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On type 2 degenerate Bernoulli and Euler polynomials of complex variable

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Abstract

Recently, Masjed-Jamei, Beyki, and Koepf studied the so-called new type Euler polynomials without using Euler polynomials of complex variable. Here we study the type 2 degenerate cosine-Euler and type 2 degenerate sine-Euler polynomials, which are type 2 degenerate versions of these new type Euler polynomials, by considering the degenerate Euler polynomials of complex variable and by treating the real and imaginary parts separately. In addition, we investigate the corresponding ones for Bernoulli polynomials in the same manner. We derive some explicit expressions for those new polynomials and some identities relating to them. Here we note that the idea of separating the real and imaginary parts separately gives an affirmative answer to the question asked by Hacène Belbachir.

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1 Introduction

As is known, the type 2 Bernoulli polynomials $B_n(x)$, $(n \geq 0)$ and the type 2 Euler polynomials $E_n(x)$, $(n \geq 0)$ are respectively defined by

$$e^{xt}\frac{t}{2}\operatorname{csch}\frac{t}{2} = \frac{t}{e^{\frac{t}{2}} - e^{-\frac{t}{2}}}e^{xt} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} \tag{1.1}$$

and

$$e^{xt}\operatorname{sech}\frac{t}{2} = \frac{2}{e^{\frac{t}{2}} + e^{-\frac{t}{2}}}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} \quad \text{(see [5, 20])}. \tag{1.2}$$

When $x = 0$, $B_n = B_n(0)$ (or $E_n = E_n(0)$) are called the type 2 Bernoulli (or type 2 Euler) numbers.

For $n \geq 0$, the central factorial numbers of the second kind are defined by the generating function to be

$$\frac{1}{k!}\left(e^{\frac{t}{2}} - e^{-\frac{t}{2}}\right)^k = \sum_{n=k}^{\infty} T(n,k)\frac{t^n}{n!} \quad \text{(see [3, 7, 19])}. \tag{1.3}$$

From (1.3), we note that

$$x^n = \sum_{k=0}^{n} T(n,k)x^{[k]} \quad (n \geq 0), \text{ (see [12])}, \tag{1.4}$$

where $x^{[0]} = 1$, $x^{[n]} = x(x + \frac{n}{2} - 1)(x + \frac{n}{2} - 2) \cdots (x - \frac{n}{2} + 1)$, $(n \geq 1)$. For $\lambda \in \mathbb{R}$, the degenerate exponential functions are defined as

$$e_\lambda^x(t) = (1 + \lambda t)^{\frac{x}{\lambda}}, \qquad e_\lambda(t) = e_\lambda^1(t) = (1 + \lambda t)^{\frac{1}{\lambda}}. \tag{1.5}$$

By (1.5), we get

$$e_\lambda^x(t) = \sum_{n=0}^{\infty} (x)_{n,\lambda} \frac{t^n}{n!} \quad \text{(see [10, 11, 13, 14])}, \tag{1.6}$$

where

$$(x)_{0,\lambda} = 1, \qquad (x)_{n,\lambda} = x(x - \lambda) \cdots \big(x - (n-1)\lambda\big), \quad (n \geq 1). \tag{1.7}$$

In [1, 2], Carlitz considered the degenerate Bernoulli polynomials given by

$$\frac{t}{e_\lambda(t) - 1} e_\lambda^x(t) = \sum_{n=0}^{\infty} \beta_{n,\lambda}(x) \frac{t^n}{n!}. \tag{1.8}$$

When $x = 0$, $\beta_{n,\lambda} = \beta_{n,\lambda}(0)$ are called the degenerate Bernoulli numbers. In [12], Kim and Kim introduced the degenerate central factorial polynomials of the second kind given by

$$\frac{1}{k!}\big(e_\lambda^{\frac{1}{2}}(t) - e_\lambda^{-\frac{1}{2}}(t)\big)^k e_\lambda^x(t) = \sum_{n=k}^{\infty} T_\lambda(n,k|x) \frac{t^n}{n!}, \tag{1.9}$$

where k is a nonnegative integer. When $x = 0$, $T_\lambda(n,k) = T_\lambda(n,k|0)$ are called the degenerate central factorial numbers of the second kind.

Recently, as a degenerate version of (1.1), the type 2 degenerate Bernoulli polynomials have been defined by

$$\frac{t}{e_\lambda^{\frac{1}{2}}(t) - e_\lambda^{-\frac{1}{2}}(t)} e_\lambda^x(t) = \sum_{n=0}^{\infty} B_{n,\lambda}(x) \frac{t^n}{n!} \quad \text{(see [5])}. \tag{1.10}$$

When $x = 0$, $B_{n,\lambda} = B_{n,\lambda}(0)$ are the type 2 degenerate Bernoulli numbers. By the same motivation as (1.10), the type 2 Euler polynomials are defined by

$$\frac{2}{e_\lambda^{\frac{1}{2}}(t) + e_\lambda^{-\frac{1}{2}}(t)} e_\lambda^x(t) = \sum_{n=0}^{\infty} E_{n,\lambda}(x) \frac{t^n}{n!} \quad \text{(see [5])}. \tag{1.11}$$

When $x = 0$, $E_{n,\lambda} = E_{n,\lambda}(0)$ are the type 2 degenerate Euler numbers.

Recently, several authors studied the degenerate Bernoulli and degenerate Euler numbers and polynomials (see [1, 2, 4, 5, 8, 10–18, 21]). In addition, Jeong, Kang, and Rim introduced symmetry identities for Changhee polynomials of type 2 closely related to the type 2 degenerate Euler polynomials (see [6]), and Zhang and Lin obtained some interesting identities involving trigonometric functions and Bernoulli numbers (see [21]).

