1}C_1\int_0^w e^{-\lambda(w-s)}I^{\nu}\big|\varphi(s)\big|\,ds+Ł_{f_2}C_1\int_0^w e^{-\lambda(w-s)}(\mathcal{D}+\lambda)\big|\varphi(s)\big|\,ds\\
&+Ł_{f_1}C_1\frac{M_k}{\Gamma(p+1)}\int_0^{\eta}(\eta-s)^p e^{-\lambda(\eta-s)}I^{\nu}\big|\varphi(s))\big|\,ds\\
&+Ł_{f_2}C_1M_k\int_0^T e^{-\lambda(T-s)}(\mathcal{D}+\lambda)\big|\varphi(s)\big|\,ds\\
&+Ł_{f_2}C_1\frac{M_k}{\Gamma(p+1)}\int_0^{\eta}(\eta-s)^p e^{-\lambda(\eta-s)}(\mathcal{D}+\lambda)\big|\varphi(s)\big|\,ds
\end{aligned}
$$

$$
\begin{aligned}
&+ Ł_{f_1} C_1 M_k \int_0^T e^{-\lambda(T-s)} I^\nu \left|\varphi(s)\right| ds \\
&+ Ł_{f_1} C_1 N_k \int_0^{w_k} e^{-\lambda(w_k-s)} I^\nu \left|\mu(s) - \upsilon(s)\right| ds \\
&+ Ł_{f_2} C_1 N_k \int_0^{w_k} e^{-\lambda(w_k-s)} (\mathcal{D} + \lambda) \left|\varphi(s)\right| ds + N_k Ł_{g_k} C_2 \psi \\
\le{}& Ł_{f_1} C_1 \left(\int_0^w e^{-\lambda(w-s)}\, ds\right) \left(\int_0^w I^\nu \big(\varphi(s)\big)\, ds\right) \\
&+ Ł_{f_2} C_1 \left(\int_0^w e^{-\lambda(w-s)}\, ds\right) \left(\int_0^w (\mathcal{D} + \lambda) \big(\varphi(s)\big)\, ds\right) \\
&+ Ł_{f_1} C_1 \frac{M_k}{\Gamma(p+1)} \left(\int_0^\eta (\eta - s)^p\, ds\right) \left(\int_0^\eta e^{-\lambda(\eta-s)}\, ds\right) \left(\int_0^\eta I^\nu \big(\varphi(s)\big)\, ds\right) \\
&+ Ł_{f_2} C_1 \frac{M_k}{\Gamma(p+1)} \left(\int_0^\eta (\eta - s)^p\, ds\right) \left(\int_0^\eta e^{-\lambda(\eta-s)}\, ds\right) \left(\int_0^\eta (\mathcal{D} + \lambda) \big(\varphi(s)\big)\, ds\right) \\
&+ Ł_{f_1} C_1 M_k \left(\int_0^T e^{-\lambda(T-s)}\, ds\right) \left(\int_0^T I^\nu \big(\varphi(s)\big)\, ds\right) \\
&+ Ł_{f_1} C_1 N_k \left(\int_0^{w_k} e^{-\lambda(w_k-s)}\, ds\right) \left(\int_0^{w_k} I^\nu \big(\varphi(s)\big)\, ds\right) \\
&+ Ł_{f_2} C_1 M_k \left(\int_0^T e^{-\lambda(T-s)}\, ds\right) \left(\int_0^T (\mathcal{D} + \lambda) \big(\varphi(s)\big)\, ds\right) \\
&+ Ł_{f_2} C_1 N_k \left(\int_0^{w_k} e^{-\lambda(w_k-s)}\, ds\right) \left(\int_0^{w_k} (\mathcal{D} + \lambda) \big(\varphi(s)\big)\, ds\right) + N_k Ł_{g_k} C_2 \psi \\
\le{}& Ł_{f_1} C_1 \left(\frac{1 - e^{-\lambda w}}{\lambda}\right) C_\varphi \varphi(w) + Ł_{f_2} C_1 \left(\frac{1 - e^{-\lambda w}}{\lambda}\right) C_\gamma \varphi(w) \\
&+ Ł_{f_1} C_1 N_k \left(\frac{1 - e^{-\lambda w_k}}{\lambda}\right) C_\varphi \varphi(w) \\
&+ Ł_{f_1} C_1 \frac{M_k}{\Gamma(p+1)} \frac{\eta^{p+1}}{p+1} \left(\frac{1 - e^{-\lambda \eta}}{\lambda}\right) C_\varphi \varphi(w) \\
&+ Ł_{f_2} C_1 \frac{M_k}{\Gamma(p+1)} \frac{\eta^{p+1}}{p+1} \left(\frac{1 - e^{-\lambda \eta}}{\lambda}\right) C_\gamma \varphi(w) \\
&+ Ł_{f_1} C_1 M_k \left(\frac{1 - e^{-\lambda T}}{\lambda}\right) C_\varphi \varphi(w) + Ł_{f_2} C_1 M_k \left(\frac{1 - e^{-\lambda T}}{\lambda}\right) C_\gamma \varphi(w) \\
&+ Ł_{f_2} C_1 N_k \left(\frac{1 - e^{-\lambda w_k}}{\lambda}\right) C_\gamma \varphi(w) + N_k Ł_{g_k} C_2 \psi \\
={}& C_1 \left(\frac{1 - e^{-\lambda w}}{\lambda}\right) \varphi(w) (Ł_{f_1} C_\varphi + Ł_{f_2} C_\gamma) \\
&+ C_1 \frac{M_k}{\Gamma(p+1)} \frac{\eta^{p+1}}{p+1} \left(\frac{1 - e^{-\lambda \eta}}{\lambda}\right) \varphi(w) (Ł_{f_1} C_\varphi + Ł_{f_2} C_\gamma) \\
&+ C_1 M_k \left(\frac{1 - e^{-\lambda T}}{\lambda}\right) \varphi(w) (Ł_{f_1} C_\varphi + Ł_{f_2} C_\gamma) \\
&+ C_1 N_k \left(\frac{1 - e^{-\lambda w_k}}{\lambda}\right) \varphi(w) (Ł_{f_1} C_\varphi + Ł_{f_2} C_\gamma) + N_k Ł_{g_k} C_2 \psi
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\left(\left(\frac{1-e^{-\lambda w}}{\lambda} \right) + \frac{M_k}{\Gamma(p+1)} \frac{\eta^{p+1}}{p+1} \left(\frac{1-e^{-\lambda\eta}}{\lambda} \right) + M_k \left(\frac{1-e^{-\lambda T}}{\lambda} \right) + N_k \left(\frac{1-e^{-\lambda w_k}}{\lambda} \right) \right) \right. \\
&\quad \times C_1 \varphi(w) (Ł_{f_1} C_\varphi + Ł_{f_2} C_\gamma) + N_k Ł_{gk} C_2 \psi \\
&\leq \left\{ \left(\left(\frac{1-e^{-\lambda w}}{\lambda} \right) + \frac{M_k}{\Gamma(p+1)} \frac{\eta^{p+1}}{p+1} \left(\frac{1-e^{-\lambda\eta}}{\lambda} \right) + M_k \left(\frac{1-e^{-\lambda T}}{\lambda} \right) \right. \right. \\
&\quad \left. \left. + N_k \left(\frac{1-e^{-\lambda w_k}}{\lambda} \right) \right) (Ł_{f_1} C_\varphi + Ł_{f_2} C_\gamma) + N_k Ł_{gk} \right\} (C_1 + C_2) \big(\varphi(w) + \psi \big).
\end{aligned}
$$

