Alsaedi et al. *Advances in Difference Equations* (2019) 2019:474
https://doi.org/10.1186/s13662-019-2412-x

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH **Open Access**

On a coupled system of higher order nonlinear Caputo fractional differential equations with coupled Riemann–Stieltjes type integro-multipoint boundary conditions

Ahmed Alsaedi[1], Bashir Ahmad[1*], Ymnah Alruwaily[1] and Sotiris K. Ntouyas[1,2]

*Correspondence:
bashirahmad_qau@yahoo.com
[1]Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah, Saudi Arabia
Full list of author information is available at the end of the article

Abstract
We study a coupled system of Caputo fractional differential equations with coupled non-conjugate Riemann–Stieltjes type integro-multipoint boundary conditions. Existence and uniqueness results for the given boundary value problem are obtained by applying the Leray–Schauder nonlinear alternative, the Krasnoselskii fixed point theorem and Banach's contraction mapping principle. Examples are constructed to illustrate the obtained results.

MSC: 26A33; 34B15

Keywords: Caputo derivative; System; Riemann–Stieltjes integral; Multipoint boundary conditions; Existence

1 Introduction

Fractional-order differential systems constitute the mathematical models of many real world problems. Examples include disease models [1–3], anomalous diffusion [4, 5], synchronization of chaotic systems [6, 7], ecological models [8]. For applications in bioengineering, chaos and financial economics, we refer the reader to [9–11]. The details about rheological models in the context of local fractional derivatives can be found in [12]. In view of the extensive applications of such systems, many researchers turned to investigation of the theoretical aspects of fractional differential equations. In particular, there was a special attention on proving the existence and uniqueness of solutions for fractional differential systems supplemented with a variety of classical and non-classical (nonlocal) boundary conditions with the aid of modern methods of functional analysis. For details and examples, see [13–22] and the references cited therein. It is imperative to mention that fractional-order models are more practical and informative than their integer-order counterparts. It has been mainly due to the fact that fractional-order operators can describe the hereditary properties of the processes and phenomena under investigation.

In this paper, we study the existence of solutions for a nonlinear coupled system of Liouville–Caputo type fractional differential equations on an arbitrary domain:

$$\begin{cases} {}^cD^{\alpha}x(t) = f(t,x(t),y(t)), & 3 < \alpha \leq 4, t \in [a,b], \\ {}^cD^{\beta}y(t) = g(t,x(t),y(t)), & 3 < \beta \leq 4, t \in [a,b], \end{cases} \tag{1.1}$$

equipped with coupled non-conjugate Riemann–Stieltjes integro-multipoint boundary conditions:

$$\begin{cases} x'(a) = 0, \qquad x(b) = 0, \qquad x'(b) = 0, \\ x(a) = \int_a^b y(s)\,dA(s) + \sum_{i=1}^{n-2}\alpha_i y(\xi_i), \\ y'(a) = 0, \qquad y(b) = 0, \qquad y'(b) = 0, \\ y(a) = \int_a^b x(s)\,dA(s) + \sum_{i=1}^{n-2}\beta_i x(\xi_i), \end{cases} \tag{1.2}$$

where ${}^cD^{\varrho}$ denotes the Caputo fractional derivative of order ϱ with $(\varrho = \alpha,\beta)$, $f,g : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions, $a < \xi_1 < \xi_2 < \cdots < \xi_{n-2} < b$, $\alpha_i, \beta_i \in \mathbb{R}$, $i = 1,2,\ldots,n-2$ and A is a function of bounded variation. In passing we remark that the present work is motivated by a recent paper [23], where the authors studied the existence and stability of solutions for a fractional-order differential equation with non-conjugate Riemann–Stieltjes integro-multipoint boundary conditions.

We arrange the rest of the paper as follows. Section 2 contains an auxiliary result that plays a key role in analyzing the given problem. Existence results for the problem (1.1)–(1.2) with the illustrative examples are presented in Sect. 3, while the uniqueness of solutions is discussed in Sect. 4.

2 Auxiliary result

Before giving an auxiliary result for system (1.1)–(1.2), we recall some necessary definitions of fractional calculus [24, 25].

Definition 2.1 Let ϑ be a locally integrable real-valued function on $-\infty \leq a < s < b \leq +\infty$. The Riemann–Liouville fractional integral I_a^{ϑ} of order $\vartheta \in \mathbb{R}$ $(\vartheta > 0)$ for the function q is defined as

$$I_a^{\vartheta}q(s) = (q * K_{\vartheta})(s) = \frac{1}{\Gamma(\vartheta)}\int_a^s (s-u)^{\vartheta-1}q(u)\,du,$$

where $K_{\vartheta} = \frac{t^{\vartheta-1}}{\Gamma(\vartheta)}$, Γ denotes the Euler gamma function.

Definition 2.2 The Caputo derivative of fractional order ϑ for an $(m-1)$-times absolutely continuous function $q : [a,\infty) \longrightarrow \mathbb{R}$ is defined as

$${}^cD^{\vartheta}q(s) = \frac{1}{\Gamma(m-\vartheta)}\int_a^s (s-u)^{m-\vartheta-1}q^{(m)}(u)\,du, \quad m-1 < \vartheta \leq m, m = [\vartheta]+1,$$

where $[\vartheta]$ denotes the integer part of the real number ϑ.

Lemma 2.3 ([24]) *The general solution of the fractional differential equation* ${}^cD^{\vartheta}x(s) = 0, m-1 < \vartheta < m, s \in [a,b]$, *is*

$$x(s) = \upsilon_0 + \upsilon_1(s-a) + \upsilon_2(s-a)^2 + \cdots + \upsilon_{m-1}(s-a)^{m-1},$$

where $\upsilon_i \in \mathbb{R}, i = 0,1,\ldots,m-1$. *Furthermore,*

$$I^{\vartheta}\, {}^cD^{\vartheta}x(s) = x(s) + \sum_{i=0}^{m-1}\upsilon_i(s-a)^i.$$

Lemma 2.4 *For* $\widehat{f},\widehat{g} \in C([a,b],\mathbb{R})$, *the solution of the linear system of fractional differential equations*:

$$\begin{cases} {}^cD^{\alpha}x(t) = \widehat{f}(t), & 3 < \alpha \leq 4, t \in [a,b], \\ {}^cD^{\beta}y(t) = \widehat{g}(t), & 3 < \beta \leq 4, t \in [a,b], \end{cases} \tag{2.1}$$

supplemented with the boundary conditions (1.2) *is equivalent to the system of integral equations*:

$$\begin{aligned} x(t) ={}& \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\widehat{f}(s)\,ds - \phi_1(t)\int_a^b \frac{(b-s)^{\alpha-1}}{\Gamma(\alpha)}\widehat{f}(s)\,ds - \phi_6(t)\int_a^b \frac{(b-s)^{\alpha-2}}{\Gamma(\alpha-1)}\widehat{f}(s)\,ds \\ &+ \phi_2(t)\left[\int_a^b \left(\int_a^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}\widehat{f}(u)\,du\right)dA(s) + \sum_{i=1}^{n-2}\beta_i\int_a^{\xi_i}\frac{(\xi_i-s)^{\alpha-1}}{\Gamma(\alpha)}\widehat{f}(s)\,ds\right] \\ &- \phi_3(t)\int_a^b \frac{(b-s)^{\beta-1}}{\Gamma(\beta)}\widehat{g}(s)\,ds - \phi_4(t)\int_a^b \frac{(b-s)^{\beta-2}}{\Gamma(\beta-1)}\widehat{g}(s)\,ds \\ &+ \phi_5(t)\left[\int_a^b \left(\int_a^s \frac{(s-u)^{\beta-1}}{\Gamma(\beta)}\widehat{g}(u)\,du\right)dA(s)\right. \\ &\left.+ \sum_{i=1}^{n-2}\alpha_i\int_a^{\xi_i}\frac{(\xi_i-s)^{\beta-1}}{\Gamma(\beta)}\widehat{g}(s)\,ds\right], \end{aligned} \tag{2.2}$$

$$\begin{aligned} y(t) ={}& \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\widehat{g}(s)\,ds - \psi_1(t)\int_a^b \frac{(b-s)^{\alpha-1}}{\Gamma(\alpha)}\widehat{f}(s)\,ds - \psi_2(t)\int_a^b \frac{(b-s)^{\alpha-2}}{\Gamma(\alpha-1)}\widehat{f}(s)\,ds \\ &+ \phi_5(t)\left[\int_a^b \left(\int_a^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}\widehat{f}(u)\,du\right)dA(s) + \sum_{i=1}^{n-2}\beta_i\int_a^{\xi_i}\frac{(\xi_i-s)^{\alpha-1}}{\Gamma(\alpha)}\widehat{f}(s)\,ds\right] \\ &- \psi_4(t)\int_a^b \frac{(b-s)^{\beta-1}}{\Gamma(\beta)}\widehat{g}(s)\,ds - \psi_5(t)\int_a^b \frac{(b-s)^{\beta-2}}{\Gamma(\beta-1)}\widehat{g}(s)\,ds \\ &+ \psi_3(t)\left[\int_a^b \left(\int_a^s \frac{(s-u)^{\beta-1}}{\Gamma(\beta)}\widehat{g}(u)\,du\right)dA(s)\right. \\ &\left.+ \sum_{i=1}^{n-2}\alpha_i\int_a^{\xi_i}\frac{(\xi_i-s)^{\beta-1}}{\Gamma(\beta)}\widehat{g}(s)\,ds\right], \end{aligned} \tag{2.3}$$

where

$$\phi_i(t) = \nu_i + (t-a)^2\mu_i + (t-a)^3\delta_i, \quad i = 1,\ldots,6, \tag{2.4}$$

$$\psi_j(t) = \omega_j + (t-a)^2\rho_j + (t-a)^3\lambda_j, \quad j = 1,\dots,5, \tag{2.5}$$

$$\mu_k = \frac{-3(b-a)\delta_k}{2}, \quad k = 1,2,3,4,5, \qquad \mu_6 = \frac{1-3(b-a)^2\delta_6}{2(b-a)}, \tag{2.6}$$

$$\rho_m = \frac{-3(b-a)\lambda_m}{2}, \quad m = 1,2,3,4, \qquad \rho_5 = \frac{1-3(b-a)^2\lambda_5}{2(b-a)}, \tag{2.7}$$

$$\delta_1 = \frac{2\nu_1 - 2}{(b-a)^3}, \delta_n = \frac{2\nu_n}{(b-a)^3}, \quad n = 2,3,4,5, \qquad \delta_6 = \frac{2\nu_6 + (b-a)}{(b-a)^3}, \tag{2.8}$$