In [9], the authors considered the degenerate Bernoulli and degenerate Euler polynomials of complex variable. By treating the real and imaginary parts separately, they were able to introduce the degenerate cosine-Bernoulli polynomials, degenerate sine-Bernoulli polynomials, degenerate cosine-Euler polynomials, and degenerate sine-Euler polynomials and derived some interesting results for them.

In this paper, we study the type 2 degenerate Bernoulli and type 2 degenerate Euler polynomials of complex variable, of which the latter are type 2 degenerate versions of the new type Euler polynomials studied in [16]. By treating the real and imaginary parts separately, the type 2 degenerate cosine-Bernoulli and type 2 degenerate sine-Bernoulli polynomials are introduced. We derive some explicit expressions for those polynomials and some identities related to them. Moreover, the type 2 degenerate cosine-Euler and type 2 degenerate sine-Euler polynomials are investigated, and analogous results to the type 2 degenerate cosine-Bernoulli and type 2 degenerate sine-Bernoulli polynomials are obtained for them.

2 Type 2 degenerate Bernoulli and Euler polynomials of complex variable

From (1.10), we define the type 2 degenerate Bernoulli polynomials of complex variable by

$$\frac{t}{e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t)}e_\lambda^{x+iy}(t)=\sum_{n=0}^{\infty}B_{n,\lambda}(x+iy)\frac{t^n}{n!} \tag{2.1}$$

and

$$\frac{t}{e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t)}e_\lambda^{x-iy}(t)=\sum_{n=0}^{\infty}B_{n,\lambda}(x-iy)\frac{t^n}{n!}, \tag{2.2}$$

where $i=\sqrt{-1}$. As is known, the degenerate cosine and sine functions are defined by

$$\cos_\lambda^{(y)}(t)=\cos\left(\frac{y}{\lambda}\log(1+\lambda t)\right) \tag{2.3}$$

and

$$\sin_\lambda^{(y)}(t)=\sin\left(\frac{y}{\lambda}\log(1+\lambda t)\right),\quad \text{(see [9])}. \tag{2.4}$$

Note that $\lim_{\lambda\to 0}\cos_\lambda^{(y)}(t)=\cos yt$, $\lim_{\lambda\to 0}\sin_\lambda^{(y)}(t)=\sin yt$. From (2.1) and (2.2), we can derive the following equations:

$$\sum_{n=0}^{\infty}\left(\frac{B_{n,\lambda}(x+iy)+B_{n,\lambda}(x-iy)}{2}\right)\frac{t^n}{n!}=\frac{t}{e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t)}e_\lambda^{x}(t)\cos_\lambda^{(y)}(t) \tag{2.5}$$

and

$$\sum_{n=0}^{\infty}\left(\frac{B_{n,\lambda}(x+iy)-B_{n,\lambda}(x-iy)}{2i}\right)\frac{t^n}{n!}=\frac{t}{e_{\lambda}^{\frac{1}{2}}(t)-e_{\lambda}^{-\frac{1}{2}}(t)}e_{\lambda}^{x}(t)\sin_{\lambda}^{(y)}(t). \tag{2.6}$$

Now, we define the type 2 degenerate cosine-Bernoulli and sine-Bernoulli polynomials by the generating functions as follows:

$$\frac{t}{e_{\lambda}^{\frac{1}{2}}(t)-e_{\lambda}^{-\frac{1}{2}}(t)}e_{\lambda}^{x}(t)\cos_{\lambda}^{(y)}(t)=\sum_{n=0}^{\infty}B_{n,\lambda}^{(c)}(x,y)\frac{t^n}{n!} \tag{2.7}$$

and

$$\frac{t}{e_{\lambda}^{\frac{1}{2}}(t)-e_{\lambda}^{-\frac{1}{2}}(t)}e_{\lambda}^{x}(t)\sin_{\lambda}^{(y)}(t)=\sum_{n=0}^{\infty}B_{n,\lambda}^{(s)}(x,y)\frac{t^n}{n!}. \tag{2.8}$$

Therefore, by (2.5), (2.6), (2.7), and (2.8), we obtain the following theorem.

Theorem 2.1 *For $n\geq 0$, we have*

$$\frac{B_{n,\lambda}(x+iy)+B_{n,\lambda}(x-iy)}{2}=B_{n,\lambda}^{(c)}(x,y)$$

and

$$\frac{B_{n,\lambda}(x+iy)-B_{n,\lambda}(x-iy)}{2i}=B_{n,\lambda}^{(s)}(x,y).$$

From (1.10), (2.3), and (2.4), we note that

$$\begin{aligned}
&\frac{t}{e_{\lambda}^{\frac{1}{2}}(t)-e_{\lambda}^{-\frac{1}{2}}(t)}e_{\lambda}^{x}(t)\cos_{\lambda}^{(y)}(t)\\
&\quad=\sum_{l=0}^{\infty}B_{l,\lambda}(x)\frac{t^l}{l!}\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m)!}\left(\frac{y}{\lambda}\right)^{2m}\bigl(\log(1+\lambda t)\bigr)^{2m}\\
&\quad=\sum_{l=0}^{\infty}B_{l,\lambda}(x)\frac{t^l}{l!}\sum_{m=0}^{\infty}(-1)^m y^{2m}\lambda^{-2m}\sum_{k=2m}^{\infty}S_1(k,2m)\lambda^k\frac{t^k}{k!}\\
&\quad=\sum_{l=0}^{\infty}B_{l,\lambda}(x)\frac{t^l}{l!}\sum_{k=0}^{\infty}\left(\sum_{m=0}^{[\frac{k}{2}]}(-1)^m y^{2m}\lambda^{k-2m}S_1(k,2m)\right)\frac{t^k}{k!}\\
&\quad=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{m=0}^{[\frac{k}{2}]}\binom{n}{k}B_{n-k,\lambda}(x)(-1)^m y^{2m}\lambda^{k-2m}S_1(k,2m)\right)\frac{t^n}{n!},
\end{aligned} \tag{2.9}$$

