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On an initial inverse problem for a diffusion equation with a conformable derivative

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Abstract

In this paper, we consider the initial inverse problem for a diffusion equation with a conformable derivative in a general bounded domain. We show that the backward problem is ill-posed, and we propose a regularizing scheme using a fractional Landweber regularization method. We also present error estimates between the regularized solution and the exact solution using two parameter choice rules.

MSC: 35K05; 35K99; 47J06; 47H10

Keywords: Inverse problem; Diffusion equation; Conformable derivative

1 Introduction

In this paper we consider the following diffusion equation:

$$D^0_\gamma u(t,x) - \mathfrak{B}u(t,x) = F(t,x), \quad (t,x) \in (0,\mathscr{T}_o) \times \boldsymbol{\Omega}, \tag{1}$$

subject to the boundary conditions

$$u(t,x) = 0, \quad (t,x) \in (0,\mathscr{T}_o) \times \partial\boldsymbol{\Omega}, \tag{2}$$

and the initial condition

$$u(0,x) = u_0(x), \quad x \in \boldsymbol{\Omega} \tag{3}$$

where the domain $\boldsymbol{\Omega}$ is a subset of a d-dimensional space $\mathbb{R}^d$ ($d = 1,2,3$ is the dimension of $\boldsymbol{\Omega}$), which is a bounded domain with sufficient smooth boundary $\partial\boldsymbol{\Omega}$, F is the source term, $\mathscr{T}_o > 0$ is a fixed value, and D^0_γ is the conformable derivative [4, 14, 16].

Fractional differential equations are successful models of real life phenomenon and many authors studied fractional partial differential equations (see e.g. [34–36]). This gives one motivation to study and discuss some of the well known classical differential equations, when some classical derivatives are replaced by fractional derivatives. One of the classical equations is the diffusion equation with the conformable derivative and because of the relationship between the conformable derivative and the classical derivative our equation could be considered as a modified classical diffusion equation.

Khalid et al. [16] introduced the conformable derivative and it was developed in [2–4, 6–9, 31, 33]. Applications of this derivative were given in [5, 14, 25, 32]. In [22] the existence of solutions to conformable nonlinear differential equations with constant coefficients under mild conditions on the nonlinear term was discussed and in [29] the authors presented the iterative learning control for conformable differential equations. Recently, Machado et al. [1] and Baleanu et al. [15, 30] mentioned the critical analysis of the conformable derivative.

If the initial data u_0 and the source term F are given, Problem (1) satisfying (2) and (3) is called the direct problem. The inverse problem for (1) is less well known. Inverse problems occur when we do not know all the given data. However, by adding some given data, we can discuss inverse problems such as the backward problem (recovering the initial data) or the source identification problem (recovering the source function). Initial inverse problems for fractional Riemann–Liouville or Caputo diffusion equations were discussed in the literature [20, 26, 27]. However, little is known on the initial inverse problem for the diffusion equation with a conformable derivative.

Motivated by the above, in this paper, we study the initial inverse problem of the diffusion equation with a conformable derivative (1) satisfying (2) and we reconstruct the initial data $g(x) = u(0, x)$ from the additional data

$$u(\mathscr{T}_o, x) = h(x), \quad x \in \boldsymbol{\Omega}. \tag{4}$$

Note we cannot observe the data (h, F), so we only get approximate data $(h^\varepsilon, F^\varepsilon)$ such that

$$\left\| h - h^\varepsilon \right\|_{L^2(\boldsymbol{\Omega})} + \left\| F - F^\varepsilon \right\|_{L^\infty(0,\mathscr{T}_o;L^2(\boldsymbol{\Omega}))} \le \varepsilon, \tag{5}$$

where $\varepsilon > 0$ is the noise level (in this paper we will also let $\| \cdot \|$ denote the $L^2(\boldsymbol{\Omega})$ norm).

We show that this problem is ill-posed, i.e., the solution (if it exists) does not depend continuously on the given data. Indeed, a small error of the given observation can result in that the solution may have a large error. Some regularization method is required for constructing stable approximations for a sought solution.

We use the Landweber method to find a regularized solution and the idea is based on iterative sequences. Using this method, some authors established a fractional method for solving some linear ill-posed models; see, for example, [13, 23]. We will consider regularized solutions and regularity for the regularized solution. Also, we present an error estimate of the Landweber regularized solution to the exact solution under an a priori assumption using an a priori regularization parameter choice rule, which depends on the noise level ε and the a priori bound condition **E** of the unknown solution. That means the a priori choice of the regularization parameter depends on the a priori bound of the unknown solution. However, an a priori bound cannot be known exactly in practice, and working with an incorrect value may lead to a bad regularized solution. Therefore, we provide an a posteriori choice of the regularization parameter. We also present a regularized problem and consider the well-posedness of the regularized solution and an error estimate under two parameter choice rules are considered.

The structure of this paper is as follows. First, we give some preliminaries which are needed for this paper in Sect. 2. Next, in Sect. 3, we construct an approximate regularized solution by using the Landweber regularization method. Finally, we estimate the error between the approximation and the sought solution under two parameter choice rules in Sect. 4.

2 Preliminaries

2.1 Some basic results

In this section, we introduce some spaces and some basic definitions associated with conformable derivatives.

Definition 2.1 (Conformable Derivative) Let $f : [0,\infty) \to \mathbb{R}$. Then

$$D^0_\gamma f(s) = \lim_{\varepsilon\to 0} \frac{f(s+\varepsilon s^{1-\gamma}) - f(s)}{\varepsilon}, \quad s > 0, \tag{6}$$

is called the conformable derivative of f of order $\gamma \in (0,1]$.

Some properties of the conformable derivative can be found in [2, 4, 14] and the references therein.

Consider the operator $-\mathfrak{B}$ on $L^2(\boldsymbol{\Omega})$ with domain $D(-\mathfrak{B}) \subset H^1_0(\boldsymbol{\Omega}) \cap H^2(\boldsymbol{\Omega})$. Assume that $-\mathfrak{B}$ has eigenvalues $\{\widetilde{a}_m\}$ satisfying

$$0 < \widetilde{a}_1 \le \widetilde{a}_2 \le \widetilde{a}_3 \le \cdots \le \widetilde{a}_m \le \cdots$$

and $\widetilde{a}_m \to \infty$ as $m \to \infty$ with corresponding eigenfunctions $e_m \in H^1_0(\boldsymbol{\Omega}) \cap H^2(\boldsymbol{\Omega})$. Now

$$\begin{cases} \mathfrak{B} e_m(x) = -\widetilde{a}_m e_m(x), & x \in \boldsymbol{\Omega}, \\ e_m(x) = 0, & x \in \partial\boldsymbol{\Omega}, \end{cases}$$

and we note that there exists a positive constant C such that $\widetilde{a}_m \ge C m^{\frac{2}{d}}$ for $m \in \mathbb{N}$ and $m \ge 1$, where d is the dimension of the domain $\boldsymbol{\Omega}$; see [10].

For $r \ge 0$, consider the Hilbert scale space (see [21])

$$\mathcal{H}^r(\boldsymbol{\Omega}) = \left\{ \nu \in L^2(\Omega) : \sum_{k=1}^{\infty} \widetilde{a}^r_m \left| \langle \nu, e_m \rangle \right|^2 < +\infty \right\}, \tag{7}$$

with the norm

$$\|\nu\|_{\mathcal{H}^r(\boldsymbol{\Omega})} = \left(\sum_{k=1}^{\infty} \widetilde{a}^r_m \left| \langle \nu, e_m \rangle \right|^2 \right)^{\frac{1}{2}}.$$

If $r = 0$, we have $\mathcal{H}^0(\boldsymbol{\Omega}) = L^2(\boldsymbol{\Omega})$.

For a given real number $p \ge 1$, let $L^p(0,\mathscr{T}_o; L^2(\boldsymbol{\Omega}))$ be the space of all functions such that

$$\|\nu\|_{L^p(0,\mathscr{T}_o;L^2(\boldsymbol{\Omega}))} := \left(\int_0^{\mathscr{T}_o} \|\nu(t)\|^p_{L^2(\boldsymbol{\Omega})}\, dt \right)^{\frac{1}{p}} < +\infty.$$

We give two lemmas, which will be needed later.

Lemma 2.1 ([19, 28]) *For* $0 < \lambda < 1$, $r > 0$ *and* $k \in \mathbb{N}$, *we have*

$$(1-\lambda)^k \lambda^r \le r^r (k+1)^{-r} < r^r k^{-r}.$$

Lemma 2.2 *For* $\widetilde{a}_m > 0$, $\gamma > 0$, $\alpha \in (\frac{1}{2}, 1]$ *and* $0 < \mu \exp(-2\widetilde{a}_m \frac{\mathscr{T}_o^\gamma}{\gamma}) < 1$, *we get*

$$\sup_{\widetilde{a}_m > 0} \exp\left(\widetilde{a}_m \frac{\mathscr{T}_o^\gamma}{\gamma}\right)\left[1 - \left(1 - \mu \exp\left(-2\widetilde{a}_m \frac{\mathscr{T}_o^\gamma}{\gamma}\right)\right)^k\right]^\alpha \le \mu^{\frac{1}{2}} k^{\frac{1}{2}}.$$

Proof First, we obtain

$$\begin{aligned} &\exp\left(\widetilde{a}_m \frac{\mathscr{T}_o^\gamma}{\gamma}\right)\left[1 - \left(1 - \mu \exp\left(-2\widetilde{a}_m \frac{\mathscr{T}_o^\gamma}{\gamma}\right)\right)^k\right]^\alpha \\ &\quad = \mu^{\frac{1}{2}}\left[\mu^{\frac{1}{2}} \exp\left(-\widetilde{a}_m \frac{\mathscr{T}_o^\gamma}{\gamma}\right)\right]^{-1}\left[1 - \left(1 - \mu \exp\left(-2\widetilde{a}_m \frac{\mathscr{T}_o^\gamma}{\gamma}\right)\right)^k\right]^\alpha. \end{aligned} \tag{8}$$

Let

$$\Psi(\nu) := \nu^{-2}\left[1 - \left(1 - \nu^2\right)^k\right]^{2\alpha},$$

where $\mu := \mu^{\frac{1}{2}} \exp(-\widetilde{a}_m \frac{\mathscr{T}_o^\gamma}{\gamma})$. We note that $0 < \mu < \frac{1}{\|\mathcal{K}\|^2}$ (see [18]), and this implies that $0 < \mu \exp(-2\widetilde{a}_m \frac{\mathscr{T}_o^\gamma}{\gamma}) < 1$. Hence, this function is continuous in $[0, +\infty)$ when $\mu \in (0, 1)$.

For $\alpha \in (\frac{1}{2}, 1)$ and $\mu \in (0, 1)$, from Lemma 3.3 in [18], we obtain

$$\Psi(\nu) \le k. \tag{9}$$

Combining (8) and (9), we deduce that

$$\sup_{\widetilde{a}_m > 0} \exp\left(\widetilde{a}_m \frac{\mathscr{T}_o^\gamma}{\gamma}\right)\left[1 - \left(1 - \mu \exp\left(-2\widetilde{a}_m \frac{\mathscr{T}_o^\gamma}{\gamma}\right)\right)^k\right]^\alpha \le \mu^{\frac{1}{2}} k^{\frac{1}{2}}.$$ □

2.2 Solution for the fractional diffusion equation with conformable derivative

Assume that Problem (1) satisfying (2) and (3) (i.e. the direct problem) has a solution u as follows:

$$u(t,x) = \sum_{m=1}^{\infty} \langle u(t,x), e_m(x)\rangle e_m(x).$$

Note

$$\begin{cases} D_\gamma^0 \langle u(t,x), e_m(x)\rangle - \widetilde{a}_m \langle u(t,x), e_m(x)\rangle = \langle F(t,x), e_m(x)\rangle, & (t,x) \in (0, \mathscr{T}_o) \times \boldsymbol{\Omega}, \\ \langle u(0,x), e_m(x)\rangle = \langle u_0(x), e_m(x)\rangle, & x \in \boldsymbol{\Omega}, \end{cases} \tag{10}$$

where $\langle \mathfrak{B} u(t,x), e_m(x)\rangle = -\widetilde{a}_m \langle u(t,x), e_m(x)\rangle$. Using the result in [14] and [22], the solution of the latter problem is

$$\begin{aligned} u(t,x) = \sum_{m=1}^{\infty} \Bigg[&\exp\left(-\widetilde{a}_m \frac{t^\gamma}{\gamma}\right)\langle u_0(x), e_m(x)\rangle \\ &+ \int_0^t \tau^{\gamma-1} \exp\left(-\widetilde{a}_m \frac{t^\gamma - \tau^\gamma}{\gamma}\right)\langle F(\tau,x), e_m(x)\rangle \, d\tau \Bigg] e_m(x). \end{aligned}$$

Let $\mathrm{t} = \mathscr{T}_o$ and we get (with $\mathrm{h_m} = \langle \mathrm{h}, \mathrm{e_m}\rangle$ and $\mathrm{F_m}(\tau) = \langle \mathrm{F}(\tau,.), \mathrm{e_m}\rangle$)

$$\begin{aligned}\mathrm{h_m}(\mathrm{x}) = {}& \exp\biggl(-\widetilde{\mathrm{a}}_\mathrm{m}\frac{\mathscr{T}_o^\gamma}{\gamma}\biggr)\bigl\langle \mathrm{u}_0(\mathrm{x}), \mathrm{e_m}(\mathrm{x})\bigr\rangle \\ &+ \int_0^{\mathscr{T}_o} \tau^{\gamma-1}\exp\biggl(-\widetilde{\mathrm{a}}_\mathrm{m}\frac{\mathscr{T}_o^\gamma - \tau^\gamma}{\gamma}\biggr)\mathrm{F_m}(\tau)\,d\tau.\end{aligned}$$

