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Local and global bifurcation of steady states to a general Brusselator model

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Abstract

In this paper, we consider the local and global bifurcation of nonnegative nonconstant solutions of a general Brusselator model

$$\begin{cases} -d_1\Delta u = a-(b+1)f(u)+u^2v, & x\in\Omega, \\ -d_2\Delta v = bf(u)-u^2v, & x\in\Omega, \\ \frac{\partial u}{\partial n}=\frac{\partial v}{\partial n}=0, & x\in\partial\Omega, \end{cases}$$

where $d_1,d_2,a>0$ are fixed parameters with $d_2>d_1$, $b>0$ is a bifurcation parameter; $f\in C([0,\infty),[0,\infty))$ is a strictly increasing function and $f'(f^{-1}(a))\in(0,\infty)$. Moreover, via the Rabinowitz bifurcation theorem, we obtain the global structure of nonconstant solutions under the condition that $\frac{f(s)}{s^2}$ is nonincreasing in $(0,\infty)$.

MSC: 34B15; 34C23; 35B45

Keywords: Brusselator model; Nonconstant steady state solutions; Local bifurcation; Global bifurcation

1 Introduction

In 1968, Prigogine and Lefever [15] introduced firstly the Brusselator model for a chemical reaction-diffusion of self-catalysis as follows:

$$\begin{cases} \frac{\partial u}{\partial t}-d_1\Delta u = a-(b+1)u+u^2v, & x\in\Omega, t>0, \\ \frac{\partial v}{\partial t}-d_2\Delta v = bu-u^2v, & x\in\Omega, t>0, \\ \frac{\partial u}{\partial n}=\frac{\partial v}{\partial n}=0, & x\in\partial\Omega, t>0, \end{cases} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^N$ $(N\geq 1)$ is a smooth and bounded domain, n denotes the outward unit normal vector on $\partial\Omega$, u and v represent the concentration of two intermediary reactants having the diffusion rates $d_1,d_2\in(0,\infty)$ with $d_2>d_1$, $a,b>0$ are the fixed concentrations. Indeed, (1.1) has been extensively investigated in the last decades from both analytical and numerical point of view (see [1–8, 11–14, 17, 18]). Most of the authors are interested in finding spatially nonconstant solutions of the equilibrium problem

$$\begin{cases} -d_1\triangle u = a - (b+1)u + u^2 v, & x \in \Omega, \\ -d_2\triangle v = bu - u^2 v, & x \in \Omega, \\ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0, & x \in \partial\Omega. \end{cases} \tag{1.2}$$

In [3, 4, 13, 14], they obtained the existence or nonexistence of the nonconstant solutions of (1.2) by a priori estimate and topological degree theory. Peng and Wang [13] considered the following problem:

$$\begin{cases} -\theta\triangle u = \lambda(1 - (b+1)u + bu^2 v), & x \in \Omega, \\ -\triangle v = \lambda a^2(u - u^2 v), & x \in \Omega, \\ \partial_n u = \partial_n v = 0, & x \in \partial\Omega, \end{cases} \tag{1.3}$$

and proved the nonexistence for nonconstant of (1.3) for either small λ, large θ, or small b. Note that [3, 4, 13, 14] only studied the existence and nonexistence of nonnegative nonconstant solutions of (1.2). They could not get the global structure of the nonconstant solutions due to the limitations of the tools used. Ma and Hu [11] applied the Rabinowitz bifurcation theorem to get the global structure of nonconstant solutions of (1.2). Inspired by [11], we will consider the new, more general form of the Brusselator model:

$$\begin{cases} -d_1\triangle u = a - (b+1)f(u) + u^2 v, & x \in \Omega, \\ -d_2\triangle v = bf(u) - u^2 v, & x \in \Omega, \\ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0, & x \in \partial\Omega, \end{cases} \tag{1.4}$$

where $d_1, d_2, a > 0$ are fixed parameters and $d_2 > d_1$, $b > 0$ is a bifurcation parameter. Clearly, $f(u) = \frac{f(u)}{u} \cdot u$, then (1.4) is seen to be equivalent to

$$\begin{cases} -d_1\triangle u = a - (b+1)\frac{f(u)}{u} \cdot u + u^2 v, & x \in \Omega, \\ -d_2\triangle v = b\frac{f(u)}{u} \cdot u - u^2 v, & x \in \Omega, \\ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0, & x \in \partial\Omega. \end{cases} \tag{1.5}$$

Compared with problem (1.2), $\frac{f(u)}{u}$ can be regarded as a variable coefficient. It is well known that the linear terms $(b+1)u$ and bu in (1.2) cannot withstand any small perturbation. In fact, (1.5) has been widely applied in chemical and biological fields. We will study the local and global behavior of nonnegative nonconstant solutions of (1.4) under the following assumptions:

(H1) $f \in C([0,\infty), [0,\infty))$ is a strictly increasing function.
(H2) $f'(f^{-1}(a)) \in (0,\infty)$.
(H3) $\frac{f(s)}{s^2}$ is nonincreasing in $(0,\infty)$.