where $S_1(k,l)$ are the Stirling numbers of the first kind. By the same method as in (2.9), we get

$$\begin{aligned}
&\frac{t}{e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t)} e_\lambda^x(t)\sin_\lambda^{(y)}(t) \\
&= \sum_{l=0}^\infty B_{l,\lambda}(x)\frac{t^l}{l!}\sum_{k=1}^\infty\left(\sum_{m=0}^{[\frac{k-1}{2}]}(-1)^m y^{2m+1}\lambda^{k-2m-1}S_1(k,2m+1)\right)\frac{t^k}{k!} \\
&= \sum_{n=1}^\infty\left(\sum_{k=1}^n\sum_{m=0}^{[\frac{k-1}{2}]}\binom{n}{k}B_{n-k,\lambda}(x)(-1)^m y^{2m+1}\lambda^{k-2m-1}S_1(k,2m+1)\right)\frac{t^n}{n!}.
\end{aligned} \tag{2.10}$$

Therefore, by (2.7), (2.8), (2.9), and (2.10), we obtain the following theorem.

Theorem 2.2 *For $n \in \mathbb{N}\cup\{0\}$, we have*

$$B_{n,\lambda}^{(c)}(x,y) = \sum_{k=0}^n\sum_{m=0}^{[\frac{k}{2}]}\binom{n}{k}B_{n-k,\lambda}(x)(-1)^m y^{2m}\lambda^{k-2m}S_1(k,2m).$$

In addition,

$$\begin{aligned}
&B_{0,\lambda}^{(s)}(x,y) = 0, \\
&B_{n,\lambda}^{(s)}(x,y) = \sum_{k=1}^n\sum_{m=0}^{[\frac{k-1}{2}]}\binom{n}{k}B_{n-k,\lambda}(x)(-1)^m y^{2m+1}\lambda^{k-2m-1}S_1(k,2m+1),
\end{aligned}$$

where n is a positive integer.

We observe that

$$\begin{aligned}
\sum_{n=0}^\infty B_{n,\lambda}^{(c)}(x,0)\frac{t^n}{n!} &= \frac{t}{e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t)}e_\lambda^x(t) \\
&= \frac{t}{e_\lambda(t)-1}e_\lambda^{x+\frac{1}{2}}(t) \\
&= \sum_{n=0}^\infty \beta_{n,\lambda}\left(x+\frac{1}{2}\right)\frac{t^n}{n!}.
\end{aligned} \tag{2.11}$$

Therefore, by (2.11), we obtain the following theorem.

Theorem 2.3 *For $n \geq 0$, we have*

$$B_{n,\lambda}^{(c)}(x,0) = \beta_{n,\lambda}\left(x+\frac{1}{2}\right).$$

From (2.7), we note that

$$\begin{aligned} & e_\lambda^x(t)\cos_\lambda^{(y)}(t) \\ &= \frac{1}{t}\left(e_\lambda^{\frac{1}{2}}(t) - e_\lambda^{-\frac{1}{2}}(t)\right)\sum_{l=0}^{\infty} B_{l,\lambda}^{(c)}(x,y)\frac{t^l}{l!} \\ &= \frac{1}{t}\sum_{n=1}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}\left(\left(\frac{1}{2}\right)_{n-l,\lambda} - \left(-\frac{1}{2}\right)_{n-l,\lambda}\right)B_{l,\lambda}(x,y)\right)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left\{\frac{1}{n+1}\sum_{l=0}^{n+1}\binom{n+1}{l}\left(\left(\frac{1}{2}\right)_{n+1-l,\lambda} - \left(-\frac{1}{2}\right)_{n+1-l,\lambda}\right)B_{l,\lambda}(x,y)\right\}\frac{t^n}{n!}. \end{aligned} \tag{2.12}$$

On the other hand,

$$\begin{aligned} e_\lambda^x(t)\cos_\lambda^{(y)}(t) &= \sum_{l=0}^{\infty}(x)_{l,\lambda}\frac{t^l}{l!}\cos_\lambda^{(y)}(t) \\ &= \sum_{l=0}^{\infty}(x)_{l,\lambda}\frac{t^l}{l!}\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m)!}\left(\frac{y}{\lambda}\right)^{2m}\left(\log(1+\lambda t)\right)^{2m} \\ &= \sum_{l=0}^{\infty}(x)_{l,\lambda}\frac{t^l}{l!}\sum_{m=0}^{\infty}(-1)^m\lambda^{-2m}y^{2m}\sum_{k=2m}^{\infty}S_1(k,2m)\lambda^k\frac{t^k}{k!} \\ &= \sum_{l=0}^{\infty}(x)_{l,\lambda}\frac{t^l}{l!}\sum_{k=0}^{\infty}\left(\sum_{m=0}^{[\frac{k}{2}]}(-1)^m\lambda^{k-2m}y^{2m}S_1(k,2m)\right)\frac{t^k}{k!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{m=0}^{[\frac{k}{2}]}\binom{n}{k}(x)_{n-k,\lambda}(-1)^m\lambda^{k-2m}y^{2m}S_1(k,2m)\right)\frac{t^n}{n!}. \end{aligned} \tag{2.13}$$

Therefore, by (2.12) and (2.13), we obtain the following theorem.