This implies that

$$\begin{aligned}\mathrm{u}(\mathrm{t},\mathrm{x}) = {}& \sum_{\mathrm{m}=1}^{\infty}\biggl[\exp\biggl(-\widetilde{\mathrm{a}}_\mathrm{m}\frac{\mathrm{t}^\gamma - \mathscr{T}_o^\gamma}{\gamma}\biggr) \\ &\times\biggl(\mathrm{h_m} - \int_0^{\mathscr{T}_o}\tau^{\gamma-1}\exp\biggl(-\widetilde{\mathrm{a}}_\mathrm{m}\frac{\mathscr{T}_o^\gamma - \tau^\gamma}{\gamma}\biggr)\mathrm{F_m}(\tau)\,d\tau\biggr) \\ &+ \int_0^{\mathrm{t}}\tau^{\gamma-1}\exp\biggl(-\widetilde{\mathrm{a}}_\mathrm{m}\frac{\mathrm{t}^\gamma - \tau^\gamma}{\gamma}\biggr)\mathrm{F_m}(\tau)\,d\tau\biggr]\mathrm{e_m}(\mathrm{x}). \end{aligned} \tag{11}$$

Let

$$\begin{aligned}\mathcal{A}_\gamma^1(\mathrm{t},\mathscr{T}_o)\mathrm{v} &:= \sum_{\mathrm{m}=1}^{\infty}\exp\biggl(-\widetilde{\mathrm{a}}_\mathrm{m}\frac{\mathrm{t}^\gamma - \mathscr{T}_o^\gamma}{\gamma}\biggr)\langle \mathrm{v}, \mathrm{e_m}\rangle\mathrm{e_m}, \\ \mathcal{A}_\gamma^2(\mathrm{t})\mathrm{v} &:= \sum_{\mathrm{m}=1}^{\infty}\int_0^{\mathrm{t}}\tau^{\gamma-1}\exp\biggl(-\widetilde{\mathrm{a}}_\mathrm{m}\frac{\mathrm{t}^\gamma - \tau^\gamma}{\gamma}\biggr)\langle \mathrm{v}, \mathrm{e_m}\rangle\,d\tau\,\mathrm{e_m},\end{aligned}$$

for $\mathrm{v} \in L^2(\boldsymbol{\Omega})$, and $0 \le \tau \le \mathrm{t} \le \mathscr{T}_o$. Then it follows from (11) that

$$\mathrm{u}(\mathrm{t}) = \mathcal{A}_\gamma^1(\mathrm{t},\mathscr{T}_o)\bigl[\mathrm{h} - \mathcal{A}_\gamma^2(\mathscr{T}_o)\mathrm{F}(\mathrm{t})\bigr] + \mathcal{A}_\gamma^2(\mathrm{t})\mathrm{F}(\mathrm{t}).$$

Recall for any $\mathrm{n} > 0$, there exists a positive constant $\mathcal{P}_{1,\mathrm{n}}$ (from elementary calculus note we can take $\mathcal{P}_{1,\mathrm{n}}$ to be $\mathrm{n}^\mathrm{n}\exp(-\mathrm{n})$) such that

$$\exp(-z) \le \mathcal{P}_{1,\mathrm{n}}z^{-\mathrm{n}}, \quad z \ge 0 \tag{12}$$

and if $0 < \mathrm{s} < 1$, there exists a positive constant $\mathcal{P}_{2,\mathrm{s}}$ (we can take $\mathcal{P}_{2,s}$ to be $(1-s)^{1-s} \times \exp(s-1)$) such that

$$\exp(-z) \le \mathcal{P}_{2,\mathrm{s}}z^{\mathrm{s}-1}, \quad z \ge 0. \tag{13}$$

Lemma 2.3 *Given $0 \le \tau \le \mathscr{T}_o$ and $\boldsymbol{\Omega} \subset \mathbb{R}^d$ for any $1 \le d \le 3$.*

(a) *If $w \in L^2(\boldsymbol{\Omega})$, then $\|\mathcal{A}_\gamma^2(t)w\| \in L^2(\boldsymbol{\Omega})$ and*

$$\bigl\|\mathcal{A}_\gamma^2(t)w\bigr\|_{L^2(\boldsymbol{\Omega})} \le \mathcal{R}\|w\|_{L^2(\boldsymbol{\Omega})} \quad \text{where } \mathcal{R} = \Biggl(\sum_{m=1}^{\infty}\frac{1}{C^2m^{\frac{4}{d}}}\Biggr)^{\frac{1}{2}}.$$

(b) *If $w \in L^2(\boldsymbol{\Omega})$ for $0 < n \ne 1$, then $\|\mathcal{A}_\gamma^2(t)w\| \in \mathcal{H}^n(\boldsymbol{\Omega})$ and*

$$\bigl\|\mathcal{A}_\gamma^2(t)w\bigr\|^2_{\mathcal{H}^n(\boldsymbol{\Omega})} \le \mathcal{P}_{1,n}\frac{\mathscr{T}_o^{\gamma(1-n)}}{\gamma(1-n)}\int_0^t\tau^{\gamma-1}\bigl\|w(\tau)\bigr\|^2_{L^2(\boldsymbol{\Omega})}\,d\tau.$$

(c) *If* $w \in \mathcal{H}^{q+1}(\boldsymbol{\Omega})$ *for* $0 < q < 1$ *and* $0 \le t_1 < t_2 \le \mathscr{T}_o$, *then* $\|\mathcal{A}^2_\gamma(t)w\| \in L^2(\boldsymbol{\Omega})$ *and*

$$\begin{aligned}
&\big\|\big(\mathcal{A}^2_\gamma(t_1) - \mathcal{A}^2_\gamma(t_2)\big)w\big\|_{L^2(\boldsymbol{\Omega})} \\
&\quad\le \left(\mathcal{P}_{2,q}\frac{|t_1^\gamma - t_2^\gamma|^2}{\gamma^{q+1}}\frac{\mathscr{T}_o^{q\gamma}}{q\gamma}\int_0^{t_1}\tau^{\gamma-1}\|w\|^2_{\mathcal{H}^{q+1}(\boldsymbol{\Omega})}\,d\tau\right)^{\frac{1}{2}} \\
&\qquad + \left(\frac{t_2^\gamma - t_1^\gamma}{\gamma}\int_{t_1}^{t_2}\tau^{\gamma-1}\|w\|^2_{L^2(\boldsymbol{\Omega})}\,d\tau\right)^{\frac{1}{2}}.
\end{aligned}$$

Proof (a) Note

$$\begin{aligned}
\big\|\mathcal{A}^2_\gamma(\mathrm{t})\mathrm{w}\big\|_{L^2(\boldsymbol{\Omega})} &= \sqrt{\sum_{\mathrm{m}=1}^{\infty}\left[\int_0^{\mathrm{t}}\tau^{\gamma-1}\exp\left(-\widetilde{\mathrm{a}}_{\mathrm{m}}\frac{\mathrm{t}^\gamma - \tau^\gamma}{\gamma}\right)\langle \mathrm{w}(\tau), \mathrm{e}_{\mathrm{m}}\rangle\,d\tau\right]^2} \\
&\le \|\mathrm{w}\|_{L^2(\boldsymbol{\Omega})}\sqrt{\sum_{\mathrm{m}=1}^{\infty}\left[\int_0^{\mathrm{t}}\tau^{\gamma-1}\exp\left(-\widetilde{\mathrm{a}}_{\mathrm{m}}\frac{\mathrm{t}^\gamma - \tau^\gamma}{\gamma}\right)d\tau\right]^2}. \qquad (14)
\end{aligned}$$

By the change variable $\xi = \frac{\tau^\gamma}{\gamma}$, we obtain

$$\begin{aligned}
\int_0^{\mathrm{t}}\tau^{\gamma-1}\exp\left(-\widetilde{\mathrm{a}}_{\mathrm{m}}\frac{\mathrm{t}^\gamma - \tau^\gamma}{\gamma}\right)d\tau &= \int_0^{\frac{\mathrm{t}^\gamma}{\gamma}}\exp\left(-\widetilde{\mathrm{a}}_{\mathrm{m}}\frac{\mathrm{t}^\gamma}{\gamma} + \widetilde{\mathrm{a}}_{\mathrm{m}}\xi\right)d\xi \\
&= \frac{1 - \exp(-\widetilde{\mathrm{a}}_{\mathrm{m}}\frac{\mathrm{t}^\gamma}{\gamma})}{\widetilde{\mathrm{a}}_{\mathrm{m}}} \le \frac{1}{\widetilde{\mathrm{a}}_{\mathrm{m}}}. \qquad (15)
\end{aligned}$$

Since $\widetilde{\mathrm{a}}_{\mathrm{m}} \ge C\mathrm{m}^{\frac{2}{d}}$, where d is the dimension of the domain $\boldsymbol{\Omega}$, we know that

$$\sum_{\mathrm{m}=1}^{\infty}\frac{1}{\widetilde{\mathrm{a}}^2_{\mathrm{m}}} \le \sum_{\mathrm{m}=1}^{\infty}\frac{1}{C^2\mathrm{m}^{\frac{4}{d}}} = \mathcal{R}^2. \qquad (16)$$

Combining (14), (15) and (16), we deduce that

$$\big\|\mathcal{A}^2_\gamma(\mathrm{t})\mathrm{w}\big\|_{L^2(\boldsymbol{\Omega})} \le \mathcal{R}\|\mathrm{w}\|_{L^\infty(0,\mathscr{T}_o;L^2(\boldsymbol{\Omega}))}.$$

(b) For n > 0, using (12) we have

$$\exp\left(-\widetilde{\mathrm{a}}_{\mathrm{m}}\frac{\mathrm{t}^\gamma - \tau^\gamma}{\gamma}\right) \le \mathcal{P}_{1,\mathrm{n}}\widetilde{\mathrm{a}}_{\mathrm{m}}^{-\mathrm{n}}\gamma^{\mathrm{n}}\big(\mathrm{t}^\gamma - \tau^\gamma\big)^{-\mathrm{n}}.$$

Therefore

$$\begin{aligned}
\big\|\mathcal{A}^2_\gamma(\mathrm{t})\mathrm{w}\big\|^2_{\mathcal{H}^{\mathrm{n}}(\boldsymbol{\Omega})} &= \sum_{\mathrm{m}=1}^{\infty}\widetilde{\mathrm{a}}_{\mathrm{m}}^{\mathrm{n}}\left[\int_0^{\mathrm{t}}\tau^{\gamma-1}\exp\left(-\widetilde{\mathrm{a}}_{\mathrm{m}}\frac{\mathrm{t}^\gamma - \tau^\gamma}{\gamma}\right)\langle \mathrm{w}(\tau), \mathrm{e}_{\mathrm{m}}\rangle\,d\tau\right]^2 \\
&\le \sum_{\mathrm{m}=1}^{\infty}\widetilde{\mathrm{a}}_{\mathrm{m}}^{\mathrm{n}}\left[\int_0^{\mathrm{t}}\tau^{\gamma-1}\exp\left(-\widetilde{\mathrm{a}}_{\mathrm{m}}\frac{\mathrm{t}^\gamma - \tau^\gamma}{\gamma}\right)d\tau\right] \\
&\qquad \times \left[\int_0^{\mathrm{t}}\tau^{\gamma-1}\exp\left(-\widetilde{\mathrm{a}}_{\mathrm{m}}\frac{\mathrm{t}^\gamma - \tau^\gamma}{\gamma}\right)\langle \mathrm{w}(\tau), \mathrm{e}_{\mathrm{m}}\rangle^2\,d\tau\right]
\end{aligned}$$

$$
\begin{aligned}
&\leq \mathcal{P}_{1,n}\left[\int_0^t \tau^{\gamma-1}\left(t^\gamma-\tau^\gamma\right)^{-n}\,d\tau\right]\sum_{m=1}^{\infty}\left[\int_0^t \tau^{\gamma-1}\left\langle w(\tau),e_m\right\rangle^2\,d\tau\right]\\
&\leq \mathcal{P}_{1,n}\frac{\mathscr{T}_o^{\gamma(1-n)}}{\gamma(1-n)}\int_0^t \tau^{\gamma-1}\left\|w(\tau)\right\|^2_{L^2(\boldsymbol{\Omega})}\,d\tau,
\end{aligned}
$$

where we have noted that

$$
\begin{aligned}
\int_0^t \tau^{\gamma-1}\left(t^\gamma-\tau^\gamma\right)^{-n}\,d\tau &= \frac{1}{\gamma}\int_0^t \left(t^\gamma-\xi\right)^{-n}\,d\xi\\
&= \frac{t^{\gamma(1-n)}}{\gamma(1-n)} \leq \frac{\mathscr{T}_o^{\gamma(1-n)}}{\gamma(1-n)}.
\end{aligned}
$$

