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Positive solutions to n-dimensional $\alpha_1 + \alpha_2$ order fractional differential system with p-Laplace operator

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Abstract

In this paper, we study an n-dimensional fractional differential system with p-Laplace operator, which involves multi-strip integral boundary conditions. By using the Leggett–Williams fixed point theorem, the existence results of at least three positive solutions are established. Besides, we also get the nonexistence results of positive solutions. Finally, two examples are presented to validate the main results.

Keywords: Positive solutions; Fractional differential equation; n-dimensional; p-Laplace operator; Multi-strip boundary conditions; The fixed point theorem

1 Introduction

Recently, there has been a rapid increase in researching fractional differential equations since their practical applications in various fields of physics, engineering, control theory, economics, etc. Fractional differential models can always make the description more accurate, and make the physical significance of parameters more explicit than the integer order ones. So, many monographs and literature works have appeared on fractional calculus and fractional differential equations, see [1–6].

It is well known that p-Laplace operator has deep background in analyzing mechanics, chemical physics, dynamic systems, etc. In the last ten years, fractional boundary value problems with p-Laplace operator have been widely studied, and there have been some excellent results on the existence, nonexistence, uniqueness, multiplicity of the solutions and positive solutions, we refer the readers to [7–14] and the references therein.

Meanwhile, boundary value problems with integral boundary conditions arise in lots of applied models [15–17] and some scholars have been interested in the BVP with the Riemann–Stieltjes integral boundary conditions, see [18–20]. Specially, multi-strip integral boundary value problems have drawn the attention of many scholars and have been extensively used in semiconductor, blood flow, hydrodynamics, etc., see [21–25].

In [23], Ahmad *et al.* investigated the following fractional differential equation:

$$^{c}D^{q}x(t) = f(t, x(t), {}^{c}D^{\beta}x(t)), \quad 0 < \beta < 1, 1 < q \le 2, t \in [0, 1], \tag{1.1}$$

supplemented with the boundary conditions of the form

$$\begin{cases} ax(0) + bx(1) = \sum_{i=1}^{m-2} \alpha_i x(\sigma_i) + \sum_{j=1}^{p-2} r_j \int_{\xi_j}^{\eta_j} x(s)\,ds, \\ cx'(0) + dx'(1) = \sum_{i=1}^{m-2} \delta_i x'(\sigma_i) + \sum_{j=1}^{p-2} \gamma_j \int_{\xi_j}^{\eta_j} x'(s)\,ds, \\ 0 < \sigma_1 < \cdots < \sigma_{m-2} < \cdots < \xi_1 < \eta_1 < \cdots < \xi_{p-2} < \eta_{p-2} < 1, \end{cases} \tag{1.2}$$

where ${}^cD^q$, ${}^cD^\beta$ denote the Caputo fractional derivatives of order q and β, respectively, f is a given continuous function, a, b, c, d are real constants, and α_i, δ_i $(i = 1, 2, \ldots, m-2)$, r_j, γ_j $(j = 1, 2, \ldots, p-2)$ are positive real constants. Several existence and uniqueness results are established by applying the tools of fixed-point theory.

Furthermore, n-dimensional differential systems are high generalizations of differential equations, which have broad application prospects and profound practical significance. However, n-dimensional differential systems have not been fully studied, and only a few results have been obtained (see [26–29] for instance); and the studies of n-dimensional fractional differential system boundary value problems are even fewer, see [29].

In [27], Feng *et al.* considered the following fourth-order n-dimensional m-Laplace system:

$$\begin{cases} \phi_m(\boldsymbol{x}''(t))'' = \boldsymbol{\Psi}(t)\boldsymbol{f}(t, \boldsymbol{x}(t)), \quad 0 < t < 1, \\ \boldsymbol{x}(0) = \boldsymbol{x}(1) = \int_0^1 \boldsymbol{g}(s)\boldsymbol{x}(s)\,ds, \\ \phi_m(\boldsymbol{x}''(0)) = \phi_m(\boldsymbol{x}''(1)) = \int_0^1 \boldsymbol{h}(s)\phi_m(\boldsymbol{x}''(s))\,ds, \end{cases} \tag{1.3}$$

where the vector-valued function $\boldsymbol{x}$ is defined by $\boldsymbol{x} = [x_1, x_2, \ldots, x_n]^T$. The authors investigated the existence, multiplicity, and nonexistence of symmetric positive solutions by the fixed point theorem in a cone and the inequality technique.

Inspired by the above achievements, we consider the following $\alpha_1 + \alpha_2$ fractional order n-dimensional p-Laplace system:

$$\begin{cases} D_{0+}^{\alpha_2}(\Phi_p(D_{0+}^{\alpha_1}\boldsymbol{u}(t))) = \kappa \boldsymbol{f}(t, \boldsymbol{u}(t), D_{0+}^{\alpha_1}\boldsymbol{u}(t)), \quad t \in (0,1), \\ \boldsymbol{u}(0) = \boldsymbol{0}, \qquad \boldsymbol{u}(1) = \sum_{i=1}^m b_i \int_0^{\xi_i} \boldsymbol{u}(s)\,d\boldsymbol{A}(s), \\ D_{0+}^{\alpha_1}\boldsymbol{u}(0) = \boldsymbol{0}, \qquad \Phi_p(D_{0+}^{\alpha_1}\boldsymbol{u}(1)) = \lambda \Phi_p(D_{0+}^{\alpha_1}\boldsymbol{u}(\eta)), \end{cases} \tag{1.4}$$

where $1 < \alpha_k \leq 2$, D^{α_k} is the standard Riemann–Liouville fractional derivative of order α_k for $k = 1, 2$; $\Phi_p(s) = |s|^{p-2}s$, $p > 1$; $\kappa > 0$; $0 < \xi_i < 1$, $b_i \geq 0$, $\int_0^{\xi_i} \boldsymbol{u}(s)\,d\boldsymbol{A}(s)$ denotes a Riemann–Stieltjes integral and $\boldsymbol{A}(s)$ is a matrix composed of functions of bounded variations for $i = 1, 2, \ldots, m$; $\lambda > 0$; $0 < \eta < 1$ and

$$\begin{aligned} &\boldsymbol{u}(t) = \bigl(u_1(t), u_2(t), \ldots, u_n(t)\bigr)^T, \\ &\boldsymbol{f}\bigl(t, \boldsymbol{u}, D_{0+}^{\alpha_1}\boldsymbol{u}\bigr) = \bigl(f_1\bigl(t, \boldsymbol{u}, D_{0+}^{\alpha_1}\boldsymbol{u}\bigr), f_2\bigl(t, \boldsymbol{u}, D_{0+}^{\alpha_1}\boldsymbol{u}\bigr), \ldots, f_n\bigl(t, \boldsymbol{u}, D_{0+}^{\alpha_1}\boldsymbol{u}\bigr)\bigr)^T, \\ &\Phi_p\bigl(D_{0+}^{\alpha_1}\boldsymbol{u}(t)\bigr) = \bigl(\Phi_p\bigl(D_{0+}^{\alpha_1}u_1(t)\bigr), \Phi_p\bigl(D_{0+}^{\alpha_1}u_2(t)\bigr), \ldots, \Phi_p\bigl(D_{0+}^{\alpha_1}u_n(t)\bigr)\bigr)^T, \\ &\boldsymbol{A}(s) = \operatorname{diag}\bigl[A_1(s), A_2(s), \ldots, A_n(s)\bigr]. \end{aligned}$$

Here, we should understand that $f_j(t, \boldsymbol{u}, D_{0+}^{\alpha_1}\boldsymbol{u})$ means $f_j(t, u_1, u_2, \ldots, u_n, D_{0+}^{\alpha_1}u_1, D_{0+}^{\alpha_1}u_2, \ldots, D_{0+}^{\alpha_1}u_n)$ for $j = 1, 2, \ldots, n$.

Therefore, system (1.4) means that

$$\begin{cases} \begin{pmatrix} D_{0+}^{\alpha_2}(\Phi_p(D_{0+}^{\alpha_1}u_1(t))) \\ D_{0+}^{\alpha_2}(\Phi_p(D_{0+}^{\alpha_1}u_2(t))) \\ \vdots \\ D_{0+}^{\alpha_2}(\Phi_p(D_{0+}^{\alpha_1}u_n(t))) \end{pmatrix} = \kappa \begin{pmatrix} f_1(t,\mathbf{u},D_{0+}^{\alpha_1}\mathbf{u}) \\ f_2(t,\mathbf{u},D_{0+}^{\alpha_1}\mathbf{u}) \\ \vdots \\ f_n(t,\mathbf{u},D_{0+}^{\alpha_1}\mathbf{u}) \end{pmatrix}, \quad \begin{pmatrix} u_1(0) \\ u_2(0) \\ \vdots \\ u_n(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \\ \begin{pmatrix} u_1(1) \\ u_2(1) \\ \vdots \\ u_n(1) \end{pmatrix} = \sum_{i=1}^m b_i \int_0^{\xi_i} \begin{pmatrix} u_1(s) \\ u_2(s) \\ \vdots \\ u_n(s) \end{pmatrix} d \begin{pmatrix} A_1(s) & 0 & \dots & 0 \\ 0 & A_2(s) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_n(s) \end{pmatrix}, \\ \begin{pmatrix} D_{0+}^{\alpha_1}u_1(0) \\ D_{0+}^{\alpha_1}u_2(0) \\ \vdots \\ D_{0+}^{\alpha_1}u_n(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \Phi_p(D_{0+}^{\alpha_1}u_1(1)) \\ \Phi_p(D_{0+}^{\alpha_1}u_2(1)) \\ \vdots \\ \Phi_p(D_{0+}^{\alpha_1}u_n(1)) \end{pmatrix} = \lambda \begin{pmatrix} \Phi_p(D_{0+}^{\alpha_1}u_1(\eta)) \\ \Phi_p(D_{0+}^{\alpha_1}u_2(\eta)) \\ \vdots \\ \Phi_p(D_{0+}^{\alpha_1}u_n(\eta)) \end{pmatrix}. \end{cases} \tag{1.5}$$