Theorem 2.4 *For $n \geq 0$, we have*

$$\begin{aligned} & \frac{1}{n+1}\sum_{l=0}^{n+1}\binom{n+1}{l}\left(\left(\frac{1}{2}\right)_{n+1-l,\lambda} - \left(-\frac{1}{2}\right)_{n+1-l,\lambda}\right)B_{l,\lambda}^{(c)}(x,y) \\ &= \sum_{k=0}^{n}\sum_{m=0}^{[\frac{k}{2}]}\binom{n}{k}(x)_{n-k,\lambda}(-1)^m\lambda^{k-2m}y^{2m}S_1(k,2m). \end{aligned}$$

Furthermore, for $n \in \mathbb{N}$, we have

$$\begin{aligned} & \frac{1}{n+1}\sum_{l=0}^{n+1}\binom{n+1}{l}\left(\left(\frac{1}{2}\right)_{n+1-l,\lambda} - \left(-\frac{1}{2}\right)_{n+1-l,\lambda}\right)B_{l,\lambda}^{(s)}(x,y) \\ &= \sum_{k=1}^{n}\sum_{m=0}^{[\frac{k-1}{2}]}\binom{n}{k}(x)_{n-k,\lambda}(-1)^m\lambda^{k-2m-1}y^{2m+1}S_1(k,2m+1). \end{aligned}$$

By replacing t by $\frac{1}{\lambda}(e^{\lambda t}-1)$ in (2.7), we get

$$\begin{aligned}
&\frac{1}{\lambda t}\left(e^{\lambda t}-1\right)\left(\frac{t}{e^{\frac{t}{2}}-e^{-\frac{t}{2}}}e^{xt}\cos yt\right)\\
&=\sum_{k=0}^{\infty}B_{k,\lambda}^{(c)}(x,y)\frac{1}{k!}\left(e^{\lambda t}-1\right)^{k}\lambda^{-k}\\
&=\sum_{k=0}^{\infty}B_{k,\lambda}^{(c)}(x,y)\lambda^{-k}\sum_{n=k}^{\infty}S_2(n,k)\lambda^{n}\frac{t^n}{n!}\\
&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\lambda^{n-k}B_{k,\lambda}^{(c)}(x,y)S_2(n,k)\right)\frac{t^n}{n!},
\end{aligned} \tag{2.14}$$

where $S_2(n,k)$ are the Stirling numbers of the second kind. On the other hand,

$$\begin{aligned}
&\frac{1}{\lambda t}\left(e^{\lambda t}-1\right)\left(\frac{t}{e^{\frac{t}{2}}-e^{-\frac{t}{2}}}e^{xt}\cos yt\right)\\
&=\sum_{l=0}^{\infty}\frac{\lambda^{l}}{l+1}\frac{t^l}{l!}\sum_{m=0}^{\infty}\left(\sum_{l=0}^{[\frac{m}{2}]}\binom{m}{2l}(-1)^{l}y^{2l}B_{m-2l}(x)\right)\frac{t^m}{m!}\\
&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\frac{\lambda^{n-m}}{n-m+1}\binom{n}{m}\sum_{l=0}^{[\frac{m}{2}]}\binom{m}{2l}(-1)^{l}y^{2l}B_{m-2l}(x)\right)\frac{t^n}{n!}.
\end{aligned} \tag{2.15}$$

Therefore, by (2.14) and (2.15), we obtain the following theorem.

Theorem 2.5 *For $n\geq 0$, we have*

$$\sum_{k=0}^{n}\lambda^{n-k}B_{k,\lambda}^{(c)}(x,y)S_2(n,k)=\sum_{m=0}^{n}\sum_{l=0}^{[\frac{m}{2}]}\frac{\lambda^{n-m}}{n-m+1}\binom{n}{m}\binom{m}{2l}(-1)^{l}y^{2l}B_{m-2l}(x).$$

Let us replace t by $\frac{1}{\lambda}\log(1+\lambda t)$ in (1.1). Then we have

$$\begin{aligned}
\frac{\log(1+\lambda t)}{\lambda t}\frac{t}{e_{\lambda}^{\frac{1}{2}}(t)-e_{\lambda}^{-\frac{1}{2}}(t)}e_{\lambda}^{x+iy}(t)&=\sum_{k=0}^{\infty}B_k(x+iy)\lambda^{-k}\frac{(\log(1+\lambda t))^k}{k!}\\
&=\sum_{k=0}^{\infty}B_k(x+iy)\lambda^{-k}\sum_{n=k}^{\infty}S_1(n,k)\lambda^{n}\frac{t^n}{n!}\\
&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\lambda^{n-k}B_k(x+iy)S_1(n,k)\right)\frac{t^n}{n!}.
\end{aligned} \tag{2.16}$$