Also, for $w \in (w_k, s_k]$ and $s \in (s_{k-1}, w_k]$, we have

$$
\begin{aligned}
&\big| (\Lambda\mu)(w) - (\Lambda\upsilon)(w) \big| \\
&= \bigg| \int_0^w e^{-\lambda(w-s)} I^\nu f\big(s, \mu(s), {}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)\mu(s)\big)\, ds \\
&\quad - \int_0^w e^{-\lambda(w-s)} I^\nu f\big(s, \upsilon(s), {}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)\upsilon(s)\big)\, ds \\
&\quad + \frac{M_k}{\Gamma(p+1)} \int_0^\eta (\eta-s)^p e^{-\lambda(\eta-s)} I^\nu f\big(s, \mu(s), {}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)\mu(s)\big)\, ds \\
&\quad - M_k \int_0^T e^{-\lambda(T-s)} I^\nu f\big(s, \mu(s), {}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)\mu(s)\big)\, ds \\
&\quad + N_k \int_0^{w_k} e^{-\lambda(w_k-s)} I^\nu f\big(s, \mu(s), {}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)\mu(s)\big)\, ds \\
&\quad - \frac{M_k}{\Gamma(p+1)} \int_0^\eta (\eta-s)^p e^{-\lambda(\eta-s)} I^\nu f\big(s, \upsilon(s), {}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)\upsilon(s)\big)\, ds \\
&\quad + M_k \int_0^T e^{-\lambda(T-s)} I^\nu f\big(s, \upsilon(s), {}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)\upsilon(s)\big)\, ds \\
&\quad - N_k \int_0^{w_k} e^{-\lambda(w_k-s)} I^\nu f\big(s, \upsilon(s), {}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)\upsilon(s)\big)\, ds \\
&\quad + N_k g_k\big(w_k, \mu(w_k)\big) - N_k g_k\big(w_k, \nu(w_k)\big) \bigg| \\
&\leq \bigg| \int_0^w e^{-\lambda(w-s)} I^\nu f\big(s, \mu(s), {}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)\mu(s)\big)\, ds \\
&\quad - \int_0^w e^{-\lambda(w-s)} I^\nu f\big(s, \upsilon(s), {}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)\upsilon(s)\big)\, ds \bigg| \\
&\quad + \bigg| \frac{M_k}{\Gamma(p+1)} \int_0^\eta (\eta-s)^p e^{-\lambda(\eta-s)} I^\nu f\big(s, \mu(s), {}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)\mu(s)\big)\, ds \\
&\quad - \frac{M_k}{\Gamma(p+1)} \int_0^\eta (\eta-s)^p e^{-\lambda(\eta-s)} I^\nu f\big(s, \upsilon(s), {}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)\upsilon(s)\big)\, ds \bigg| \\
&\quad + \bigg| M_k \int_0^T e^{-\lambda(T-s)} I^\nu f\big(s, \upsilon(s), {}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)\upsilon(s)\big)\, ds \\
&\quad - M_k \int_0^T e^{-\lambda(T-s)} I^\nu f\big(s, \mu(s), {}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)\mu(s)\big)\, ds \bigg| \\
&\quad + \bigg| N_k \int_0^{w_k} e^{-\lambda(w_k-s)} I^\nu f\big(s, \mu(s), {}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)\mu(s)\big)\, ds
\end{aligned}
$$