$$\lambda_r = \frac{2\omega_r}{(b-a)^3}, \quad r = 1,2,3, \qquad \lambda_4 = \frac{2\omega_4 - 2}{(b-a)^3}, \qquad \lambda_5 = \frac{2\omega_5 + (b-a)}{(b-a)^3}, \tag{2.9}$$

$$\begin{cases} \nu_1 = \nu_2(A_4 - \frac{2\gamma_3}{(b-a)^3}), & \nu_2 = \frac{2(b-a)^3\gamma_1}{(b-a)^6 - 4\gamma_1\gamma_3}, \\ \nu_3 = \nu_5(A_1 - \frac{2\gamma_1}{(b-a)^3}), & \nu_4 = \nu_5(\gamma_2 + \frac{\gamma_1}{(b-a)^2}), \\ \nu_5 = \frac{(b-a)^6}{(b-a)^6 - 4\gamma_1\gamma_3}, & \nu_6 = \nu_2(\gamma_4 + \frac{\gamma_3}{(b-a)^2}), \end{cases} \tag{2.10}$$

$$\begin{cases} \omega_1 = \nu_5(A_4 - \frac{2\gamma_3}{(b-a)^3}), & \omega_2 = \nu_5(\gamma_4 + \frac{\gamma_3}{(b-a)^2}), \\ \omega_3 = \frac{2(b-a)^3\gamma_3}{(b-a)^6 - 4\gamma_1\gamma_3}, & \omega_4 = \omega_3(A_1 - \frac{2\gamma_1}{(b-a)^3}), \\ \omega_5 = \omega_3(\gamma_2 + \frac{\gamma_1}{(b-a)^2}), \end{cases} \tag{2.11}$$

$$\begin{cases} \gamma_1 = \frac{(b-a)^3A_1 - 3(b-a)A_2 + 2A_3}{2}, & \gamma_2 = \frac{A_2 - (b-a)^2A_1}{2(b-a)}, \\ \gamma_3 = \frac{(b-a)^3A_4 - 3(b-a)A_5 + 2A_6}{2}, & \gamma_4 = \frac{A_5 - (b-a)^2A_4}{2(b-a)}, \end{cases} \tag{2.12}$$

$$\begin{cases} A_1 = \int_a^b dA(s) + \sum_{i=1}^{n-2}\alpha_i, & A_2 = \int_a^b (s-a)^2\, dA(s) + \sum_{i=1}^{n-2}\alpha_i(\xi_i - a)^2, \\ A_3 = \int_a^b (s-a)^3\, dA(s) + \sum_{i=1}^{n-2}\alpha_i(\xi_i - a)^3, & A_4 = \int_a^b dA(s) + \sum_{i=1}^{n-2}\beta_i, \\ A_5 = \int_a^b (s-a)^2\, dA(s) + \sum_{i=1}^{n-2}\beta_i(\xi_i - a)^2, \\ A_6 = \int_a^b (s-a)^3\, dA(s) + \sum_{i=1}^{n-2}\beta_i(\xi_i - a)^3. \end{cases} \tag{2.13}$$

Proof By Lemma 2.3, the general solutions of fractional differential equations in (2.1) can be written as

$$x(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\widehat{f}(s)\, ds + c_0 + c_1(t-a) + c_2(t-a)^2 + c_3(t-a)^3, \tag{2.14}$$

$$y(t) = \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\widehat{g}(s)\, ds + b_0 + b_1(t-a) + b_2(t-a)^2 + b_3(t-a)^3, \tag{2.15}$$

where $c_i, b_i \in \mathbb{R}, i = 0,1,2,3$ are unknown arbitrary constants.

Using the boundary conditions (1.2) in (2.14), (2.15), we obtain $c_1 = 0, b_1 = 0$, and

$$c_0 + (b-a)^2c_2 + (b-a)^3c_3 = I_1, \tag{2.16}$$

$$b_0 + (b-a)^2b_2 + (b-a)^3b_3 = K_1, \tag{2.17}$$

$$2(b-a)c_2 + 3(b-a)^2c_3 = I_2, \tag{2.18}$$

$$2(b-a)b_2 + 3(b-a)^2b_3 = K_2, \tag{2.19}$$

$$c_0 - A_1b_0 - A_2b_2 - A_3b_3 = K_3, \tag{2.20}$$

$$b_0 - A_4c_0 - A_5c_2 - A_6c_3 = I_3, \tag{2.21}$$

where A_i $(i = 1,\dots,6)$ are given by (2.13) and I_i, K_i $(i = 1,2,3)$ are defined by

$$I_1 = -\int_a^b \frac{(b-s)^{\alpha-1}}{\Gamma(\alpha)}\widehat{f}(s)\,ds, \qquad I_2 = -\int_a^b \frac{(b-s)^{\alpha-2}}{\Gamma(\alpha-1)}\widehat{f}(s)\,ds,$$
$$I_3 = \int_a^b \left(\int_a^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}\widehat{f}(u)\,du\right) dA(s) + \sum_{i=1}^{n-2}\beta_i \int_a^{\xi_i} \frac{(\xi_i-s)^{\alpha-1}}{\Gamma(\alpha)}\widehat{f}(s)\,ds,$$
$$K_1 = -\int_a^b \frac{(b-s)^{\beta-1}}{\Gamma(\beta)}\widehat{g}(s)\,ds, \qquad K_2 = -\int_a^b \frac{(b-s)^{\beta-2}}{\Gamma(\beta-1)}\widehat{g}(s)\,ds,$$
$$K_3 = \int_a^b \left(\int_a^s \frac{(s-u)^{\beta-1}}{\Gamma(\beta)}\widehat{g}(u)\,du\right) dA(s) + \sum_{i=1}^{n-2}\alpha_i \int_a^{\xi_i} \frac{(\xi_i-s)^{\beta-1}}{\Gamma(\beta)}\widehat{g}(s)\,ds.$$

From (2.16) and (2.18), we find that

$$c_3 = \frac{2}{(b-a)^3}\left(c_0 - I_1 + \frac{(b-a)}{2}I_2\right). \tag{2.22}$$

Eliminating b_2 from (2.17) and (2.19), we get

$$b_3 = \frac{2}{(b-a)^3}\left(b_0 - K_1 + \frac{(b-a)}{2}K_2\right). \tag{2.23}$$

Combining (2.19), (2.20) and (2.23) yields

$$c_0 = \gamma_1 b_3 + A_1 K_1 + \gamma_2 K_2 + K_3. \tag{2.24}$$

From (2.18), (2.21) and (2.22), we have

$$b_0 = \gamma_3 c_3 + A_4 I_1 + \gamma_4 I_2 + I_3, \tag{2.25}$$

where γ_i $(i = 1,\dots,4)$ are given by (2.12).

Eliminating b_3 from (2.23) and (2.24), and c_3 from (2.22) and (2.25), we obtain

$$c_0 = \frac{2\gamma_1}{(b-a)^3}b_0 + \left(A_1 - \frac{2\gamma_1}{(b-a)^3}\right)K_1 + \left(\gamma_2 + \frac{\gamma_1}{(b-a)^2}\right)K_2 + K_3, \tag{2.26}$$
$$b_0 = \frac{2\gamma_3}{(b-a)^3}c_0 + \left(A_4 - \frac{2\gamma_3}{(b-a)^3}\right)I_1 + \left(\gamma_4 + \frac{\gamma_3}{(b-a)^2}\right)I_2 + I_3. \tag{2.27}$$

Solving (2.26) and (2.27) simultaneously for c_0 and b_0, we obtain

$$c_0 = \nu_1 I_1 + \nu_6 I_2 + \nu_2 I_3 + \nu_3 K_1 + \nu_4 K_2 + \nu_5 K_3, \tag{2.28}$$
$$b_0 = \omega_1 I_1 + \omega_2 I_2 + \nu_5 I_3 + \omega_4 K_1 + \omega_5 K_2 + \omega_3 K_3, \tag{2.29}$$

where ν_i $(i = 1,\dots,6)$ and ω_i $(i = 1,\dots,5)$ are defined by (2.10) and (2.11), respectively. Using (2.28) in (2.22) and (2.29) in (2.23), we get

$$c_3 = \delta_1 I_1 + \delta_6 I_2 + \delta_2 I_3 + \delta_3 K_1 + \delta_4 K_2 + \delta_5 K_3,$$

$$b_3 = \lambda_1 I_1 + \lambda_2 I_2 + \delta_5 I_3 + \lambda_4 K_1 + \lambda_5 K_2 + \lambda_3 K_3,$$

where δ_i ($i = 1, \dots, 6$) and λ_i ($i = 1, \dots, 5$) are given by (2.8) and (2.9), respectively.

Substituting the values of c_3 and b_3 in (2.18) and 2.19), respectively, we find that

$$c_2 = \mu_1 I_1 + \mu_6 I_2 + \mu_2 I_3 + \mu_3 K_1 + \mu_4 K_2 + \mu_5 K_3,$$

$$b_2 = \rho_1 I_1 + \rho_2 I_2 + \mu_5 I_3 + \rho_4 K_1 + \rho_5 K_2 + \rho_3 K_3,$$

where μ_i ($i = 1, \dots, 6$) and ρ_i ($i = 1, \dots, 5$) are, respectively, defined by (2.6) and (2.7). Inserting the values of c_0, c_1, c_2, c_3 in (2.14) and b_0, b_1, b_2, b_3 in (2.15), we obtain (2.2) and (2.3). By direct computation, one can obtain the converse. The proof is complete. □

3 Existence results

We define space $\mathcal{A} = \{x | x \in C([a,b], \mathbb{R})\}$ equipped with the norm $\|x\| = \sup_{t\in[a,b]} |x(t)|$. Obviously $(\mathcal{A}, \|\cdot\|)$ is a Banach space and consequently, the product space $(\mathcal{A} \times \mathcal{A}, \|\cdot\|)$ is a Banach space with norm $\|(x,y)\| = \|x\| + \|y\|$ for $(x,y) \in \mathcal{A} \times \mathcal{A}$.