We recall here that the Bernoulli numbers of the second kind are given by

$$\frac{t}{\log(1+t)}=\sum_{n=0}^{\infty}b_n\frac{t^n}{n!}. \tag{2.17}$$

Then, from (2.7), (2.8), and (2.16), we have

$$\begin{aligned}\sum_{n=0}^{\infty}B_{n,\lambda}^{(c)}(x,y)\frac{t^n}{n!} &= \sum_{l=0}^{\infty}b_l\lambda^l\frac{t^l}{l!}\sum_{m=0}^{\infty}\left(\sum_{k=0}^{m}\lambda^{m-k}S_1(m,k)\frac{B_k(x+iy)+B_k(x-iy)}{2}\right)\frac{t^m}{m!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{k=0}^{m}\binom{n}{m}b_{n-m}\lambda^{n-k}S_1(m,k)\frac{B_k(x+iy)+B_k(x-iy)}{2}\right)\frac{t^n}{n!}\end{aligned} \tag{2.18}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty}B_{n,\lambda}^{(s)}(x,y)\frac{t^n}{n!} &= \sum_{l=0}^{\infty}b_l\lambda^l\frac{t^l}{l!}\sum_{m=0}^{\infty}\sum_{k=0}^{m}\lambda^{m-k}S_1(m,k)\left(\frac{B_k(x+iy)-B_k(x-iy)}{2i}\right)\frac{t^m}{m!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{k=0}^{m}\binom{n}{m}b_{n-m}\lambda^{n-k}S_1(m,k)\frac{B_k(x+iy)-B_k(x-iy)}{2i}\right)\frac{t^n}{n!}.\end{aligned} \tag{2.19}$$

From (1.1), we note that

$$\begin{aligned}\sum_{n=0}^{\infty}\left(\frac{B_n(x+iy)+B_n(x-iy)}{2}\right)\frac{t^n}{n!} &= \frac{t}{e^{\frac{t}{2}}-e^{-\frac{t}{2}}}e^{xt}\cos yt \\ &= \sum_{l=0}^{\infty}B_l(x)\frac{t^l}{l!}\sum_{m=0}^{\infty}y^{2m}(-1)^m\frac{t^{2m}}{(2m)!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{[\frac{n}{2}]}\binom{n}{2m}B_{n-2m}(x)y^{2m}(-1)^m\right)\frac{t^n}{n!}.\end{aligned} \tag{2.20}$$

Comparing the coefficients on both sides of (2.20), we have

$$\frac{B_n(x+iy)+B_n(x-iy)}{2} = \sum_{m=0}^{[\frac{n}{2}]}\binom{n}{2m}B_{n-2m}(x)y^{2m}(-1)^m, \tag{2.21}$$

where n is a positive integer. By the same method as in (2.21), we get

$$\frac{B_n(x+iy)-B_n(x-iy)}{2i} = \sum_{m=0}^{[\frac{n-1}{2}]}\binom{n}{2m+1}B_{n-2m-1}(x)y^{2m+1}(-1)^m, \tag{2.22}$$

where n is a positive integer. Therefore, by (2.18), (2.19), (2.21), and (2.22), we obtain the following theorem.

Theorem 2.6 *For $n \geq 0$, we have*

$$B_{n,\lambda}^{(c)}(x,y) = \sum_{m=0}^{n}\sum_{k=0}^{m}\sum_{l=0}^{[\frac{k}{2}]}\binom{n}{m}\binom{k}{2l}(-1)^l\lambda^{n-k}S_1(m,k)b_{n-m}B_{k-2l}(x)y^{2l}.$$

Furthermore, for $n \in \mathbb{N}$, we have

$$B_{n,\lambda}^{(s)}(x,y) = \sum_{m=0}^{n}\sum_{k=0}^{m}\sum_{l=0}^{[\frac{k-1}{2}]}\binom{n}{m}\binom{k}{2l+1}(-1)^l\lambda^{n-k}S_1(m,k)b_{n-m}B_{k-2l-1}(x)y^{2l+1}.$$

For $\alpha \in \mathbb{R}$, the type 2 degenerate Bernoulli polynomials of order α are defined by

$$\left(\frac{t}{e_\lambda^{\frac{1}{2}}(t) - e_\lambda^{-\frac{1}{2}}(t)}\right)^\alpha e_\lambda^x(t) = \sum_{n=0}^{\infty} B_{n,\lambda}^{(\alpha)}(x)\frac{t^n}{n!}. \tag{2.23}$$

When $x = 0$, $B_{n,\lambda}^{(\alpha)} = B_{n,\lambda}^{(\alpha)}(0)$ are called the type 2 degenerate Bernoulli numbers of order α. For $k \in \mathbb{N}$, let $\alpha = -k$ and $x = 0$. Then we have

$$\begin{aligned}\sum_{n=0}^{\infty} B_{n,\lambda}^{(-k)}\frac{t^n}{n!} &= \frac{1}{t^k}\left(e_\lambda^{\frac{1}{2}}(t) - e_\lambda^{-\frac{1}{2}}(t)\right)^k \\ &= \frac{k!}{t^k}\sum_{n=k}^{\infty} T_\lambda(n,k)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\frac{T_\lambda(n+k,k)}{\binom{n+k}{k}}\frac{t^n}{n!}.\end{aligned} \tag{2.24}$$

Thus, by (2.24), we get

$$\binom{n+k}{k}B_{n,\lambda}^{(-k)} = T_\lambda(n+k,k),$$

where n, k are nonnegative integers.