$$
\begin{aligned}
&- N_k \int_0^{w_k} e^{-\lambda(w_k-s)} I^\nu f\big(s, \upsilon(s), {}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)\upsilon(s)\big)\,ds\bigg| \\
&+ \big|N_k g_k\big(w_k, \mu(w_k)\big) - N_k g_k\big(w_k, \nu(w_k)\big)\big| \\
\leq{}& \int_0^{w} e^{-\lambda(w-s)} I^\nu \big|f\big(s, \mu(s), {}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)\mu(s)\big) - f\big(s, \upsilon(s), {}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)\upsilon(s)\big)\big|\,ds \\
&+ \frac{M_k}{\Gamma(p+1)} \int_0^{\eta} (\eta-s)^p e^{-\lambda(\eta-s)} I^\nu \big|f\big(s, \mu(s), {}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)\mu(s)\big) \\
&- f\big(s, \upsilon(s), {}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)\upsilon(s)\big)\big|\,ds \\
&+ M_k \int_0^{T} e^{-\lambda(T-s)} I^\nu \big|f\big(s, \upsilon(s), {}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)\upsilon(s)\big) - f\big(s, \mu(s), {}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)\mu(s)\big)\big|\,ds \\
&+ N_k \int_0^{w_k} e^{-\lambda(w_k-s)} I^\nu \big|f\big(s, \mu(s), {}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)\mu(s)\big) - f\big(s, \upsilon(s), {}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)\upsilon(s)\big)\big|\,ds \\
&+ N_k Ł_{gk} C_2 \psi \\
\leq{}& Ł_{f_1} \int_0^{w} e^{-\lambda(w-s)} I^\nu \big|\mu(s) - \upsilon(s)\big|\,ds \\
&+ Ł_{f_1} \frac{M_k}{\Gamma(p+1)} \int_0^{\eta} (\eta-s)^p e^{-\lambda(\eta-s)} I^\nu \big|\mu(s) - \upsilon(s)\big|\,ds \\
&+ Ł_{f_2} \int_0^{w} e^{-\lambda(w-s)} I^\nu \big|{}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)\mu(s) - {}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)\upsilon(s)\big|\,ds \\
&+ Ł_{f_2} \frac{M_k}{\Gamma(p+1)} \int_0^{\eta} (\eta-s)^p e^{-\lambda(\eta-s)} I^\nu \big|{}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)\mu(s) - {}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)\upsilon(s)\big|\,ds \\
&+ Ł_{f_1} M_k \int_0^{T} e^{-\lambda(T-s)} I^\nu \big|\upsilon(s) - \mu(s)\big|\,ds + Ł_{f_1} N_k \int_0^{w_k} e^{-\lambda(w_k-s)} I^\nu \big|\mu(s) - \upsilon(s)\big|\,ds \\
&+ Ł_{f_2} M_k \int_0^{T} e^{-\lambda(T-s)} I^\nu \big|{}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)\upsilon(s) - {}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)\mu(s)\big|\,ds \\
&+ Ł_{f_2} N_k \int_0^{w_k} e^{-\lambda(w_k-s)} I^\nu \big|{}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)\mu(s) - {}^c\mathcal{D}^\nu(\mathcal{D}+\lambda)\upsilon(s)\big|\,ds + N_k Ł_{gk} C_2 \psi \\
={}& Ł_{f_1} \int_0^{w} e^{-\lambda(w-s)} I^\nu \big|\mu(s) - \upsilon(s)\big|\,ds + Ł_{f_2} \int_0^{w} e^{-\lambda(w-s)} I^{\nu\,c}\mathcal{D}^\nu(\mathcal{D}+\lambda)\big|\mu(s) - \upsilon(s)\big|\,ds \\
&+ Ł_{f_1} \frac{M_k}{\Gamma(p+1)} \int_0^{\eta} (\eta-s)^p e^{-\lambda(\eta-s)} I^\nu \big|\mu(s) - \upsilon(s)\big|\,ds + N_k Ł_{gk} C_2 \psi \\
&+ Ł_{f_2} \frac{M_k}{\Gamma(p+1)} \int_0^{\eta} (\eta-s)^p e^{-\lambda(\eta-s)} I^{\nu\,c}\mathcal{D}^\nu(\mathcal{D}+\lambda)\big|\mu(s) - \upsilon(s)\big|\,ds \\
&+ Ł_{f_1} M_k \int_0^{T} e^{-\lambda(T-s)} I^\nu \big|\upsilon(s) - \mu(s)\big|\,ds \\
&+ Ł_{f_2} M_k \int_0^{T} e^{-\lambda(T-s)} I^{\nu\,c}\mathcal{D}^\nu(\mathcal{D}+\lambda)\big|\upsilon(s) - \mu(s)\big|\,ds \\
&+ Ł_{f_1} N_k \int_0^{w_k} e^{-\lambda(w_k-s)} I^\nu \big|\mu(s) - \upsilon(s)\big|\,ds \\
&+ Ł_{f_2} N_k \int_0^{w_k} e^{-\lambda(w_k-s)} I^{\nu\,c}\mathcal{D}^\nu(\mathcal{D}+\lambda)\big|\mu(s) - \upsilon(s)\big|\,ds \\
\leq{}& Ł_{f_1} C_2 \psi \int_0^{w} e^{-\lambda(w-s)} I^\nu(1)\,ds + Ł_{f_2} C_2 \psi \int_0^{w} e^{-\lambda(w-s)} (\mathcal{D}+\lambda)(1)\,ds
\end{aligned}
$$