Remark 1.1 If $f(u) = u$, then (1.4) will reduce to (1.2). However, if $f(u) = u + u^2$, a perturbation term is added to bu and $(b+1)u$ in (1.2). It is easy to see that this small perturbation leads to the results in [11] that are not available.

The rest of the paper is organized as follows: In Sect. 2, we give a priori estimate and some preliminary results. Section 3 is devoted to studying the local bifurcation of nonnegative nonconstant solutions of (1.4) with N = 1 under conditions (H1)–(H2). Finally, in Sect. 4, we add condition (H3) to obtain the global bifurcation of nonnegative nonconstant solutions of (1.4) with N = 1.

2 Preliminary results

At first, let us look for the constant solution of (1.4). To get it, it suffices to look for the constant solution of the following problem:

$$\begin{cases} a-(b+1)f(u(x))+u^2(x)v(x)=0, & x\in\Omega,\\ bf(u(x))-u^2(x)v(x)=0, & x\in\Omega. \end{cases} \tag{2.1}$$

By (H1), problem (2.1) has a unique solution $(f^{-1}(a),\frac{ab}{[f^{-1}(a)]^2})$. Obviously, this is the unique solution of (1.4).

Basic to a priori estimate of the solutions of (1.4) is the following result which is due to Lou and Ni (see [9, Proposition 2.2] or [10, Lemma 2.1]).

Lemma 2.1 *Let $g\in C^1(\bar{\Omega}\times\mathbb{R})$.*

(1) *If $w\in C^2(\Omega)\cap C^1(\bar{\Omega})$ satisfies*

$$\triangle w+g(x,w)\geq 0 \quad in\ \Omega, \qquad \frac{\partial w}{\partial n}\leq 0 \quad on\ \partial\Omega,$$

and $w(x_0)=\max_{\bar{\Omega}} w$, then $g(x_0,w(x_0))\geq 0$.

(2) *If $w\in C^2(\Omega)\cap C^1(\bar{\Omega})$ satisfies*

$$\triangle w+g(x,w)\leq 0 \quad in\ \Omega, \qquad \frac{\partial w}{\partial n}\geq 0 \quad on\ \partial\Omega,$$

and $w(x_0)=\min_{\bar{\Omega}} w$, then $g(x_0,w(x_0))\leq 0$.

Now, we will give a priori estimate of the nonnegative nonconstant solutions of (1.4).

Lemma 2.2 *Let* (H1), (H2), *and* (H3) *hold. Then any nonnegative nonconstant solution (u,v) of* (1.4) *satisfies*

$$f^{-1}\left(\frac{a}{b+1}\right)\leq u(x)\leq f^{-1}(a)+\frac{d_2}{d_1}\cdot\frac{ab}{(b+1)[f^{-1}(\frac{a}{b+1})]^2}, \quad x\in\Omega,$$

$$\frac{bf(f^{-1}(a)+\frac{d_2}{d_1}\cdot\frac{ab}{(b+1)[f^{-1}(\frac{a}{b+1})]^2})}{[f^{-1}(a)+\frac{d_2}{d_1}\cdot\frac{ab}{(b+1)[f^{-1}(\frac{a}{b+1})]^2}]^2}\leq v(x)\leq\frac{ab}{(b+1)[f^{-1}(\frac{a}{b+1})]^2}, \quad x\in\Omega.$$

Proof Let $x_0\in\bar{\Omega}$ be the minimum point of u. From (2) of Lemma 2.1, we have

$$a-(b+1)f\bigl(u(x_0)\bigr)+u^2(x_0)v(x_0)\leq 0,$$

$$a-(b+1)f\bigl(u(x_0)\bigr)\leq 0,$$

$$f\big(u(x_0)\big) \geq \frac{a}{b+1}.$$

Then $u(x_0) \geq f^{-1}(\frac{a}{b+1})$ by (H1), and so

$$u(x) \geq u(x_0) \geq f^{-1}\left(\frac{a}{b+1}\right), \quad x \in \Omega. \tag{2.2}$$