In view of Lemma 2.4, we define an operator $\mathcal{F} : \mathcal{A} \times \mathcal{A} \to \mathcal{A} \times \mathcal{A}$ as

$$\mathcal{F}(x,y)(t) := \big(\mathcal{F}_1(x,y)(t), \mathcal{F}_2(x,y)(t)\big), \tag{3.1}$$

where

$$\begin{aligned}
\mathcal{F}_1(x,y)(t) = {} & \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\big(s,x(s),y(s)\big)\,ds - \phi_1(t) \int_a^b \frac{(b-s)^{\alpha-1}}{\Gamma(\alpha)} f\big(s,x(s),y(s)\big)\,ds \\
& - \phi_6(t) \int_a^b \frac{(b-s)^{\alpha-2}}{\Gamma(\alpha-1)} f\big(s,x(s),y(s)\big)\,ds + \phi_2(t) \Bigg[\int_a^b \bigg(\int_a^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} \\
& \times f\big(u,x(u),y(u)\big)\,du \bigg)\,dA(s) + \sum_{i=1}^{n-2} \beta_i \int_a^{\xi_i} \frac{(\xi_i-s)^{\alpha-1}}{\Gamma(\alpha)} f\big(s,x(s),y(s)\big)\,ds \Bigg] \\
& - \phi_3(t) \int_a^b \frac{(b-s)^{\beta-1}}{\Gamma(\beta)} g\big(s,x(s),y(s)\big)\,ds \\
& - \phi_4(t) \int_a^b \frac{(b-s)^{\beta-2}}{\Gamma(\beta-1)} g\big(s,x(s),y(s)\big)\,ds \\
& + \phi_5(t) \Bigg[\int_a^b \bigg(\int_a^s \frac{(s-u)^{\beta-1}}{\Gamma(\beta)} g\big(u,x(u),y(u)\big)\,du \bigg)\,dA(s) \\
& + \sum_{i=1}^{n-2} \alpha_i \int_a^{\xi_i} \frac{(\xi_i-s)^{\beta-1}}{\Gamma(\beta)} g\big(s,x(s),y(s)\big)\,ds \Bigg],
\end{aligned} \tag{3.2}$$

$$\begin{aligned}
\mathcal{F}_2(x,y)(t) = {} & \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} g\big(s,x(s),y(s)\big)\,ds - \psi_1(t) \int_a^b \frac{(b-s)^{\alpha-1}}{\Gamma(\alpha)} f\big(s,x(s),y(s)\big)\,ds \\
& - \psi_2(t) \int_a^b \frac{(b-s)^{\alpha-2}}{\Gamma(\alpha-1)} f\big(s,x(s),y(s)\big)\,ds + \phi_5(t) \Bigg[\int_a^b \bigg(\int_a^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} \\
& \times f\big(u,x(u),y(u)\big)\,du \bigg)\,dA(s) + \sum_{i=1}^{n-2} \beta_i \int_a^{\xi_i} \frac{(\xi_i-s)^{\alpha-1}}{\Gamma(\alpha)} f\big(s,x(s),y(s)\big)\,ds \Bigg]
\end{aligned}$$

$$
\begin{aligned}
&- \psi_4(t)\int_a^b \frac{(b-s)^{\beta-1}}{\Gamma(\beta)} g\big(s,x(s),y(s)\big)\,ds \\
&- \psi_5(t)\int_a^b \frac{(b-s)^{\beta-2}}{\Gamma(\beta-1)} g\big(s,x(s),y(s)\big)\,ds \\
&+ \psi_3(t)\Bigg[\int_a^b \bigg(\int_a^s \frac{(s-u)^{\beta-1}}{\Gamma(\beta)} g\big(u,x(u),y(u)\big)\,du\bigg)\,dA(s) \\
&+ \sum_{i=1}^{n-2}\alpha_i \int_a^{\xi_i} \frac{(\xi_i-s)^{\beta-1}}{\Gamma(\beta)} g\big(s,x(s),y(s)\big)\,ds\Bigg],
\end{aligned}
\tag{3.3}
$$

and $\phi_i(t), i = 1,\dots,6$ and $\psi_j(t), j = 1,\dots,5$ are given by (2.4) and (2.5), respectively.

In the forthcoming analysis, we assume that $f,g : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions satisfying the following conditions:

$(\mathcal{O}_1)$ $\forall t \in [a,b]$ and $x_j, y_j \in \mathbb{R}, j = 1,2$, there exist $L_i, i = 1,2$ such that

$$
\begin{aligned}
&\big|f(t,x_1,y_1) - f(t,x_2,y_2)\big| \le L_1\big(|x_1-x_2| + |y_1-y_2|\big), \\
&\big|g(t,x_1,y_1) - g(t,x_2,y_2)\big| \le L_2\big(|x_1-x_2| + |y_1-y_2|\big);
\end{aligned}
$$

$(\mathcal{O}_2)$ $\forall t \in [a,b], x, y \in \mathbb{R}$ there exist real constants $\varepsilon_i, \kappa_i \ge 0, i = 1,2, \varepsilon_0, \kappa_0 > 0$ such that

$$
\begin{aligned}
&\big|f(t,x,y)\big| \le \varepsilon_0 + \varepsilon_1|x| + \varepsilon_2|y|, \\
&\big|g(t,x,y)\big| \le \kappa_0 + \kappa_1|x| + \kappa_2|y|.
\end{aligned}
$$

For computational convenience, we introduce the following notations:

$$
\begin{aligned}
\Lambda_1 &= \frac{(b-a)^\alpha}{\Gamma(\alpha+1)} + \widetilde{\phi}_1\frac{(b-a)^\alpha}{\Gamma(\alpha+1)} + \widetilde{\phi}_6\frac{(b-a)^{\alpha-1}}{\Gamma(\alpha)} \\
&\quad + \widetilde{\phi}_2\bigg(\int_a^b \frac{(s-a)^\alpha}{\Gamma(\alpha+1)}\,dA(s) + \sum_{i=1}^{n-2}|\beta_i|\frac{(\xi_i-a)^\alpha}{\Gamma(\alpha+1)}\bigg), \\
\Lambda_2 &= \widetilde{\phi}_3\frac{(b-a)^\beta}{\Gamma(\beta+1)} + \widetilde{\phi}_4\frac{(b-a)^{\beta-1}}{\Gamma(\beta)} \\
&\quad + \widetilde{\phi}_5\bigg(\int_a^b \frac{(s-a)^\beta}{\Gamma(\beta+1)}\,dA(s) + \sum_{i=1}^{n-2}|\alpha_i|\frac{(\xi_i-a)^\beta}{\Gamma(\beta+1)}\bigg), \\
\Lambda_3 &= \widetilde{\psi}_1\frac{(b-a)^\alpha}{\Gamma(\alpha+1)} + \widetilde{\psi}_2\frac{(b-a)^{\alpha-1}}{\Gamma(\alpha)} \\
&\quad + \widetilde{\phi}_5\bigg(\int_a^b \frac{(s-a)^\alpha}{\Gamma(\alpha+1)}\,dA(s) + \sum_{i=1}^{n-2}|\beta_i|\frac{(\xi_i-a)^\alpha}{\Gamma(\alpha+1)}\bigg), \\
\Lambda_4 &= \frac{(b-a)^\beta}{\Gamma(\beta+1)} + \widetilde{\psi}_4\frac{(b-a)^\beta}{\Gamma(\beta+1)} + \widetilde{\psi}_5\frac{(b-a)^{\beta-1}}{\Gamma(\beta)} \\
&\quad + \widetilde{\psi}_3\bigg(\int_a^b \frac{(s-a)^\beta}{\Gamma(\beta+1)}\,dA(s) + \sum_{i=1}^{n-2}|\alpha_i|\frac{(\xi_i-a)^\beta}{\Gamma(\beta+1)}\bigg),
\end{aligned}
\tag{3.4}
$$

where $\widetilde{\phi}_i = \sup_{t\in[a,b]} |\phi_i(t)|, i = 1, \dots, 6$ and $\widetilde{\psi}_j = \sup_{t\in[a,b]} |\psi_j(t)|, j = 1, \dots, 5$,

$$N_1 = \sup_{t\in[a,b]} \bigl|f(t,0,0)\bigr| < \infty, \qquad N_2 = \sup_{t\in[a,b]} \bigl|g(t,0,0,)\bigr| < \infty, \tag{3.5}$$

$$\begin{aligned} &\Delta = \Lambda_1 L_1 + \Lambda_2 L_2, \qquad \overline{\Delta} = \Lambda_3 L_1 + \Lambda_4 L_2, \\ &M = \Lambda_1 N_1 + \Lambda_2 N_2, \qquad \overline{M} = \Lambda_3 N_1 + \Lambda_4 N_2, \end{aligned} \tag{3.6}$$

$$\Omega_0 = (\Lambda_1 + \Lambda_3)\varepsilon_0 + (\Lambda_2 + \Lambda_4)\kappa_0, \tag{3.7}$$

$$\begin{aligned} &\Omega_1 = (\Lambda_1 + \Lambda_3)\varepsilon_1 + (\Lambda_2 + \Lambda_4)\kappa_1, \qquad \Omega_2 = (\Lambda_1 + \Lambda_3)\varepsilon_2 + (\Lambda_2 + \Lambda_4)\kappa_2, \\ &\Omega = \max\{\Omega_1, \Omega_2\}. \end{aligned} \tag{3.8}$$

Now we present our main results. The first result, based on the Leray–Schauder alternative, deals with the existence of solution for system (1.1)–(1.2).

Lemma 3.1 (Leray–Schauder alternative [26]) *Let $\mathfrak{J} : \mathcal{U} \longrightarrow \mathcal{U}$ be a completely continuous operator (i.e., a map that restricted to any bounded set in $\mathcal{U}$ is compact). Let $\mathcal{Q}(\mathfrak{J}) = \{x \in \mathcal{U} : x = \eta\mathfrak{J}(x)$ for some $0 < \eta < 1\}$. Then either the set $\mathcal{Q}(\mathfrak{J})$ is unbounded, or $\mathfrak{J}$ has at lest one fixed point.*

Theorem 3.2 *Assume that $f, g : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions satisfying assumption $(\mathcal{O}_2)$. Then system (1.1)–(1.2) has at least one solution on $[a,b]$ if $\Omega < 1$, where Ω is given by (3.8).*

Proof In the first step, we show that the operator $\mathcal{F} : \mathcal{A} \times \mathcal{A} \to \mathcal{A} \times \mathcal{A}$ is completely continuous. Notice that the operator $\mathcal{F}$ is continuous in view of continuity of the functions f and g.