And then it follows respectively from (1.5) that

$$\begin{cases} D_{0+}^{\alpha_2}(\Phi_p(D_{0+}^{\alpha_1}u_1(t))) = \kappa f_1(t,u_1,u_2,\dots,u_n,D_{0+}^{\alpha_1}u_1,D_{0+}^{\alpha_1}u_2,\dots,D_{0+}^{\alpha_1}u_n), \\ D_{0+}^{\alpha_2}(\Phi_p(D_{0+}^{\alpha_1}u_2(t))) = \kappa f_2(t,u_1,u_2,\dots,u_n,D_{0+}^{\alpha_1}u_1,D_{0+}^{\alpha_1}u_2,\dots,D_{0+}^{\alpha_1}u_n), \\ \vdots \\ D_{0+}^{\alpha_2}(\Phi_p(D_{0+}^{\alpha_1}u_n(t))) = \kappa f_n(t,u_1,u_2,\dots,u_n,D_{0+}^{\alpha_1}u_1,D_{0+}^{\alpha_1}u_2,\dots,D_{0+}^{\alpha_1}u_n), \end{cases} \tag{1.6}$$

$$\begin{cases} u_1(0) = 0, \qquad u_1(1) = \sum_{i=1}^m b_i \int_0^{\xi_i} u_1(s)\,dA_1(s), \\ u_2(0) = 0, \qquad u_2(1) = \sum_{i=1}^m b_i \int_0^{\xi_i} u_2(s)\,dA_2(s), \\ \vdots \\ u_n(0) = 0, \qquad u_n(1) = \sum_{i=1}^m b_i \int_0^{\xi_i} u_n(s)\,dA_n(s), \end{cases} \tag{1.7}$$

$$\begin{cases} D_{0+}^{\alpha_1}u_1(0) = 0, \qquad \Phi_p(D_{0+}^{\alpha_1}u_1(1)) = \lambda\Phi_p(D_{0+}^{\alpha_1}u_1(\eta)), \\ D_{0+}^{\alpha_1}u_2(0) = 0, \qquad \Phi_p(D_{0+}^{\alpha_1}u_2(1)) = \lambda\Phi_p(D_{0+}^{\alpha_1}u_2(\eta)), \\ \vdots \\ D_{0+}^{\alpha_1}u_n(0) = 0, \qquad \Phi_p(D_{0+}^{\alpha_1}u_n(1)) = \lambda\Phi_p(D_{0+}^{\alpha_1}u_n(\eta)). \end{cases} \tag{1.8}$$

Our model has the following characteristics. Firstly, the equations are fractional derivative differential, if α_1 and α_2 both equal to 2, our equations degenerate into the model in [27]. Secondly, the nonlinear terms of the equations are related not only to the vector-valued function, but also to the derivative of vector-valued function. Thirdly, the boundary conditions are multi-point and multi-strip mixed boundary conditions.

In addition, we give the following assumptions ahead:

(F1) $f_j : [0,1] \times \mathbb{R}_+^n \times \mathbb{R}_-^n \to \mathbb{R}_+$ is continuous for $j = 1,2,\dots,n$;

(F2) $A_j(s)$ is a monotone nondecreasing function for $j = 1,2,\dots,n$;
Let $1 - \sum_{i=1}^m b_i \int_0^{\xi_i} s^{\alpha_1-1}\,dA_j(s) = \Delta_j$ satisfying $0 < \Delta_j < 1$ for $j = 1,2,\dots,n$;

(F3) $\lambda \ge 0$ with $0 < \lambda\eta^{\alpha_2-1} < 1$.

The structure of this paper is as follows. In Sect. 2, we give some necessary preliminaries, which will be used in the main proof. In Sect. 3, we establish the existence results of positive solutions by using the Leggett–Williams fixed point theorem. In Sect. 4, we investigate the nonexistence results of positive solutions. In Sect. 5, we illustrate two examples to demonstrate the main results.

2 Preliminaries

In this section, we consider the n-dimensional fractional order system (1.4) and put forward some indispensable definitions and theorems in advance.

Definition 2.1 The Riemann–Liouville fractional integral of order $\alpha > 0$ of a function $f : (0,\infty) \to \mathbb{R}$ is given by

$$I^{\alpha}_{0+}f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\,ds,$$

provided the right-hand side is pointwise defined on $(0,\infty)$, where $\Gamma(\alpha)$ is the Euler gamma function defined by $\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1}e^{-t}\,dt$ for $\alpha > 0$.

Definition 2.2 The Riemann–Liouville fractional derivative of order $\alpha > 0$ for a function $f : (0,\infty) \to \mathbb{R}$ is given by

$$D^{\alpha}_{0+}f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} f(s)\,ds,$$

where $n = [\alpha] + 1$, $[\alpha]$ stands for the largest integer not greater than α.

According to the definition of Riemann–Liouville's derivative, the following lemmas can be achieved.

Lemma 2.1 *For $\alpha > 0$, if we assume that $u \in C[0,\infty) \cap L^1(0,1)$, then we have*

$$I^{\alpha}_{0+}\bigl(D^{\alpha}_{0+}u(t)\bigr) = u(t) + m_1 t^{\alpha-1} + m_2 t^{\alpha-2} + \cdots + m_n t^{\alpha-n}$$

for some $m_i \in \mathbb{R}, i = 1,2,\ldots,n$, while n is the smallest integer greater than or equal to α.

Definition 2.3 Let E be a real Banach space. A nonempty, closed, and convex set $K \subset E$ is a cone if the following two conditions are satisfied:

(1) if $x \in K$ and $\mu \geq 0$, then $\mu x \in K$;

(2) if $x \in K$ and $-x \in K$, then $x = 0$.

Every cone $K \subset E$ induces the ordering in E given by $x_1 \leq x_2$ if and only if $x_2 - x_1 \in K$.

Definition 2.4 The map γ is said to be a continuous nonnegative convex (concave) function on a cone K of a real Banach space E provided that $\gamma : K \to [0,\infty)$ is continuous and

$$\gamma\bigl(tx + (1-t)y\bigr) \leq (\geq) t\gamma(x) + (1-t)\gamma(y), \quad x,y \in K, t \in [0,1].$$

For $h_j(t) \in C(0,1) \cap L^1(0,1)$, $j = 1, 2, \ldots, n$, we consider a component of the corresponding linearization problem according to (1.6)–(1.8):

$$\begin{cases} D_{0+}^{\alpha_2}(\Phi_p(D_{0+}^{\alpha_1} u_j(t))) = h_j(t), & t \in (0,1), \\ u_j(0) = 0, & u_j(1) = \sum_{i=1}^m b_i \int_0^{\xi_i} u_j(s)\,dA_j(s), \\ D_{0+}^{\alpha_1} u_j(0) = 0, & \Phi_p(D_{0+}^{\alpha_1} u_j(1)) = \lambda \Phi_p(D_{0+}^{\alpha_1} u_j(\eta)). \end{cases} \tag{2.1}$$

By means of the transformation

$$\Phi_p\bigl(D_{0+}^{\alpha_1} u_j(t)\bigr) = -v_j(t), \tag{2.2}$$

we can convert equation (2.1) into

$$\begin{cases} D_{0+}^{\alpha_1} u_j(t) = -\Phi_q(v_j(t)), & t \in (0,1), \\ u_j(0) = 0, & u_j(1) = \sum_{i=1}^m b_i \int_0^{\xi_i} u_j(s)\,dA_j(s) \end{cases} \tag{2.3}$$

and

$$\begin{cases} D_{0+}^{\alpha_2} v_j(t) = -h_j(t), & t \in (0,1), \\ v_j(0) = 0, & v_j(1) = \lambda v_j(\eta), \end{cases} \tag{2.4}$$

where $\Phi_q = \Phi_p^{-1}$, $\frac{1}{p} + \frac{1}{q} = 1$.

For $k = 1, 2$, define the Green's function as follows:

$$G_k(t,s) = \frac{1}{\Gamma(\alpha_k)} \begin{cases} t^{\alpha_k - 1}(1-s)^{\alpha_k - 1} - (t-s)^{\alpha_k - 1}, & 0 \le s \le t \le 1, \\ t^{\alpha_k - 1}(1-s)^{\alpha_k - 1}, & 0 \le t \le s \le 1. \end{cases} \tag{2.5}$$

Lemma 2.2 *Boundary value problem* (2.3) *has a unique solution*

$$u_j(t) = \int_0^1 H_j(t,s)\Phi_q\bigl(v_j(s)\bigr)\,ds, \tag{2.6}$$

where

$$H_j(t,s) = G_1(t,s) + \frac{t^{\alpha_1 - 1}}{\Delta_j} \sum_{i=1}^m b_i \int_0^{\xi_i} G_1(\tau,s)\,dA_j(\tau), \tag{2.7}$$

and $G_1(t,s)$ *is given by* (2.5) *for* $k = 1$.

Proof From Lemma 2.1, we can reduce $D_{0+}^{\alpha_1} u_j(t) = -\Phi_q(v_j(t))$ to the following equivalent equation:

$$u_j(t) = -\frac{1}{\Gamma(\alpha_1)} \int_0^t (t-s)^{\alpha_1 - 1}\Phi_q\bigl(v_j(s)\bigr)\,ds + c_1 t^{\alpha_1 - 1} + c_2 t^{\alpha_1 - 2}, \tag{2.8}$$

where c_1 and c_2 are arbitrary real constants.