Let $x_1 \in \bar{\Omega}$ be the maximum point of v. Similarly, we can get that

$$bf\big(u(x_1)\big) - u^2(x_1)v(x_1) \geq 0.$$

Then

$$v(x) \leq v(x_1) \leq \frac{bf(u(x_1))}{u^2(x_1)}, \quad x \in \Omega.$$

Combining this with (2.2), from (H3), we show

$$v(x) \leq \frac{bf(u(x_1))}{u^2(x_1)} \leq \frac{ab}{(b+1)[f^{-1}(\frac{a}{b+1})]^2}, \quad x \in \Omega. \tag{2.3}$$

Let $w = d_1 u + d_2 v$. Then it follows from (1.4) that

$$\begin{cases} -\triangle w(x) = a - f(u(x)), & x \in \Omega, \\ \frac{\partial w}{\partial n} = 0, & x \in \partial\Omega. \end{cases}$$

Now, let $x_2 \in \bar{\Omega}$ be the maximum point of w. By (1) of Lemma 2.1, $a - f(u(x_2)) \geq 0$. Then, from (H1), it is easy to see $u(x_2) \leq f^{-1}(a)$. Combining this with (2.3), we know that, for any $x \in \bar{\Omega}$,

$$d_1 u(x) \leq w(x) \leq w(x_2) \leq d_1 f^{-1}(a) + d_2 \cdot \frac{ab}{(b+1)[f^{-1}(\frac{a}{b+1})]^2},$$

then

$$u(x) \leq f^{-1}(a) + \frac{d_2}{d_1} \cdot \frac{ab}{(b+1)[f^{-1}(\frac{a}{b+1})]^2}, \quad x \in \Omega.$$

From (2) of Lemma 2.1, if $x_3 \in \bar{\Omega}$ is the minimum point of v, then

$$bf\big(u(x_3)\big) - u^2(x_3)v(x_3) \leq 0,$$

and

$$v(x) \geq v(x_3) \geq \frac{bf(u(x_3))}{u^2(x_3)} \geq \frac{bf(f^{-1}(a) + \frac{d_2}{d_1} \cdot \frac{ab}{(b+1)[f^{-1}(\frac{a}{b+1})]^2})}{[f^{-1}(a) + \frac{d_2}{d_1} \cdot \frac{ab}{(b+1)[f^{-1}(\frac{a}{b+1})]^2}]^2}, \quad x \in \Omega.$$

Consequently, the proof is completed. □

For any fixed $l > 0$. It is well known that

$$\begin{cases} -\varphi'' = \mu\varphi, & x \in (0,l), \\ \varphi'(0) = \varphi'(l) = 0 \end{cases}$$

has a sequence of simple eigenvalues

$$\mu_j = \left(\frac{j\pi}{l}\right)^2, \quad j = 0, 1, 2, \ldots,$$

the corresponding eigenfunctions are

$$\varphi_j(x) = \begin{cases} 1, & j = 0, \\ \cos(\frac{j\pi x}{l}), & j > 0. \end{cases} \tag{2.4}$$

Let

$$X := \{(u,v) : u, v \in C^2[0,l], u'(0) = u'(l) = v'(0) = v'(l) = 0\}, \qquad Y := L^2(0,l) \times L^2(0,l).$$

X constitutes the Banach space in C^2 norm and Y is a Hilbert space based on the inner product

$$(w_1, w_2)_Y = (u_1, u_2)_{L^2(0,l)} + (v_1, v_2)_{L^2(0,l)},$$

where $w_1 = (u_1, v_1)$, $w_2 = (u_2, v_2) \in Y$.

3 Local bifurcation

For simplicity, let us consider (1.4) with $N = 1$ and $\Omega = (0,l)$,

$$\begin{cases} -d_1 u'' = a - (b+1)f(u) + u^2 v, & x \in (0,l), \\ -d_2 v'' = bf(u) - u^2 v, & x \in (0,l), \\ u'(0) = u'(l) = v'(0) = v'(l) = 0. \end{cases} \tag{3.1}$$

Clearly, $\bar{w} := (f^{-1}(a), \frac{ab}{[f^{-1}(a)]^2})$ is the unique constant solution of (3.1).