For $\alpha \in \mathbb{R}$, let us define the type 2 degenerate cosine-Bernoulli polynomials of order α and the type 2 degenerate sine-Bernoulli polynomials of order α, respectively, by

$$\left(\frac{t}{e_\lambda^{\frac{1}{2}}(t) - e_\lambda^{-\frac{1}{2}}(t)}\right)^\alpha e_\lambda^x(t)\cos_\lambda^{(y)}(t) = \sum_{n=0}^{\infty} B_{n,\lambda}^{(c,\alpha)}(x,y)\frac{t^n}{n!} \tag{2.25}$$

and

$$\left(\frac{t}{e_\lambda^{\frac{1}{2}}(t) - e_\lambda^{-\frac{1}{2}}(t)}\right)^\alpha e_\lambda^x(t)\sin_\lambda^{(y)}(t) = \sum_{n=0}^{\infty} B_{n,\lambda}^{(s,\alpha)}(x,y)\frac{t^n}{n!}. \tag{2.26}$$

Then we note that

$$B_{n,\lambda}^{(c,\alpha)}(x,y) = \frac{B_{n,\lambda}^{(\alpha)}(x+iy) + B_{n,\lambda}^{(\alpha)}(x-iy)}{2}, \tag{2.27}$$

where n is a nonnegative integer.

$$B_{n,\lambda}^{(s,\alpha)}(x,y)=\frac{B_{n,\lambda}^{(\alpha)}(x+iy)-B_{n,\lambda}^{(\alpha)}(x-iy)}{2i}, \tag{2.28}$$

where n is a positive integer. Proceeding just as in (2.9) and (2.10), we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}\left(\frac{B_{n,\lambda}^{(\alpha)}(x+iy)+B_{n,\lambda}^{(\alpha)}(x-iy)}{2}\right)\frac{t^n}{n!} \\
&\quad=\left(\frac{t}{e_{\lambda}^{\frac{1}{2}}(t)-e_{\lambda}^{-\frac{1}{2}}(t)}\right)^{\alpha}e_{\lambda}^{x}(t)\cos_{\lambda}^{(y)}(t) \\
&\quad=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{m=0}^{[\frac{k}{2}]}\binom{n}{k}B_{n-k,\lambda}^{(\alpha)}(x)(-1)^m\lambda^{k-2m}y^{2m}S_1(k,2m)\right)\frac{t^n}{n!}
\end{aligned} \tag{2.29}$$

and

$$\begin{aligned}
&\sum_{n=0}^{\infty}\left(\frac{B_{n,\lambda}^{(\alpha)}(x+iy)-B_{n,\lambda}^{(\alpha)}(x-iy)}{2i}\right)\frac{t^n}{n!} \\
&\quad=\left(\frac{t}{e_{\lambda}^{\frac{1}{2}}(t)-e_{\lambda}^{-\frac{1}{2}}(t)}\right)^{\alpha}e_{\lambda}^{x}(t)\sin_{\lambda}^{(y)}(t) \\
&\quad=\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\sum_{m=0}^{[\frac{k-1}{2}]}\binom{n}{k}B_{n-k,\lambda}^{(\alpha)}(x)(-1)^m\lambda^{k-2m-1}y^{2m+1}S_1(k,2m+1)\right)\frac{t^n}{n!}.
\end{aligned} \tag{2.30}$$

Therefore, by (2.27), (2.28), (2.29), and (2.30), we obtain the following theorem.

Theorem 2.7 *For $n\geq 0$, we have*

$$B_{n,\lambda}^{(c,\alpha)}(x,y)=\sum_{k=0}^{n}\sum_{m=0}^{[\frac{k}{2}]}\binom{n}{k}B_{n-k,\lambda}^{(\alpha)}(x)(-1)^m\lambda^{k-2m}y^{2m}S_1(k,2m).$$

Furthermore, for $n\in\mathbb{N}$, we have

$$\begin{aligned}
&B_{n,\lambda}^{(s,\alpha)}(x,y) \\
&\quad=\sum_{k=1}^{n}\sum_{m=0}^{[\frac{k-1}{2}]}\binom{n}{k}B_{n-k,\lambda}^{(\alpha)}(x)(-1)^m\lambda^{k-2m-1}y^{2m+1}S_1(k,2m+1).
\end{aligned}$$

For $k \in \mathbb{N}$, let $\alpha = -k$. Then, by (2.25), we get

$$\begin{aligned}
&\sum_{n=0}^{\infty} B_{n,\lambda}^{(c,-k)}(x,y)\frac{t^n}{n!} \\
&= \frac{k!}{t^k}\frac{1}{k!}\left(e_\lambda^{\frac{1}{2}}(t) - e_\lambda^{-\frac{1}{2}}(t)\right)^k e_\lambda^x(t)\cos_\lambda^{(y)}(t) \\
&= \sum_{l=0}^{\infty} \frac{T_\lambda(l+k,k|x)}{\binom{l+k}{k}}\frac{t^l}{l!}\sum_{j=0}^{\infty}\left(\sum_{m=0}^{[\frac{j}{2}]}(-1)^m y^{2m}\lambda^{j-2m}S_1(j,2m)\right)\frac{t^j}{j!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\sum_{m=0}^{[\frac{j}{2}]}\frac{\binom{n}{j}}{\binom{n-j+k}{k}}T_\lambda(n-j+k,k|x)(-1)^m y^{2m}\lambda^{j-2m}S_1(j,2m)\right)\frac{t^n}{n!}.
\end{aligned} \tag{2.31}$$

Therefore, by (2.31), we obtain the following theorem.