According to $u_j(0) = 0$, we have $c_2 = 0$, thus

$$u_j(1) = -\frac{1}{\Gamma(\alpha_1)} \int_0^1 (1-s)^{\alpha_1 - 1} \Phi_q\big(v_j(s)\big)\, ds + c_1,$$

with $u_j(1) = \sum_{i=1}^m b_i \int_0^{\xi_i} u_j(s)\, dA_j(s)$, we get

$$c_1 = \frac{1}{\Gamma(\alpha_1)} \int_0^1 (1-s)^{\alpha_1 - 1} \Phi_q\big(v_j(s)\big)\, ds + \sum_{i=1}^m b_i \int_0^{\xi_i} u_j(s)\, dA_j(s)$$

and

$$\begin{aligned} u_j(t) &= -\frac{1}{\Gamma(\alpha_1)} \int_0^t (t-s)^{\alpha_1 - 1} \Phi_q\big(v_j(s)\big)\, ds \\ &\quad + \frac{1}{\Gamma(\alpha_1)} \int_0^1 t^{\alpha_1 - 1}(1-s)^{\alpha_1 - 1} \Phi_q\big(v_j(s)\big)\, ds \\ &\quad + t^{\alpha_1 - 1} \sum_{i=1}^m b_i \int_0^{\xi_i} u_j(s)\, dA_j(s) \\ &= \int_0^1 G_1(t,s) \Phi_q\big(v_j(s)\big)\, ds + t^{\alpha_1 - 1} \sum_{i=1}^m b_i \int_0^{\xi_i} u_j(s)\, dA_j(s), \end{aligned} \tag{2.9}$$

where $G_1(t,s)$ is given by (2.5). Because of

$$\begin{aligned} &\sum_{i=1}^m b_i \int_0^{\xi_i} u_j(\tau)\, dA_j(\tau) \\ &\quad = \sum_{i=1}^m b_i \int_0^{\xi_i} \left[\int_0^1 G_1(\tau,s) \Phi_q\big(v_j(s)\big)\, ds + \tau^{\alpha_1 - 1} \sum_{i=1}^m b_i \int_0^{\xi_i} u_j(s)\, dA_j(s) \right] dA_j(\tau) \\ &\quad = \sum_{i=1}^m b_i \int_0^{\xi_i} \int_0^1 G_1(\tau,s) \Phi_q\big(v_j(s)\big)\, ds\, dA_j(\tau) \\ &\qquad + \sum_{i=1}^m b_i \int_0^{\xi_i} \tau^{\alpha_1 - 1}\, dA_j(\tau) \sum_{i=1}^m b_i \int_0^{\xi_i} u_j(s)\, dA_j(s), \end{aligned}$$

we get

$$\begin{aligned} &\sum_{i=1}^m b_i \int_0^{\xi_i} u_j(\tau)\, dA_j(\tau) \\ &\quad = \frac{1}{\Delta_j} \sum_{i=1}^m b_i \int_0^{\xi_i} \int_0^1 G_1(\tau,s) \Phi_q\big(v_j(s)\big)\, ds\, dA_j(\tau). \end{aligned} \tag{2.10}$$

Thus, we have

$$u_j(t) = \int_0^1 G_1(t,s) \Phi_q\big(v_j(s)\big)\, ds + \frac{t^{\alpha_1 - 1}}{\Delta_j} \sum_{i=1}^m b_i \int_0^{\xi_i} \int_0^1 G_1(\tau,s) \Phi_q\big(v_j(s)\big)\, ds\, dA_j(\tau)$$

$$= \int_0^1 H_j(t,s)\Phi_q\big(v_j(s)\big)\,ds, \tag{2.11}$$

where $H_j(t,s)$ is given by (2.7).

This completes the proof of the lemma. □

Lemma 2.3 *Boundary value problem* (2.4) *has a unique solution*

$$v_j(t) = \int_0^1 H(t,s)h_j(s)\,ds, \tag{2.12}$$

where

$$H(t,s) = G_2(t,s) + \frac{\lambda t^{\alpha_2-1}}{1-\lambda\eta^{\alpha_2-1}}G_2(\eta,s), \tag{2.13}$$

and $G_2(t,s)$ *is given by* (2.5) *for* $k=2$.

Proof From Lemma 2.1, we can reduce $D_{0+}^{\alpha_2}v_j(t) = -h_j(t)$ to

$$v_j(t) = -\frac{1}{\Gamma(\alpha_2)}\int_0^t (t-s)^{\alpha_2-1}h_j(s)\,ds + d_1t^{\alpha_2-1} + d_2t^{\alpha_2-2}, \tag{2.14}$$

where d_1 and d_2 are arbitrary real constants.

According to $v_j(0)=0$, we have $d_2=0$. Thus

$$v_j(1) = -\frac{1}{\Gamma(\alpha_2)}\int_0^1 (1-s)^{\alpha_2-1}h_j(s)\,ds + d_1,$$

with $v_j(1) = \lambda v_j(\eta)$, we get

$$d_1 = \frac{1}{\Gamma(\alpha_2)}\int_0^1 (1-s)^{\alpha_2-1}h_j(s)\,ds + \lambda v_j(\eta)$$

and

$$\begin{aligned} v_j(t) &= -\frac{1}{\Gamma(\alpha_2)}\int_0^t (t-s)^{\alpha_2-1}h_j(s)\,ds + \frac{1}{\Gamma(\alpha_2)}\int_0^1 t^{\alpha_2-1}(1-s)^{\alpha_2-1}h_j(s)\,ds + \lambda t^{\alpha_2-1}v_j(\eta) \\ &= \int_0^1 G_2(t,s)h_j(s)\,ds + \lambda t^{\alpha_2-1}v_j(\eta), \end{aligned} \tag{2.15}$$

where $G_2(t,s)$ is given by (2.5).

From $v_j(\eta) = \frac{1}{1-\lambda\eta^{\alpha_2-1}}\int_0^1 G_2(\eta,s)h_j(s)\,ds$, we get

$$\begin{aligned} v_j(t) &= \int_0^1 G_2(t,s)h_j(s)\,ds + \frac{\lambda t^{\alpha_2-1}}{1-\lambda\eta^{\alpha_2-1}}\int_0^1 G_2(\eta,s)h_j(s)\,ds \\ &= \int_0^1 H(t,s)h_j(s)\,ds, \end{aligned} \tag{2.16}$$

where $H(t,s)$ is given by (2.13).

This completes the proof of the lemma. □

Above all, equation (2.1) has the unique solution

$$u_j(t) = \int_0^1 H_j(t,s)\Phi_q\left(\int_0^1 H(s,\tau)h_j(\tau)\,d\tau\right)ds, \quad j = 1,2,\ldots,n. \tag{2.17}$$

Next, we present some properties of $G_1(t,s)$, $G_2(t,s)$, $H_j(t,s)$, and $H(t,s)$.

Lemma 2.4 *Suppose that θ is a positive constant satisfying $0 < \theta < \frac{1}{2} < 1-\theta < 1$, then $G_1(t,s)$, $G_2(t,s)$, $H_j(t,s)$, and $H(t,s)$ satisfy the following properties*:

(a) *For $t,s \in [0,1]$, $k = 1,2$, $0 \le G_k(t,s) \le \frac{1}{\Gamma(\alpha_k)}(1-s)^{\alpha_k-1}$;*

(b) *For $t,s \in [\theta,1-\theta]$, $k = 1,2$,*

$$\frac{1}{\Gamma(\alpha_k-1)}t^{\alpha_k-1}(1-s)^{\alpha_k-1}(1-t)s \le G_k(t,s) \le \frac{1}{\Gamma(\alpha_k)}(1-s)^{\alpha_k-1};$$

(c) *For $t,s \in [\theta,1-\theta]$, $j = 1,2,\ldots,n$,*

$$\rho_j\frac{M_j}{\Gamma(\alpha_1)}(1-s)^{\alpha_1-1} \le H_j(t,s) \le \frac{M_j}{\Gamma(\alpha_1)}(1-s)^{\alpha_1-1},$$
$$\frac{1}{\Gamma(\alpha_2-1)}t^{\alpha_2-1}(1-s)^{\alpha_2-1}(1-t)s \le H(t,s) \le \frac{M}{\Gamma(\alpha_2)}(1-s)^{\alpha_2-1},$$

where

$$M = 1 + \frac{\lambda}{1-\lambda\eta^{\alpha_2-1}}, \qquad \rho_j = \frac{\alpha_1-1}{M_j}\theta^2(1-\theta),$$
$$M_j = 1 + \frac{1}{\Delta_j}\sum_{i=1}^m b_i\int_0^{\xi_i} dA_j(s), \quad j = 1,2,\ldots,n.$$

Proof (a) For $0 \le s \le t \le 1$, we have

$$\begin{aligned} G_k(t,s) &= \frac{1}{\Gamma(\alpha_k)}\left[t^{\alpha_k-1}(1-s)^{\alpha_k-1} - (t-s)^{\alpha_k-1}\right] \\ &= \frac{1}{\Gamma(\alpha_k)}\left[(t-ts)^{\alpha_k-1} - (t-s)^{\alpha_k-1}\right] \\ &\ge 0; \\ G_k(t,s) &= \frac{1}{\Gamma(\alpha_k)}\left[t^{\alpha_k-1}(1-s)^{\alpha_k-1} - (t-s)^{\alpha_k-1}\right] \\ &\le \frac{1}{\Gamma(\alpha_k)}t^{\alpha_k-1}(1-s)^{\alpha_k-1} \\ &\le \frac{1}{\Gamma(\alpha_k)}(1-s)^{\alpha_k-1}. \end{aligned}$$

For $0 \le t \le s \le 1$, we have

$$G_k(t,s) = \frac{1}{\Gamma(\alpha_k)}t^{\alpha_k-1}(1-s)^{\alpha_k-1} \ge 0;$$

$$
\begin{aligned}
G_k(t,s) &= \frac{1}{\Gamma(\alpha_k)} t^{\alpha_k-1}(1-s)^{\alpha_k-1} \\
&\leq \frac{1}{\Gamma(\alpha_k)}(1-s)^{\alpha_k-1}.
\end{aligned}
$$

(b) For $\theta \leq s \leq t \leq 1-\theta$, we have

$$
\begin{aligned}
G_k(t,s) &= \frac{1}{\Gamma(\alpha_k)}\bigl[t^{\alpha_k-1}(1-s)^{\alpha_k-1} - (t-s)^{\alpha_k-1}\bigr] \\
&= \frac{\alpha_k-1}{\Gamma(\alpha_k)}\int_{t-s}^{t(1-s)} x^{\alpha_k-2}\,dx \\
&\geq \frac{1}{\Gamma(\alpha_k-1)}\bigl[t(1-s)\bigr]^{\alpha_k-2}\bigl[t(1-s)-(t-s)\bigr] \\
&\geq \frac{1}{\Gamma(\alpha_k-1)} t^{\alpha_k-1}(1-s)^{\alpha_k-1}(1-t)s.
\end{aligned}
$$

For $\theta \leq t \leq s \leq 1-\theta$, we have

$$
\begin{aligned}
G_k(t,s) &= \frac{1}{\Gamma(\alpha_k)} t^{\alpha_k-1}(1-s)^{\alpha_k-1} \\
&\geq \frac{1}{\Gamma(\alpha_k)} t^{\alpha_k-1}(1-s)^{\alpha_k-1}(1-t)s \\
&\geq \frac{1}{\Gamma(\alpha_k-1)} t^{\alpha_k-1}(1-s)^{\alpha_k-1}(1-t)s.
\end{aligned}
$$