$$
\begin{aligned}
&+ Ł_{f_1} C_2 \psi \frac{M_k}{\Gamma(p+1)} \int_0^\eta (\eta-s)^p e^{-\lambda(\eta-s)} I^\nu(1)\,ds + Ł_{f_1} C_2 \psi M_k \int_0^T e^{-\lambda(T-s)} I^\nu(1)\,ds \\
&+ Ł_{f_2} C_2 \psi \frac{M_k}{\Gamma(p+1)} \int_0^\eta (\eta-s)^p e^{-\lambda(\eta-s)} (\mathcal{D}+\lambda)(1)\,ds \\
&+ Ł_{f_2} C_2 \psi M_k \int_0^T e^{-\lambda(T-s)} (\mathcal{D}+\lambda)(1)\,ds \\
&+ Ł_{f_1} C_2 \psi N_k \int_0^{w_k} e^{-\lambda(w_k-s)} I^\nu(1)\,ds + Ł_{f_2} C_2 \psi N_k \int_0^{w_k} e^{-\lambda(w_k-s)} (\mathcal{D}+\lambda)(1)\,ds \\
&+ N_k Ł_{gk} C_2 \psi \\
={}& Ł_{f_1} C_2 \psi \int_0^w e^{-\lambda(w-s)} I^\nu(1)\,ds + Ł_{f_2} C_2 \psi \int_0^w e^{-\lambda(w-s)} \big(\mathcal{D}(1)+\lambda(1)\big)\,ds \\
&+ Ł_{f_1} C_2 \psi \frac{M_k}{\Gamma(p+1)} \int_0^\eta (\eta-s)^p e^{-\lambda(\eta-s)} I^\nu(1)\,ds + Ł_{f_1} C_2 \psi M_k \int_0^T e^{-\lambda(T-s)} I^\nu(1)\,ds \\
&+ Ł_{f_2} C_2 \psi \frac{M_k}{\Gamma(p+1)} \int_0^\eta (\eta-s)^p e^{-\lambda(\eta-s)} \big(\mathcal{D}(1)+\lambda(1)\big)\,ds \\
&+ Ł_{f_2} C_2 \psi M_k \int_0^T e^{-\lambda(T-s)} \big(\mathcal{D}(1)+\lambda(1)\big)\,ds + Ł_{f_1} C_2 \psi N_k \int_0^{w_k} e^{-\lambda(w_k-s)} I^\nu(1)\,ds \\
&+ Ł_{f_2} C_2 \psi N_k \int_0^{w_k} e^{-\lambda(w_k-s)} \big(\mathcal{D}(1)+\lambda(1)\big)\,ds + N_k Ł_{gk} C_2 \psi \\
={}& Ł_{f_1} C_2 \psi \int_0^w e^{-\lambda(w-s)} I^\nu(1)\,ds \\
&+ Ł_{f_2} C_2 \lambda \psi \int_0^w e^{-\lambda(w-s)}\,ds + Ł_{f_2} C_2 \psi \lambda N_k \int_0^{w_k} e^{-\lambda(w_k-s)}\,ds \\
&+ Ł_{f_1} C_2 \psi \frac{M_k}{\Gamma(p+1)} \int_0^\eta (\eta-s)^p e^{-\lambda(\eta-s)} I^\nu(1)\,ds \\
&+ Ł_{f_2} C_2 \psi \lambda \frac{M_k}{\Gamma(p+1)} \int_0^\eta (\eta-s)^p e^{-\lambda(\eta-s)}\,ds \\
&+ Ł_{f_1} C_2 \psi M_k \int_0^T e^{-\lambda(T-s)} I^\nu(1)\,ds + Ł_{f_2} C_2 \psi \lambda M_k \int_0^T e^{-\lambda(T-s)}\,ds \\
&+ Ł_{f_1} C_2 \psi N_k \int_0^{w_k} e^{-\lambda(w_k-s)} I^\nu(1)\,ds + N_k Ł_{gk} C_2 \psi \\
\le{}& Ł_{f_1} C_2 \psi \left(\frac{1-e^{-\lambda w}}{\lambda}\right)\left(\frac{w^\nu}{\Gamma(\nu+1)}\right) + Ł_{f_2} C_2 \lambda \psi \left(\frac{1-e^{-\lambda w}}{\lambda}\right) \\
&+ Ł_{f_2} C_2 \lambda \psi N_k \left(\frac{1-e^{-\lambda w_k}}{\lambda}\right) \\
&+ Ł_{f_1} C_2 \psi \frac{M_k}{\Gamma(p+1)} \left(\frac{\eta^{p+1}}{p+1}\right)\left(\frac{1-e^{-\lambda\eta}}{\lambda}\right)\left(\frac{\eta^\nu}{\Gamma(\nu+1)}\right) + Ł_{f_2} C_2 \lambda \psi M_k \left(\frac{1-e^{-\lambda T}}{\lambda}\right) \\
&+ Ł_{f_2} C_2 \lambda \psi \frac{M_k}{\Gamma(p+1)} \left(\frac{\eta^{p+1}}{p+1}\right)\left(\frac{1-e^{-\lambda\eta}}{\lambda}\right) + Ł_{f_1} C_2 \psi M_k \left(\frac{1-e^{-\lambda T}}{\lambda}\right)\left(\frac{T^\nu}{\Gamma(\nu+1)}\right) \\
&+ Ł_{f_1} C_2 \psi N_k \left(\frac{1-e^{-\lambda w_k}}{\lambda}\right)\left(\frac{w_k^\nu}{\Gamma(\nu+1)}\right) + N_k Ł_{gk} C_2 \psi \\
={}& \left\{ Ł_{f_1} \left(\frac{1-e^{-\lambda w}}{\lambda}\right)\left(\frac{w^\nu}{\Gamma(\nu+1)}\right) + Ł_{f_2} \lambda \left(\frac{1-e^{-\lambda w}}{\lambda}\right) \right.
\end{aligned}
$$