Let $\mathcal{V} \subset \mathcal{A} \times \mathcal{A}$ be bounded. Then there exist positive constants θ_1 and θ_2 such that $|f(t,x(t),y(t))| \le \theta_1, |g(t,x(t),y(t))| \le \theta_2, \forall (x,y) \in \mathcal{V}$. So, for any $(x,y) \in \mathcal{V}$, we have

$$\begin{aligned} \bigl|\mathcal{F}_1(x,y)(t)\bigr| \le{}& \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\theta_1\,ds + \bigl|\phi_1(t)\bigr| \int_a^b \frac{(b-s)^{\alpha-1}}{\Gamma(\alpha)}\theta_1\,ds \\ &+ \bigl|\phi_6(t)\bigr| \int_a^b \frac{(b-s)^{\alpha-2}}{\Gamma(\alpha-1)}\theta_1\,ds + \bigl|\phi_2(t)\bigr| \Biggl[\int_a^b \biggl(\int_a^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}\theta_1\biggr)\,dA(s) \\ &+ \sum_{i=1}^{n-2} |\beta_i| \int_a^{\xi_i} \frac{(\xi_i-s)^{\alpha-1}}{\Gamma(\alpha)}\theta_1\,ds\Biggr] \\ &+ \bigl|\phi_3(t)\bigr| \int_a^b \frac{(b-s)^{\beta-1}}{\Gamma(\beta)}\theta_2\,ds + \bigl|\phi_4(t)\bigr| \int_a^b \frac{(b-s)^{\beta-2}}{\Gamma(\beta-1)}\theta_2\,ds \\ &+ \bigl|\phi_5(t)\bigr| \Biggl[\int_a^b \biggl(\int_a^s \frac{(s-u)^{\beta-1}}{\Gamma(\beta)}\theta_2\,du\biggr)\,dA(s) + \sum_{i=1}^{n-2} \alpha_i \int_a^{\xi_i} \frac{(\xi_i-s)^{\beta-1}}{\Gamma(\beta)}\theta_2\,ds\Biggr] \\ \le{}& \Lambda_1\theta_1 + \Lambda_2\theta_2. \end{aligned}$$

Thus,

$$\bigl\|\mathcal{F}_1(x,y)\bigr\| \le \Lambda_1\theta_1 + \Lambda_2\theta_2. \tag{3.9}$$

Similarly, we can get

$$\left\|\mathcal{F}_2(x,y)\right\| \le \Lambda_3\theta_1 + \Lambda_4\theta_2. \tag{3.10}$$

Hence, from (3.9) and (3.10), it follows that $\mathcal{F}$ is uniformly bounded, since $\|\mathcal{F}(x,y)\| \le (\Lambda_1+\Lambda_3)\theta_1 + (\Lambda_2+\Lambda_4)\theta_2$.

Next, we show that the operator $\mathcal{F}$ is equicontinuous. For $t_1, t_2 \in [a,b]$ with $t_1 < t_2$, we obtain

$$\begin{aligned}
&\left|\mathcal{F}_1(x,y)(t_2) - \mathcal{F}_1(x,y)(t_1)\right| \\
&\quad\le \left|\int_a^{t_1} \frac{[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}]}{\Gamma(\alpha)} f\big(s,x(s),y(s)\big)\,ds\right| \\
&\qquad + \left|\int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} f\big(s,x(s),y(s)\big)\,ds\right| \\
&\qquad + \left|\phi_1(t_2)-\phi_1(t_1)\right| \int_a^b \frac{(b-s)^{\alpha-1}}{\Gamma(\alpha)} \left|f\big(s,x(s),y(s)\big)\right|ds \\
&\qquad + \left|\phi_6(t_2)-\phi_6(t_1)\right| \int_a^b \frac{(b-s)^{\alpha-2}}{\Gamma(\alpha-1)} \left|f\big(s,x(s),y(s)\big)\right|ds \\
&\qquad + \left|\phi_2(t_2)-\phi_2(t_1)\right| \left[\int_a^b \left(\int_a^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} \left|f\big(u,x(u),y(u)\big)\right|du\right) dA(s)\right. \\
&\qquad \left. + \sum_{i=1}^{n-2} |\beta_i| \int_a^{\xi_i} \frac{(\xi_i-s)^{\alpha-1}}{\Gamma(\alpha)} \left|f\big(s,x(s),y(s)\big)\right|ds\right] \\
&\qquad + \left|\phi_3(t_2)-\phi_3(t_1)\right| \int_a^b \frac{(b-s)^{\beta-1}}{\Gamma(\beta)} \left|g\big(s,x(s),y(s)\big)\right|ds \\
&\qquad + \left|\phi_4(t_2)-\phi_4(t_1)\right| \int_a^b \frac{(b-s)^{\beta-2}}{\Gamma(\beta-1)} \left|g\big(s,x(s),y(s)\big)\right|ds \\
&\qquad + \left|\phi_5(t_2)-\phi_5(t_1)\right| \left[\int_a^b \left(\int_a^s \frac{(s-u)^{\beta-1}}{\Gamma(\beta)} \left|g\big(u,x(u),y(u)\big)\right|du\right) dA(s)\right. \\
&\qquad \left. + \sum_{i=1}^{n-2} |\alpha_i| \int_a^{\xi_i} \frac{(\xi_i-s)^{\beta-1}}{\Gamma(\beta)} \left|g\big(s,x(s),y(s)\big)\right|ds\right] \\
&\quad\le \frac{\theta_1}{\Gamma(\alpha+1)}\left[2(t_2-t_1)^\alpha + \left|(t_2-a)^\alpha - (t_1-a)^\alpha\right|\right] \\
&\qquad + \theta_1\left|\phi_1(t_2)-\phi_1(t_1)\right| \int_a^b \frac{(b-s)^{\alpha-1}}{\Gamma(\alpha)}\,ds + \theta_1\left|\phi_6(t_2)-\phi_6(t_1)\right| \int_a^b \frac{(b-s)^{\alpha-2}}{\Gamma(\alpha-1)}\,ds \\
&\qquad + \theta_1\left|\phi_2(t_2)-\phi_2(t_1)\right| \left[\int_a^b \left(\int_a^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}\,du\right) dA(s)\right. \\
&\qquad \left. + \sum_{i=1}^{n-2} |\beta_i| \int_a^{\xi_i} \frac{(\xi_i-s)^{\alpha-1}}{\Gamma(\alpha)}\,ds\right] + \theta_2\left|\phi_3(t_2)-\phi_3(t_1)\right| \int_a^b \frac{(b-s)^{\beta-1}}{\Gamma(\beta)}\,ds \\
&\qquad + \theta_2\left|\phi_4(t_2)-\phi_4(t_1)\right| \int_a^b \frac{(b-s)^{\beta-2}}{\Gamma(\beta-1)}\,ds
\end{aligned}$$

$$+ \theta_2 \left|\phi_5(t_2) - \phi_5(t_1)\right| \left[\int_a^b \left(\int_a^s \frac{(s-u)^{\beta-1}}{\Gamma(\beta)} du \right) dA(s) + \sum_{i=1}^{n-2} |\alpha_i| \int_a^{\xi_i} \frac{(\xi_i - s)^{\beta-1}}{\Gamma(\beta)} ds \right].$$

In consequence, $\|\mathcal{F}_1(x,y) - \mathcal{F}_1(x,y)\| \to 0$ independent of x and y as $t_2 \to t_1$. Also, we can obtain

$$\begin{aligned}
&\left|\mathcal{F}_2(x,y)(t_2) - \mathcal{F}_2(x,y)(t_1)\right| \\
&\quad \le \frac{\theta_2}{\Gamma(\beta+1)} \left[2(t_2 - t_1)^\beta + \left|(t_2 - a)^\beta - (t_1 - a)^\beta\right| \right] \\
&\qquad + \theta_1 \left|\psi_1(t_2) - \psi_1(t_1)\right| \int_a^b \frac{(b-s)^{\alpha-1}}{\Gamma(\alpha)} ds + \theta_1 \left|\psi_2(t_2) - \psi_2(t_1)\right| \int_a^b \frac{(b-s)^{\alpha-2}}{\Gamma(\alpha-1)} ds \\
&\qquad + \theta_1 \left|\phi_5(t_2) - \phi_5(t_1)\right| \left[\int_a^b \left(\int_a^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} du \right) dA(s) + \sum_{i=1}^{n-2} |\beta_i| \int_a^{\xi_i} \frac{(\xi_i - s)^{\alpha-1}}{\Gamma(\alpha)} ds \right] \\
&\qquad + \theta_2 \left|\psi_4(t_2) - \psi_4(t_1)\right| \int_a^b \frac{(b-s)^{\beta-1}}{\Gamma(\beta)} ds + \theta_2 \left|\psi_5(t_2) - \psi_5(t_1)\right| \int_a^b \frac{(b-s)^{\beta-2}}{\Gamma(\beta-1)} ds \\
&\qquad + \theta_2 \left|\psi_3(t_2) - \psi_3(t_1)\right| \left[\int_a^b \left(\int_a^s \frac{(s-u)^{\beta-1}}{\Gamma(\beta)} du \right) dA(s) + \sum_{i=1}^{n-2} |\alpha_i| \int_a^{\xi_i} \frac{(\xi_i - s)^{\beta-1}}{\Gamma(\beta)} ds \right],
\end{aligned}$$

which imply that $\|\mathcal{F}_2(x,y) - \mathcal{F}_2(x,y)\| \to 0$ independent of x and y as $t_2 \to t_1$. Therefore, the operator $\mathcal{F}(x,y)$ is equicontinuous. Then, by Arzelá–Ascoli theorem, the operator $\mathcal{F}$ is completely continuous.