(c) Since $0 \leq t_1 < t_2 \leq \mathscr{T}_o$ we have

$$
\begin{aligned}
&\left(\mathcal{A}_\gamma^2(t_1)-\mathcal{A}_\gamma^2(t_2)\right)w\\
&\quad= \sum_{m=1}^{\infty}\int_0^{t_1}\left[\exp\left(-\widetilde{a}_m\frac{t_1^\gamma}{\gamma}\right)-\exp\left(-\widetilde{a}_m\frac{t_2^\gamma}{\gamma}\right)\right]\tau^{\gamma-1}\exp\left(\widetilde{a}_m\frac{\tau^\gamma}{\gamma}\right)\langle w,e_m\rangle\,d\tau\, e_m\\
&\qquad- \sum_{m=1}^{\infty}\int_{t_1}^{t_2}\tau^{\gamma-1}\exp\left(-\widetilde{a}_m\frac{t_2^\gamma-\tau^\gamma}{\gamma}\right)\langle w,e_m\rangle\,d\tau\, e_m\\
&\quad= \mathfrak{D}_1(m,t,\gamma)-\mathfrak{D}_2(m,t,\gamma),
\end{aligned}
$$

where $\mathfrak{D}_2(m,t,\gamma) = \sum_{m=1}^{\infty}\int_{t_1}^{t_2}\tau^{\gamma-1}\exp(-\widetilde{a}_m\frac{t_2^\gamma-\tau^\gamma}{\gamma})\langle w,e_m\rangle\,d\tau\, e_m$. Also note $|\exp(-a)-\exp(-b)| \leq |a-b|\max\{\exp(-a),\exp(-b)\}$ and for $t_1 < t_2$ then $t_1^\gamma < t_2^\gamma$ and

$$
\left|\exp\left(-\widetilde{a}_m\frac{t_1^\gamma}{\gamma}\right)-\exp\left(-\widetilde{a}_m\frac{t_2^\gamma}{\gamma}\right)\right| \leq \frac{\widetilde{a}_m}{\gamma}\left|t_1^\gamma-t_2^\gamma\right|\exp\left(-\widetilde{a}_m\frac{t_1^\gamma}{\gamma}\right),
$$

and this together with Hölder's inequality yields

$$
\begin{aligned}
&\left\|\mathfrak{D}_1(m,t,\gamma)\right\|^2_{L^2(\boldsymbol{\Omega})}\\
&\quad\leq \left\|\sum_{m=1}^{\infty}\int_0^{t_1}\frac{\widetilde{a}_m}{\gamma}\left|t_1^\gamma-t_2^\gamma\right|\tau^{\gamma-1}\exp\left(\widetilde{a}_m\frac{\tau^\gamma-t_1^\gamma}{\gamma}\right)\langle w,e_m\rangle\,d\tau\, e_m\right\|^2_{L^2(\boldsymbol{\Omega})}\\
&\quad\leq \frac{|t_1^\gamma-t_2^\gamma|^2}{\gamma^2}\sum_{m=1}^{\infty}\widetilde{a}_m^2\left[\int_0^{t_1}\tau^{\gamma-1}\exp\left(\widetilde{a}_m\frac{\tau^\gamma-t_1^\gamma}{\gamma}\right)d\tau\right]\\
&\qquad\times\left[\int_0^{t_1}\tau^{\gamma-1}\exp\left(\widetilde{a}_m\frac{\tau^\gamma-t_1^\gamma}{\gamma}\right)\left|\langle w,e_m\rangle\right|^2\,d\tau\right].
\end{aligned}
$$

For $0 < q < 1$, using (13) we obtain

$$
\exp\left(\widetilde{a}_m\frac{\tau^\gamma-t_1^\gamma}{\gamma}\right) \leq \mathcal{P}_{2,q}\widetilde{a}_m^{q-1}\gamma^{1-q}\left(t_1^\gamma-\tau^\gamma\right)^{q-1}.
$$

This implies that

$$\begin{aligned}
&\left\|\mathfrak{D}_1(\mathrm{m},\mathrm{t},\gamma)\right\|^2_{L^2(\boldsymbol{\Omega})}\\
&\quad\le \mathcal{P}_{2,\mathrm{q}}\frac{|\mathrm{t}_1^\gamma-\mathrm{t}_2^\gamma|^2}{\gamma^{\mathrm{q}+1}}\int_0^{\mathrm{t}_1}\tau^{\gamma-1}\big(\mathrm{t}_1^\gamma-\tau^\gamma\big)^{\mathrm{q}-1}\,d\tau\sum_{\mathrm{m}=1}^\infty\widetilde{\mathrm{a}}_{\mathrm{m}}^{\mathrm{q}+1}\int_0^{\mathrm{t}_1}\tau^{\gamma-1}\big|\langle\mathrm{w},\mathrm{e}_{\mathrm{m}}\rangle\big|^2\,d\tau.
\end{aligned}$$

We note that

$$\int_0^{\mathrm{t}_1}\tau^{\gamma-1}\big(\mathrm{t}_1^\gamma-\tau^\gamma\big)^{\mathrm{q}-1}\,d\tau=\frac{1}{\gamma}\int_0^{\mathrm{t}_1^\gamma}\big(\mathrm{t}^\gamma-\xi\big)^{\mathrm{q}-1}\,d\xi=\frac{\mathrm{t}_1^{\mathrm{q}\gamma}}{\mathrm{q}\gamma}\le\frac{\mathscr{T}_o^{\mathrm{q}\gamma}}{\mathrm{q}\gamma}.$$

Therefore

$$\left\|\mathfrak{D}_1(\mathrm{m},\mathrm{t},\gamma)\right\|^2_{L^2(\boldsymbol{\Omega})}\le\mathcal{P}_{2,\mathrm{q}}\frac{|\mathrm{t}_1^\gamma-\mathrm{t}_2^\gamma|^2}{\gamma^{\mathrm{q}+1}}\frac{\mathscr{T}_o^{\mathrm{q}\gamma}}{\mathrm{q}\gamma}\int_0^{\mathrm{t}_1}\tau^{\gamma-1}\|\mathrm{w}\|^2_{\mathcal{H}^{\mathrm{q}+1}(\boldsymbol{\Omega})}\,d\tau.$$

Using Hölder's inequality, we have

$$\begin{aligned}
&\left\|\mathfrak{D}_2(\mathrm{m},\mathrm{t},\gamma)\right\|^2_{L^2(\boldsymbol{\Omega})}\\
&\quad=\left\|\sum_{\mathrm{m}=1}^\infty\int_{\mathrm{t}_1}^{\mathrm{t}_2}\tau^{\gamma-1}\exp\left(-\widetilde{\mathrm{a}}_{\mathrm{m}}\frac{\mathrm{t}_2^\gamma-\tau^\gamma}{\gamma}\right)\langle\mathrm{w},\mathrm{e}_{\mathrm{m}}\rangle\,d\tau\,\mathrm{e}_{\mathrm{m}}\right\|^2_{L^2(\boldsymbol{\Omega})}\\
&\quad=\sum_{\mathrm{m}=1}^\infty\left[\int_{\mathrm{t}_1}^{\mathrm{t}_2}\tau^{\gamma-1}\exp\left(-\widetilde{\mathrm{a}}_{\mathrm{m}}\frac{\mathrm{t}_2^\gamma-\tau^\gamma}{\gamma}\right)\langle\mathrm{w},\mathrm{e}_{\mathrm{m}}\rangle\,d\tau\right]^2\\
&\quad\le\sum_{\mathrm{m}=1}^\infty\left[\int_{\mathrm{t}_1}^{\mathrm{t}_2}\tau^{\gamma-1}\exp\left(-\widetilde{\mathrm{a}}_{\mathrm{m}}\frac{\mathrm{t}_2^\gamma-\tau^\gamma}{\gamma}\right)d\tau\right]\left[\int_{\mathrm{t}_1}^{\mathrm{t}_2}\tau^{\gamma-1}\exp\left(-\widetilde{\mathrm{a}}_{\mathrm{m}}\frac{\mathrm{t}_2^\gamma-\tau^\gamma}{\gamma}\right)\big|\langle\mathrm{w},\mathrm{e}_{\mathrm{m}}\rangle\big|^2\,d\tau\right]\\
&\quad\le\sum_{\mathrm{m}=1}^\infty\left[\int_{\mathrm{t}_1}^{\mathrm{t}_2}\tau^{\gamma-1}\,d\tau\right]\left[\int_{\mathrm{t}_1}^{\mathrm{t}_2}\tau^{\gamma-1}\big|\langle\mathrm{w},\mathrm{e}_{\mathrm{m}}\rangle\big|^2\,d\tau\right]\\
&\quad\le\frac{\mathrm{t}_2^\gamma-\mathrm{t}_1^\gamma}{\gamma}\int_{\mathrm{t}_1}^{\mathrm{t}_2}\tau^{\gamma-1}\|\mathrm{w}\|^2_{L^2(\boldsymbol{\Omega})}\,d\tau.
\end{aligned}$$

From the above results, we get

$$\begin{aligned}
&\left\|\big(\mathcal{A}_\gamma^2(\mathrm{t}_1)-\mathcal{A}_\gamma^2(\mathrm{t}_2)\big)\mathrm{w}\right\|_{L^2(\boldsymbol{\Omega})}\\
&\quad\le\left(\mathcal{P}_{2,\mathrm{q}}\frac{|\mathrm{t}_1^\gamma-\mathrm{t}_2^\gamma|^2}{\gamma^{\mathrm{q}+1}}\frac{\mathscr{T}_o^{\mathrm{q}\gamma}}{\mathrm{q}\gamma}\int_0^{\mathrm{t}_1}\tau^{\gamma-1}\|\mathrm{w}\|^2_{\mathcal{H}^{\mathrm{q}+1}(\boldsymbol{\Omega})}\,d\tau\right)^{\frac{1}{2}}\\
&\qquad+\left(\frac{\mathrm{t}_2^\gamma-\mathrm{t}_1^\gamma}{\gamma}\int_{\mathrm{t}_1}^{\mathrm{t}_2}\tau^{\gamma-1}\|\mathrm{w}\|^2_{L^2(\boldsymbol{\Omega})}\,d\tau\right)^{\frac{1}{2}}.
\end{aligned}$$

□

2.3 Ill-posedness of determining initial data and stability estimate

Recall that the solution $\mathfrak{u}(\mathrm{t},\mathrm{x})$ of Problem (4) (i.e. the inverse problem) is given by (11). Let

$$\mathfrak{g}(\mathrm{x}):=\mathfrak{u}(0,\mathrm{x})=\sum_{\mathrm{m}=1}^\infty\exp\left(\widetilde{\mathrm{a}}_{\mathrm{m}}\frac{\mathscr{T}_o^\gamma}{\gamma}\right)\boldsymbol{\Upsilon}_{\mathrm{m}}\mathrm{e}_{\mathrm{m}}(\mathrm{x}),\tag{17}$$

where

$$\boldsymbol{\Upsilon}_{\mathrm{m}} = \langle \boldsymbol{\Upsilon}, \mathrm{e}_{\mathrm{m}} \rangle, \quad \text{with } \boldsymbol{\Upsilon} = \mathrm{h} - \mathcal{A}_\gamma^2(\mathscr{T}_o)\mathrm{F}(\tau). \tag{18}$$

In this paper, our main purpose is to determine the initial value $\mathfrak{g}(\mathrm{x})$ from the final data $\mathrm{h}(\mathrm{x})$ and the source term $\mathrm{F}(\mathrm{t},\mathrm{x})$. We can find $\mathfrak{g}(\mathrm{x})$ by solving an operator equation as follows:

$$\mathcal{K}\mathfrak{g} = \boldsymbol{\Upsilon},$$

where $\mathcal{K} : L^2(\boldsymbol{\Omega}) \to L^2(\boldsymbol{\Omega})$ is the integral operator defined by

$$(\mathcal{K}\mathfrak{g})(\mathrm{x}) := \sum_{\mathrm{m}=1}^{\infty} \exp\left(-\widetilde{\mathrm{a}}_{\mathrm{m}} \frac{\mathscr{T}_o^\gamma}{\gamma}\right) \langle \mathfrak{g}, \mathrm{e}_{\mathrm{m}} \rangle \mathrm{e}_{\mathrm{m}} = \int_{\boldsymbol{\Omega}} \varrho(\xi, \mathrm{x}) \mathfrak{g}(\xi)\, d\xi,$$

with kernel $\varrho(\cdot,\cdot)$ given by

$$\varrho(\xi, \mathrm{x}) := \sum_{\mathrm{m}=1}^{\infty} \exp\left(-\widetilde{\mathrm{a}}_{\mathrm{m}} \frac{\mathscr{T}_o^\gamma}{\gamma}\right) \mathrm{e}_{\mathrm{m}}(\xi) \mathrm{e}_{\mathrm{m}}(\mathrm{x}).$$

Since $\varrho(\xi, \mathrm{x}) = \varrho(\mathrm{x}, \xi)$, we know that the operator $\mathcal{K}$ is self-adjoint. Assume that $\boldsymbol{\Upsilon} \in L^2(\boldsymbol{\Omega})$.

Lemma 2.4 *Let $h \in L^2(\boldsymbol{\Omega})$ and $F \in L^\infty(0, \mathscr{T}_o; L^2(\boldsymbol{\Omega}))$. Then $\boldsymbol{\Upsilon}$ as in (18) belongs to $L^2(\boldsymbol{\Omega})$ and*

$$\|\boldsymbol{\Upsilon}\|_{L^2(\boldsymbol{\Omega})} \le \|h\|_{L^2(\boldsymbol{\Omega})} + \mathcal{R}\|F\|_{L^\infty(0,\mathscr{T}_o;L^2(\boldsymbol{\Omega}))}.$$

Proof From the definition of $\boldsymbol{\Upsilon}$, we get

$$\begin{aligned}\|\boldsymbol{\Upsilon}\|_{L^2(\boldsymbol{\Omega})} &\le \left\|\mathrm{h} - \mathcal{A}_\gamma^2(\mathscr{T}_o)\mathrm{F}(\tau)\right\|_{L^2(\boldsymbol{\Omega})} \\ &\le \|\mathrm{h}\|_{L^2(\boldsymbol{\Omega})} + \left\|\mathcal{A}_\gamma^2(\mathscr{T}_o)\mathrm{F}(\tau)\right\|_{L^2(\boldsymbol{\Omega})}.\end{aligned}$$

Using Lemma 2.3, we deduce that

$$\left\|\mathcal{A}_\gamma^2(\mathscr{T}_o)\mathrm{F}(\mathrm{t})\right\|_{L^2(\boldsymbol{\Omega})} \le \mathcal{R}\left\|\mathrm{F}(\mathrm{t})\right\|_{L^2(\boldsymbol{\Omega})} \le \mathcal{R}\|\mathrm{F}\|_{L^\infty(0,\mathscr{T}_o;L^2(\boldsymbol{\Omega}))}. \tag{19}$$

Therefore

$$\|\boldsymbol{\Upsilon}\|_{L^2(\boldsymbol{\Omega})} \le \|\mathrm{h}\|_{L^2(\boldsymbol{\Omega})} + \mathcal{R}\|\mathrm{F}\|_{L^\infty(0,\mathscr{T}_o;L^2(\boldsymbol{\Omega}))}.$$

This completes the proof. □

Therefore, $\mathcal{K} : L^2(\boldsymbol{\Omega}) \to L^2(\boldsymbol{\Omega})$ is compact operator of infinite rank. Hence $\mathcal{K}$ does not have a continuous inverse [24].