Theorem 2.8 *For $k \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{0\}$, we have*

$$B_{n,\lambda}^{(c,-k)}(x,y) = \sum_{j=0}^{n}\sum_{m=0}^{[\frac{j}{2}]}\frac{\binom{n}{j}}{\binom{n-j+k}{k}}T_\lambda(n-j+k,k|x)(-1)^m y^{2m}\lambda^{j-2m}S_1(j,2m).$$

From (1.11), we define the type 2 degenerate Euler polynomials of complex variable by

$$\frac{2}{e_\lambda^{\frac{1}{2}}(t) + e_\lambda^{-\frac{1}{2}}(t)}e_\lambda^{x+iy}(t) = \sum_{n=0}^{\infty} E_{n,\lambda}(x+iy)\frac{t^n}{n!}. \tag{2.32}$$

From (2.32), we have

$$\sum_{n=0}^{\infty}\left(\frac{E_{n,\lambda}(x+iy) + E_{n,\lambda}(x-iy)}{2}\right)\frac{t^n}{n!} = \frac{2e_\lambda^x(t)}{e_\lambda^{\frac{1}{2}}(t) + e_\lambda^{-\frac{1}{2}}(t)}\cos_\lambda^{(y)}(t), \tag{2.33}$$

and

$$\sum_{n=0}^{\infty}\left(\frac{E_{n,\lambda}(x+iy) - E_{n,\lambda}(x-iy)}{2i}\right)\frac{t^n}{n!} = \frac{2e_\lambda^x(t)}{e_\lambda^{\frac{1}{2}}(t) + e_\lambda^{-\frac{1}{2}}(t)}\sin_\lambda^{(y)}(t). \tag{2.34}$$

Now, we define the type 2 degenerate cosine-Euler and type 2 degenerate sine-Euler polynomials as follows:

$$\frac{2}{e_\lambda^{\frac{1}{2}}(t) + e_\lambda^{-\frac{1}{2}}(t)}e_\lambda^x(t)\cos_\lambda^{(y)}(t) = \sum_{n=0}^{\infty} E_{n,\lambda}^{(c)}(x,y)\frac{t^n}{n!} \tag{2.35}$$

and

$$\frac{2}{e_\lambda^{\frac{1}{2}}(t) + e_\lambda^{-\frac{1}{2}}(t)}e_\lambda^x(t)\sin_\lambda^{(y)}(t) = \sum_{n=0}^{\infty} E_{n,\lambda}^{(s)}(x,y)\frac{t^n}{n!}. \tag{2.36}$$

By (1.11), we see that

$$\begin{aligned}
&\frac{2}{e_\lambda^{\frac{1}{2}}(t)+e_\lambda^{-\frac{1}{2}}(t)}e_\lambda^x(t)\cos_\lambda^{(y)}(t) \\
&\quad=\sum_{l=0}^{\infty}E_{l,\lambda}(x)\frac{t^l}{l!}\cos_\lambda^{(y)}(t) \\
&\quad=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{m=0}^{[\frac{k}{2}]}\binom{n}{k}E_{n-k,\lambda}(x)(-1)^m\lambda^{k-2m}y^{2m}S_1(k,2m)\right)\frac{t^n}{n!}
\end{aligned}\tag{2.37}$$

and

$$\begin{aligned}
&\frac{2}{e_\lambda^{\frac{1}{2}}(t)+e_\lambda^{-\frac{1}{2}}(t)}e_\lambda^x(t)\sin_\lambda^{(y)}(t) \\
&\quad=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{m=0}^{[\frac{k-1}{2}]}\binom{n}{k}E_{n-k,\lambda}(x)(-1)^m\lambda^{k-2m-1}y^{2m+1}S_1(k,2m+1)\right)\frac{t^n}{n!}.
\end{aligned}\tag{2.38}$$

Therefore, by (2.35), (2.36), (2.37), and (2.38), we obtain the following theorem.

Theorem 2.9 *For $n\in\mathbb{N}\cup\{0\}$, we have*

$$E_{n,\lambda}^{(c)}(x,y)=\sum_{k=0}^{n}\sum_{m=0}^{[\frac{k}{2}]}\binom{n}{k}E_{n-k,\lambda}(x)(-1)^m\lambda^{k-2m}y^{2m}S_1(k,2m).$$

Moreover, for $n\in\mathbb{N}$,

$$E_{n,\lambda}^{(s)}(x,y)=\sum_{k=0}^{n}\sum_{m=0}^{[\frac{k-1}{2}]}\binom{n}{k}E_{n-k,\lambda}(x)(-1)^m\lambda^{k-2m-1}y^{2m+1}S_1(k,2m+1).$$

By replacing t by $\frac{1}{\lambda}(e^{\lambda t}-1)$ in (2.32), we get

$$\begin{aligned}
\frac{2}{e^{\frac{t}{2}}+e^{-\frac{t}{2}}}e^{(x+iy)t}&=\sum_{k=0}^{\infty}E_{k,\lambda}(x+iy)\lambda^{-k}\frac{1}{k!}\left(e^{\lambda t}-1\right)^k \\
&=\sum_{k=0}^{\infty}E_{k,\lambda}(x+iy)\lambda^{-k}\sum_{n=k}^{\infty}S_2(n,k)\lambda^n\frac{t^n}{n!} \\
&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}E_{k,\lambda}(x+iy)S_2(n,k)\lambda^{n-k}\right)\frac{t^n}{n!}.
\end{aligned}\tag{2.39}$$

On the other hand,

$$\frac{2}{e^{\frac{t}{2}}+e^{-\frac{t}{2}}}e^{(x+iy)t}=\sum_{n=0}^{\infty}E_n(x+iy)\frac{t^n}{n!}.\tag{2.40}$$

Therefore, by (2.39) and (2.40), we obtain the following theorem.