Next, we show that the set $\mathcal{P} = \{(x,y) \in \mathcal{A} \times \mathcal{A} | (x,y) = \sigma \mathcal{F}(x,y), 0 \le \sigma \le 1\}$ is bounded. Let $(x,y) \in \mathcal{P}$, then $(x,y) = \sigma \mathcal{F}(x,y)$ and for any $t \in [a,b]$, we have

$$x(t) = \sigma \mathcal{F}_1(x,y)(t), \qquad y(t) = \sigma \mathcal{F}_2(x,y)(t).$$

In consequence, we have

$$\left|x(t)\right| \le \Lambda_1 \left(\varepsilon_0 + \varepsilon_1 |x| + \varepsilon_2 |y|\right) + \Lambda_2 \left(\kappa_0 + \kappa_1 |x| + \kappa_2 |y|\right),$$

which leads to

$$\|x\| \le \Lambda_1 \varepsilon_0 + \Lambda_2 \kappa_0 + (\Lambda_1 \varepsilon_1 + \Lambda_2 \kappa_1)\|x\| + (\Lambda_1 \varepsilon_2 + \Lambda_2 \kappa_2)\|y\|. \tag{3.11}$$

In a similar manner, we can find that

$$\|y\| \le \Lambda_3 \varepsilon_0 + \Lambda_4 \kappa_0 + (\Lambda_3 \varepsilon_1 + \Lambda_4 \kappa_1)\|x\| + (\Lambda_3 \varepsilon_2 + \Lambda_4 \kappa_2)\|y\|. \tag{3.12}$$

From (3.11) and (3.12) together with the notations (3.7) and (3.8), we get

$$\begin{aligned}
\|x\| + \|y\| \le{}& \left[(\Lambda_1 + \Lambda_3)\varepsilon_0 + (\Lambda_2 + \Lambda_4)\kappa_0\right] + \left[(\Lambda_1 + \Lambda_3)\varepsilon_1 + (\Lambda_2 + \Lambda_4)\kappa_1\right]\|x\| \\
&+ \left[(\Lambda_1 + \Lambda_3)\varepsilon_2 + (\Lambda_2 + \Lambda_4)\kappa_2\right]\|y\|.
\end{aligned}$$

Thus,

$$\left\|(x,y)\right\| \le \Omega_0 + \max\{\Omega_1, \Omega_2\} \left\|(x,y)\right\| \le \Omega_0 + \Omega \left\|(x,y)\right\|,$$

which can alternatively be written as

$$\|(x,y)\| \le \frac{\Omega_0}{1-\Omega}.$$

This show that the set $\mathcal{P}$ is bounded. Therefore, by Lemma 3.1 (Leray–Schauder alternative theorem), the operator $\mathcal{F}$ has at least one fixed point, which implies that there exist at least one solution for the system (1.1)–(1.2) on $[a,b]$. □

Example 3.3 Consider the coupled system of fractional differential equations given by

$$\begin{cases} {}^cD^{10/3}x(t) = \frac{9\sin(t)}{3+t^3} + \frac{14x(t)|y(t)|}{61(1+|y(t)|)} + \frac{2\sin y(t)|\tan^{-1}x(t)|}{\pi\sqrt{4+t^2}}, \\ {}^cD^{15/4}y(t) = \frac{2t}{15\sqrt{9+t^4}} + \frac{5\sin(x(t))}{8(t^2+2)} + \frac{2y(t)}{\sqrt{121+2t^2}} + \sin t, \quad t\in[0,1], \end{cases} \tag{3.13}$$

with the boundary conditions

$$\begin{cases} x'(0)=0, \quad x(1)=0, \quad x'(1)=0, \\ x(0)=\int_0^1 y(s)\,dA(s) + \sum_{i=1}^4 \alpha_i y(\xi_i), \\ y'(0)=0, \quad y(1)=0, \quad y'(1)=0, \\ y(0)=\int_0^1 x(s)\,dA(s) + \sum_{i=1}^4 \beta_i x(\xi_i), \end{cases} \tag{3.14}$$

where $a=0, b=1, \alpha=\frac{10}{3}, \beta=\frac{15}{4}, \alpha_1=\frac{-1}{5}, \alpha_2=1, \alpha_3=\frac{3}{2}, \alpha_4=2, \beta_1=\frac{-1}{3}, \beta_2=0, \beta_3=\frac{1}{2}, \beta_4=\frac{5}{2}, \xi_1=\frac{1}{8}, \xi_2=\frac{1}{3}, \xi_3=\frac{1}{2}, \xi_4=\frac{2}{3}, f(t,x(t),y(t))=\frac{9\sin(t)}{3+t^3}+\frac{14x(t)|y(t)|}{61(1+|y(t)|)}+\frac{2\sin y(t)|\tan^{-1}x(t)|}{\pi\sqrt{4+t^2}}$, and $g(t,x(t),y(t))=\frac{2t}{15\sqrt{9+t^4}}+\frac{5\sin(x(t))}{8(t^2+2)}+\frac{2y(t)}{\sqrt{121+2t^2}}+\sin t$.

Let us take $A(s)=\frac{23(s^2+1)}{2}$. Using the given data, we have $A_1\simeq 15.8000, A_2\simeq 7.12188, A_3\simeq 5.41674, A_4\simeq 14.1667, A_5\simeq 6.98090, A_6\simeq 5.40259, \gamma_1\simeq 2.63395, \gamma_2\simeq -4.33906, \gamma_3\simeq 2.01460, \gamma_4\simeq -3.59290, \nu_1\simeq -2.64041, \nu_2\simeq -0.260460, \nu_3\simeq -0.520737, \nu_4\simeq 0.084305, \nu_5\simeq -0.049442, \nu_6\simeq 0.411084, \omega_1\simeq -0.501226, \omega_2\simeq 0.078035, \omega_3\simeq -0.199215, \omega_4\simeq -2.09815, \omega_5\simeq 0.339683, \delta_1\simeq -7.28082, \delta_2\simeq -0.520920, \delta_3\simeq -1.041470, \delta_4\simeq 0.168611, \delta_5\simeq -0.098885, \delta_6\simeq 1.82217, \lambda_1\simeq -1.00245, \lambda_2\simeq 0.156071, \lambda_3\simeq -0.398430, \lambda_4\simeq -6.19630, \lambda_5\simeq 1.67937, \mu_1\simeq 10.9212, \mu_2\simeq 0.781380, \mu_3\simeq 1.56220, \mu_4\simeq -0.252916, \mu_5\simeq 0.148328, \mu_6\simeq -2.23326, \rho_1\simeq 1.50368, \rho_2\simeq -0.234106, \rho_3\simeq 0.597645, \rho_4\simeq 9.29445, \rho_5\simeq -2.01906, \widetilde{\phi}_1\simeq 2.64041, \widetilde{\phi}_2\simeq 0.260460, \widetilde{\phi}_3\simeq 0.520737, \widetilde{\phi}_4\simeq 0.084305, \widetilde{\phi}_5\simeq 0.049442, \widetilde{\phi}_6\simeq 0.411084, \widetilde{\psi}_1\simeq 0.501226, \widetilde{\psi}_2\simeq 0.078035, \widetilde{\psi}_3\simeq 0.199215, \widetilde{\psi}_4\simeq 2.09815, \widetilde{\psi}_5\simeq 0.339683, \Lambda_1\simeq 0.681978, \Lambda_2\simeq 0.064064, \Lambda_3\simeq 0.108960, \Lambda_4\simeq 0.318420$.

Clearly,

$$\begin{aligned} \bigl|f\bigl(t,x(t),y(t)\bigr)\bigr| &\le 3+\frac{14}{61}\|x\|+\frac{1}{2}\|y\|, \\ \bigl|g\bigl(t,x(t),y(t)\bigr)\bigr| &\le \frac{2}{45}+\frac{5}{16}\|x\|+\frac{2}{11}\|y\|, \end{aligned}$$

with $\varepsilon_0=3, \varepsilon_1=\frac{14}{61}, \varepsilon_2=\frac{1}{2}, \kappa_0=\frac{2}{45}, \kappa_1=\frac{5}{16}$, and $\kappa_2=\frac{2}{11}$. Using (3.8), we find that $\Omega_1\simeq 0.301053, \Omega_2\simeq 0.465012$ and $\Omega=\max\{\Omega_1,\Omega_2\}\simeq 0.465012<1$. Therefore, by Theorem 3.2, the problem (3.13)–(3.14) has at least one solution on $[0,1]$.

Our next result is based on the Krasnoselskii fixed point theorem.

Lemma 3.4 (Krasnoselskii) *Let $\mathcal{X}$ be a closed, bounded, convex and nonempty subset of a Banach space $\mathcal{Y}$. Let $\mathcal{H}_1, \mathcal{H}_2$ be operators mapping $\mathcal{X}$ to $\mathcal{Y}$ such that*

(a) $\mathcal{H}_1 z_1 + \mathcal{H}_2 z_2 \in \mathcal{X}$ *where* $z_1, z_2 \in \mathcal{X}$;
(b) $\mathcal{H}_1$ *is compact and continuous*;
(c) $\mathcal{H}_2$ *is a contraction mapping.*

Then there exists $z \in \mathcal{X}$ such that $z = \mathcal{H}_1 z_1 + \mathcal{H}_2 z_2$.

Theorem 3.5 *Assume that $f, g : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions satisfying the condition $(\mathcal{O}_1)$. Furthermore, we assume that there exist two positive constants B_1, B_2 such that $\forall t \in [a,b]$ and $x, y \in \mathbb{R}$,*

$$\left|f(t,x,y)\right| \leq B_1 \quad \text{and} \quad \left|g(t,x,y)\right| \leq B_2. \tag{3.15}$$

Then system (1.1)–(1.2) has at least one solution on $[a,b]$, if

$$(Q_1 L_1 + \Lambda_2 L_2) + (\Lambda_3 L_1 + Q_2 L_2) < 1, \tag{3.16}$$

where $Q_1 = \Lambda_1 - \frac{(b-a)^\alpha}{\Gamma(\alpha+1)}$ and $Q_2 = \Lambda_4 - \frac{(b-a)^\beta}{\Gamma(\beta+1)}$.

Proof Define a closed ball $S_\eta = \{(x,y) \in \mathcal{A} \times \mathcal{A} : \|(x,y)\| \leq \eta\}$ which is bounded and convex subset of the Banach space $\mathcal{A} \times \mathcal{A}$ and select

$$\eta \geq \max\{\Lambda_1 B_1 + \Lambda_2 B_2, \Lambda_3 B_1 + \Lambda_4 B_2\}. \tag{3.17}$$