To illustrate the ill-posedness of the backward problem, we give an example. Let $(\mathrm{h}, \mathrm{F}) = (0, 0)$ and $(\overline{\mathrm{h}}, \overline{\mathrm{F}}) = (\frac{1}{\sqrt{\widetilde{\mathrm{a}}_{\mathrm{l}}}}\mathrm{e}_{\mathrm{l}}, \frac{1}{\sqrt{\widetilde{\mathrm{a}}_{\mathrm{l}}}}\mathrm{e}_{\mathrm{l}})$. It is easy to see that

$$\|\overline{\mathrm{h}} - \mathrm{h}\| = \frac{1}{\sqrt{\widetilde{\mathrm{a}}_{\mathrm{l}}}}, \quad \text{and} \quad \|\overline{\mathrm{F}} - \mathrm{F}\| = \frac{1}{\sqrt{\widetilde{\mathrm{a}}_{\mathrm{l}}}}.$$

Hence

$$\lim_{l\to\infty} \|\overline{h} - h\| = 0, \quad \text{and} \quad \lim_{l\to\infty} \|\overline{F} - F\| = 0, \tag{20}$$

so $(\overline{h}, \overline{F})$ is an approximation of (h, F) when l is large enough. Using $(\overline{h}, \overline{F})$, we get the corresponding initial data $\overline{\mathfrak{g}}$ and the equation $\overline{\boldsymbol{\Upsilon}}$ as follows:

$$\overline{\boldsymbol{\Upsilon}}(x) = \sum_{m=1}^{\infty}\left[\overline{h}_m - \int_0^{\mathscr{T}_o} \tau^{\gamma-1} \exp\left(-\widetilde{a}_m \frac{\mathscr{T}_o^{\gamma} - \tau^{\gamma}}{\gamma}\right)\overline{F}_m(\tau)\,d\tau\right] e_m(x),$$

$$\overline{\mathfrak{g}}(x) = \sum_{m=1}^{\infty} \exp\left(\widetilde{a}_m \frac{\mathscr{T}_o^{\gamma}}{\gamma}\right)\left[\overline{h}_m - \int_0^{\mathscr{T}_o} \tau^{\gamma-1} \exp\left(-\widetilde{a}_m \frac{\mathscr{T}_o^{\gamma} - \tau^{\gamma}}{\gamma}\right)\overline{F}_m(\tau)\,d\tau\right] e_m(x).$$

From Parseval's equality and (15), we get

$$\begin{aligned}
\|\overline{\boldsymbol{\Upsilon}} - \boldsymbol{\Upsilon}\|^2 &= \sum_{m=1}^{\infty}\left[\langle \overline{h}_m - \overline{h}, e_m\rangle - \int_0^{\mathscr{T}_o} \tau^{\gamma-1} \exp\left(-\widetilde{a}_m \frac{\mathscr{T}_o^{\gamma} - \tau^{\gamma}}{\gamma}\right)\langle \overline{F}_m - F, e_m\rangle\,d\tau\right]^2 \\
&= \left[\frac{1}{\sqrt{\widetilde{a}_l}} - \frac{1}{\sqrt{\widetilde{a}_l}} \int_0^{\mathscr{T}_o} \tau^{\gamma-1} \exp\left(-\widetilde{a}_l \frac{\mathscr{T}_o^{\gamma} - \tau^{\gamma}}{\gamma}\right) d\tau\right]^2 \\
&= \frac{1}{\widetilde{a}_l}\left[1 - \frac{1 - \exp(-\widetilde{a}_l \frac{\mathscr{T}_o^{\gamma}}{\gamma})}{\widetilde{a}_l}\right]^2.
\end{aligned}$$

This gives

$$\lim_{l\to\infty} \|\overline{\boldsymbol{\Upsilon}} - \boldsymbol{\Upsilon}\| = 0.$$

On the other hand, we have

$$\begin{aligned}
&\|\overline{\mathfrak{g}} - \mathfrak{g}\|^2 \\
&\quad = \sum_{m=1}^{\infty} \exp\left(\widetilde{a}_m \frac{\mathscr{T}_o^{\gamma}}{\gamma}\right)\Bigg[\langle \overline{h}_m - \overline{h}, e_m\rangle \\
&\qquad - \int_0^{\mathscr{T}_o} \tau^{\gamma-1} \exp\left(-\widetilde{a}_m \frac{\mathscr{T}_o^{\gamma} - \tau^{\gamma}}{\gamma}\right)\langle \overline{F}_m - F, e_m\rangle\,d\tau\Bigg]^2 \\
&\quad = \exp\left(\widetilde{a}_l \frac{\mathscr{T}_o^{\gamma}}{\gamma}\right)\left[\frac{1}{\sqrt{\widetilde{a}_l}} - \frac{1}{\sqrt{\widetilde{a}_l}} \int_0^{\mathscr{T}_o} \tau^{\gamma-1} \exp\left(-\widetilde{a}_l \frac{\mathscr{T}_o^{\gamma} - \tau^{\gamma}}{\gamma}\right) d\tau\right]^2 \\
&\quad = \exp\left(\widetilde{a}_l \frac{\mathscr{T}_o^{\gamma}}{\gamma}\right)\frac{1}{\widetilde{a}_l}\left[1 - \frac{1 - \exp(-\widetilde{a}_l \frac{\mathscr{T}_o^{\gamma}}{\gamma})}{\widetilde{a}_l}\right]^2,
\end{aligned}$$

so

$$\lim_{l\to+\infty} \|\overline{\mathfrak{g}} - \mathfrak{g}\| = +\infty. \tag{21}$$

We conclude that the backward problem is ill-posed in the Hadamard sense. Hence a regularization method is necessary. We will use the Landweber method to deal with the ill-posed problem. Before doing that, we impose an a priori bound on the initial data; that is,

$$\sum_{m=1}^{\infty} \exp\left(2r\widetilde{a}_m \frac{\mathscr{T}_o^\gamma}{\gamma}\right)\left|\langle \mathfrak{g}, e_m\rangle\right|^2 \le \mathbf{E}^2, \quad \text{for any } r \ge 0, \tag{22}$$

where $\mathbf{E}$ is a positive constant. The a priori bound of the exact solution is necessary for any ill-posed problem, otherwise, the rate of convergence is very slow or the regularization solution is not convergent (see [12]).

3 Landweber regularization method and regularity of the regularized solution

In this section, we present a regularized problem by using the Landweber regularization method, and also we consider the well- posedness of the regularized solution. From [17], the operator equation $\mathcal{K}\mathfrak{g} = \boldsymbol{\Upsilon}$ is equivalent to the following equation:

$$\mathfrak{g} = \left(I - \mu\mathcal{K}^*\mathcal{K}\right)\mathfrak{g} + \mu\mathcal{K}^*\boldsymbol{\Upsilon}, \tag{23}$$

for any $\mu > 0$. Here, $\mathcal{K}^*$ is the adjoint operator of $\mathcal{K}$, and $\mu > 0$ satisfies $0 < \mu < \frac{1}{\|\mathcal{K}\|^2}$. The iterative implementation of the Landweber method was constructed in [18]. Denote the Landweber regularization solution by

$$\mathfrak{g}_{k,\alpha}(x) = \sum_{m=1}^{\infty} \exp\left(\widetilde{a}_m \frac{\mathscr{T}_o^\gamma}{\gamma}\right)\left[1 - \left(1 - \mu\exp\left(-2\widetilde{a}_m \frac{\mathscr{T}_o^\gamma}{\gamma}\right)\right)^k\right]^\alpha \langle \boldsymbol{\Upsilon}, e_m\rangle e_m, \tag{24}$$

and the Landweber regularization solution with noisy data by

$$\mathfrak{g}_{k,\alpha}^{\varepsilon}(x) = \sum_{m=1}^{\infty} \exp\left(\widetilde{a}_m \frac{\mathscr{T}_o^\gamma}{\gamma}\right)\left[1 - \left(1 - \mu\exp\left(-2\widetilde{a}_m \frac{\mathscr{T}_o^\gamma}{\gamma}\right)\right)^k\right]^\alpha \langle \boldsymbol{\Upsilon}^{\varepsilon}, e_m\rangle e_m, \tag{25}$$

where $\alpha \in (\frac{1}{2}, 1]$ is called the fractional parameter, and $k = 1, 2, 3, \ldots$ is a regularization parameter. When $\alpha = 1$, this is the classical Landweber method.

Hence, we get the Landweber regularization solution of Problem (4) (i.e. the inverse problem):

$$\begin{aligned} u_{k,\alpha}(t,x) = \sum_{m=1}^{\infty}\Bigg[&\exp\left(-\widetilde{a}_m \frac{t^\gamma - \mathscr{T}_o^\gamma}{\gamma}\right) \\ &\times \left[1 - \left(1 - \mu\exp\left(-2\widetilde{a}_m \frac{\mathscr{T}_o^\gamma}{\gamma}\right)\right)^k\right]^\alpha \langle \boldsymbol{\Upsilon}, e_m\rangle \\ &+ \int_0^t \tau^{\gamma-1}\exp\left(-\widetilde{a}_m \frac{t^\gamma - \tau^\gamma}{\gamma}\right) F_m(\tau)\,d\tau\Bigg] e_m(x). \end{aligned} \tag{26}$$

Let us consider the operator

$$\begin{aligned} \mathcal{A}_\gamma^3(t, \mathscr{T}_o)v := \sum_{m=1}^{\infty} &\exp\left(-\widetilde{a}_m \frac{t^\gamma - \mathscr{T}_o^\gamma}{\gamma}\right) \\ &\times \left[1 - \left(1 - \mu\exp\left(-2\widetilde{a}_m \frac{\mathscr{T}_o^\gamma}{\gamma}\right)\right)^k\right]^\alpha \langle v, e_m\rangle e_m, \end{aligned}$$

for $v \in L^2(\boldsymbol{\Omega})$, and $0 \le t \le \mathscr{T}_o$.

Lemma 3.1 *Given* $\mu > 0$ *and* $k \in \mathbb{N}$ *with* $k \geq 1$.

(a) *If* $v \in L^2(\boldsymbol{\Omega})$, *then* $\|\mathcal{A}_\gamma^3(t, \mathscr{T}_o)v\| \in L^2(\boldsymbol{\Omega})$ *and*

$$\left\|\mathcal{A}_\gamma^3(t, \mathscr{T}_o)v\right\|_{L^2(\boldsymbol{\Omega})} \leq \mu^{\frac{1}{2}} k^{\frac{1}{2}} \|v\|_{L^2(\boldsymbol{\Omega})}.$$

(b) *If* $v \in L^2(\boldsymbol{\Omega})$ *for* $n > 0$, *then* $\|\mathcal{A}_\gamma^3(t, \mathscr{T}_o)v\| \in \mathcal{H}^n(\boldsymbol{\Omega})$ *and*

$$\left\|\mathcal{A}_\gamma^3(t, \mathscr{T}_o)v\right\|_{\mathcal{H}^n(\boldsymbol{\Omega})} \leq \mu^{\frac{1}{2}} k^{\frac{1}{2}} \mathcal{P}_{1,n}^{\frac{1}{2}} \left(\frac{2}{\gamma}\right)^{-\frac{n}{2}} t^{-\frac{n\gamma}{2}} \|v\|_{L^2(\boldsymbol{\Omega})}.$$

(c) *If* $v \in \mathcal{H}^s(\boldsymbol{\Omega})$ *for* $0 < s < 1$ *and* $0 \leq t_1 < t_2 \leq \mathscr{T}_o$, *then* $\|\mathcal{A}_\gamma^3(t, \mathscr{T}_o)v\| \in L^2(\boldsymbol{\Omega})$ *and*

$$\left\|\left(\mathcal{A}_\gamma^3(t_1, \mathscr{T}_o) - \mathcal{A}_\gamma^3(t_2, \mathscr{T}_o)\right)v\right\|_{L^2(\boldsymbol{\Omega})} \leq \mu^{\frac{1}{2}} k^{\frac{1}{2}} \frac{\mathcal{P}_{2,s}}{s\gamma^s} \left[t_2^\gamma - t_1^\gamma\right]^s \|v\|_{\mathcal{H}^s(\boldsymbol{\Omega})}.$$

Proof (a) First, using Lemma 2.2, we obtain

$$\begin{aligned}
&\left\|\mathcal{A}_\gamma^3(\mathrm{t}, \mathscr{T}_o)\mathrm{v}\right\|_{L^2(\boldsymbol{\Omega})} \\
&= \left\|\sum_{\mathrm{m}=1}^{\infty} \exp\left(-\widetilde{\mathrm{a}}_\mathrm{m} \frac{\mathrm{t}^\gamma - \mathscr{T}_o^\gamma}{\gamma}\right)\left[1 - \left(1 - \mu \exp\left(-2\widetilde{\mathrm{a}}_\mathrm{m} \frac{\mathscr{T}_o^\gamma}{\gamma}\right)\right)^{\mathrm{k}}\right]^\alpha \langle \mathrm{v}, \mathrm{e}_\mathrm{m}\rangle \mathrm{e}_\mathrm{m}\right\|_{L^2(\boldsymbol{\Omega})} \\
&\leq \left\|\mu^{\frac{1}{2}} \mathrm{k}^{\frac{1}{2}} \sum_{\mathrm{m}=1}^{\infty} \exp\left(-\widetilde{\mathrm{a}}_\mathrm{m} \frac{\mathrm{t}^\gamma}{\gamma}\right) \langle \mathrm{v}, \mathrm{e}_\mathrm{m}\rangle \mathrm{e}_\mathrm{m}\right\|_{L^2(\boldsymbol{\Omega})} \\
&\leq \mu^{\frac{1}{2}} \mathrm{k}^{\frac{1}{2}} \|\mathrm{v}\|_{L^2(\boldsymbol{\Omega})}.
\end{aligned}$$