$$
\begin{aligned}
&+ Ł_{f_2}\lambda \frac{M_k}{\Gamma(p+1)}\left(\frac{\eta^{p+1}}{p+1}\right)\left(\frac{1-e^{-\lambda\eta}}{\lambda}\right) \\
&+ Ł_{f_1}\frac{M_k}{\Gamma(p+1)}\left(\frac{\eta^{p+1}}{p+1}\right)\left(\frac{1-e^{-\lambda\eta}}{\lambda}\right)\left(\frac{\eta^{\nu}}{\Gamma(\nu+1)}\right) \\
&+ Ł_{f_1}M_k\left(\frac{1-e^{-\lambda T}}{\lambda}\right)\left(\frac{T^{\nu}}{\Gamma(\nu+1)}\right) \\
&+ Ł_{f_2}\lambda M_k\left(\frac{1-e^{-\lambda T}}{\lambda}\right) + Ł_{f_1}N_k\left(\frac{1-e^{-\lambda w_k}}{\lambda}\right)\left(\frac{w_k^{\nu}}{\Gamma(\nu+1)}\right) \\
&+ Ł_{f_2}\lambda N_k\left(\frac{1-e^{-\lambda w_k}}{\lambda}\right) + N_k Ł_{gk}\bigg\}C_2\psi \\
\le{}& \bigg\{Ł_{f_1}\left(\frac{1-e^{-\lambda w}}{\lambda}\right)\left(\frac{w^{\nu}}{\Gamma(\nu+1)}\right) + Ł_{f_2}\lambda\left(\frac{1-e^{-\lambda w}}{\lambda}\right) \\
&+ Ł_{f_2}\lambda \frac{M_k}{\Gamma(p+1)}\left(\frac{\eta^{p+1}}{p+1}\right)\left(\frac{1-e^{-\lambda\eta}}{\lambda}\right) \\
&+ Ł_{f_1}\frac{M_k}{\Gamma(p+1)}\left(\frac{\eta^{p+1}}{p+1}\right)\left(\frac{1-e^{-\lambda\eta}}{\lambda}\right)\left(\frac{\eta^{\nu}}{\Gamma(\nu+1)}\right) \\
&+ Ł_{f_1}M_k\left(\frac{1-e^{-\lambda T}}{\lambda}\right)\left(\frac{T^{\nu}}{\Gamma(\nu+1)}\right) \\
&+ Ł_{f_2}\lambda M_k\left(\frac{1-e^{-\lambda T}}{\lambda}\right) + Ł_{f_1}iN_k\left(\frac{1-e^{-\lambda w_k}}{\lambda}\right)\left(\frac{w_k^{\nu}}{\Gamma(\nu+1)}\right) \\
&+ Ł_{f_2}\lambda N_k\left(\frac{1-e^{-\lambda w_k}}{\lambda}\right) + N_k Ł_{gk}\bigg\}(C_1+C_2)\big(\varphi(w)+\psi\big).
\end{aligned}
$$

From the above we have

$$\big|(\Lambda\mu)(w)-(\Lambda\upsilon)(w)\big| \le Ł(C_1+C_2)\big(\varphi(w)+\psi\big), \quad w\in[0,\tau],$$

that is,

$$d(\Lambda\mu,\Lambda\upsilon) \le Ł(C_1+C_2)\big(\varphi(w)+\psi\big).$$

Hence, we conclude that

$$d(\Lambda\mu,\Lambda\upsilon) \le Ł\, d(\mu,\upsilon)$$

for any $\mu,\upsilon\in V$, since condition (4.5) is strictly contractive property is shown.