Define the mapping $P : (0,\infty) \times X \to Y$,

$$P(b,w) = \begin{pmatrix} d_1 u'' + a - (b+1)f(u) + u^2 v \\ d_2 v'' + bf(u) - u^2 v \end{pmatrix}.$$

For the fixed $b > 0$, $w = (u,v)$ is a solution of (3.1) if and only if (b,w) is a zero-point of P. Note that $P(b,\bar{w}) = \binom{0}{0}$ for any $b > 0$, since $\bar{w}$ is the constant solution of (3.1). Let

$$\begin{cases} u = f^{-1}(a) + \sum_{k=1}^{\infty} \varepsilon^k u_k, \\ v = \frac{ab}{[f^{-1}(a)]^2} + \sum_{k=1}^{\infty} \varepsilon^k v_k, \end{cases} \tag{3.2}$$

and

$$b = b_0 + \sum_{k=1}^{\infty} \varepsilon^k b_k. \tag{3.3}$$

We also have to Taylor expand f at the point $f^{-1}(a)$. The purpose of the rest of this section is to solve b_0 and prove that $(b_0, \bar{w})$ is the bifurcation point of $P(b,w) = \binom{0}{0}$. First of all, we substitute (3.2) and (3.3) into (3.1) and let the higher-order term of ε be equal to 0. Then we can get the problem

$$\begin{cases} -d_1 u_1'' = (-b_0 f'(f^{-1}(a)) - f'(f^{-1}(a)) + \frac{2ab_0}{f^{-1}(a)})u_1 + [f^{-1}(a)]^2 v_1, & x \in (0,l), \\ -d_2 v_1'' = (b_0 f'(f^{-1}(a)) - \frac{2ab_0}{f^{-1}(a)})u_1 - [f^{-1}(a)]^2 v_1, & x \in (0,l), \\ u_1'(0) = u_1'(l) = v_1'(0) = v_1'(l) = 0. \end{cases} \tag{3.4}$$

In (3.4), by using the undetermined coefficient method, it follows that

$$\begin{aligned} b_0 &= \frac{f^{-1}(a)\{d_1 d_2 \mu_j^2 + d_2 f'(f^{-1}(a))\mu_j + d_1[f^{-1}(a)]^2 \mu_j + [f^{-1}(a)]^2 f'(f^{-1}(a))\}}{2ad_2\mu_j - d_2 f^{-1}(a) f'(f^{-1}(a))\mu_j} \\ &:= b_0^j, \quad j = 1, 2, \ldots. \end{aligned}$$

Moreover, it is not difficult to prove that (3.4) has a nontrivial solution (u_1, v_1),

$$\begin{cases} u_1 = c_1(j)\cos(\frac{j\pi}{l}x) = c_1(j)\varphi_j(x), \qquad c_1(j) = -\frac{d_2\mu_j}{d_1\mu_j + f'(f^{-1}(a))}, \\ v_1 = \cos(\frac{j\pi}{l}x) = \varphi_j(x). \end{cases}$$

Next, we substitute (3.2) and (3.3) into (3.1) and let the higher-order term of ε^2 be equal to 0, then (3.1) becomes the following system:

$$\begin{cases} d_1 u_2'' + (-b_0 f'(f^{-1}(a)) - f'(f^{-1}(a)) + \frac{2ab_0}{f^{-1}(a)})u_2 + [f^{-1}(a)]^2 v_2 = -F_1, & x \in (0,l), \\ d_2 v_2'' + (b_0 f'(f^{-1}(a)) - \frac{2ab_0}{f^{-1}(a)})u_2 - [f^{-1}(a)]^2 v_2 = F_1, & x \in (0,l), \\ u_2'(0) = u_2'(l) = v_2'(0) = v_2'(l) = 0, \end{cases} \tag{3.5}$$

where

$$F_1 = \left(\frac{2ab_1}{f^{-1}(a)} - b_1 f'\big(f^{-1}(a)\big)\right)u_1 + 2f^{-1}(a)u_1 v_1 + \frac{ab_0}{[f^{-1}(a)]^2}u_1^2.$$

In order to solve b_1 from (3.5), let us consider the following adjoint system of the homogeneous system related to (3.5):

$$\begin{cases} d_1 y_2'' + (-b_0 f'(f^{-1}(a)) - f'(f^{-1}(a)) + \frac{2ab_0}{f^{-1}(a)})y_2 \\ \quad + (b_0 f'(f^{-1}(a)) - \frac{2ab_0}{f^{-1}(a)})z_2 = 0, & x \in (0,l), \\ d_2 z_2'' + [f^{-1}(a)]^2 y_2 - [f^{-1}(a)]^2 z_2 = 0, & x \in (0,l), \\ y_2'(0) = y_2'(l) = z_2'(0) = z_2'(l) = 0. \end{cases} \tag{3.6}$$