(b) For n > 0, using (12) we have

$$\begin{aligned}
&\left\|\mathcal{A}_\gamma^3(\mathrm{t}, \mathscr{T}_o)\mathrm{v}\right\|_{\mathcal{H}^\mathrm{n}(\boldsymbol{\Omega})} \\
&= \left\|\sum_{\mathrm{m}=1}^{\infty} \exp\left(-\widetilde{\mathrm{a}}_\mathrm{m} \frac{\mathrm{t}^\gamma - \mathscr{T}_o^\gamma}{\gamma}\right)\left[1 - \left(1 - \mu \exp\left(-2\widetilde{\mathrm{a}}_\mathrm{m} \frac{\mathscr{T}_o^\gamma}{\gamma}\right)\right)^{\mathrm{k}}\right]^\alpha \langle \mathrm{v}, \mathrm{e}_\mathrm{m}\rangle \mathrm{e}_\mathrm{m}\right\|_{\mathcal{H}^\mathrm{n}} \\
&\leq \left\|\mu^{\frac{1}{2}} \mathrm{k}^{\frac{1}{2}} \sum_{\mathrm{m}=1}^{\infty} \exp\left(-\widetilde{\mathrm{a}}_\mathrm{m} \frac{\mathrm{t}^\gamma}{\gamma}\right) \langle \mathrm{v}, \mathrm{e}_\mathrm{m}\rangle \mathrm{e}_\mathrm{m}\right\|_{\mathcal{H}^\mathrm{n}(\boldsymbol{\Omega})} \\
&\leq \mu^{\frac{1}{2}} \mathrm{k}^{\frac{1}{2}} \sqrt{\sum_{\mathrm{m}=1}^{\infty} \widetilde{\mathrm{a}}_\mathrm{m}^\mathrm{n} \exp\left(-2\widetilde{\mathrm{a}}_\mathrm{m} \frac{\mathrm{t}^\gamma}{\gamma}\right) \left|\langle \mathrm{v}, \mathrm{e}_\mathrm{m}\rangle\right|^2} \\
&\leq \mu^{\frac{1}{2}} \mathrm{k}^{\frac{1}{2}} \mathcal{P}_{1,\mathrm{n}}^{\frac{1}{2}} \left(\frac{2}{\gamma}\right)^{-\frac{\mathrm{n}}{2}} \mathrm{t}^{-\frac{\mathrm{n}\gamma}{2}} \|\mathrm{v}\|_{L^2(\boldsymbol{\Omega})}.
\end{aligned}$$

(c) From the definition of $\mathcal{A}_\gamma^3(t, \mathscr{T}_o)$, we get

$$\begin{aligned}
&\big\|\big(\mathcal{A}_\gamma^3(t_1, \mathscr{T}_o) - \mathcal{A}_\gamma^3(t_2, \mathscr{T}_o)\big)v\big\|_{L^2(\Omega)} \\
&\quad = \left\| \sum_{m=1}^{\infty} \left[\exp\left(-\frac{\widetilde{a}_m t_1^\gamma}{\gamma}\right) - \exp\left(-\frac{\widetilde{a}_m t_2^\gamma}{\gamma}\right)\right] \exp\left(\widetilde{a}_m \frac{\mathscr{T}_o^\gamma}{\gamma}\right) \right. \\
&\qquad \left. \times \left[1 - \left(1 - \mu \exp\left(-2\widetilde{a}_m \frac{\mathscr{T}_o^\gamma}{\gamma}\right)\right)^k\right]^\alpha \langle v, e_m\rangle e_m \right\|_{L^2(\Omega)}.
\end{aligned}$$

Using Lemma 2.2, we get

$$\begin{aligned}
&\big\|\big(\mathcal{A}_\gamma^3(t_1, \mathscr{T}_o) - \mathcal{A}_\gamma^3(t_2, \mathscr{T}_o)\big)v\big\|_{L^2(\Omega)} \\
&\quad \le \left(\mu k \sum_{m=1}^{\infty} \left[\exp\left(\frac{-\widetilde{a}_m t_1^\gamma}{\gamma}\right) - \exp\left(\frac{-\widetilde{a}_m t_2^\gamma}{\gamma}\right)\right]^2 \big|\langle v, e_m\rangle\big|^2\right)^{\frac{1}{2}}.
\end{aligned}$$

For $0 < s < 1$, using (13) we obtain

$$\begin{aligned}
\exp\left(\frac{-\widetilde{a}_m t_1^\gamma}{\gamma}\right) - \exp\left(\frac{-\widetilde{a}_m t_2^\gamma}{\gamma}\right) &= \int_{t_1}^{t_2} \exp\left(-\frac{\widetilde{a}_m t^\gamma}{\gamma}\right) d\left(\frac{\widetilde{a}_m t^\gamma}{\gamma}\right) \\
&\le \mathcal{P}_{2,s} \int_{t_1}^{t_2} \left(\frac{\widetilde{a}_m t^\gamma}{\gamma}\right)^{s-1} d\left(\frac{\widetilde{a}_m t^\gamma}{\gamma}\right) \\
&\le \frac{\mathcal{P}_{2,s}}{s} \left(\frac{\widetilde{a}_m}{\gamma}\right)^s \big[t_2^\gamma - t_1^\gamma\big]^s.
\end{aligned}$$

From the above results, we deduce that

$$\begin{aligned}
&\big\|\big(\mathcal{A}_\gamma^3(t_1, \mathscr{T}_o) - \mathcal{A}_\gamma^3(t_2, \mathscr{T}_o)\big)v\big\|_{L^2(\Omega)} \\
&\quad \le \mu^{\frac{1}{2}} k^{\frac{1}{2}} \frac{\mathcal{P}_{2,s}}{s\gamma^s} \big[t_2^\gamma - t_1^\gamma\big]^s \left(\sum_{m=1}^{\infty} \widetilde{a}_m^s \big|\langle v, e_m\rangle\big|^2\right)^{\frac{1}{2}} \\
&\quad \le \mu^{\frac{1}{2}} k^{\frac{1}{2}} \frac{\mathcal{P}_{2,s}}{s\gamma^s} \big[t_2^\gamma - t_1^\gamma\big]^s \|v\|_{\mathcal{H}^s(\Omega)}.
\end{aligned}$$

□

The Landweber regularization solution of Problem (4) can be transformed into the form

$$u_{k,\alpha}(t) = \mathcal{A}_\gamma^3(t, \mathscr{T}_o)\big[h - \mathcal{A}_\gamma^2(\mathscr{T}_o)F(t)\big] + \mathcal{A}_\gamma^2(t)F(t). \tag{27}$$

and the Landweber regularization solution of Problem (4) with noisy data:

$$\begin{aligned}
u_{k,\alpha}^\varepsilon(t) &= \mathcal{A}_\gamma^3(t, \mathscr{T}_o)\big[h^\varepsilon - \mathcal{A}_\gamma^2(\mathscr{T}_o)F^\varepsilon(t)\big] + \mathcal{A}_\gamma^2(t)F^\varepsilon(t) \\
&= \varXi_1(t) + \varXi_2(t) + \varXi_3(t).
\end{aligned} \tag{28}$$

Next, we consider the regularity of the solution $u_{k,\alpha}^\varepsilon$. We give a result which establishes regularity of the regularized solution.

Theorem 3.1

(a) *Let* $h^\varepsilon \in L^2(\boldsymbol{\Omega})$ *and* $F \in L^\infty(0,\mathscr{T}_o;L^2(\boldsymbol{\Omega}))$. *If* $\boldsymbol{\Omega} \subset \mathbb{R}^d$ *for any* $1 \le d \le 3$ *then*

$$\|u^\varepsilon_{k,\alpha}\|_{L^\infty(0,\mathscr{T}_o;L^2(\boldsymbol{\Omega}))} \le \mu^{\frac{1}{2}} k^{\frac{1}{2}} \|h^\varepsilon\|_{L^2(\boldsymbol{\Omega})} + \big(\mu^{\frac{1}{2}} k^{\frac{1}{2}} + 1\big)\mathcal{R}\|F^\varepsilon\|_{L^\infty(0,\mathscr{T}_o;L^2(\boldsymbol{\Omega}))}.$$

(b) *Given* $0 < n \neq 1$ *and* $1 < p < \min(\frac{2}{n\gamma}, \frac{1}{1-\gamma})$. *Let* $h^\varepsilon \in L^2(\boldsymbol{\Omega})$ *and* $F \in L^{\frac{2p}{p-1}}(0,\mathscr{T}_o;L^2(\boldsymbol{\Omega})) \cap L^\infty(0,\mathscr{T}_o;L^2(\boldsymbol{\Omega}))$. *Then there exists* $\mathfrak{M}$ *depends only* $\mathscr{T}_o$, p, n, μ, k *and* (*the regularized solution*) $u^\varepsilon_{k,\alpha} \in L^p(0,\mathscr{T}_o,\mathcal{H}^n(\boldsymbol{\Omega})) \cap L^\infty(0,\mathscr{T}_o;L^2(\boldsymbol{\Omega}))$ *such that*

$$\|u^\varepsilon_{k,\alpha}\|_{L^p(0,\mathscr{T}_o,\mathcal{H}^n(\boldsymbol{\Omega}))} \le \mathfrak{M}\Big(\|h^\varepsilon\|_{L^2(\boldsymbol{\Omega})} + \|F^\varepsilon\|_{L^\infty(0,\mathscr{T}_o;L^2(\boldsymbol{\Omega}))} + \|F^\varepsilon\|_{L^{\frac{2p}{p-1}}(0,\mathscr{T}_o;L^2(\boldsymbol{\Omega}))}\Big),$$

(c) *Let* $h^\varepsilon \in \mathcal{H}^s(\boldsymbol{\Omega})$ *for* $0 < s < 1$ *and* $F \in L^{\frac{2p}{p-1}}(0,\mathscr{T}_o;\mathcal{H}^{q+1}(\boldsymbol{\Omega}))$ *for* $1 < p < \min(\frac{2}{\gamma}, \frac{1}{1-\gamma})$ *and* $0 < q < 1$. *Then* $u^\varepsilon_{k,\alpha} \in C([0,\mathscr{T}_o];L^2(\boldsymbol{\Omega}))$.

Proof (a) Since $h^\varepsilon \in L^2(\boldsymbol{\Omega})$ so part (a) of Lemma 3.1 yields

$$\|\Xi_1(t)\|_{L^2(\boldsymbol{\Omega})} = \|\mathcal{A}^3_\gamma(t,\mathscr{T}_o)h^\varepsilon\|_{L^2(\boldsymbol{\Omega})} \le \mu^{\frac{1}{2}} k^{\frac{1}{2}} \|h^\varepsilon\|_{L^2(\boldsymbol{\Omega})}.$$

By a similar method, we obtain

$$\|\Xi_2(t)\|_{L^2(\boldsymbol{\Omega})} = \|\mathcal{A}^3_\gamma(t,\mathscr{T}_o)\big[\mathcal{A}^2_\gamma(\mathscr{T}_o)F^\varepsilon(t)\big]\|_{L^2(\boldsymbol{\Omega})} \le \mu^{\frac{1}{2}} k^{\frac{1}{2}} \|\mathcal{A}^2_\gamma(\mathscr{T}_o)F^\varepsilon(t)\|_{L^2(\boldsymbol{\Omega})}.$$

Assume $F \in L^\infty(0,\mathscr{T}_o;L^2(\boldsymbol{\Omega}))$. Now

$$\big|\langle F^\varepsilon(t), e_m\rangle\big|^2 \le \operatorname*{ess\,sup}_{0\le t\le \mathscr{T}_o} \sum_{m=1}^\infty \big|\langle F^\varepsilon(t), e_m\rangle\big|^2 = \|F^\varepsilon\|^2_{L^\infty(0,\mathscr{T}_o;L^2(\boldsymbol{\Omega}))}.$$

Using Lemma 2.3, we deduce that

$$\|\mathcal{A}^2_\gamma(\mathscr{T}_o)F^\varepsilon(t)\|_{L^2(\boldsymbol{\Omega})} \le \mathcal{R}\|F^\varepsilon(t)\|_{L^2(\boldsymbol{\Omega})} \le \mathcal{R}\|F^\varepsilon\|_{L^\infty(0,\mathscr{T}_o;L^2(\boldsymbol{\Omega}))}. \tag{29}$$

Hence

$$\|\Xi_2(t)\|_{L^2(\boldsymbol{\Omega})} \le \mu^{\frac{1}{2}} k^{\frac{1}{2}} \mathcal{R}\|F^\varepsilon\|_{L^\infty(0,\mathscr{T}_o;L^2(\boldsymbol{\Omega}))}.$$

On the other hand

$$\|\Xi_3(t)\|_{L^2(\boldsymbol{\Omega})} = \|\mathcal{A}^2_\gamma(t)F^\varepsilon(t)\|_{L^2(\boldsymbol{\Omega})} \le \mathcal{R}\|F^\varepsilon\|_{L^\infty(0,\mathscr{T}_o;L^2(\boldsymbol{\Omega}))}.$$

Combining the above results, we get

$$\begin{aligned}\|u^\varepsilon_{k,\alpha}(t)\|_{L^2(\boldsymbol{\Omega})} &= \|\Xi_1(t)(t)\|_{L^2(\boldsymbol{\Omega})} + \|\Xi_2(t)\|_{L^2(\boldsymbol{\Omega})} + \|\Xi_3(t)\|_{L^2(\boldsymbol{\Omega})} \\ &\le \mu^{\frac{1}{2}} k^{\frac{1}{2}} \|h^\varepsilon\|_{L^2(\boldsymbol{\Omega})} + \big(\mu^{\frac{1}{2}} k^{\frac{1}{2}} + 1\big)\mathcal{R}\|F^\varepsilon\|_{L^\infty(0,\mathscr{T}_o;L^2(\boldsymbol{\Omega}))}.\end{aligned} \tag{30}$$

This implies that $u^\varepsilon_{k,\alpha} \in L^\infty(0,\mathscr{T}_o;L^2(\boldsymbol{\Omega}))$.