Now we take $\mu_0\in V$. From the piecewise continuous property of μ_0 and $\Lambda\mu_0$, it follows that there exists a constant $0<G_1<\infty$ such that

$$
\begin{aligned}
\big|(\Lambda\mu_0)(w)-\mu_0(w)\big| \le{}& \bigg|\int_0^w e^{-\lambda(w-s)} I^{\nu} f\big(s,\mu_0(s),{}^c\mathcal{D}^{\nu}(\mathcal{D}+\lambda)\mu_0(s)\big)\,ds \\
&+ \frac{A_{11}}{\Gamma(p+1)}\int_0^{\eta}(\eta-s)^p e^{-\lambda(\eta-s)} I^{\nu} f\big(s,\mu_0(s),{}^c\mathcal{D}^{\nu}(\mathcal{D}+\lambda)\mu_0(s)\big)\,ds \\
&- A_{11}\int_0^T e^{-\lambda(T-s)} I^{\nu} f\big(s,\mu_0(s),{}^c\mathcal{D}^{\nu}(\mathcal{D}+\lambda)\mu_0(s)\big)\,ds
\end{aligned}
$$

$$
\begin{aligned}
&+\big(A_{11}\big(\eta^{p}E_{(1,p+1)}(aw)-e^{\lambda T}\big)+e^{\lambda T}\big)z_0-\mu_0(w)\Big| \\
&\le G_1\varphi(w)\le G_1\big(\varphi(w)+\psi\big),\quad w\in(0,s_0].
\end{aligned}
$$

There exists a constant $0 < G_2 < \infty$ such that

$$
\begin{aligned}
&\big|(\Lambda\mu_0)(w)-\mu_0(w)\big| = \big|g_k\big(w,\mu_0(w)\big)-\mu_0(w)\big| \le G_2\psi \le G_2\big(\varphi(w)+\psi\big),\\
&\quad w\in(s_{k-1},w_k], k=1,2,\dots,m.
\end{aligned}
$$

Also we can find a constant $0 < G_3 < \infty$ such that

$$
\begin{aligned}
\big|(\Lambda\mu_0)(w)-\mu_0(w)\big| \le{}& \bigg|\int_0^w e^{-\lambda(w-s)}I^{\nu}f\big(s,\mu_0(s),{}^{c}\mathcal{D}^{\nu}(\mathcal{D}+\lambda)\mu_0(s)\big)\,ds\\
&+\frac{M_k}{\Gamma(p+1)}\int_0^{\eta}(\eta-s)^{p}e^{-\lambda(\eta-s)}I^{\nu}f\big(s,\mu_0(s),{}^{c}\mathcal{D}^{\nu}(\mathcal{D}+\lambda)\mu_0(s)\big)\,ds\\
&-M_k\int_0^{T}e^{-\lambda(T-s)}I^{\nu}f\big(s,\mu_0(s),{}^{c}\mathcal{D}^{\nu}(\mathcal{D}+\lambda)\mu_0(s)\big)\,ds\\
&+N_k\int_0^{w_k}e^{-\lambda(w_k-s)}I^{\nu}f\big(s,\mu_0(s),{}^{c}\mathcal{D}^{\nu}(\mathcal{D}+\lambda)\mu_0(s)\big)\,ds\\
&+N_k g_k\big(w_k,\mu_0(w_k)\big)-\mu_0(w)\bigg|\\
\le{}& G_3\varphi(w)\le G_3\big(\varphi(w)+\psi\big),\quad w\in(w_k,s_k], k=1,2,\dots,m.
\end{aligned}
$$

Since f, g_k, and μ_0 are bounded on J and $\varphi(.)>0$, thus (4.6) implies that $d(\Lambda\mu_0,\mu_0)<\infty$.

By using the Banach fixed point theorem, there exists a continuous function $z : J \to \mathbb{R}$ such that $\Lambda^n\mu_0 \to z_0$ in (V,d) as $n\to\infty$ and $\Lambda z = z_0$, that is, z_0 satisfies Eq. (4.3) for every $w\in J$.

Now we show that $\{\mu\in V$ such that $d(\mu_0,\mu)<\infty\} = V$. For any $g\in V$, since μ and μ_0 are bounded on J and $\min_{w\in J}(\varphi(w)+\psi)>0$, there exists a constant $0<C_\mu<\infty$ such that $|\mu_0(w)-\mu(w)|\le C_\mu(\varphi(w)+\psi)$ for any $w\in J$. Hence, we have $d(\mu_0,\mu)<\infty$ for all $\mu\in V$, that is, $\{\mu\in V$ such that $d(\mu_0,\mu)<\infty\}=V$. Thus, we determine that z is the unique continuous function with Eq. (4.3). From (3.2) and (H_4), we can write

$$
\begin{aligned}
d(z,\Lambda z)\le{}& \bigg(\bigg(\frac{1-e^{-\lambda w}}{\lambda}\bigg)+\frac{M_k}{\Gamma(p+1)}\frac{\eta^{p+1}}{p+1}\bigg(\frac{1-e^{-\lambda\eta}}{\lambda}\bigg)\\
&+M_k\bigg(\frac{1-e^{-\lambda T}}{\lambda}\bigg)+N_k\bigg(\frac{1-e^{-\lambda w_k}}{\lambda}\bigg)\bigg)(Ł_{f_1}C_{\varphi}+Ł_{f_2}C_{\gamma})+N_kŁ_{gk}.
\end{aligned}
$$