(c) For $t,s \in [0,1]$, we have

$$
\begin{aligned}
H_j(t,s) &= G_1(t,s) + \frac{t^{\alpha_1-1}}{\Delta_j}\sum_{i=1}^{m} b_i \int_0^{\xi_i} G_1(\tau,s)\,dA_j(\tau) \\
&\leq \frac{1}{\Gamma(\alpha_1)}(1-s)^{\alpha_1-1} + \frac{1}{\Delta_j}\sum_{i=1}^{m} b_i \int_0^{\xi_i} \frac{1}{\Gamma(\alpha_1)}(1-s)^{\alpha_1-1}\,dA_j(\tau) \\
&= \frac{M_j}{\Gamma(\alpha_1)}(1-s)^{\alpha_1-1},
\end{aligned}
\tag{2.18}
$$

for $t,s \in [\theta, 1-\theta]$, we have

$$
\begin{aligned}
H_j(t,s) &= G_1(t,s) + \frac{t^{\alpha_1-1}}{\Delta_j}\sum_{i=1}^{m} b_i \int_0^{\xi_i} G_1(\tau,s)\,dA_j(\tau) \\
&\geq G_1(t,s) \geq \frac{\alpha_1-1}{\Gamma(\alpha_1)} t^{\alpha_1-1}(1-s)^{\alpha_1-1}(1-t)s \\
&\geq \frac{\alpha_1-1}{M_j} t(1-t)s \frac{M_j}{\Gamma(\alpha_1)}(1-s)^{\alpha_1-1} \\
&\geq \frac{\alpha_1-1}{M_j}\theta^2(1-\theta)\frac{M_j}{\Gamma(\alpha_1)}(1-s)^{\alpha_1-1} \\
&= \frac{\rho_j M_j}{\Gamma(\alpha_1)}(1-s)^{\alpha_1-1}.
\end{aligned}
\tag{2.19}
$$

And for $t, s \in [0,1]$, we have

$$\begin{aligned} H(t,s) &= G_2(t,s) + \frac{\lambda t^{\alpha_2-1}}{1-\lambda\eta^{\alpha_2-1}} G_2(\eta,s) \\ &\leq \frac{1}{\Gamma(\alpha_2)}(1-s)^{\alpha_2-1} + \frac{\lambda}{1-\lambda\eta^{\alpha_2-1}} \frac{1}{\Gamma(\alpha_2)}(1-s)^{\alpha_2-1} \\ &\leq \frac{M}{\Gamma(\alpha_2)}(1-s)^{\alpha_2-1}, \end{aligned}$$

for $t, s \in [\theta, 1-\theta]$, we have

$$\begin{aligned} H(t,s) &= G_2(t,s) + \frac{\lambda t^{\alpha_2-1}}{1-\lambda\eta^{\alpha_2-1}} G_2(\eta,s) \\ &\geq G_2(t,s) \geq \frac{1}{\Gamma(\alpha_2-1)} t^{\alpha_2-1}(1-s)^{\alpha_2-1}(1-t)s. \end{aligned}$$

Then the proof is completed. □

Let $J = [0,1]$, $I = [\theta, 1-\theta]$, $E = \{u_j(t) | u_j(t) \in C[0,1]$ and $D_{0^+}^{\alpha_1} u_j(t) \in C[0,1], j = 1,2,\ldots,n\}$, $U = \underbrace{E \times E \times \cdots \times E}_{n}$ for all $\boldsymbol{u} = (u_1, u_2, \ldots, u_n)^T \in U$, define the norms as follows:

$$\|\boldsymbol{u}\|_1 = \sum_{j=1}^{n} \sup_{t\in J} |u_j(t)|, \qquad \|\boldsymbol{u}\|_2 = \sum_{j=1}^{n} \sup_{t\in J} |D_{0^+}^{\alpha_1} u_j(t)|, \qquad \|\boldsymbol{u}\| = \max\{\|\boldsymbol{u}\|_1, \|\boldsymbol{u}\|_2\}.$$

Then $(U, \|\cdot\|)$ is a real Banach space.

Define set K in U by

$$K = \left\{ \boldsymbol{u} \in U : u_j(t) \geq 0, D_{0^+}^{\alpha_1} u_j(t) \leq 0, \min_{t\in I} \sum_{j=1}^{n} u_j(t) \geq \rho \|\boldsymbol{u}\|_1, j = 1,2,\ldots,n \right\}, \tag{2.20}$$

where

$$\rho_j = \frac{\alpha_1 - 1}{M_j} \theta^2 (1-\theta), \qquad \rho = \min_{1\leq j\leq n} \rho_j.$$

From $M_j \geq 1$, we can get $0 < \rho < 1$.

For $\boldsymbol{u}, \boldsymbol{v} \in K$ and $m_1, m_2 \geq 0$, it is not difficult to see that

$$m_1 \boldsymbol{u}(t) + m_2 \boldsymbol{v}(t) \geq 0, \qquad D_{0^+}^{\alpha_1}\bigl(m_1 \boldsymbol{u}(t) + m_2 \boldsymbol{v}(t)\bigr) = m_1 D_{0^+}^{\alpha_1} \boldsymbol{u}(t) + m_2 D_{0^+}^{\alpha_1} \boldsymbol{v}(t) \leq 0,$$

and

$$\begin{aligned} &\min_{t\in I} \left\{ \sum_{j=1}^{n} m_1 u_j(t) + \sum_{j=1}^{n} m_2 v_j(t) \right\} \\ &\quad \geq \min_{t\in I} \left\{ \sum_{j=1}^{n} m_1 u_j(t) \right\} + \min_{t\in I} \left\{ \sum_{j=1}^{n} m_2 v_j(t) \right\} \end{aligned}$$

$$\geq \rho m_1\|\boldsymbol{u}\|_1 + \rho m_2\|\boldsymbol{v}\|_1 = \rho\big(m_1\|\boldsymbol{u}\|_1 + m_2\|\boldsymbol{v}\|_1\big)$$
$$\geq \rho\|m_1\boldsymbol{u} + m_2\boldsymbol{v}\|_1. \tag{2.21}$$

Thus, for $\boldsymbol{u},\boldsymbol{v} \in K$ and $m_1,m_2 \geq 0$, $m_1\boldsymbol{u} + m_2\boldsymbol{v} \in K$. And if $\boldsymbol{u} \in K$, $\boldsymbol{u} \neq 0$, it is easy to prove that $-\boldsymbol{u} \notin K$. Therefore, K is a cone in U.

Let $\boldsymbol{T} : K \to U$ be a map with components $T_1,\dots,T_j,\dots,T_n$. Here we understand $\boldsymbol{Tu} = (T_1\boldsymbol{u},\dots,T_j\boldsymbol{u},\dots,T_n\boldsymbol{u})^T$, where

$$(T_j\boldsymbol{u})(t) = \int_0^1 H_j(t,s)\Phi_q\left(\int_0^1 H(s,\tau)\kappa f_j\big(\tau,\boldsymbol{u}(\tau),D_{0^+}^{\alpha_1}\boldsymbol{u}(\tau)\big)\,d\tau\right)ds. \tag{2.22}$$

From Lemma 2.2 and Lemma 2.3, we have the following remark.

Remark 2.1 From (2.22), we know that $\boldsymbol{u} \in U$ is a solution of system (1.4) if and only if $\boldsymbol{u}$ is a fixed point of the map $\boldsymbol{T}$.

Lemma 2.5 $\boldsymbol{T} : K \to K$ *is completely continuous.*

Proof For all $\boldsymbol{u} \in K$, by the continuity and nonnegativity of $f_j(t,\boldsymbol{u}(t),D_{0^+}^{\alpha_1}\boldsymbol{u}(t))$, $H_j(t,s)$ and $H(t,s)$, $\boldsymbol{T}$ is continuous and $(\boldsymbol{Tu})(t) \geq 0$, $D_{0^+}^{\alpha_1}(\boldsymbol{Tu})(t) \leq 0$. Furthermore, from (2.18) and (2.19), we have

$$\begin{aligned}
&\min_{t\in I}\sum_{j=1}^n (T_j\boldsymbol{u})(t)\\
&= \min_{t\in I}\sum_{j=1}^n \int_0^1 H_j(t,s)\Phi_q\left(\int_0^1 H(s,\tau)\kappa f_j\big(\tau,\boldsymbol{u}(\tau),D_{0^+}^{\alpha_1}\boldsymbol{u}(\tau)\big)\,d\tau\right)ds\\
&\geq \sum_{j=1}^n \int_0^1 \min_{t\in I} H_j(t,s)\Phi_q\left(\int_0^1 H(s,\tau)\kappa f_j\big(\tau,\boldsymbol{u}(\tau),D_{0^+}^{\alpha_1}\boldsymbol{u}(\tau)\big)\,d\tau\right)ds\\
&= \sum_{j=1}^n \int_0^1 \frac{\rho_j M_j}{\Gamma(\alpha_1)}(1-s)^{\alpha_1-1}\Phi_q\left(\int_0^1 H(s,\tau)\kappa f_j\big(\tau,\boldsymbol{u}(\tau),D_{0^+}^{\alpha_1}\boldsymbol{u}(\tau)\big)\,d\tau\right)ds\\
&\geq \sum_{j=1}^n \int_0^1 \rho\sup_{t\in J} H_j(t,s)\Phi_q\left(\int_0^1 H(s,\tau)\kappa f_j\big(\tau,\boldsymbol{u}(\tau),D_{0^+}^{\alpha_1}\boldsymbol{u}(\tau)\big)\,d\tau\right)ds\\
&\geq \rho\sum_{j=1}^n \sup_{t\in J}\big|T_j\boldsymbol{u}(t)\big| = \rho\big\|T_j\boldsymbol{u}(t)\big\|_1.
\end{aligned} \tag{2.23}$$

We can get $(T_j\boldsymbol{u})(K) \subseteq K$ for $j = 1,2,\dots,n$, thus $(\boldsymbol{Tu})(K) \subseteq K$.