(b) Since $h^\varepsilon \in L^2(\boldsymbol{\Omega})$ so Lemma 3.1 with $0 < n$ yields

$$\|\Xi_1(t)\|_{\mathcal{H}^n(\boldsymbol{\Omega})} = \|\mathcal{A}_\gamma^3(t,\mathscr{T}_o)h^\varepsilon\|_{\mathcal{H}^n(\boldsymbol{\Omega})} \le \mu^{\frac{1}{2}}k^{\frac{1}{2}}\mathcal{P}_{1,n}^{\frac{1}{2}}\left(\frac{2}{\gamma}\right)^{-\frac{n}{2}} t^{-\frac{n\gamma}{2}}\|h^\varepsilon\|_{L^2(\boldsymbol{\Omega})}.$$

Since $p < \frac{2}{n\gamma}$, we get

$$1 - \frac{pn\gamma}{2} = \frac{2 - pn\gamma}{2} > 0.$$

Therefore,

$$\begin{aligned}
\|\Xi_1\|_{L^p(0,\mathscr{T}_o,\mathcal{H}^n(\boldsymbol{\Omega}))} &= \left(\int_0^{\mathscr{T}_o} \|\Xi_1(t)\|_{\mathcal{H}^n(\boldsymbol{\Omega})}^p \, dt\right)^{\frac{1}{p}} \\
&\le \mu^{\frac{1}{2}}k^{\frac{1}{2}}\mathcal{P}_{1,n}^{\frac{1}{2}}\left(\frac{2}{\gamma}\right)^{-\frac{n}{2}}\left(\int_0^{\mathscr{T}_o} t^{-\frac{pn\gamma}{2}}\, dt\right)^{\frac{1}{p}}\|h^\varepsilon\|_{L^2(\boldsymbol{\Omega})} \\
&\le \mu^{\frac{1}{2}}k^{\frac{1}{2}}\mathcal{P}_{1,n}^{\frac{1}{2}}\left(\frac{2}{\gamma}\right)^{-\frac{n}{2}}\left(\frac{2}{2-pn\gamma}\right)^{\frac{1}{p}}\mathscr{T}_o^{\frac{2-pn\gamma}{2p}}\|h^\varepsilon\|_{L^2(\boldsymbol{\Omega})} \\
&= \mathfrak{M}_1(\mu,k,p,n)\|h^\varepsilon\|_{L^2(\boldsymbol{\Omega})},
\end{aligned} \tag{31}$$

where

$$\mathfrak{M}_1(\mu,k,p,n) = \mu^{\frac{1}{2}}k^{\frac{1}{2}}\mathcal{P}_{1,n}^{\frac{1}{2}}\left(\frac{2}{\gamma}\right)^{-\frac{n}{2}}\left(\frac{2}{2-pn\gamma}\right)^{\frac{1}{p}}\mathscr{T}_o^{\frac{2-pn\gamma}{2p}}.$$

By a similar method, we obtain

$$\begin{aligned}
\|\Xi_2(t)\|_{\mathcal{H}^n(\boldsymbol{\Omega})} &= \|\mathcal{A}_\gamma^3(t,\mathscr{T}_o)[\mathcal{A}_\gamma^2(\mathscr{T}_o)F^\varepsilon(t)]\|_{\mathcal{H}^n(\boldsymbol{\Omega})} \\
&\le \mu^{\frac{1}{2}}k^{\frac{1}{2}}\mathcal{P}_{1,n}^{\frac{1}{2}}\left(\frac{2}{\gamma}\right)^{-\frac{n}{2}} t^{-\frac{n\gamma}{2}}\|\mathcal{A}_\gamma^2(\mathscr{T}_o)F^\varepsilon(t)\|_{L^2(\boldsymbol{\Omega})}.
\end{aligned}$$

From (29) we obtain

$$\|\Xi_2(t)\|_{\mathcal{H}^n(\boldsymbol{\Omega})} \le \mu^{\frac{1}{2}}k^{\frac{1}{2}}\mathcal{P}_{1,n}^{\frac{1}{2}}\left(\frac{2}{\gamma}\right)^{-\frac{n}{2}} t^{-\frac{n\gamma}{2}}\mathcal{R}\|F^\varepsilon\|_{L^\infty(0,\mathscr{T}_o;L^2(\boldsymbol{\Omega}))}.$$

Therefore

$$\begin{aligned}
\|\Xi_2\|_{L^p(0,\mathscr{T}_o,\mathcal{H}^n(\boldsymbol{\Omega}))} &= \left(\int_0^{\mathscr{T}_o} \|\Xi_2(t)\|_{\mathcal{H}^n(\boldsymbol{\Omega})}^p \, dt\right)^{\frac{1}{p}} \\
&\le \mu^{\frac{1}{2}}k^{\frac{1}{2}}\mathcal{P}_{1,n}^{\frac{1}{2}}\left(\frac{2}{\gamma}\right)^{-\frac{n}{2}}\mathcal{R}\left(\int_0^{\mathscr{T}_o} t^{-\frac{pn\gamma}{2}}\, dt\right)^{\frac{1}{p}}\|F^\varepsilon\|_{L^\infty(0,\mathscr{T}_o;L^2(\boldsymbol{\Omega}))} \\
&\le \mu^{\frac{1}{2}}k^{\frac{1}{2}}\mathcal{P}_{1,n}^{\frac{1}{2}}\left(\frac{2}{\gamma}\right)^{-\frac{n}{2}}\mathcal{R}\left(\frac{2}{2-pn\gamma}\right)^{\frac{1}{p}}\mathscr{T}_o^{\frac{2-pn\gamma}{2p}}\|F^\varepsilon\|_{L^\infty(0,\mathscr{T}_o;L^2(\boldsymbol{\Omega}))} \\
&= \mathfrak{M}_2(\mu,k,p,n)\|F^\varepsilon\|_{L^\infty(0,\mathscr{T}_o;L^2(\boldsymbol{\Omega}))},
\end{aligned} \tag{32}$$

where

$$\mathfrak{M}_2(\mu,k,p,n) = \mu^{\frac{1}{2}} k^{\frac{1}{2}} \mathcal{P}_{1,n}^{\frac{1}{2}} \mathcal{R}\left(\frac{2}{\gamma}\right)^{-\frac{n}{2}} \left(\frac{2}{2-pn\gamma}\right)^{\frac{1}{p}} \mathscr{T}_o^{\frac{2-pn\gamma}{2p}}.$$

For $n \neq 1$ using Lemma 2.3 we have

$$\begin{aligned}\|\Xi_3(t)\|_{\mathcal{H}^n(\boldsymbol{\Omega})} &= \|\mathcal{A}_\gamma^2(t)F^\varepsilon(t)\|_{\mathcal{H}^n(\boldsymbol{\Omega})} \\ &\leq \left(\frac{\mathcal{P}_{1,n}}{\gamma(1-n)}\right)^{\frac{1}{2}} \mathscr{T}_o^{\frac{\gamma(1-n)}{2}} \left(\int_0^t \tau^{\gamma-1}\|F^\varepsilon(\tau)\|_{L^2(\boldsymbol{\Omega})}^2\,d\tau\right)^{\frac{1}{2}}.\end{aligned}$$

Hölder's inequality gives

$$\begin{aligned}\int_0^t \tau^{\gamma-1}\|F^\varepsilon(\tau)\|_{L^2(\boldsymbol{\Omega})}^2\,d\tau &= \left(\int_0^t \tau^{p(\gamma-1)}\,d\tau\right)^{\frac{1}{p}} \left(\int_0^t \|F^\varepsilon(\tau)\|_{L^2(\boldsymbol{\Omega})}^{\frac{2p}{p-1}}\,d\tau\right)^{\frac{p-1}{p}} \\ &\leq \frac{\mathscr{T}_o^{\gamma-1+\frac{1}{p}}}{(\gamma p-p+1)^{\frac{1}{p}}}\|F^\varepsilon\|^2_{L^{\frac{2p}{p-1}}(0,\mathscr{T}_o;L^2(\boldsymbol{\Omega}))}.\end{aligned}$$

From the above results we have

$$\|\Xi_3(t)\|_{\mathcal{H}^n(\boldsymbol{\Omega})} \leq \left(\frac{\mathcal{P}_{1,n}}{\gamma(1-n)}\right)^{\frac{1}{2}} \mathscr{T}_o^{\frac{\gamma(1-n)}{2}} \frac{\mathscr{T}_o^{\frac{(\gamma-1)p+1}{2p}}}{(\gamma p-p+1)^{\frac{1}{2p}}}\|F^\varepsilon\|_{L^{\frac{2p}{p-1}}(0,\mathscr{T}_o;L^2(\boldsymbol{\Omega}))}.$$

Hence

$$\begin{aligned}&\|\Xi_3\|_{L^p(0,\mathscr{T}_o,\mathcal{H}^n(\boldsymbol{\Omega}))} \\ &\quad\leq \left(\frac{\mathcal{P}_{1,n}}{\gamma(1-n)(\gamma p-p+1)^{\frac{1}{p}}}\right)^{\frac{1}{2}} \mathscr{T}_o^{\gamma-\frac{\gamma n+1}{2}+\frac{3}{2p}}\|F^\varepsilon\|_{L^{\frac{2p}{p-1}}(0,\mathscr{T}_o;L^2(\boldsymbol{\Omega}))} \\ &\quad\leq \mathfrak{M}_3(\mu,k,p,n)\|F^\varepsilon\|_{L^{\frac{2p}{p-1}}(0,\mathscr{T}_o;L^2(\boldsymbol{\Omega}))},\end{aligned} \tag{33}$$

where

$$\mathfrak{M}_3(\mu,k,p,n) = \left(\frac{\mathcal{P}_{1,n}}{\gamma(1-n)(\gamma p-p+1)^{\frac{1}{p}}}\right)^{\frac{1}{2}} \mathscr{T}_o^{\gamma-\frac{\gamma n+1}{2}+\frac{3}{2p}}.$$

Combining (31), (32) and (33) we get

$$\|u_{k,\alpha}^\varepsilon\|_{L^p(0,\mathscr{T}_o,\mathcal{H}^n(\boldsymbol{\Omega}))} \leq \mathfrak{M}\left(\|h^\varepsilon\|_{L^2(\boldsymbol{\Omega})} + \|F^\varepsilon\|_{L^\infty(0,\mathscr{T}_o;L^2(\boldsymbol{\Omega}))} + \|F^\varepsilon\|_{L^{\frac{2p}{p-1}}(0,\mathscr{T}_o;L^2(\boldsymbol{\Omega}))}\right),$$

where

$$\mathfrak{M} = \max\left\{\mathfrak{M}_i(\mu,k,p,n) : i = \overline{1,3}\right\}.$$

(c) We obtain

$$\begin{aligned}\left\|u^{\varepsilon}_{k,\alpha}(t+\eta)-u^{\varepsilon}_{k,\alpha}(t)\right\|_{L^2(\boldsymbol{\Omega})} &\le \left\|\Xi_1(t+\eta)-\Xi_1(t)\right\|_{L^2(\boldsymbol{\Omega})} \\ &\quad+\left\|\Xi_2(t+\eta)-\Xi_2(t)\right\|_{L^2(\boldsymbol{\Omega})}+\left\|\Xi_3(t+\eta)-\Xi_3(t)\right\|_{L^2(\boldsymbol{\Omega})}.\end{aligned}$$

Now part (c) of Lemma 3.1 yields

$$\begin{aligned}\left\|\Xi_1(t+\eta)-\Xi_1(t)\right\|_{L^2(\boldsymbol{\Omega})} &= \left\|\left(\mathcal{A}^3_\gamma(t+\eta,\mathscr{T}_o)-\mathcal{A}^3_\gamma(t,\mathscr{T}_o)\right)h^\varepsilon\right\|_{L^2(\boldsymbol{\Omega})} \\ &\le \mu^{\frac12}k^{\frac12}\frac{\mathcal{P}_{2,s}}{s\gamma^s}\left[(t+\eta)^\gamma-t^\gamma\right]^s\left\|h^\varepsilon\right\|_{\mathcal{H}^s(\boldsymbol{\Omega})}.\end{aligned}$$

We use the inequality $(a_1+a_2)^\vartheta \le a_1^\vartheta + a_2^\vartheta$ for $0<\vartheta\le 1$ to get

$$(t+\eta)^\gamma - t^\gamma \le \eta^\gamma, \quad 0<\gamma\le 1. \tag{34}$$

Therefore

$$\left\|\Xi_1(t+\eta)-\Xi_1(t)\right\|_{L^2(\boldsymbol{\Omega})} \le \mu^{\frac12}k^{\frac12}\frac{\mathcal{P}_{2,s}}{s\gamma^s}\eta^{\gamma s}\left\|h^\varepsilon\right\|_{\mathcal{H}^s(\boldsymbol{\Omega})}.$$

Applying a similar method gives

$$\begin{aligned}\left\|\Xi_2(t+\eta)-\Xi_2(t)\right\|_{L^2(\boldsymbol{\Omega})} &= \left\|\left(\mathcal{A}^3_\gamma(t+\eta,\mathscr{T}_o)-\mathcal{A}^3_\gamma(t,\mathscr{T}_o)\right)\left[\mathcal{A}^2_\gamma(\mathscr{T}_o)F^\varepsilon(t)\right]\right\|_{L^2(\boldsymbol{\Omega})} \\ &\le \mu^{\frac12}k^{\frac12}\frac{\mathcal{P}_{2,s}}{s\gamma^s}\eta^{\gamma s}\left\|\mathcal{A}^2_\gamma(\mathscr{T}_o)F^\varepsilon(t)\right\|_{\mathcal{H}^s(\boldsymbol{\Omega})}.\end{aligned}$$