In order to verify the hypotheses of Lemma 3.4, we decompose the operator $\mathcal{F}$ into four operators $\mathcal{F}_{1,1}, \mathcal{F}_{1,2}, \mathcal{F}_{2,1}$ and $\mathcal{F}_{2,2}$ on S_η as follows:

$$\begin{aligned}
\mathcal{F}_{1,1}(x,y)(t) &= \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\bigl(s,x(s),y(s)\bigr)\,ds, \\
\mathcal{F}_{1,2}(x,y)(t) &= -\phi_1(t) \int_a^b \frac{(b-s)^{\alpha-1}}{\Gamma(\alpha)} f\bigl(s,x(s),y(s)\bigr)\,ds \\
&\quad - \phi_6(t) \int_a^b \frac{(b-s)^{\alpha-2}}{\Gamma(\alpha-1)} f\bigl(s,x(s),y(s)\bigr)\,ds + \phi_2(t) \Biggl[\int_a^b \biggl(\int_a^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} \\
&\quad \times f\bigl(u,x(u),y(u)\bigr)\,du \biggr)\,dA(s) + \sum_{i=1}^{n-2} \beta_i \int_a^{\xi_i} \frac{(\xi_i-s)^{\alpha-1}}{\Gamma(\alpha)} f\bigl(s,x(s),y(s)\bigr)\,ds \Biggr] \\
&\quad - \phi_3(t) \int_a^b \frac{(b-s)^{\beta-1}}{\Gamma(\beta)} g\bigl(s,x(s),y(s)\bigr)\,ds \\
&\quad - \phi_4(t) \int_a^b \frac{(b-s)^{\beta-2}}{\Gamma(\beta-1)} g\bigl(s,x(s),y(s)\bigr)\,ds \\
&\quad + \phi_5(t) \Biggl[\int_a^b \biggl(\int_a^s \frac{(s-u)^{\beta-1}}{\Gamma(\beta)} g\bigl(u,x(u),y(u)\bigr)\,du \biggr)\,dA(s) \\
&\quad + \sum_{i=1}^{n-2} \alpha_i \int_a^{\xi_i} \frac{(\xi_i-s)^{\beta-1}}{\Gamma(\beta)} g\bigl(s,x(s),y(s)\bigr)\,ds \Biggr],
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{F}_{2,1}(x,y)(t) &= \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} g\big(s,x(s),y(s)\big)\,ds,\\
\mathcal{F}_{2,2}(x,y)(t) &= -\psi_1(t)\int_a^b \frac{(b-s)^{\alpha-1}}{\Gamma(\alpha)} f\big(s,x(s),y(s)\big)\,ds\\
&\quad - \psi_2(t)\int_a^b \frac{(b-s)^{\alpha-2}}{\Gamma(\alpha-1)} f\big(s,x(s),y(s)\big)\,ds + \phi_5(t)\Bigg[\int_a^b\bigg(\int_a^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}\\
&\quad \times f\big(u,x(u),y(u)\big)\,du\bigg)\,dA(s) + \sum_{i=1}^{n-2}\beta_i\int_a^{\xi_i}\frac{(\xi_i-s)^{\alpha-1}}{\Gamma(\alpha)} f\big(s,x(s),y(s)\big)\,ds\Bigg]\\
&\quad - \psi_4(t)\int_a^b \frac{(b-s)^{\beta-1}}{\Gamma(\beta)} g\big(s,x(s),y(s)\big)\,ds\\
&\quad - \psi_5(t)\int_a^b \frac{(b-s)^{\beta-2}}{\Gamma(\beta-1)} g\big(s,x(s),y(s)\big)\,ds\\
&\quad + \psi_3(t)\Bigg[\int_a^b\bigg(\int_a^s \frac{(s-u)^{\beta-1}}{\Gamma(\beta)} g\big(u,x(u),y(u)\big)\,du\bigg)\,dA(s)\\
&\quad + \sum_{i=1}^{n-2}\alpha_i\int_a^{\xi_i}\frac{(\xi_i-s)^{\beta-1}}{\Gamma(\beta)} g\big(s,x(s),y(s)\big)\,ds\Bigg].
\end{aligned}$$

Notice that $\mathcal{F}_1(x,y)(t) = \mathcal{F}_{1,1}(x,y)(t) + \mathcal{F}_{1,2}(x,y)(t)$ and $\mathcal{F}_2(x,y)(t) = \mathcal{F}_{2,1}(x,y)(t) + \mathcal{F}_{2,2}(x,y)(t)$ on S_η. For verifying condition (a) of Lemma 3.4 we use (3.17) to show that $\mathcal{F}S_\eta \subset S_\eta$. Setting $x = (x_1,x_2), y = (y_1,y_2), \hat{x} = (\hat{x}_1,\hat{x}_2)$ and $\hat{y} = (\hat{y}_1,\hat{y}_2) \in S_\eta$, and using condition (3.15), we obtain

$$\begin{aligned}
&\big\|\mathcal{F}_{1,1}(x,y) + \mathcal{F}_{1,2}(\hat{x},\hat{y})\big\|\\
&\quad \le \sup_{t\in[a,b]}\Bigg\{\int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}B_1\,ds + \big|\phi_1(t)\big|\int_a^b \frac{(b-s)^{\alpha-1}}{\Gamma(\alpha)}B_1\,ds\\
&\qquad + \big|\phi_6(t)\big|\int_a^b \frac{(b-s)^{\alpha-2}}{\Gamma(\alpha-1)}B_1\,ds + \big|\phi_2(t)\big|\Bigg[\int_a^b\bigg(\int_a^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}B_1\,du\bigg)\,dA(s)\\
&\qquad + \sum_{i=1}^{n-2}|\beta_i|\int_a^{\xi_i}\frac{(\xi_i-s)^{\alpha-1}}{\Gamma(\alpha)}B_1\,ds\Bigg] + \big|\phi_3(t)\big|\int_a^b \frac{(b-s)^{\beta-1}}{\Gamma(\beta)}B_2\,ds\\
&\qquad + \big|\phi_4(t)\big|\int_a^b \frac{(b-s)^{\beta-2}}{\Gamma(\beta-1)}B_2\,ds + \big|\phi_5(t)\big|\Bigg[\int_a^b\bigg(\int_a^s \frac{(s-u)^{\beta-1}}{\Gamma(\beta)}B_2\,du\bigg)\,dA(s)\\
&\qquad + \sum_{i=1}^{n-2}|\alpha_i|\int_a^{\xi_i}\frac{(\xi_i-s)^{\beta-1}}{\Gamma(\beta)}B_2\,ds\Bigg]\Bigg\}\\
&\quad = \Lambda_1B_1 + \Lambda_2B_2 \le \eta.
\end{aligned}$$

Similarly, we can find that

$$\big\|\mathcal{F}_{2,1}(x,y) + \mathcal{F}_{2,2}(\hat{x},\hat{y})\big\| \le \Lambda_3B_1 + \Lambda_4B_2 \le \eta.$$

Clearly the above two inequalities lead to the fact that $\mathcal{F}_1(x,y)+\mathcal{F}_2(\hat{x},\hat{y})\in S_\eta$.

Now we prove that the operator $(\mathcal{F}_{1,2},\mathcal{F}_{2,2})$ is a contraction satisfying condition (c) of Lemma 3.4. For $(x_1,y_1),(x_2,y_2)\in S_\eta$, we have

$$\begin{aligned}
&\bigl\|\mathcal{F}_{1,2}(x_1,y_1)-\mathcal{F}_{1,2}(x_2,y_2)\bigr\| \\
&\quad=\sup_{t\in[a,b]}\bigl|\mathcal{F}_{1,2}(x_1,y_1)(t)-\mathcal{F}_{1,2}(x_2,y_2)(t)\bigr| \\
&\quad\le\sup_{t\in[a,b]}\Biggl\{\bigl|\phi_1(t)\bigr|\int_a^b\frac{(b-s)^{\alpha-1}}{\Gamma(\alpha)}\bigl|f\bigl(s,x_1(s),y_1(s)\bigr)-f\bigl(s,x_2(s),y_2(s)\bigr)\bigr|\,ds \\
&\qquad+\bigl|\phi_6(t)\bigr|\int_a^b\frac{(b-s)^{\alpha-2}}{\Gamma(\alpha-1)}\bigl|f\bigl(s,x_1(s),y_1(s)\bigr)-f\bigl(s,x_2(s),y_2(s)\bigr)\bigr|\,ds \\
&\qquad+\bigl|\phi_2(t)\bigr|\Biggl[\int_a^b\biggl(\int_a^s\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}\bigl|f\bigl(u,x_1(u),y_1(u)\bigr)-f\bigl(u,x_2(u),y_2(u)\bigr)\bigr|\,du\biggr)\,dA(s) \\
&\qquad+\sum_{i=1}^{n-2}|\beta_i|\int_a^{\xi_i}\frac{(\xi_i-s)^{\alpha-1}}{\Gamma(\alpha)}\bigl|f\bigl(s,x_1(s),y_1(s)\bigr)-f\bigl(s,x_2(s),y_2(s)\bigr)\bigr|\,ds\Biggr] \\
&\qquad+\bigl|\phi_3(t)\bigr|\int_a^b\frac{(b-s)^{\beta-1}}{\Gamma(\beta)}\bigl|g\bigl(s,x_1(s),y_1(s)\bigr)-g\bigl(s,x_2(s),y_2(s)\bigr)\bigr|\,ds \\
&\qquad+\bigl|\phi_4(t)\bigr|\int_a^b\frac{(b-s)^{\beta-2}}{\Gamma(\beta-1)}\bigl|g\bigl(s,x_1(s),y_1(s)\bigr)-g\bigl(s,x_2(s),y_2(s)\bigr)\bigr|\,ds \\
&\qquad+\bigl|\phi_5(t)\bigr|\Biggl[\int_a^b\biggl(\int_a^s\frac{(s-u)^{\beta-1}}{\Gamma(\beta)}\bigl|g\bigl(u,x_1(u),y_1(u)\bigr)-g\bigl(u,x_2(u),y_2(u)\bigr)\bigr|\,du\biggr)\,dA(s) \\
&\qquad+\sum_{i=1}^{n-2}|\alpha_i|\int_a^{\xi_i}\frac{(\xi_i-s)^{\beta-1}}{\Gamma(\beta)}\bigl|g\bigl(s,x_1(s),y_1(s)\bigr)-g\bigl(s,x_2(s),y_2(s)\bigr)\bigr|\,ds\Biggr]\Biggr\} \\
&\quad\le\Biggl[\biggl(\widetilde{\phi}_1\frac{(b-a)^\alpha}{\Gamma(\alpha+1)}+\widetilde{\phi}_6\frac{(b-a)^{\alpha-1}}{\Gamma(\alpha)}+\widetilde{\phi}_2\biggl(\int_a^b\frac{(s-a)^\alpha}{\Gamma(\alpha+1)}\,dA(s) \\
&\qquad+\sum_{i=1}^{n-2}|\beta_i|\frac{(\xi_i-a)^\alpha}{\Gamma(\alpha+1)}\biggr)\biggr)L_1+\biggl(\widetilde{\phi}_3\frac{(b-a)^\beta}{\Gamma(\beta+1)}+\widetilde{\phi}_4\frac{(b-a)^{\beta-1}}{\Gamma(\beta)} \\
&\qquad+\widetilde{\phi}_5\biggl(\int_a^b\frac{(s-a)^\beta}{\Gamma(\beta+1)}\,dA(s)+\sum_{i=1}^{n-2}|\beta_i|\frac{(\xi_i-a)^\beta}{\Gamma(\beta+1)}\biggr)\biggr)L_2\Biggr]\bigl(\|x_1-x_2\|+\|y_1-y_2\|\bigr) \\
&\quad=(Q_1L_1+\Lambda_2L_2)\bigl(\|x_1-x_2\|+\|y_1-y_2\|\bigr)
\end{aligned}\tag{3.18}$$

and

$$\bigl\|\mathcal{F}_{2,2}(x_1,y_1)-\mathcal{F}_{2,2}(x_2,y_2)\bigr\|\le(\Lambda_3L_1+Q_2L_2)\bigl(\|x_1-x_2\|+\|y_1-y_2\|\bigr).\tag{3.19}$$

It follows from (3.18) and (3.19) that

$$\begin{aligned}
&\bigl\|(\mathcal{F}_{1,2},\mathcal{F}_{2,2})(x_1,y_1)-(\mathcal{F}_{1,2},\mathcal{F}_{2,2})(x_2,y_2)\bigr\| \\
&\quad\le\bigl[(Q_1L_1+\Lambda_2L_2)+(\Lambda_3L_1+Q_2L_2)\bigr]\bigl(\|x_1-x_2\|+\|y_1-y_2\|\bigr),
\end{aligned}$$

then by using (3.16) the operator $(\mathcal{F}_{1,2}, \mathcal{F}_{2,2})$ is a contraction. Therefore, the condition (c) of Lemma 3.4 is satisfied.