Then, in order to show $\boldsymbol{T}$ is uniformly bounded, we show T_j is uniformly bounded. Let D be a bounded closed convex set in K, i.e., there exists a positive constant l such that $\|\boldsymbol{u}\| \leq l$. Let $M_0^j = \sup_{t\in J}\{f_j(t,\boldsymbol{u}(t),D_{0^+}^{\alpha_1}\boldsymbol{u}(t))|\boldsymbol{u} \in U, \|\boldsymbol{u}\| \leq l\} > 0$. For all $(\boldsymbol{u}_m)_{m\in\mathbb{N}} \in D$, we have

$$\begin{aligned}
&\big|(T_j\boldsymbol{u}_m)(t)\big|\\
&= \left|\int_0^1 H_j(t,s)\Phi_q\left(\int_0^1 H(s,\tau)\kappa f_j\big(\tau,\boldsymbol{u}_m(\tau),D_{0^+}^{\alpha_1}\boldsymbol{u}_m(\tau)\big)\,d\tau\right)ds\right|
\end{aligned}$$

$$
\begin{aligned}
&\leq \int_0^1 \left|H_j(t,s)\right| \Phi_q\left(\int_0^1 \left|H(s,\tau)\right|\left|\kappa f_j\left(\tau, \boldsymbol{u}_m(\tau), D_{0^+}^{\alpha_1}\boldsymbol{u}_m(\tau)\right)\right| d\tau\right) ds \\
&\leq \int_0^1 \frac{M_j}{\Gamma(\alpha_1)}(1-s)^{\alpha_1-1} \Phi_q\left(\int_0^1 \frac{M}{\Gamma(\alpha_2)}(1-\tau)^{\alpha_2-1}\kappa M_0^j \, d\tau\right) ds \\
&\leq \frac{M_j}{\Gamma(\alpha_1)}\left(\frac{\kappa M M_0^j}{\Gamma(\alpha_2)}\right)^{q-1} \int_0^1 (1-s)^{\alpha_1-1} \Phi_q\left(\int_0^1 (1-\tau)^{\alpha_2-1} \, d\tau\right) ds \\
&= \frac{M_j}{\Gamma(\alpha_1+1)}\left(\frac{\kappa M M_0^j}{\Gamma(\alpha_2+1)}\right)^{q-1} := N_1^j.
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
&\left|D_{0^+}^{\alpha_1}(T_j\boldsymbol{u}_m)(t)\right| \\
&\quad = \left|-\Phi_q\left(\int_0^1 H(t,s)\kappa f_j\left(s, \boldsymbol{u}_m(s), D_{0^+}^{\alpha_1}\boldsymbol{u}_m(s)\right) ds\right)\right| \\
&\quad = \Phi_q\left(\int_0^1 H(t,s)\kappa f_j\left(s, \boldsymbol{u}_m(s), D_{0^+}^{\alpha_1}\boldsymbol{u}_m(s)\right) ds\right) \\
&\quad \leq \Phi_q\left(\int_0^1 \frac{M}{\Gamma(\alpha_2)}(1-s)^{\alpha_2-1}\kappa M_0^j \, ds\right) \\
&\quad = \left(\frac{\kappa M M_0^j}{\Gamma(\alpha_2+1)}\right)^{q-1} := N_2^j.
\end{aligned}
$$

Thus, $\|T_j\boldsymbol{u}_m\| \leq \max\{N_1^j, N_2^j\}$, which implies that $T_j(D)$ is uniformly bounded.

Then we show $(T_j\boldsymbol{u}_m)(t)_{m\in\mathbb{N}}$ is equicontinuous. Because $H_j(t,s)$ is continuous on $J\times J$, $H_j(t,s)$ is uniformly continuous on $J\times J$, so for any $\varepsilon > 0$, there exists $\delta_1 > 0$ such that, for $t_1, t_2 \in J$ with $|t_1 - t_2| < \delta_1$, $|H_j(t_1,s) - H_j(t_2,s)| < \varepsilon_1[\frac{\kappa M M_0^j}{\Gamma(\alpha_2+1)}]^{1-q}$. We can infer that

$$
\begin{aligned}
&\left|(T_j\boldsymbol{u}_m)(t_2) - (T_j\boldsymbol{u}_m)(t_1)\right| \\
&\quad \leq \int_0^1 \left|H_j(t_2,s) - H_j(t_1,s)\right| \Phi_q\left(\int_0^1 H(s,\tau)\kappa f_j\left(\tau, \boldsymbol{u}_m(\tau)\right), D_{0^+}^{\alpha_1}\boldsymbol{u}_m(\tau)\right) d\tau) \, ds \\
&\quad \leq \left[\frac{\kappa M M_0^j}{\Gamma(\alpha_2+1)}\right]^{q-1} \int_0^1 \left|H_j(t_2,s) - H_j(t_1,s)\right| ds \\
&\quad < \varepsilon_1.
\end{aligned}
$$

On the other hand, from $H(t,s)$ is continuous on $J\times J$, we know $H(t,s)$ is uniformly continuous on $J\times J$, then for any $\varepsilon > 0$, there exists $\delta_2 > 0$ such that, for any $t_1, t_2 \in J$ and $|t_1 - t_2| < \delta_2$, we have $|H(t_1,s) - H(t_2,s)| < \delta_3(\kappa M_0^j)^{-1}$. Hence,

$$
\begin{aligned}
&\left|\int_0^1 H(t_2,s)\kappa f_j\left(s, \boldsymbol{u}_m(s), D_{0^+}^{\alpha_1}\boldsymbol{u}_m(s)\right) ds - \int_0^1 H(t_1,s)\kappa f_j\left(s, \boldsymbol{u}_m(s), D_{0^+}^{\alpha_1}\boldsymbol{u}_m(s)\right) ds\right| \\
&\quad \leq \int_0^1 \left|H(t_2,s) - H(t_1,s)\right| \kappa f_j\left(s, \boldsymbol{u}_m(s), D_{0^+}^{\alpha_1}\boldsymbol{u}_m(s)\right) ds \\
&\quad \leq \delta_3.
\end{aligned}
$$

Because $\Phi_q(s)$ is continuous, when $|s_2 - s_1| < \delta_3$, we have $|\Phi_p(s_2) - \Phi_p(s_1)| < \varepsilon_2$, thus,

$$\begin{aligned}
&\left|D_{0^+}^{\alpha_1}(T_j\boldsymbol{u}_m)(t_2) - D_{0^+}^{\alpha_1}(T_j\boldsymbol{u}_m)(t_1)\right| \\
&\quad = \left|-\Phi_q\left(\int_0^1 H(t_2,s)\kappa f_j\bigl(s,\boldsymbol{u}_m(s),D_{0^+}^{\alpha_1}\boldsymbol{u}_m(s)\bigr)\,ds\right)\right. \\
&\qquad \left.+ \Phi_q\left(\int_0^1 H(t_1,s)\kappa f_j\bigl(s,\boldsymbol{u}_m(s),D_{0^+}^{\alpha_1}\boldsymbol{u}_m(s)\bigr)\,ds\right)\right| \\
&\quad < \varepsilon_2.
\end{aligned}$$

Therefore, it follows from the Arzelà–Ascoli theorem that $(T_j\boldsymbol{u}_m)_{m\in\mathbb{N}}$ is compact on J.

Finally, we will prove the continuity of T_j. Let $(\boldsymbol{u}_m)_{m\in\mathbb{N}}$ be any sequence converging on K to $\boldsymbol{u} \in K$, and let $S > 0$ be such that $\|\boldsymbol{u}_m\| \le S$ for all $m \in \mathbb{N}$. Note that $f_j(t,\boldsymbol{u},D_{0^+}^{\alpha_1}\boldsymbol{u})$ is continuous on $J \times K_S$. It is easy to see that the dominated convergence theorem guarantees that

$$\lim_{m\to\infty}(T_j\boldsymbol{u}_m)(t) = (T_j\boldsymbol{u})(t) \tag{2.24}$$

and

$$\lim_{m\to\infty} D_{0^+}^{\alpha_1}(T_j\boldsymbol{u}_m)(t) = D_{0^+}^{\alpha_1}(T_j\boldsymbol{u})(t), \tag{2.25}$$

for each $t \in J$. Moreover, the compactness of T_j implies that $(T_j\boldsymbol{u}_m)(t)$ converges uniformly to $(T_j\boldsymbol{u})(t)$ on J. If not, then there exist $\varepsilon_0 > 0$ and a subsequence $(\boldsymbol{u}_{m_k})_{k\in\mathbb{N}}$ of $(\boldsymbol{u}_m)_{m\in\mathbb{N}}$ such that

$$\sup_{t\in J}\left|(T_j\boldsymbol{u}_{m_k})(t) - (T_j\boldsymbol{u})(t)\right| \ge \varepsilon_0, \quad k \in \mathbb{N}. \tag{2.26}$$

Now, it follows from the compactness of T_j that there exists a subsequence of $\boldsymbol{u}_{m_k}$ (without loss of generality, assume that the subsequence is $\boldsymbol{u}_{m_k}$) such that $T_j\boldsymbol{u}_{m_k}$ converges uniformly to $y_0 \in C[0,1]$. Thus,we easily see that

$$\sup_{t\in J}\left|y_0(t) - (T_j\boldsymbol{u})(t)\right| \ge \varepsilon_0, \quad k \in \mathbb{N}. \tag{2.27}$$

On the other hand, from the pointwise convergence (2.24) we obtain

$$y_0(t) = (T_j\boldsymbol{u})(t), \quad t \in J.$$

This is a contradiction to (2.27). Similarly, we can get that $D_{0^+}^{\alpha_1}(T_j\boldsymbol{u}_m)(t)$ converges uniformly to $D_{0^+}^{\alpha_1}(T_j\boldsymbol{u})(t)$. Therefore, T_j is continuous.

Thus, we assert that $T_j : K \to K$ is completely continuous for $j = 1,2,\dots,n$. This completes the proof of Lemma 2.5. □

3 Existence results

In this section, by using Lemmas 2.1–2.5, we show the existence of at least three positive solutions for system (1.4).

Before the main results, we give the Leggett–Williams fixed point theorem.