Since $0<s<1$ using Lemma 2.3 we get

$$\begin{aligned}\left\|\mathcal{A}^2_\gamma(\mathscr{T}_o)F^\varepsilon(t)\right\|_{\mathcal{H}^s(\boldsymbol{\Omega})} &\le \left(\frac{\mathcal{P}_{2,s}}{\gamma(1-s)}\right)^{\frac12}\mathscr{T}_o^{\frac{\gamma(1-s)}{2}}\left(\int_0^{\mathscr{T}_o}\tau^{\gamma-1}\left\|F^\varepsilon(\tau)\right\|^2_{L^2(\boldsymbol{\Omega})}\,d\tau\right)^{\frac12} \\ &\le \left(\frac{\mathcal{P}_{2,s}}{\gamma(1-s)(\gamma p-p+1)^{\frac1p}}\right)^{\frac12}\mathscr{T}_o^{\gamma-\frac{\gamma s+1}{2}+\frac{1}{2p}}\left\|F^\varepsilon\right\|^2_{L^{\frac{2p}{p-1}}(0,\mathscr{T}_o;L^2(\boldsymbol{\Omega}))}.\end{aligned}$$

This implies that

$$\begin{aligned}&\left\|\Xi_2(t+\eta)-\Xi_2(t)\right\|_{L^2(\boldsymbol{\Omega})} \\ &\quad\le \mu^{\frac12}k^{\frac12}\frac{\mathcal{P}_{2,s}}{s\gamma^s}\eta^{\gamma s}\left(\frac{\mathcal{P}_{2,s}}{\gamma(1-s)(\gamma p-p+1)^{\frac1p}}\right)^{\frac12}\mathscr{T}_o^{\gamma-\frac{\gamma s+1}{2}+\frac{1}{2p}}\left\|F^\varepsilon\right\|^2_{L^{\frac{2p}{p-1}}(0,\mathscr{T}_o;L^2(\boldsymbol{\Omega}))}.\end{aligned}$$

Using Lemma 2.3 and $0<q<1$ so we have

$$\begin{aligned}&\left\|\Xi_3(t+\eta)-\Xi_3(t)\right\|_{L^2(\boldsymbol{\Omega})} \\ &\quad= \left\|\left(\mathcal{A}^2_\gamma(t+\eta)-\mathcal{A}^2_\gamma(t)\right)F^\varepsilon(t)\right\|_{L^2(\boldsymbol{\Omega})}\end{aligned}$$

$$\leq \left(\mathcal{P}_{2,q}\frac{|(t+\eta)^{\gamma}-t^{\gamma}|^{2}}{\gamma^{q+1}}\frac{\mathscr{T}_o^{q\gamma}}{q\gamma}\int_0^t \tau^{\gamma-1}\left\|F^{\varepsilon}(\tau)\right\|^2_{\mathcal{H}^{q-2\kappa+1}(\boldsymbol{\Omega})}\,d\tau\right)^{\frac{1}{2}} + \left(\frac{(t+\eta)^{\gamma}-t^{\gamma}}{\gamma}\int_t^{t+\eta}\tau^{\gamma-1}\left\|F^{\varepsilon}(\tau)\right\|^2_{L^2(\boldsymbol{\Omega})}\,d\tau\right)^{\frac{1}{2}}. \tag{35}$$

Apply Hölder's inequality and we have

$$\begin{aligned}\int_0^t \tau^{\gamma-1}\left\|F^{\varepsilon}(\tau)\right\|^2_{\mathcal{H}^{q+1}(\boldsymbol{\Omega})}\,d\tau &\leq \left(\int_0^t \tau^{p(\gamma-1)}\,d\tau\right)^{\frac{1}{p}}\left(\int_0^t \left\|F^{\varepsilon}(\tau)\right\|^{\frac{2p}{p-1}}_{\mathcal{H}^{q+1}(\boldsymbol{\Omega})}\,d\tau\right)^{\frac{p-1}{p}} \\ &\leq \frac{\mathscr{T}_o^{\gamma-1+\frac{1}{p}}}{(\gamma p-p+1)^{\frac{1}{p}}}\left\|F^{\varepsilon}\right\|^2_{L^{\frac{2p}{p-1}}(0,\mathscr{T}_o;\mathcal{H}^{q+1}(\boldsymbol{\Omega}))}\end{aligned} \tag{36}$$

and

$$\begin{aligned}\int_t^{t+\eta} \tau^{\gamma-1}\left\|F^{\varepsilon}(\tau)\right\|^2_{L^2(\boldsymbol{\Omega})}\,d\tau &\leq \left(\int_t^{t+\eta} \tau^{p(\gamma-1)}\,d\tau\right)^{\frac{1}{p}}\left(\int_t^{t+\eta} \left\|F^{\varepsilon}(\tau)\right\|^{\frac{2p}{p-1}}_{L^2(\boldsymbol{\Omega})}\,d\tau\right)^{\frac{p-1}{p}} \\ &\leq \left(\int_0^{\mathscr{T}_o} \tau^{p(\gamma-1)}\,d\tau\right)^{\frac{1}{p}}\left(\int_0^{\mathscr{T}_o} \left\|F^{\varepsilon}(\tau)\right\|^{\frac{2p}{p-1}}_{L^2(\boldsymbol{\Omega})}\,d\tau\right)^{\frac{p-1}{p}} \\ &\leq \frac{\mathscr{T}_o^{\gamma-1+\frac{1}{p}}}{(\gamma p-p+1)^{\frac{1}{p}}}\left\|F^{\varepsilon}\right\|^2_{L^{\frac{2p}{p-1}}(0,\mathscr{T}_o;L^2(\boldsymbol{\Omega}))}.\end{aligned} \tag{37}$$

Combining (34), (35), (36) and (37) and we get

$$\begin{aligned}\left\|\Xi_3(t+\eta)-\Xi_3(t)\right\|_{L^2(\boldsymbol{\Omega})} &\leq \left(\mathcal{P}_{2,q}\frac{\eta^{2\gamma}}{\gamma^{q+2}}\frac{\mathscr{T}_o^{q\gamma+\gamma-1+\frac{1}{p}}}{q(\gamma p-p+1)^{\frac{1}{p}}}\right)^{\frac{1}{2}}\left\|F^{\varepsilon}\right\|_{L^{\frac{2p}{p-1}}(0,\mathscr{T}_o;\mathcal{H}^{q+1}(\boldsymbol{\Omega}))} \\ &\quad + \left(\frac{\eta^{\gamma}}{\gamma}\frac{\mathscr{T}_o^{\gamma-1+\frac{1}{p}}}{(\gamma p-p+1)^{\frac{1}{p}}}\right)^{\frac{1}{2}}\left\|F^{\varepsilon}\right\|_{L^{\frac{2p}{p-1}}(0,\mathscr{T}_o;L^2(\boldsymbol{\Omega}))}.\end{aligned}$$

Therefore, we conclude that $u^{\varepsilon}_{k,\alpha} \in C([0,\mathscr{T}_o];L^2(\boldsymbol{\Omega}))$. □

4 Convergence analysis and error estimate under two parameter choice rules

In this section, we choose a regularization parameter $k := k(\varepsilon)$ such that $\|u-u^{\varepsilon}_{k,\alpha}\|_{L^2(\boldsymbol{\Omega})} \to 0$ as $\varepsilon \to 0$, and we also consider the convergence analysis between the regularized solution $u^{\varepsilon}_{k,\alpha}$ and the exact solution u.

4.1 The a priori parameter choice

Theorem 4.1 *Let $h \in L^2(\boldsymbol{\Omega})$ and $F \in L^{\infty}(0,\mathscr{T}_o;L^2(\boldsymbol{\Omega}))$. Assume the a priori bound condition (22) holds. If we choose the regularization parameter*

$$k = \left\lfloor \left(\frac{\boldsymbol{E}}{\varepsilon}\right)^{\frac{2}{r+1}} \right\rfloor,$$

then we get the following error estimate between the exact solution and its regularization solution with noisy data:

$$\|u - u^{\varepsilon}_{k,\alpha}\|_{L^2(\boldsymbol{\Omega})} \le \mu^{-\frac{r}{2}}\left(\frac{r}{2}\right)^{\frac{r}{2}} \boldsymbol{E}^{\frac{1}{r+1}}\varepsilon^{\frac{r}{r+1}} + \mu^{\frac{1}{2}}(1+\mathcal{R})\boldsymbol{E}^{\frac{1}{r+1}}\varepsilon^{\frac{r}{r+1}} + \varepsilon\mathcal{R},$$

where $\lfloor k \rfloor$ *denotes the largest integer less than or equal to* k.

Proof From the triangle inequality, we obtain

$$\|u - u^{\varepsilon}_{k,\alpha}\|_{L^2(\boldsymbol{\Omega})} \le \|u - u_{k,\alpha}\|_{L^2(\boldsymbol{\Omega})} + \|u_{k,\alpha} - u^{\varepsilon}_{k,\alpha}\|_{L^2(\boldsymbol{\Omega})}. \tag{38}$$

Apply part (a) of Theorem 3.1 and we get

$$\begin{aligned}\|u_{k,\alpha} - u^{\varepsilon}_{k,\alpha}\|_{L^2(\boldsymbol{\Omega})} &\le \|u_{k,\alpha} - u^{\varepsilon}_{k,\alpha}\|_{L^\infty(0,\mathscr{T}_o;L^2(\boldsymbol{\Omega}))} \\ &\le \mu^{\frac{1}{2}}k^{\frac{1}{2}}\|h - h^{\varepsilon}\|_{L^2(\boldsymbol{\Omega})} + \big(\mu^{\frac{1}{2}}k^{\frac{1}{2}} + 1\big)\mathcal{R}\|F - F^{\varepsilon}\|_{L^\infty(0,\mathscr{T}_o;L^2(\boldsymbol{\Omega}))} \\ &\le \mu^{\frac{1}{2}}k^{\frac{1}{2}}\varepsilon(1+\mathcal{R}) + \varepsilon\mathcal{R}.\end{aligned} \tag{39}$$

On the other hand, we have

$$\|u - u_{k,\alpha}\|_{L^2(\boldsymbol{\Omega})} = \big\|\big(\mathcal{A}^1_{\gamma}(t,\mathscr{T}_o) - \mathcal{A}^3_{\gamma}(t,\mathscr{T}_o)\big)\boldsymbol{\Upsilon}\big\|_{L^2(\boldsymbol{\Omega})}.$$

Note $\alpha \in (\frac{1}{2}, 1]$ and $0 < \mu < \frac{1}{\|\mathcal{K}\|^2}$, so it follows that

$$\begin{aligned}&\|u - u_{k,\alpha}\|_{L^2(\boldsymbol{\Omega})} \\ &\quad = \Bigg\|\sum_{m=1}^{\infty}\exp\left(-\widetilde{a}_m\frac{t^{\gamma} - \mathscr{T}_o^{\gamma}}{\gamma}\right)\left(1 - \left[1 - \left(1 - \mu\exp\left(-2\widetilde{a}_m\frac{\mathscr{T}_o^{\gamma}}{\gamma}\right)\right)^k\right]^{\alpha}\right) \\ &\qquad \times \langle\boldsymbol{\Upsilon}, e_m\rangle e_m\Bigg\|_{L^2(\boldsymbol{\Omega})} \\ &\quad \le \Bigg\|\sum_{m=1}^{\infty}\exp\left(-\widetilde{a}_m\frac{t^{\gamma} - \mathscr{T}_o^{\gamma}}{\gamma}\right)\left(1 - \mu\exp\left(-2\widetilde{a}_m\frac{\mathscr{T}_o^{\gamma}}{\gamma}\right)\right)^k\langle\boldsymbol{\Upsilon}, e_m\rangle e_m\Bigg\|_{L^2(\boldsymbol{\Omega})}.\end{aligned}$$

From the definition of $\boldsymbol{\Upsilon}$ in (17), we get

$$\begin{aligned}\|u - u_{k,\alpha}\|_{L^2(\boldsymbol{\Omega})} &\le \Bigg\|\sum_{m=1}^{\infty}\exp\left(-\widetilde{a}_m\frac{t^{\gamma}}{\gamma}\right)\left(1 - \mu\exp\left(-2\widetilde{a}_m\frac{\mathscr{T}_o^{\gamma}}{\gamma}\right)\right)^k\langle\mathfrak{g}, e_m\rangle e_m\Bigg\|_{L^2(\boldsymbol{\Omega})} \\ &\le \mu^{-\frac{r}{2}}\sup_{\widetilde{a}_m>0}\left(1 - \mu\exp\left(-2\widetilde{a}_m\frac{\mathscr{T}_o^{\gamma}}{\gamma}\right)\right)^k\left(\mu\exp\left(-2\widetilde{a}_m\frac{\mathscr{T}_o^{\gamma}}{\gamma}\right)\right)^{\frac{r}{2}} \\ &\quad \times \sqrt{\sum_{m=1}^{\infty}\exp\left(2r\widetilde{a}_m\frac{\mathscr{T}_o^{\gamma}}{\gamma}\right)\big|\langle\mathfrak{g}, e_m\rangle\big|^2}.\end{aligned}$$

Apply Lemma 2.1 and we obtain

$$\|u-u_{k,\alpha}\|_{L^2(\Omega)} \le \mu^{-\frac{r}{2}}\left(\frac{r}{2}\right)^{\frac{r}{2}} k^{-\frac{r}{2}}\mathbf{E}. \tag{40}$$

Combining (38), (39) and (40) we obtain

$$\left\|u-u^{\varepsilon}_{k,\alpha}\right\|_{L^2(\Omega)} \le \mu^{-\frac{r}{2}}\left(\frac{r}{2}\right)^{\frac{r}{2}} k^{-\frac{r}{2}}\mathbf{E} + \mu^{\frac{1}{2}}k^{\frac{1}{2}}\varepsilon(1+\mathcal{R}) + \varepsilon\mathcal{R}.$$

Choosing the regularization parameter k,

$$k = \left\lfloor \left(\frac{\mathbf{E}}{\varepsilon}\right)^{\frac{2}{r+1}} \right\rfloor,$$

we obtain the error estimate

$$\left\|u-u^{\varepsilon}_{k,\alpha}\right\|_{L^2(\Omega)} \le \mu^{-\frac{r}{2}}\left(\frac{r}{2}\right)^{\frac{r}{2}} \mathbf{E}^{\frac{1}{r+1}}\varepsilon^{\frac{r}{r+1}} + \mu^{\frac{1}{2}}(1+\mathcal{R})\mathbf{E}^{\frac{1}{r+1}}\varepsilon^{\frac{r}{r+1}} + \varepsilon\mathcal{R}.$$ □

4.2 A posteriori parameter choice rule and convergence estimate

In the above result, we obtained an error estimate between the exact solution and its regularization solution with noisy data by choosing the a priori parameter k, and this k depends on the noise level ε and the a priori bound condition **E**. Now, from results in Morozov's discrepancy principal [11], we choose the regularization parameter k by using an a posteriori choice rule.