Summarizing, we have

$$
\begin{aligned}
d(z_0,z)\le{}& \frac{d(\Lambda z,z)}{1-Ł}\\
\le{}& \bigg\{\bigg(\bigg(\frac{1-e^{-\lambda w}}{\lambda}\bigg)+\frac{M_k}{\Gamma(p+1)}\frac{\eta^{p+1}}{p+1}\bigg(\frac{1-e^{-\lambda\eta}}{\lambda}\bigg)\\
&+M_k\bigg(\frac{1-e^{-\lambda T}}{\lambda}\bigg)+N_k\bigg(\frac{1-e^{-\lambda w_k}}{\lambda}\bigg)\bigg)(Ł_{f_1}C_{\varphi}+Ł_{f_2}C_{\gamma})+N_kŁ_{gk}\bigg\}\bigg(\frac{1}{1-Ł}\bigg).
\end{aligned}
$$

This shows that (4.4) is true for $w \in J$. □

Finally, we give an example to illustrate our main results.

Example 4.4

$$\begin{cases} {}^c\mathcal{D}_{0,w}^{\frac{1}{2}}(\mathcal{D}+2)z(w) = \frac{|z(w)|+{}^c\mathcal{D}_{0,w}^{\frac{1}{2}}(\mathcal{D}+2)z(w)}{8+e^w+w^2+{}^c\mathcal{D}_{0,w}^{\frac{1}{2}}(\mathcal{D}+2)z(w)}, & w \in (0,1] \cup (2,3], \\ z(w) = \frac{z(w)}{(3+w^2)(1+|z(w)|)}, & w \in (1,2], \\ z(0) = \frac{\sqrt{2}}{3}, \qquad z(1) = \frac{5}{6}\int_0^{\frac{1}{4}} \frac{(\frac{1}{4}-s)}{\Gamma\frac{4}{3}}\,ds, & 0 < \eta < 1, \end{cases} \tag{4.9}$$

and

$$\begin{cases} |{}^c\mathcal{D}_{0,w}^{\frac{1}{2}}(\mathcal{D}+2)z(w) - \frac{|z(w)|+{}^c\mathcal{D}_{0,w}^{\frac{1}{2}}(\mathcal{D}+2)z(w)}{8+e^w+w^2+{}^c\mathcal{D}_{0,w}^{\frac{1}{2}}(\mathcal{D}+2)z(w)}| \le e^w, & w \in (0,1] \cup (2,3], \\ |z(w) - \frac{z(w)}{(3+w^2)(1+|z(w)|)}| \le 1, & w \in (1,2]. \end{cases}$$

Let $J = [0,3]$, $\nu = \frac{1}{2}$, $p = \frac{4}{3}$, $\eta = \frac{1}{4}$, and $0 = w_0 < s_0 = 1 < w_1 = 2 < s_1 = \tau = T = 3$. Denote $f(w,z(w)) = \frac{|z(w)|}{8+e^w+w^2}$ with $Ł_{f_1} = \frac{1}{4}, Ł_{f_2} = \frac{1}{3}$ for $w \in (0,1] \cup (2,3]$ and $g_1(w,z(w)) = \frac{z(w)}{(3+w^2)(1+|z(w)|)}$ with $L_{g_k} = 1$ for $w \in (1,2]$. Putting $Ł_f = \frac{1}{4}$, $\varphi(w) = e^w$, and $C_1 = C_\varphi = C_\gamma = 1$, we have $\int_0^w I^{\frac{1}{2}} e^s\,ds \le e^w$ and $Ł_1 \approx 0.1231$, $Ł_2 \approx 0.4741$, so $Ł \approx 0.4741 < 1$.

By Theorem 4.3, there exists a unique solution $z : [0,3] \to \mathbb{R}$ such that