Let γ and μ be nonnegative continuous convex functions on K, ω be a nonnegative concave function on K, and ψ be a nonnegative continuous function on K. For $a,b,c,d>0$, we define the following convex sets:

$$K(\gamma;d)=\{\boldsymbol{u}\in K:\gamma(\boldsymbol{u})<d\},$$
$$K(\gamma,\omega;b,d)=\{\boldsymbol{u}\in K:b\le\omega(\boldsymbol{u}),\gamma(\boldsymbol{u})\le d\},$$
$$K(\gamma,\mu,\omega;b,c,d)=\{\boldsymbol{u}\in K:b\le\omega(\boldsymbol{u}),\mu(\boldsymbol{u})\le c;\gamma(\boldsymbol{u})\le d\},$$

and a closed set

$$K(\gamma,\psi;a,d)=\{\boldsymbol{u}\in K:a\le\psi(\boldsymbol{u}),\gamma(\boldsymbol{u})\le d\}.$$

Lemma 3.1 (Leggett–Williams fixed point theorem [30]) *Let K be a cone in a real Banach space E. Let γ and μ be nonnegative continuous convex functions on K, ω be a nonnegative concave function on K, and ψ be a nonnegative continuous function on K satisfying $\psi(\zeta x)\le\zeta\psi(x)$ for $0\le\zeta\le1$ such that, for some positive numbers L and d,*

$$\omega(x)\le\psi(x),\qquad \|\boldsymbol{u}\|\le L\gamma(x) \tag{3.1}$$

for all $x\in\overline{K(\gamma;d)}$. Suppose that

$$T:\overline{K(\gamma;d)}\to\overline{K(\gamma;d)}$$

is completely continuous and there exist positive numbers a, b, and c with $a<b$ such that

(H1) $\{x\in K(\gamma,\mu,\omega;b,c,d):\omega(x)>b\}\neq\emptyset$, *and* $\omega(Tx)>b$ *for* $x\in K(\gamma,\mu,\omega;b,c,d)$;

(H2) $\omega(Tx)>b$ *for* $x\in K(\gamma,\omega;b,d)$ *with* $\mu(Tx)>c$;

(H3) $x\notin K(\gamma,\psi;a,d)$ *and* $\psi(Tx)<a$ *for* $x\in K(\gamma,\psi;a,d)$ *with* $\psi(x)=a$.

Then T has at least three fixed points $x_1,x_2,x_3\in\overline{K(\gamma;d)}$ such that

$$\gamma(x_i)\le d,\quad i=1,2,3;\qquad \omega(x_1)>b,\qquad a<\omega(x_2),\qquad \psi(x_2)<b,\qquad \psi(x_3)<a.$$

Denote the positive constants

$$J_1=\sum_{j=1}^{n}\frac{M_j}{\Gamma(\alpha_1+1)}\left[\frac{M}{\Gamma(\alpha_2+1)}\right]^{q-1},\qquad J_2=\sum_{j=1}^{n}\left[\frac{M}{\Gamma(\alpha_2+1)}\right]^{q-1},$$
$$J_3=\sum_{j=1}^{n}\frac{M_jB(q,q+1)}{\Gamma(\alpha_1)\Gamma(\alpha_2-1)^{q-1}6^q},$$

where $B(q,q+1)$ is the beta function defined by $B(P,Q)=\int_0^1 x^{P-1}(1-x)^{Q-1}\,dx$.

Define the functions as follows:

$$\gamma(\boldsymbol{u})=\|\boldsymbol{u}\|,\qquad \mu(\boldsymbol{u})=\psi(\boldsymbol{u})=\|\boldsymbol{u}\|_1,\qquad \omega(\boldsymbol{u})=\min_{t\in I}\sum_{j=1}^{n}u_j(t),$$

then γ and μ are continuous nonnegative convex functions, ω is a continuous nonnegative concave function, ψ is a continuous nonnegative function, and

$$\rho\mu(\boldsymbol{u}) \le \omega(\boldsymbol{u}) \le \mu(\boldsymbol{u}) = \psi(\boldsymbol{u}), \qquad \|\boldsymbol{u}\| \le L\gamma(\boldsymbol{u}),$$

where $L = 1$. Therefore, condition (3.1) in Lemma 3.1 is satisfied.

Theorem 3.2 *Suppose that* $(F1)$–$(F3)$ *hold, and there exist positive constants* a, b, d *with* $a < b < \rho d \min\{\frac{J_3}{J_1}, \frac{J_3}{J_2}\}$ *and* $c = \frac{b}{\rho}$, *for* $j = 1, 2, \dots, n$, *such that*

(L1) $f_j(t, \boldsymbol{u}, \boldsymbol{w}) \le \frac{1}{\kappa} \min\{\Phi_p(\frac{d}{J_1}), \Phi_p(\frac{d}{J_2})\}$ *for* $(t, \boldsymbol{u}, \boldsymbol{w}) \in J \times [0, d]^n \times [-d, 0]^n$;

(L2) $f_j(t, \boldsymbol{u}, \boldsymbol{w}) > \frac{1}{\kappa} \Phi_p(\frac{b}{\rho_j J_3})$ *for* $(t, \boldsymbol{u}, \boldsymbol{w}) \in I \times [b, \frac{b}{\rho}]^n \times [-d, 0]^n$;

(L3) $f_j(t, \boldsymbol{u}, \boldsymbol{w}) < \frac{1}{\kappa} \Phi_p(\frac{a}{J_1})$ *for* $(t, \boldsymbol{u}, \boldsymbol{w}) \in J \times [0, a]^n \times [-d, 0]^n$.

Then system (1.4) *has at least three positive solutions* $\boldsymbol{u}^1$, $\boldsymbol{u}^2$, $\boldsymbol{u}^3$ *satisfying*

$$\|\boldsymbol{u}^i\| \le d \quad (i = 1, 2, 3), \tag{3.2}$$

$$\begin{aligned} &\min_{t\in I}\left|\sum_{j=1}^n u_j^1(t)\right| > b, \qquad a < \min_{t\in I}\left|\sum_{j=1}^n u_j^2(t)\right|, \\ &\sum_{j=1}^n \sup_{t\in J}\left|u_j^2(t)\right| < b, \qquad \sum_{j=1}^n \sup_{t\in J}\left|u_j^3(t)\right| < a. \end{aligned} \tag{3.3}$$

Proof For $\boldsymbol{u} \in \overline{K(\gamma, d)}$, we have

$$\gamma(\boldsymbol{u}) = \|\boldsymbol{u}\| \le d,$$

this implies

$$\sum_{j=1}^n \sup_{t\in J}\left|u_j(t)\right| \le d, \qquad \sum_{j=1}^n \sup_{t\in J}\left|D_{0^+}^{\alpha_1} u_j(t)\right| \le d,$$

then, for $t \in J$, we have

$$0 \le \sum_{j=1}^n u_j(t) \le d, \qquad -d \le \sum_{j=1}^n D_{0^+}^{\alpha_1} u_j(t) \le 0.$$

By (L1), we have

$$\begin{aligned} \|\boldsymbol{T}\boldsymbol{u}\|_1 &= \sum_{j=1}^n \sup_{t\in J}\left|(T_j\boldsymbol{u})(t)\right| \\ &= \sum_{j=1}^n \sup_{t\in J} \int_0^1 H_j(t,s)\Phi_q\left(\int_0^1 H(s,\tau)\kappa f_j\left(\tau, \boldsymbol{u}(\tau), D_{0^+}^{\alpha_1}\boldsymbol{u}(\tau)\right) d\tau\right) ds \\ &\le \sum_{j=1}^n \int_0^1 \sup_{t\in J} H_j(t,s)\Phi_q\left(\int_0^1 H(s,\tau)\kappa f_j\left(\tau, \boldsymbol{u}(\tau), D_{0^+}^{\alpha_1}\boldsymbol{u}(\tau)\right) d\tau\right) ds \\ &\le \sum_{j=1}^n \int_0^1 \frac{M_j}{\Gamma(\alpha_1)}(1-s)^{\alpha_1-1}\Phi_q\left(\int_0^1 \frac{M}{\Gamma(\alpha_2)}(1-\tau)^{\alpha_2-1}\Phi_p\left(\frac{d}{J_1}\right) d\tau\right) ds \end{aligned}$$

$$= \frac{d}{J_1} \sum_{j=1}^{n} \frac{M_j}{\Gamma(\alpha_1+1)} \left[\frac{M}{\Gamma(\alpha_2+1)} \right]^{q-1} = d, \tag{3.4}$$

and

$$\begin{aligned}
\|\boldsymbol{Tu}\|_2 &= \sum_{j=1}^{n} \sup_{t\in J} \bigl| D_{0^+}^{\alpha_1}(T_j\boldsymbol{u})(t) \bigr| \\
&= \sum_{j=1}^{n} \sup_{t\in J} \Phi_q \biggl(\int_0^1 H(t,s)\kappa f_j\bigl(s,\boldsymbol{u}(s),D_{0^+}^{\alpha_1}\boldsymbol{u}(s)\bigr)\,ds \biggr) \\
&\le \sum_{j=1}^{n} \Phi_q \biggl(\int_0^1 \frac{M}{\Gamma(\alpha_2)}(1-s)^{\alpha_2-1}\Phi_p\biggl(\frac{d}{J_2}\biggr)\,ds \biggr) \\
&= \frac{d}{J_2} \sum_{j=1}^{n} \left[\frac{M}{\Gamma(\alpha_2+1)} \right]^{q-1} = d.
\end{aligned} \tag{3.5}$$

So,

$$\gamma(\boldsymbol{Tu}) = \|\boldsymbol{Tu}\| = \max\bigl\{\|\boldsymbol{Tu}\|_1, \|\boldsymbol{Tu}\|_2\bigr\} \le d.$$

Therefore $\boldsymbol{T}: \overline{K(\gamma,d)} \to \overline{K(\gamma,d)}$.

Let $u_j(t) = \frac{b}{n\rho}$, for $j = 1,2,\ldots,n$. Then $\boldsymbol{u}(t) \in K(\gamma,\mu,\omega;b,c,d)$ and $\sum_{j=1}^{n} u_j(t) = \frac{b}{\rho} > b$, which implies that

$$\bigl\{\boldsymbol{u} \in K(\gamma,\mu,\omega;b,c,d) : \omega(\boldsymbol{u}) > b\bigr\} \ne \varnothing.$$

For $\boldsymbol{u} \in K(\gamma,\mu,\omega;b,c,d)$, we know that $b < \sum_{j=1}^{n} u_j(t) \le c = \frac{b}{\rho}$ for $t \in I$ and $-d \le \sum_{j=1}^{n} D_{0^+}^{\alpha_1} u_j(t) \le 0$.