The general a posteriori rule can be formulated as follows:

$$\left\|\mathcal{K}\mathfrak{g}^{\varepsilon}_{k,\alpha} - \boldsymbol{\Upsilon}^{\varepsilon}\right\|_{L^2(\Omega)} \le \chi\varepsilon \le \left\|\mathcal{K}\mathfrak{g}^{\varepsilon}_{k-1,\alpha} - \boldsymbol{\Upsilon}^{\varepsilon}\right\|_{L^2(\Omega)}, \tag{41}$$

where $\|\boldsymbol{\Upsilon}^{\varepsilon}\|_{L^2(\Omega)} \ge \chi\varepsilon$, χ is a constant independent of ε and $k > 0$ is the regularization parameter which makes (41) hold at the first iteration time.

Choosing $\chi > 1$ the following lemma gives a bound for k in terms of ε and **E**.

Lemma 4.1 *Let* $\chi > 1$ *and k satisfies* (41). *Also assume the a priori bound condition of* $\mathfrak{g}$ *satisfies* (22). *Then*

$$k \le \frac{r-1}{2\mu}\left(\frac{1}{\chi-\mathcal{R}-1}\right)^{\frac{2}{r-1}}\left(\frac{E}{\varepsilon}\right)^{\frac{2}{r-1}}.$$

Proof From the definition of k we get

$$\begin{aligned}
&\left\|\mathcal{K}\mathfrak{g}^{\varepsilon}_{k-1,\alpha} - \boldsymbol{\Upsilon}^{\varepsilon}\right\|_{L^2(\Omega)} \\
&\quad = \left\|\sum_{m=1}^{\infty}\left(1-\left[1-\left(1-\mu\exp\left(-2\widetilde{a}_m\frac{\mathcal{T}_o^{\gamma}}{\gamma}\right)\right)^{k-1}\right]^{\alpha}\right)\langle\boldsymbol{\Upsilon}^{\varepsilon}, e_m\rangle e_m\right\|_{L^2(\Omega)}
\end{aligned}$$

$$\leq \left\| \sum_{m=1}^{\infty} \left(1 - \left[1 - \left(1 - \mu \exp\left(-2\widetilde{a}_m \frac{\mathscr{T}_o^\gamma}{\gamma}\right)\right)^{k-1}\right]^\alpha\right) \langle \boldsymbol{\Upsilon}^\varepsilon - \boldsymbol{\Upsilon}, e_m\rangle e_m \right\|_{L^2(\boldsymbol{\Omega})} + \left\| \sum_{m=1}^{\infty} \left(1 - \left[1 - \left(1 - \mu \exp\left(-2\widetilde{a}_m \frac{\mathscr{T}_o^\gamma}{\gamma}\right)\right)^{k-1}\right]^\alpha\right) \langle \boldsymbol{\Upsilon}, e_m\rangle e_m \right\|_{L^2(\boldsymbol{\Omega})}.$$

Since $\alpha \in (\frac{1}{2}, 1]$ and $0 < \mu < \frac{1}{\|\mathcal{K}\|^2}$ it follows that

$$\begin{aligned} &\left\| \mathcal{K} g_{k-1,\alpha}^\varepsilon - \boldsymbol{\Upsilon}^\varepsilon \right\|_{L^2(\boldsymbol{\Omega})} \\ &\quad \leq \left\| \boldsymbol{\Upsilon}^\varepsilon - \boldsymbol{\Upsilon} \right\|_{L^2(\boldsymbol{\Omega})} + \left\| \sum_{m=1}^{\infty} \left(1 - \mu \exp\left(-2\widetilde{a}_m \frac{\mathscr{T}_o^\gamma}{\gamma}\right)\right)^{k-1} \langle \boldsymbol{\Upsilon}, e_m\rangle e_m \right\|_{L^2(\boldsymbol{\Omega})}. \end{aligned}$$

Apply Lemma 2.1 and we obtain

$$\begin{aligned} &\left\| \sum_{m=1}^{\infty} \left(1 - \mu \exp\left(-2\widetilde{a}_m \frac{\mathscr{T}_o^\gamma}{\gamma}\right)\right)^{k-1} \langle \boldsymbol{\Upsilon}, e_m\rangle e_m \right\|_{L^2(\boldsymbol{\Omega})} \\ &\quad = \mu^{-\frac{1+r}{2}} \sup_{\widetilde{a}_m > 0} \left(1 - \mu \exp\left(-2\widetilde{a}_m \frac{\mathscr{T}_o^\gamma}{\gamma}\right)\right)^{k-1} \left(\mu \exp\left(-2\widetilde{a}_m \frac{\mathscr{T}_o^\gamma}{\gamma}\right)\right)^{\frac{r+1}{2}} \\ &\qquad \times \sqrt{\sum_{m=1}^{\infty} \exp\left(2r\widetilde{a}_m \frac{\mathscr{T}_o^\gamma}{\gamma}\right) \left|\langle g, e_m\rangle\right|^2} \\ &\quad \leq \mu^{-\frac{1+r}{2}} \left(\frac{r+1}{2}\right)^{\frac{r+1}{2}} k^{\frac{r+1}{2}} \mathbf{E}. \end{aligned}$$

Apply Lemma 2.4 and we obtain

$$\begin{aligned} \left\| \mathcal{K} g_{k-1,\alpha}^\varepsilon - \boldsymbol{\Upsilon}^\varepsilon \right\|_{L^2(\boldsymbol{\Omega})} &\leq \left\| h^\varepsilon - h \right\|_{L^2(\boldsymbol{\Omega})} + \mathcal{R} \left\| F^\varepsilon - F \right\|_{L^\infty(0,\mathscr{T}_o;L^2(\boldsymbol{\Omega}))} \\ &\quad + \mu^{-\frac{1+r}{2}} \left(\frac{r+1}{2}\right)^{\frac{r+1}{2}} k^{-\frac{r+1}{2}} \mathbf{E}. \end{aligned}$$

This implies that

$$\chi\varepsilon \leq (1 + \mathcal{R})\varepsilon + \mu^{-\frac{r+1}{2}} \left(\frac{r+1}{2}\right)^{\frac{r+1}{2}} k^{-\frac{r+1}{2}} \mathbf{E},$$

so

$$k \leq \frac{r+1}{2\mu} \left(\frac{1}{\chi - \mathcal{R} - 1}\right)^{\frac{2}{r+1}} \left(\frac{\mathbf{E}}{\varepsilon}\right)^{\frac{2}{r+1}}. \qquad \square$$

Theorem 4.2 *Let k be as Lemma 4.1. Assume the a priori bound condition (22) holds. Then we get*

$$\begin{aligned} \left\| u - u_{k,\alpha}^\varepsilon \right\|_{L^2(\boldsymbol{\Omega})} &\leq \mu^{\frac{1}{2}} (1 + \mathcal{R}) \left(\frac{r+1}{2\mu}\right)^{\frac{1}{2}} \left(\frac{1}{\chi - \mathcal{R} - 1}\right)^{\frac{1}{r+1}} \mathbf{E}^{\frac{1}{r+1}} \varepsilon^{\frac{r}{r+1}} \\ &\quad + \varepsilon\mathcal{R} + (1 + \mathcal{R} + \chi)^{\frac{r}{r+1}} \mathbf{E}^{\frac{1}{r+1}} \varepsilon^{\frac{r}{r+1}}. \end{aligned}$$

Proof Using the triangle inequality, we obtain

$$\left\|u-u^{\varepsilon}_{k,\alpha}\right\|_{L^2(\Omega)} \le \|u-u_{k,\alpha}\|_{L^2(\Omega)} + \left\|u_{k,\alpha}-u^{\varepsilon}_{k,\alpha}\right\|_{L^2(\Omega)}.$$

Apply the result of Theorem 4.1 and we obtain

$$\begin{aligned}\left\|u_{k,\alpha}-u^{\varepsilon}_{k,\alpha}\right\|_{L^2(\Omega)} &\le \mu^{\frac{1}{2}}k^{\frac{1}{2}}\varepsilon(1+\mathcal{R})+\varepsilon\mathcal{R}\\ &\le \mu^{\frac{1}{2}}(1+\mathcal{R})\left(\frac{r+1}{2\mu}\right)^{\frac{1}{2}}\left(\frac{1}{\chi-\mathcal{R}-1}\right)^{\frac{1}{r+1}}\mathbf{E}^{\frac{1}{r+1}}\varepsilon^{\frac{r}{r+1}}+\varepsilon\mathcal{R}. \qquad (42)\end{aligned}$$

Apply Hölder's inequality and we have

$$\begin{aligned}&\|u-u_{k,\alpha}\|_{L^2(\Omega)}\\ &\quad=\left\|\sum_{m=1}^{\infty}\exp\left(-\widetilde{a}_m\frac{t^\gamma-\mathscr{T}_o^\gamma}{\gamma}\right)\left(1-\left[1-\left(1-\mu\exp\left(-2\widetilde{a}_m\frac{\mathscr{T}_o^\gamma}{\gamma}\right)\right)^k\right]^\alpha\right)\right.\\ &\qquad\left.\times\langle\boldsymbol{\Upsilon},e_m\rangle e_m\right\|_{L^2(\Omega)}\\ &\quad\le\left\|\sum_{m=1}^{\infty}\left(1-\left[1-\left(1-\mu\exp\left(-2\widetilde{a}_m\frac{\mathscr{T}_o^\gamma}{\gamma}\right)\right)^k\right]^\alpha\right)\langle\boldsymbol{\Upsilon},e_m\rangle e_m\right\|_{L^2(\Omega)}^{\frac{r}{r+1}}\\ &\qquad\times\left\|\sum_{m=1}^{\infty}\exp\left(r\widetilde{a}_m\frac{\mathscr{T}_o^\gamma}{\gamma}\right)\left(1-\left[1-\left(1-\mu\exp\left(-2\widetilde{a}_m\frac{\mathscr{T}_o^\gamma}{\gamma}\right)\right)^k\right]^\alpha\right)\langle\mathfrak{g},e_m\rangle e_m\right\|_{L^2(\Omega)}^{\frac{1}{r+1}}.\end{aligned}$$

On the other hand, from $\alpha\in(\frac{1}{2},1]$ and $0<\mu<\frac{1}{\|\mathcal{K}\|^2}$ we get

$$\begin{aligned}&\left\|\sum_{m=1}^{\infty}\exp\left(r\widetilde{a}_m\frac{\mathscr{T}_o^\gamma}{\gamma}\right)\left(1-\left[1-\left(1-\mu\exp\left(-2\widetilde{a}_m\frac{\mathscr{T}_o^\gamma}{\gamma}\right)\right)^k\right]^\alpha\right)\langle\mathfrak{g},e_m\rangle e_m\right\|_{L^2(\Omega)}^{\frac{1}{r+1}}\\ &\quad\le\left(\sum_{m=1}^{\infty}\exp\left(2r\widetilde{a}_m\frac{\mathscr{T}_o^\gamma}{\gamma}\right)\left|\langle\mathfrak{g},e_m\rangle\right|^2\right)^{\frac{1}{2(r+1)}}\le\mathbf{E}^{\frac{1}{r+1}}.\end{aligned}$$

This implies that

$$\begin{aligned}&\|u-u_{k,\alpha}\|_{L^2(\Omega)}\\ &\quad\le\mathbf{E}^{\frac{1}{r+1}}\left(\left|\boldsymbol{\Upsilon}-\boldsymbol{\Upsilon}^\varepsilon\right|_{L^2(\Omega)}\right.\\ &\qquad\left.+\left\|\sum_{m=1}^{\infty}\left(1-\left[1-\left(1-\mu\exp\left(-2\widetilde{a}_m\frac{\mathscr{T}_o^\gamma}{\gamma}\right)\right)^k\right]^\alpha\right)\langle\boldsymbol{\Upsilon},e_m\rangle e_m\right\|_{L^2(\Omega)}\right)^{\frac{r}{r+1}}\\ &\quad\le(1+\mathcal{R}+\chi)^{\frac{r}{r+1}}\mathbf{E}^{\frac{1}{r+1}}\varepsilon^{\frac{r}{r+1}}.\end{aligned}$$

From the above we deduce that

$$\left\|u - u_{k,\alpha}^{\varepsilon}\right\|_{L^2(\Omega)} \le \mu^{\frac{1}{2}}(1+\mathcal{R})\left(\frac{r+1}{2\mu}\right)^{\frac{1}{2}}\left(\frac{1}{\chi-\mathcal{R}-1}\right)^{\frac{1}{r+1}}\mathbf{E}^{\frac{1}{r+1}}\varepsilon^{\frac{r}{r+1}} + \varepsilon\mathcal{R} + (1+\mathcal{R}+\chi)^{\frac{r}{r+1}}\mathbf{E}^{\frac{1}{r+1}}\varepsilon^{\frac{r}{r+1}}. \qquad \square$$

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