$$z_0(w) = \begin{cases} \begin{cases} \int_0^w e^{-\lambda(w-s)} I^\nu \frac{|z_0(w)|+{}^c\mathcal{D}_{0,w}^{\frac{1}{2}}(\mathcal{D}+2)z_0(w)}{8+e^w+w^2+{}^c\mathcal{D}_{0,w}^{\frac{1}{2}}(\mathcal{D}+2)z_0(w)}\,ds \\ \quad + \frac{A_{11}}{\Gamma(p+1)}\int_0^\eta (\eta-s)^p e^{-\lambda(\eta-s)} I^\nu \frac{|z_0(w)|+{}^c\mathcal{D}_{0,w}^{\frac{1}{2}}(\mathcal{D}+2)z_0(w)}{8+e^w+w^2+{}^c\mathcal{D}_{0,w}^{\frac{1}{2}}(\mathcal{D}+2)z_0(w)}\,ds \\ \quad - A_{11}\int_0^T e^{-\lambda(T-s)} I^\nu \frac{|z_0(w)|+{}^c\mathcal{D}_{0,w}^{\frac{1}{2}}(\mathcal{D}+2)z_0(w)}{8+e^w+w^2+{}^c\mathcal{D}_{0,w}^{\frac{1}{2}}(\mathcal{D}+2)z_0(w)}\,ds \\ \quad + (A_{11}(\eta^p E_{(1,p+1)}(aw) - e^{\lambda T}) + e^{\lambda T})z_0, \quad w \in [0,1], \end{cases} \\ \begin{cases} \frac{z_0(w)}{(3+w^2)(1+|z_0(w)|)}, \quad w \in (1,2], k = 1,2,\ldots,m, \end{cases} \\ \begin{cases} \int_0^w e^{-\lambda(w-s)} I^\nu \frac{|z_0(w)|+{}^c\mathcal{D}_{0,w}^{\frac{1}{2}}(\mathcal{D}+2)z_0(w)}{8+e^w+w^2+{}^c\mathcal{D}_{0,w}^{\frac{1}{2}}(\mathcal{D}+2)z_0(w)}\,ds \\ \quad + \frac{M_k}{\Gamma(p+1)}\int_0^\eta (\eta-s)^p e^{-\lambda(\eta-s)} I^\nu \frac{|z_0(w)|+{}^c\mathcal{D}_{0,w}^{\frac{1}{2}}(\mathcal{D}+2)z_0(w)}{8+e^w+w^2+{}^c\mathcal{D}_{0,w}^{\frac{1}{2}}(\mathcal{D}+2)z_0(w)}\,ds \\ \quad - M_k\int_0^T e^{-\lambda(T-s)} I^\nu \frac{|z_0(w)|+{}^c\mathcal{D}_{0,w}^{\frac{1}{2}}(\mathcal{D}+2)z_0(w)}{8+e^w+w^2+{}^c\mathcal{D}_{0,w}^{\frac{1}{2}}(\mathcal{D}+2)z_0(w)}\,ds \\ \quad + N_k\int_0^{w_k} e^{-\lambda(w_k-s)} I^\nu \frac{|z_0(w)|+{}^c\mathcal{D}_{0,w}^{\frac{1}{2}}(\mathcal{D}+2)z_0(w)}{8+e^w+w^2+{}^c\mathcal{D}_{0,w}^{\frac{1}{2}}(\mathcal{D}+2)z_0(w)}\,ds \\ \quad + N_k\frac{z_0(w)}{(3+w^2)(1+|z_0(w)|)}, \quad w \in (2,3], \end{cases} \end{cases}$$

$$\begin{aligned} \bigl|z(w) - z_0(w)\bigr| \le{}& \biggl\{\biggl(\biggl(\frac{1-e^{-\lambda w}}{\lambda}\biggr) + \frac{M_k}{\Gamma(p+1)}\frac{\eta^{p+1}}{p+1}\biggl(\frac{1-e^{-\lambda\eta}}{\lambda}\biggr) + M_k\biggl(\frac{1-e^{-\lambda T}}{\lambda}\biggr) \\ &+ N_k\biggl(\frac{1-e^{-\lambda w_k}}{\lambda}\biggr)\biggr)(Ł_{f_1}C_\varphi + Ł_{f_2}C_\gamma) + N_kŁ_{g_k}\biggr\}\biggl(\frac{\varphi(w)+\psi}{1-Ł}\biggr). \end{aligned}$$

Putting maximum of $w = w_k = T = \tau$, we obtain

$$\left|z(w) - z_0(w)\right| \leq \left\{ \left(\left(\left(\frac{1 - e^{-\lambda\tau}}{\lambda} \right) + \frac{M_k}{\Gamma(p+1)} \frac{\eta^{p+1}}{p+1} \left(\frac{1 - e^{-\lambda\eta}}{\lambda} \right) + M_k \left(\frac{1 - e^{-\lambda\tau}}{\lambda} \right) \right.\right.\right.$$
$$\left.\left.\left. + N_k \left(\frac{1 - e^{-\lambda\tau}}{\lambda} \right) \right) (Ł_{f_1} C_\varphi + Ł_{f_2} C_\gamma) + N_k Ł_{gk} \right\} \left(\frac{\varphi(w) + \psi}{1 - Ł} \right).$$

Now, putting the values, we get

$$\begin{aligned} \left|z(w) - z_0(w)\right| &\leq 0.1059 \left(\frac{e^w + 1}{1 - 0.4741} \right) \\ &\leq 0.20136 \left(e^w + 1 \right) \quad \text{for all } w \in [0, 3]. \end{aligned}$$

Thus problem (4.9) is Ulam–Hyers–Rassias stable.

5 Conclusion

In this article, we considered a nonlocal boundary value problem of nonlinear implicit impulsive Langevin equations with mixed derivatives and presented its Ulam–Hyers–Rassias stability. After introduction, we built a uniform structure to originate a formula of solutions for our proposed model. We implemented the new concept of generalized Ulam–Hyers–Rassias stability to our proposed model; finally we solved a particular example for our proposed model.

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Competing interests
All authors declare that they have no conflict of interests.

Authors' contributions
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