In view of (L2),

$$\begin{aligned}
\omega(\boldsymbol{Tu}) &= \min_{t\in I} \sum_{j=1}^{n} (T_j\boldsymbol{u})(t) \\
&= \min_{t\in I} \sum_{j=1}^{n} \int_0^1 H_j(t,s)\Phi_q\biggl(\int_0^1 H(s,\tau)\kappa f_j\bigl(\tau,\boldsymbol{u}(\tau),D_{0^+}^{\alpha_1}\boldsymbol{u}(\tau)\bigr)\,d\tau \biggr)\,ds \\
&\ge \sum_{j=1}^{n} \int_0^1 \min_{t\in I} H_j(t,s)\Phi_q\biggl(\int_0^1 H(s,\tau)\kappa f_j\bigl(\tau,\boldsymbol{u}(\tau),D_{0^+}^{\alpha_1}\boldsymbol{u}(\tau)\bigr)\,d\tau \biggr)\,ds \\
&\ge \sum_{j=1}^{n} \int_0^1 \frac{\rho_j M_j}{\Gamma(\alpha_1)}(1-s)^{\alpha_1-1} \\
&\quad \times \Phi_q\biggl(\int_0^1 \frac{1}{\Gamma(\alpha_2-1)} s^{\alpha_2-1}(1-\tau)^{\alpha_2-1}(1-s)\tau\Phi_p\biggl(\frac{b}{\rho_j J_3}\biggr)\,d\tau \biggr)\,ds \\
&\ge \sum_{j=1}^{n} \int_0^1 \frac{\rho_j M_j}{\Gamma(\alpha_1)}(1-s)\Phi_q\biggl(\int_0^1 \frac{1}{\Gamma(\alpha_2-1)} s(1-\tau)(1-s)\tau\Phi_p\biggl(\frac{b}{\rho_j J_3}\biggr)\,d\tau \biggr)\,ds \\
&= \sum_{j=1}^{n} \frac{\rho_j M_j}{\Gamma(\alpha_1)} \left[\frac{1}{\Gamma(\alpha_2-1)} \right]^{q-1} \frac{b}{\rho_j J_3} \int_0^1 (1-s)^q s^{q-1}\,ds\,\Phi_q\biggl(\int_0^1 (1-\tau)\tau\,d\tau \biggr)
\end{aligned}$$

$$> \sum_{j=1}^{n} \frac{M_j b}{\Gamma(\alpha_1)\Gamma(\alpha_2 - 1)^{q-1} J_3} B(q, q+1) 6^{-q} = b. \tag{3.6}$$

So $\omega(\boldsymbol{Tu}) > b$ for all $\boldsymbol{u} \in K(\gamma, \mu, \omega; b, c, d)$. Hence, condition (H1) in Lemma 3.1 is satisfied.

For all $\boldsymbol{u} \in K(\gamma, \omega; b, d)$ with $\mu(\boldsymbol{Tu}) > c = \frac{b}{\rho}$, from (2.23) we have

$$\omega(\boldsymbol{Tu}) \geq \rho\mu(\boldsymbol{Tu}) > \rho c = \rho \frac{b}{\rho} = b.$$

Thus, condition (H2) of Lemma 3.1 holds.

Because of $\psi(\mathbf{0}) = 0 < a$, then $\mathbf{0} \notin K(\gamma, \psi; a, d)$. For $\boldsymbol{u} \in K(\gamma, \psi; a, d)$ with $\psi(\boldsymbol{u}) = a$, we know $\gamma(\boldsymbol{u}) \leq d$, which means that $\sum_{j=1}^{n} \sup_{t\in J} u_j(t) = a$ and $-d \leq \sum_{j=1}^{n} \sup_{t\in J} D_{0^+}^{\alpha_1} u_j(t) \leq 0$.

From (L3), we can obtain

$$\begin{aligned}
\psi(\boldsymbol{Tu}) &= \sum_{j=1}^{n} \sup_{t\in J} \left|(T_j \boldsymbol{u})(t)\right| \\
&= \sum_{j=1}^{n} \sup_{t\in J} \int_0^1 H_j(t,s) \Phi_q\left(\int_0^1 H(s,\tau)\kappa f_j\left(\tau, \boldsymbol{u}(\tau), D_{0^+}^{\alpha_1}\boldsymbol{u}(\tau)\right) d\tau\right) ds \\
&\leq \sum_{j=1}^{n} \int_0^1 \sup_{t\in J} H_j(t,s) \Phi_q\left(\int_0^1 H(s,\tau)\kappa f_j\left(\tau, \boldsymbol{u}(\tau), D_{0^+}^{\alpha_1}\boldsymbol{u}(\tau)\right) d\tau\right) ds \\
&< \sum_{j=1}^{n} \int_0^1 \frac{M_j}{\Gamma(\alpha_1)} (1-s)^{\alpha_1 - 1} \Phi_q\left(\int_0^1 \frac{M}{\Gamma(\alpha_2)} (1-\tau)^{\alpha_2 - 1} \Phi_p\left(\frac{a}{J_1}\right) d\tau\right) ds \\
&= \frac{a}{J_1} \sum_{j=1}^{n} \frac{M_j}{\Gamma(\alpha_1 + 1)} \left[\frac{M}{\Gamma(\alpha_2 + 1)}\right]^{q-1} = a.
\end{aligned} \tag{3.7}$$

Therefore, condition (H3) of Lemma 3.1 is satisfied.

To sum up, the conditions of Lemma 3.1 are all verified. Hence, system (1.4) has at least three positive solutions $\boldsymbol{u}^1$, $\boldsymbol{u}^2$, $\boldsymbol{u}^3$ satisfying (3.2) and (3.3).

The proof is completed. □

4 Nonexistence results

In this section, we focus on the nonexistence results of positive solutions for system (1.4).

We introduce some notations in advance for $j = 1, 2, \ldots, n$:

$$\begin{aligned}
f_j^0 &= \liminf_{\|\boldsymbol{u}\|_1 + \|\boldsymbol{w}\|_1 \to 0} \min_{t\in I} \frac{f_j(t, \boldsymbol{u}, \boldsymbol{w})}{\Phi_p(\|\boldsymbol{u}\|_1 + \|\boldsymbol{w}\|_1)}, \\
f_j^\infty &= \liminf_{\|\boldsymbol{u}\|_1 + \|\boldsymbol{w}\|_1 \to \infty} \min_{t\in I} \frac{f_j(t, \boldsymbol{u}, \boldsymbol{w})}{\Phi_p(\|\boldsymbol{u}\|_1 + \|\boldsymbol{w}\|_1)}.
\end{aligned}$$

Then we have the following nonexistence results of positive solutions.

Theorem 4.1 *If $f_j^0 > 0$ and $f_j^\infty > 0$ for $j = 1, 2, \ldots, n$, then there exists $\kappa_0 > 0$ such that, for all $\kappa > \kappa_0$, system (1.4) has no positive solutions.*

Proof Since $f_j^0 > 0$ and $f_j^\infty > 0$, there exist positive constants $h_1, h_2, r_1, r_2, r_3, r_4$ such that $r_1 < r_3$, $r_2 < r_4$ and

$$f_j(t,\boldsymbol{u},\boldsymbol{w}) \ge h_1 \Phi_p(\|\boldsymbol{u}\|_1 + \|\boldsymbol{w}\|_1), \quad \text{for } (t,\boldsymbol{u},\boldsymbol{w}) \in I \times [0,r_1]^n \times [-r_2,0]^n,$$
$$f_j(t,\boldsymbol{u},\boldsymbol{w}) \ge h_2 \Phi_p(\|\boldsymbol{u}\|_1 + \|\boldsymbol{w}\|_1), \quad \text{for } (t,\boldsymbol{u},\boldsymbol{w}) \in I \times [r_3,\infty)^n \times (-\infty,-r_4]^n.$$

Let

$$h_3 = \min\left\{h_1, h_2, \inf_{(\boldsymbol{u},\boldsymbol{w})\in(r_1,r_3)^n\times(-r_4,-r_2)^n} \frac{\min_{t\in I} f_j(t,\boldsymbol{u},\boldsymbol{w})}{\Phi_p(\|\boldsymbol{u}\|_1 + \|\boldsymbol{w}\|_1)}\right\} > 0, \tag{4.1}$$

then we have

$$f_j(t,\boldsymbol{u},\boldsymbol{w}) \ge h_3 \Phi_p(\|\boldsymbol{u}\|_1 + \|\boldsymbol{w}\|_1), \quad \text{for } (t,\boldsymbol{u},\boldsymbol{w}) \in I \times [0,\infty)^n \times (-\infty,0]^n. \tag{4.2}$$

Suppose that $\widetilde{\boldsymbol{u}}$ is a positive solution of system (1.4); let

$$\kappa_0 = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2-1)^{q-1}6^q}{\rho_j M_j B(q,q+1)}, \tag{4.3}$$

then, for all $t \in I$, we get

$$\begin{aligned}
\|\widetilde{\boldsymbol{u}}\|_1 = \|\boldsymbol{T}\widetilde{\boldsymbol{u}}\|_1 &\ge \sum_{j=1}^n \sup_{t\in I} \int_0^1 H_j(t,s)\Phi_q\left(\int_0^1 H(s,\tau)\kappa f_j(\tau,\widetilde{\boldsymbol{u}}(\tau),D_{0^+}^{\alpha_1}\widetilde{\boldsymbol{u}}(\tau))\,d\tau\right) ds \\
&\ge \int_0^1 H_j(t,s)\Phi_q\left(\int_0^1 H(s,\tau)\kappa f_j(\tau,\widetilde{\boldsymbol{u}}(\tau),D_{0^+}^{\alpha_1}\widetilde{\boldsymbol{u}}(\tau))\,d\tau\right) ds \\
&\ge \int_0^1 \frac{\rho_j M_j}{\Gamma(\alpha_1)}(1-s)^{\alpha_1-1} \\
&\quad \times \Phi_q\left(\int_0^1 \frac{s^{\alpha_2-1}(1-\tau)^{\alpha_2-1}(1-s)\tau}{\Gamma(\alpha_2-1)}\kappa h_3 \Phi_p\left(\|\widetilde{\boldsymbol{u}}\|_1 + \left\|D_{0^+}^{\alpha_1}\widetilde{\boldsymbol{u}}\right\|_1\right) d\tau\right) ds \\
&> \frac{\kappa_0 \rho_j M_j}{\Gamma(\alpha_1)\Gamma(\alpha_2-1)^{q-1}} B(q,q+1)6^{-q}\left(\|\widetilde{\boldsymbol{u}}\|_1 + \left\|D_{0^+}^{\alpha_1}\widetilde{\boldsymbol{u}}\right\|_1\right) \\
&\ge \|\widetilde{\boldsymbol{u}}\|_1,
\end{aligned} \tag{4.4}$$

which is a contraction. Therefore, system (1.4) has no positive solution. □

5 Example

In this section, we give examples to illustrate the results.