Next we will show that the operator $(\mathcal{F}_{1,1}, \mathcal{F}_{2,1})$ satisfies the condition (b) of Lemma 3.4. By applying the continuity of the functions f, g on $[a,b] \times \mathbb{R} \times \mathbb{R}$, we can conclude that the operator $(\mathcal{F}_{1,1}, \mathcal{F}_{2,1})$ is continuous. For each $(x,y) \in S_\eta$, we can have

$$\begin{aligned} \|\mathcal{F}_{1,1}(x,y)\| &\le \sup_{t\in[a,b]} |\mathcal{F}_{1,1}(x,y)(t)| \le \sup_{t\in[a,b]} \left| \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\big(s, x(s), y(s)\big)\, ds \right| \\ &\le \frac{(b-a)^\alpha B_1}{\Gamma(\alpha+1)} = r_1 \end{aligned}$$

and

$$\begin{aligned} \|\mathcal{F}_{2,1}(x,y)\| &\le \sup_{t\in[a,b]} |\mathcal{F}_{2,1}(x,y)(t)| \le \sup_{t\in[a,b]} \left| \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} g\big(s, x(s), y(s)\big)\, ds \right| \\ &\le \frac{(b-a)^\beta B_2}{\Gamma(\beta+1)} = r_2, \end{aligned}$$

which yield

$$\|(\mathcal{F}_{1,1}, \mathcal{F}_{2,1})(x,y)\| \le r_1 + r_2.$$

Thus the set $(\mathcal{F}_{1,1}, \mathcal{F}_{2,1})S_\eta$ is uniformly bounded. In the next step, we will show that the set $(\mathcal{F}_{1,1}, \mathcal{F}_{2,1})S_\eta$ is equicontinuous. For $t_1, t_2 \in [a,b]$ with $t_1 < t_2$ and for any $(x,y) \in S_\eta$, we obtain

$$\begin{aligned} |\mathcal{F}_{1,1}(x,y)(t_2) - \mathcal{F}_{1,1}(x,y)(t_1)| &\le \left| \int_a^{t_1} \frac{[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}]}{\Gamma(\alpha)} f\big(s, x(s), y(s)\big)\, ds \right| \\ &\quad + \left| \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} f\big(s, x(s), y(s)\big)\, ds \right| \\ &\le \frac{B_1}{\Gamma(\alpha+1)} \big[2(t_2-t_1)^\alpha + |(t_2-a)^\alpha - (t_1-a)^\alpha|\big]. \end{aligned}$$

Analogously, we can get

$$|\mathcal{F}_{2,1}(x,y)(t_2) - \mathcal{F}_{2,1}(x,y)(t_1)| \le \frac{B_2}{\Gamma(\beta+1)} \big[2(t_2-t_1)^\beta + |(t_2-a)^\beta - (t_1-a)^\beta|\big].$$

Therefore, $|(\mathcal{F}_{1,1}, \mathcal{F}_{2,1})(x,y)(t_2) - (\mathcal{F}_{1,1}, \mathcal{F}_{2,1})(x,y)(t_1)|$ tends to zero as $t_1 \to t_2$ independent of $(x,y) \in S_\eta$. Hence, the set $(\mathcal{F}_{1,1}, \mathcal{F}_{2,1})S_\eta$ is equicontinuous. Thus it follows by the Arzelá–Ascoli theorem that the operator $(\mathcal{F}_{1,1}, \mathcal{F}_{2,1})$ is compact on S_η. By the conclusion of Lemma 3.4, we deduce that system (1.1)–(1.2) has at least one solution on $[a,b]$. □

Example 3.6 Consider the same problem in Example 3.3 with the coupled boundary conditions (3.14) and

$$f\big(t, x(t), y(t)\big) = \frac{2|x(t)|}{3(1+|x(t)|)} + \frac{2\cos y(t)}{\sqrt{t^2+9}} + \frac{t+2}{27}, \quad t \in [0,1],$$

and

$$g\big(t,x(t),y(t)\big)=\frac{\sin x(t)}{\sqrt{t^4+144}}+\frac{y(t)+36}{12},\quad t\in[0,1].$$

Clearly, $|f(t,x_1,y_1)-f(t,x_2,y_2)|\leq\frac{2}{3}|x_1-x_2|+\frac{2}{3}|y_1-y_2|$, $|g(t,x_1,y_1)-g(t,x_2,y_2)|\leq\frac{1}{12}|x_1-x_2|+\frac{1}{12}|y_1-y_2|$ and with the given data, we find that $Q_1\simeq 0.573992$, $Q_2\simeq 0.258129$, $\Lambda_1\simeq 0.681978$, $\Lambda_2\simeq 0.064064$, $\Lambda_3\simeq 0.108960$, $\Lambda_4\simeq 0.318420$. Note that and $(Q_1+\Lambda_3)L_1+(Q_2+\Lambda_2)L_2\simeq 0.482151<1$. Thus all the conditions of Theorem 3.5 are satisfied and consequently, its conclusion applies to the problem (3.13)–(3.14).

4 Uniqueness of solutions

This section is concerned with the uniqueness of solutions for the problem at hand and relies on the Banach contraction mapping principle.

Theorem 4.1 *Assume that the condition $(\mathcal{O}_1)$ holds. Then system (1.1)–(1.2) has a unique solution on $[a,b]$ if*

$$\Delta+\overline{\Delta}<1,\tag{4.1}$$

where Δ and $\overline{\Delta}$ are given by (3.6).

Proof Let us set $\tau>\frac{M+\overline{M}}{1-\Delta-\overline{\Delta}}$, where $\Delta,\overline{\Delta},M$ and $\overline{M}$ are given by (3.6), and show that $\mathcal{F}U_\tau\subset U_\tau$, where the operator $\mathcal{F}$ is given by (3.1) and

$$U_\tau=\big\{(x,y)\in\mathcal{A}\times\mathcal{A}:\big\|(x,y)\big\|\leq\tau\big\}.$$

In view of the assumption $(\mathcal{O}_1)$ together with (3.5), for $(x,y)\in U_\tau, t\in[a,b]$, we have

$$\begin{aligned}\big|f\big(t,x(t),y(t)\big)\big|&\leq\big|f\big(t,x(t),y(t)\big)-f(t,0,0)\big|+\big|f(t,0,0)\big|\\&\leq L_1\big(\|x\|+\|y\|\big)+N_1\leq L_1\tau+N_1,\\\big|g\big(t,x(t),y(t)\big)\big|&\leq\big|g\big(t,x(t),y(t)\big)-g(t,0,0)\big|+\big|g(t,0,0)\big|\\&\leq L_2\big(\|x\|+\|y\|\big)+N_2\leq L_2\tau+N_2.\end{aligned}$$

In view of (3.6), we obtain

$$\begin{aligned}\big|\mathcal{F}_1(x,y)(t)\big|&\leq\Lambda_1(L_1\tau+N_1)+\Lambda_2(L_2\tau+N_2)\\&=(\Lambda_1L_1+\Lambda_2L_2)\tau+(\Lambda_1N_1+\Lambda_2N_2)=\Delta\tau+M,\\\big|\mathcal{F}_2(x,y)(t)\big|&\leq\Lambda_3(L_1\tau+N_1)+\Lambda_4(L_2\tau+N_2)\\&=(\Lambda_3L_1+\Lambda_4L_2)\tau+(\Lambda_3N_1+\Lambda_4N_2)=\overline{\Delta}\tau+\overline{M},\end{aligned}$$

which imply that

$$\big\|\mathcal{F}_1(x,y)\big\|\leq\Delta\tau+M,\qquad\big\|\mathcal{F}_2(x,y)\big\|\leq\overline{\Delta}\tau+\overline{M}.\tag{4.2}$$

Thus, it follows from (4.2) that

$$\|\mathcal{F}(x,y)\| \le (\Delta\tau + M) + (\overline{\Delta}\tau + \overline{M}) \le (\Delta + \overline{\Delta})\tau + (M + \overline{M}) \le \tau.$$