Theorem 2.10 *For $n \geq 0$, we have*

$$E_n(x+iy) = \sum_{k=0}^{n} E_{k,\lambda}(x+iy) S_2(n,k) \lambda^{n-k}.$$

From (2.40), we can easily derive the following equation:

$$\begin{aligned} \sum_{n=0}^{\infty} \left(\frac{E_n(x+iy)+E_n(x-iy)}{2} \right) \frac{t^n}{n!} &= \frac{2}{e^{\frac{t}{2}}+e^{-\frac{t}{2}}} e^{xt} \cos yt \\ &= \sum_{l=0}^{\infty} E_l(x) \frac{t^l}{l!} \sum_{m=0}^{\infty} \frac{(-1)^m y^{2m}}{(2m)!} t^{2m} \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{[\frac{n}{2}]} \binom{n}{2m} E_{n-2m}(x)(-1)^m y^{2m} \right) \frac{t^n}{n!}. \end{aligned} \tag{2.41}$$

By (2.41), we get

$$\frac{E_n(x+iy)+E_n(x-iy)}{2} = \sum_{m=0}^{[\frac{n}{2}]} \binom{n}{2m} E_{n-2m}(x)(-1)^m y^{2m}, \tag{2.42}$$

where n is a nonnegative integer. From Theorem 2.10 and (2.42), we have

$$\begin{aligned} \sum_{m=0}^{[\frac{n}{2}]} \binom{n}{2m} E_{n-2m}(x)(-1)^m y^{2m} &= \sum_{k=0}^{n} S_2(n,k)\lambda^{n-k} \left(\frac{E_{n,\lambda}(x+iy)+E_{n,\lambda}(x-iy)}{2} \right) \\ &= \sum_{k=0}^{n} S_2(n,k)\lambda^{n-k} \sum_{l=0}^{k} \sum_{m=0}^{[\frac{l}{2}]} \binom{k}{l} E_{k-l,\lambda}(x)(-1)^m \lambda^{l-2m} y^{2m} S_1(l,2m) \\ &= \sum_{k=0}^{n} \sum_{l=0}^{k} \sum_{m=0}^{[\frac{l}{2}]} S_2(n,k)\lambda^{n+l-k-2m} \binom{k}{l} E_{k-l,\lambda}(x)(-1)^m y^{2m} S_1(l,2m). \end{aligned} \tag{2.43}$$

Thus, by (2.43), we get

$$\begin{aligned} \sum_{m=0}^{[\frac{n}{2}]} \binom{n}{2m} E_{n-2m}(x)(-1)^m y^{2m} &= \sum_{k=0}^{n} \sum_{l=0}^{k} \sum_{m=0}^{[\frac{l}{2}]} S_2(n,k)\lambda^{n+l-k-2m} \binom{k}{l} E_{k-l,\lambda}(x)(-1)^m y^{2m} S_1(l,2m). \end{aligned}$$

3 Conclusions

In [9], the authors considered the degenerate Bernoulli and degenerate Euler polynomials of complex variable. By treating the real and imaginary parts separately, they were able to introduce the degenerate cosine-Bernoulli polynomials, degenerate sine-Bernoulli polynomials, degenerate cosine-Euler polynomials, and degenerate sine-Euler polynomials and derived some interesting results for them. Actually, the degenerate Euler polynomials of complex variable are degenerate versions of the so-called 'new type Euler polynomials' studied by Masjed-Jamei, Beyki, and Koepf in [16]. Furthermore, the results in [9] gave an affirmative answer to the question asked by Hacène Belbachir in Mathematical Reviews (MR3808565): "Is it possible to obtain their results by considering the classical Euler polynomials of complex variable z and treating the real part and the imaginary part separately?"

Carlitz [1, 2] initiated the study of degenerate versions of Bernoulli and Euler polynomials. As it turns out (see [3–5, 9–12, 14] and the references therein), studying degenerate versions of some special polynomials and numbers has been very fruitful and promising. This idea of considering degenerate versions of some special polynomials is not only limited to polynomials but also can be extended to transcendental functions like gamma functions [11].

In Sect. 2, we studied the type 2 degenerate Bernoulli and type 2 degenerate Euler polynomials of complex variable, of which the latter are degenerate and type 2 versions of the aforementioned new type Euler polynomials studied in [16]. By treating the real and imaginary parts separately, the type 2 degenerate cosine-Bernoulli and type 2 degenerate sine-Bernoulli polynomials were introduced. They were expressed in terms of the type 2 degenerate Bernoulli polynomials and Stirling numbers of the first kind. In addition, they were represented in terms of the type 2 Bernoulli polynomials and Stirling numbers of the first kind. Identities involving the type 2 degenerate cosine-polynomials (or the type 2 degenerate sine-polynomials) and Stirling numbers of the first kind were obtained. Another identity connecting the type 2 degenerate cosine-Bernoulli polynomials, Stirling numbers of the second kind, and the type 2 Bernoulli polynomials was derived. As natural extensions of the type 2 degenerate cosine-Bernoulli and type 2 degenerate sine-Bernoulli polynomials, the type 2 degenerate cosine-Bernoulli and type 2 degenerate sine-Bernoulli polynomials of order α were introduced. They were expressed in terms of the type 2 degenerate Bernoulli polynomials of order α and Stirling numbers of the second kind. In addition, the type 2 degenerate cosine-Bernoulli polynomials of negative order were represented in terms of the degenerate central factorial polynomials of the second kind and Stirling numbers of the first kind. Moreover, the type 2 degenerate cosine-Euler and type 2 degenerate sine-Euler polynomials were investigated, and analogous results to the type 2 degenerate cosine-Bernoulli and type 2 degenerate sine-Bernoulli polynomials were obtained for them.

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Authors' contributions
TK and DSK conceived of the framework and structured the whole paper; TK wrote the paper. All the authors read and approved the final manuscript.

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