Example 5.1 Consider the following system with $n = 2$, $p = \frac{9}{5}$, $\kappa = 1$, $m = 2$:

$$\begin{cases}
D_{0^+}^{\frac{19}{10}}(\Phi_{\frac{9}{5}}(D_{0^+}^{\frac{17}{9}}\boldsymbol{u}(t))) = \boldsymbol{f}(t,\boldsymbol{u}(t),D_{0^+}^{\frac{17}{9}}\boldsymbol{u}(t)), \quad t \in (0,1), \\
\boldsymbol{u}(0) = \boldsymbol{0}, \qquad \boldsymbol{u}(1) = \frac{1}{8}\int_0^{\frac{3}{5}} \boldsymbol{u}(s)\,d\boldsymbol{A}(s) + \frac{1}{10}\int_0^{\frac{4}{5}} \boldsymbol{u}(s)\,d\boldsymbol{A}(s), \\
D_{0^+}^{\frac{17}{9}}\boldsymbol{u}(0) = \boldsymbol{0}, \qquad \Phi_{\frac{9}{5}}(D_{0^+}^{\frac{17}{9}}\boldsymbol{u}(1)) = \frac{1}{20}\Phi_{\frac{9}{5}}(D_{0^+}^{\frac{17}{9}}\boldsymbol{u}(\frac{1}{10})),
\end{cases} \tag{5.1}$$

where $\alpha_1 = \frac{17}{9}$, $\alpha_2 = \frac{19}{10}$, $b_1 = \frac{1}{8}$, $b_2 = \frac{1}{10}$, $\xi_1 = \frac{3}{5}$, $\xi_2 = \frac{4}{5}$, $\lambda = \frac{1}{20}$, $\eta = \frac{1}{10}$, and

$$\boldsymbol{A}(s) = \begin{pmatrix} A_1(s) & 0 \\ 0 & A_2(s) \end{pmatrix},$$

$$A_1(s) = \begin{cases} 0, & s \in [0, \frac{1}{2}), \\ 1, & s \in [\frac{1}{2}, \frac{3}{4}), \\ \frac{1}{2}, & s \in [\frac{3}{4}, 1), \end{cases} \qquad A_2(s) = \begin{cases} 0, & s \in [0, \frac{1}{3}), \\ \frac{3}{2}, & s \in [\frac{1}{3}, \frac{2}{3}), \\ \frac{1}{2}, & s \in [\frac{2}{3}, 1). \end{cases}$$

For $t \in J$, $\boldsymbol{w} \in \mathbb{R}^2$, set

$$\begin{pmatrix} f_1(t,\boldsymbol{u},\boldsymbol{w}) \\ f_2(t,\boldsymbol{u},\boldsymbol{w}) \end{pmatrix} = \begin{cases} \begin{pmatrix} (u_1+u_2)t \\ \frac{w_1 t}{w_1+w_2}(u_1+u_2) \end{pmatrix}, & \boldsymbol{u} \in [0, 0.1]^2, \\ \begin{pmatrix} (0.4t - \frac{\sqrt{30}}{20} - 830)(1-10u_1) + 0.2t \\ (\frac{0.4w_1 t}{w_1+w_2} - 6\sin(w_2 t + 0.15))(1-10u_2) + \frac{0.2w_1 t}{w_1+w_2} \end{pmatrix}, & \boldsymbol{u} \in (0.1, 0.15)^2, \\ \begin{pmatrix} \frac{\sqrt{u_1+u_2}}{4} + 415 \\ 3\sin(w_2 t + u_1) + 434 \end{pmatrix}, & \boldsymbol{u} \in [0.15, 3000]^2. \end{cases}$$

Thus system (5.1) is equivalent to the following problem:

$$\begin{cases} D_{0+}^{\frac{19}{10}}(\Phi_{\frac{9}{5}}(D_{0+}^{\frac{17}{9}} u_1(t))) = f_1(t, \boldsymbol{u}(t), D_{0+}^{\frac{17}{9}} \boldsymbol{u}(t)), & t \in (0,1), \\ D_{0+}^{\frac{19}{10}}(\Phi_{\frac{9}{5}}(D_{0+}^{\frac{17}{9}} u_2(t))) = f_2(t, \boldsymbol{u}(t), D_{0+}^{\frac{17}{9}} \boldsymbol{u}(t)), & t \in (0,1), \\ u_1(0) = 0, \quad u_1(1) = \frac{1}{8}(u_1(\frac{1}{2})) + \frac{1}{10}(u_1(\frac{1}{2}) - \frac{1}{2}u_1(\frac{3}{4})), \\ u_2(0) = 0, \quad u_2(1) = \frac{1}{8}(\frac{3}{2}u_2(\frac{1}{3})) + \frac{1}{10}(\frac{3}{2}u_2(\frac{1}{3}) - u_2(\frac{2}{3})), \\ D_{0+}^{\frac{17}{9}} u_1(0) = 0, \quad \Phi_{\frac{9}{5}}(D_{0+}^{\frac{17}{9}} u_1(1)) = \frac{1}{20}\Phi_{\frac{9}{5}}(D_{0+}^{\frac{17}{9}} u_1(\frac{1}{10})), \\ D_{0+}^{\frac{17}{9}} u_2(0) = 0, \quad \Phi_{\frac{9}{5}}(D_{0+}^{\frac{17}{9}} u_2(1)) = \frac{1}{20}\Phi_{\frac{9}{5}}(D_{0+}^{\frac{17}{9}} u_2(\frac{1}{10})). \end{cases} \tag{5.2}$$

Choose $a = 0.1$, $b = 0.15$, $d = 3000$, $\theta = \frac{1}{4}$, by calculations, we can obtain

$$\Delta_1 = 0.9172, \qquad \Delta_2 = 0.9426,$$
$$\rho_1 = 0.0350, \qquad \rho_2 = 0.0333,$$
$$M_1 = 1.1908, \qquad M_2 = 1.2520, \qquad M = 1.0503,$$
$$J_1 = 0.6756, \qquad J_2 = 1.0009, \qquad J_3 = 0.0023.$$

So we can check that $f_j(t,\boldsymbol{u},\boldsymbol{w})$ satisfy (for $j = 1,2$):

(L1) $f_1(t,\boldsymbol{u},\boldsymbol{w}) \le 434.3649, f_2(t,\boldsymbol{u},\boldsymbol{w}) \le 437, f_j(t,\boldsymbol{u},\boldsymbol{w}) \le \min\{\Phi_{\frac{9}{5}}(\frac{d}{J_1}), \Phi_{\frac{9}{5}}(\frac{d}{J_2})\} = 437.0214$ for $(t,\boldsymbol{u},\boldsymbol{w}) \in [0,1] \times [0,3000]^2 \times [-3000,0]^2$;

(L2) $f_1(t,\boldsymbol{u},\boldsymbol{w}) \ge 415.7506 > \Phi_{\frac{9}{5}}(\frac{b}{\rho_1 J_3}) = 413.5925, f_2(t,\boldsymbol{u},\boldsymbol{w}) \ge 431 > \Phi_{\frac{9}{5}}(\frac{b}{\rho_2 J_3}) = 430.5003$ for $(t,\boldsymbol{u},\boldsymbol{w}) \in [\frac{1}{4},\frac{3}{4}] \times [0.15, 4.5070]^2 \times [-3000,0]^2$;

(L3) $f_1(t,\boldsymbol{u},\boldsymbol{w}) \le 0.2 < \Phi_{\frac{9}{5}}(\frac{a}{J_1}) = 0.2169, f_2(t,\boldsymbol{u},\boldsymbol{w}) \le 0.2 < \Phi_{\frac{9}{5}}(\frac{a}{J_1}) = 0.2169$ for $(t,\boldsymbol{u},\boldsymbol{w}) \in [0,1] \times [0,0.1]^2 \times [-3000,0]^2$.

Thus all conditions in Theorem 3.2 are satisfied. System (5.1) has at least three positive solutions $\boldsymbol{u}^1$, $\boldsymbol{u}^2$, $\boldsymbol{u}^3$ satisfying

$$\|\boldsymbol{u}^i\| \le 3000 \quad (i = 1,2,3),$$

$$\min_{t\in I}\left|\sum_{j=1}^{n} u_j^1(t)\right| > 0.15, \qquad 0.1 < \min_{t\in I}\left|\sum_{j=1}^{n} u_j^2(t)\right|,$$
$$\sum_{j=1}^{n}\sup_{t\in J}\left|u_j^2(t)\right| < 0.15, \qquad \sum_{j=1}^{n}\sup_{t\in J}\left|u_j^3(t)\right| < 0.1.$$

Example 5.2 Consider the following system with $n = 2$, $p = \frac{9}{5}$, $\kappa = 26{,}000$, $m = 2$:

$$\begin{cases} D_{0+}^{\frac{19}{10}}(\Phi_{\frac{9}{5}}(D_{0+}^{\frac{17}{9}}\boldsymbol{u}(t))) = 26{,}000\boldsymbol{f}(t,\boldsymbol{u}(t),D_{0+}^{\frac{17}{9}}\boldsymbol{u}(t)), \quad t\in(0,1), \\ \boldsymbol{u}(0) = \boldsymbol{0}, \qquad \boldsymbol{u}(1) = \frac{1}{8}\int_0^{\frac{3}{5}}\boldsymbol{u}(s)\,d\boldsymbol{A}(s) + \frac{1}{10}\int_0^{\frac{4}{5}}\boldsymbol{u}(s)\,d\boldsymbol{A}(s), \\ D_{0+}^{\frac{17}{9}}\boldsymbol{u}(0) = \boldsymbol{0}, \qquad \Phi_{\frac{9}{5}}(D_{0+}^{\frac{17}{9}}\boldsymbol{u}(1)) = \frac{1}{20}\Phi_{\frac{9}{5}}(D_{0+}^{\frac{17}{9}}\boldsymbol{u}(\tfrac{1}{10})), \end{cases} \tag{5.3}$$

where $\alpha_1 = \frac{17}{9}$, $\alpha_2 = \frac{19}{10}$, $b_1 = \frac{1}{8}$, $b_2 = \frac{1}{10}$, $\xi_1 = \frac{3}{5}$, $\xi_2 = \frac{4}{5}$, $\lambda = \frac{1}{20}$, $\eta = \frac{1}{10}$, $\theta = \frac{1}{4}$,

$$f_j(t,\boldsymbol{u},\boldsymbol{w}) = \bigl(|u_1| + |u_2| + |w_1| + |w_2|\bigr)^2, \quad j = 1,2.$$

We can easily get that all conditions in Theorem 4.1 are satisfied. Because $\kappa_0 = 25{,}507 < \kappa$, system (5.3) has no positive solutions.

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