Consequently, $\mathcal{F}U_\tau \subset U_\tau$. Next, we show that the operator $\mathcal{F}$ is a contraction. Using the conditions $(\mathcal{O}_1)$, (3.6) and for any $(x_1,y_1),(x_2,y_2) \in \mathcal{A} \times \mathcal{A}, t \in [a,b]$, we get

$$\begin{aligned}
&\|\mathcal{F}_1(x_1,y_1) - \mathcal{F}_1(x_2,y_2)\| \\
&\quad = \sup_{t\in[a,b]} |\mathcal{F}_1(x_1,y_1)(t) - \mathcal{F}_1(x_2,y_2)(t)| \\
&\quad \le \sup_{t\in[a,b]} \bigg\{ \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \big|f\big(s,x_1(s),y_1(s)\big) - f\big(s,x_2(s),y_2(s)\big)\big|\,ds \\
&\qquad + |\phi_1(t)| \int_a^b \frac{(b-s)^{\alpha-1}}{\Gamma(\alpha)} \big|f\big(s,x_1(s),y_1(s)\big) - f\big(s,x_2(s),y_2(s)\big)\big|\,ds \\
&\qquad + |\phi_6(t)| \int_a^b \frac{(b-s)^{\alpha-2}}{\Gamma(\alpha-1)} \big|f\big(s,x_1(s),y_1(s)\big) - f\big(s,x_2(s),y_2(s)\big)\big|\,ds \\
&\qquad + |\phi_2(t)| \bigg[\int_a^b \bigg(\int_a^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} \big|f\big(u,x_1(u),y_1(u)\big) - f\big(u,x_2(u),y_2(u)\big)\big|\,du \bigg)\,dA(s) \\
&\qquad + \sum_{i=1}^{n-2} |\beta_i| \int_a^{\xi_i} \frac{(\xi_i-s)^{\alpha-1}}{\Gamma(\alpha)} \big|f\big(s,x_1(s),y_1(s)\big) - f\big(s,x_2(s),y_2(s)\big)\big|\,ds \bigg] \\
&\qquad + |\phi_3(t)| \int_a^b \frac{(b-s)^{\beta-1}}{\Gamma(\beta)} \big|g\big(s,x_1(s),y_1(s)\big) - g\big(s,x_2(s),y_2(s)\big)\big|\,ds \\
&\qquad + |\phi_4(t)| \int_a^b \frac{(b-s)^{\beta-2}}{\Gamma(\beta-1)} \big|g\big(s,x_1(s),y_1(s)\big) - g\big(s,x_2(s),y_2(s)\big)\big|\,ds \\
&\qquad + |\phi_5(t)| \bigg[\int_a^b \bigg(\int_a^s \frac{(s-u)^{\beta-1}}{\Gamma(\beta)} \big|g\big(u,x_1(u),y_1(u)\big) - g\big(u,x_2(u),y_2(u)\big)\big|\,du \bigg)\,dA(s) \\
&\qquad + \sum_{i=1}^{n-2} |\alpha_i| \int_a^{\xi_i} \frac{(\xi_i-s)^{\beta-1}}{\Gamma(\beta)} \big|g\big(s,x_1(s),y_1(s)\big) - g\big(s,x_2(s),y_2(s)\big)\big|\,ds \bigg] \bigg\} \\
&\quad \le \Lambda_1 L_1 \big(\|x_1 - x_2\| + \|y_1 - y_2\|\big) + \Lambda_2 L_2 \big(\|x_1 - x_2\| + \|y_1 - y_2\|\big) \\
&\quad = (\Lambda_1 L_1 + \Lambda_2 L_2)\big(\|x_1 - x_2\| + \|y_1 - y_2\|\big) \\
&\quad = \Delta\big(\|x_1 - x_2\| + \|y_1 - y_2\|\big).
\end{aligned}$$

Similarly

$$\begin{aligned}
\|\mathcal{F}_2(x_1,y_1) - \mathcal{F}_2(x_2,y_2)\| &= \sup_{t\in[a,b]} |\mathcal{F}_2(x_1,y_1)(t) - \mathcal{F}_2(x_2,y_2)(t)| \\
&\le (\Lambda_3 L_1 + \Lambda_4 L_2)\big(\|x_1 - x_2\| + \|y_1 - y_2\|\big) \\
&= \overline{\Delta}\big(\|x_1 - x_2\| + \|y_1 - y_2\|\big).
\end{aligned}$$

Hence we obtain

$$\left\| \mathcal{F}(x_1,y_1) - \mathcal{F}(x_2,y_2)) \right\| \le (\Delta + \overline{\Delta})\big(\|x_1 - x_2\| + \|y_1 - y_2\|\big),$$

which implies that $\mathcal{F}$ is a contraction by the assumption (4.1). Hence, we deduce by the conclusion of contraction mapping principle that there exists a unique solution for the problem (1.1)–(1.2) on $[a,b]$. □

Example 4.2 Consider the following system:

$$\begin{cases} {}^cD^{10/3}x(t) = \frac{5}{2\sqrt{81+t^2}}(\tan^{-1} x(t) + \sin y(t)) + \frac{1}{25+t^2}, \\ {}^cD^{15/4}y(t) = \frac{t^2+16}{8\sqrt{16+t^6}}(x(t) + \cos y(t)) + \frac{e^{-t}}{3\sqrt{t^2+9}}, \quad t \in [0,1], \end{cases} \tag{4.3}$$

with the coupled boundary conditions (3.14).

Clearly, $|f(t,x_1,y_1) - f(t,x_2,y_2)| \le L_1(\|x_1 - x_2\| + \|y_1 - y_2\|)$ with $L_1 = \frac{5}{18}$ and $|g(t,x_1,y_1) - g(t,x_2,y_2)| \le L_2(\|x_1 - x_2\| + \|y_1 - y_2\|)$ with $L_2 = \frac{17}{32}$. Using the given data in Example (3.3) and (4.1), we find that $\Delta + \overline{\Delta} \simeq 0.422900 < 1$. Obviously, the hypothesis of Theorem 4.1 is satisfied. Hence, by the conclusion of Theorem 4.1 there is a unique solution for the problem (3.13) on $[0,1]$.

Acknowledgements
This project was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah, under grant no. (KEP-PhD-69-130-38). The authors, therefore, acknowledge with thanks DSR technical and financial support. We also thank the reviewers for their positive remarks on our work.

Funding
This project was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah, Saudi Arabia under grant no. (KEP-PhD-69-130-38).

Abbreviations
Not applicable.

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Each of the authors, AA, BA, YA, and SKN contributed equally to each part of this work. All authors read and approved the final manuscript.

Author details
[1] Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah, Saudi Arabia. [2] Department of Mathematics, University of Ioannina, Ioannina, Greece.

Publisher's Note

Received: 3 September 2019 Accepted: 7 November 2019 Published online: 15 November 2019

References

1. Arafa, A.A.M., Rida, S.Z., Khalil, M.: Fractional modeling dynamics of HIV and CD4+ T-cells during primary infection. Nonlinear Biomed. Phys. **2012**, 6 (2012)
2. Ding, Y., Wang, Z., Ye, H.: Optimal control of a fractional-order HIV-immune system with memory. IEEE Trans. Control Syst. Technol. **20**, 763–769 (2012)
3. Carvalho, A., Pinto, C.M.A.: A delay fractional order model for the co-infection of malaria and HIV/AIDS. Int. J. Dyn. Control **5**, 168–186 (2017)

4. Sokolov, I.M., Klafter, J., Blumen, A.: Fractional kinetics. Phys. Today **55**, 48–54 (2002)
5. Yang, X.-J., Feng, Y.-Y., Cattani, C., Inc, M.: Fundamental solutions of anomalous diffusion equations with the decay exponential kernel. Math. Methods Appl. Sci. **42**(11), 4054–4060 (2019)
6. Ge, Z.M., Ou, C.Y.: Chaos synchronization of fractional order modified Duffing systems with parameters excited by a chaotic signal. Chaos Solitons Fractals **35**, 705–717 (2008)
7. Zhang, F., Chen, G., Li, C., Kurths, J.: Chaos synchronization in fractional differential systems. Philos. Trans. R. Soc. Lond. A **371**, 20120155 (2013)
8. Javidi, M., Ahmad, B.: Dynamic analysis of time fractional order phytoplankton-toxic phytoplankton–zooplankton system. Ecol. Model. **318**, 8–18 (2015)
9. Magin, R.L.: Fractional Calculus in Bioengineering. Begell House Publishers, Danbury (2006)
10. Zaslavsky, G.M.: Hamiltonian Chaos and Fractional Dynamics. Oxford University Press, Oxford (2005)
11. Fallahgoul, H.A., Focardi, S.M., Fabozzi, F.J.: Fractional Calculus and Fractional Processes with Applications to Financial Economics. Theory and Application. Elsevier/Academic Press, London (2017)
12. Yang, X.-J., Gao, F., Srivastava, H.: New rheological models within local fractional derivative. Rom. Rep. Phys. **69**(3), 113 (2017)
13. Senol, B., Yeroglu, C.: Frequency boundary of fractional order systems with nonlinear uncertainties. J. Franklin Inst. **350**, 1908–1925 (2013)
14. Ahmad, B., Ntouyas, S.K.: Existence results for a coupled system of Caputo type sequential fractional differential equations with nonlocal integral boundary conditions. Appl. Math. Comput. **266**, 615–622 (2015)
15. Henderson, J., Luca, R., Tudorache, A.: On a system of fractional differential equations with coupled integral boundary conditions. Fract. Calc. Appl. Anal. **18**, 361–386 (2015)
16. Wang, J.R., Zhang, Y.: Analysis of fractional order differential coupled systems. Math. Methods Appl. Sci. **38**, 3322–3338 (2015)
17. Ahmad, B., Ntouyas, S.K., Alsaedi, A.: On a coupled system of fractional differential equations with coupled nonlocal and integral boundary conditions. Chaos Solitons Fractals **83**, 234–241 (2016)
18. Ahmad, B., Nieto, J.J., Alsaedi, A., Aqlan, M.H.: A coupled system of Caputo-type sequential fractional differential equations with coupled (periodic/anti-periodic type) boundary conditions. Mediterr. J. Math. **14**, Article ID 227 (2017)
19. Muensawat, T., Ntouyas, S.K., Tariboon, J.: Systems of generalized Sturm–Liouville and Langevin fractional differential equations. Adv. Differ. Equ. **2017**, 63 (2017)
20. Ahmad, B., Alghamdi, N., Alsaedi, A., Ntouyas, S.K.: A system of coupled multi-term fractional differential equations with three-point coupled boundary conditions. Fract. Calc. Appl. Anal. **22**, 601–618 (2019)
21. Ahmad, B., Ntouyas, S.K., Alsaedi, A.: Fractional order differential systems involving right Caputo and left Riemann–Liouville fractional derivatives with nonlocal coupled conditions. Bound. Value Probl. **2019**, 109 (2019)
22. Ahmad, B., Alghamdi, N., Alsaedi, A., Ntouyas, S.K.: Existence theory for a system of coupled multi-term fractional differential equations with integral multi-strip coupled boundary conditions. Math. Methods Appl. Sci. (2019). https://doi.org/10.1002/mma.5788
23. Ahmad, B., Alruwaily, Y., Ntouyas, S.K., Alsaedi, A.: Existence and stability results for a fractional order differential equation with non-conjugate Riemann–Stieltjes integro-multipoint boundary conditions. Mathematics **7**, Article ID 249 (2019)
24. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. North-Holland Mathematics Studies, vol. 204. Elsevier Science B.V., Amsterdam (2006)
25. Yang, X.-J.: General Fractional Derivatives: Theory, Methods and Applications. Chapman and Hall/CRC, New York (2019)
26. Smart, D.R.: Fixed Point Theorems. Cambridge University Press, Cambridge (1980)