It is not difficult to verify that (3.6) has a solution (y_2, z_2),

$$\begin{cases} y_2 = c_2(j)\cos(\frac{j\pi}{l}x) = c_2(j)\varphi_j(x), \qquad c_2(j) = 1 + \frac{d_2\mu_j}{[f^{-1}(a)]^2}, \\ z_2 = \cos(\frac{j\pi}{l}x) = \varphi_j(x). \end{cases}$$

It is obvious that the vectors $(-F_1, F_1)$ and (y_2, z_2) should be orthogonal in $L^2(0, l)$ by virtue of the solvability condition for (3.5), i.e.,

$$\int_0^l (z_2 - y_2)F_1\,dx = 0.$$

In fact,

$$\begin{aligned} &\int_0^l (z_2 - y_2)F_1\,dx \\ &\quad = \int_0^l -\frac{d_2\mu_j}{[f^{-1}(a)]^2}\biggl[\biggl(\frac{2ab_1}{f^{-1}(a)} - b_1 f'\bigl(f^{-1}(a)\bigr)\biggr)u_1 \\ &\qquad + 2f^{-1}(a)u_1v_1 + \frac{ab_0}{[f^{-1}(a)]^2}u_1^2\biggr]\cos\biggl(\frac{j\pi x}{l}\biggr)\,dx \\ &\quad = 0. \end{aligned} \tag{3.7}$$

Let us substitute b_0, u_1, and v_1 into (3.7), then $b_1^j := b_1 = 0$, and so F_1 will reduce to

$$\begin{aligned} F_1 &= 2f^{-1}(a)c_1(j)\varphi_j^2(x) + \frac{ab_0}{[f^{-1}(a)]^2}c_1^2(j)\varphi_j^2(x) \\ &= 2f^{-1}(a)c_1(j)\cos^2\biggl(\frac{j\pi}{l}x\biggr) + \frac{ab_0}{[f^{-1}(a)]^2}c_1^2(j)\cos^2\biggl(\frac{j\pi}{l}x\biggr) \\ &= f^{-1}(a)c_1(j)\biggl(\cos\biggl(\frac{2j\pi}{l}x\biggr) + 1\biggr) + \frac{1}{2}\cdot\frac{ab_0}{[f^{-1}(a)]^2}c_1^2(j)\biggl(\cos\biggl(\frac{2j\pi}{l}x\biggr) + 1\biggr) \\ &= \frac{1}{2}\biggl[2f^{-1}(a)c_1(j) + \frac{ab_0}{[f^{-1}(a)]^2}c_1^2(j)\biggr] + \frac{1}{2}\biggl[2f^{-1}(a)c_1(j) + \frac{ab_0}{[f^{-1}(a)]^2}c_1^2(j)\biggr]\varphi_{2j}(x). \end{aligned}$$

Therefore, a particular solution (u_2, v_2) of (3.5) can be obtained as follows:

$$\begin{cases} u_2 = a_1(j) + a_2(j)\cos(\frac{2j\pi}{l}x) = a_1(j) + a_2(j)\varphi_{2j}(x), \\ v_2 = a_3(j) + a_4(j)\cos(\frac{2j\pi}{l}x) = a_3(j) + a_4(j)\varphi_{2j}(x), \end{cases}$$

where

$$\begin{aligned} a_2(j) = {}& \biggl(\bigl[f^{-1}(a)\bigr]^2 c_1 d_2\mu_{2j} + \frac{ab_0\,d_2\mu_{2j}c_1^2}{2f^{-1}(a)}\biggr) \\ &/\Bigl(d_1d_2\mu_{2j}^2 f^{-1}(a) + f'\bigl(f^{-1}(a)\bigr)f^{-1}(a)\bigl(\bigl[f^{-1}(a)\bigr]^2 + d_2\mu_{2j}(1 + b_0)\bigr) \\ &- \bigl(2ab_0d_2 - d_1\bigl[f^{-1}(a)\bigr]^3\bigr)\mu_{2j}\Bigr), \end{aligned}$$

$$a_1(j) = 0, \qquad a_3(j) = -\frac{c_1}{[f^{-1}(a)]^2}\left(f^{-1}(a) + \frac{ab_0}{2[f^{-1}(a)]^2}c_1\right),$$
$$a_4(j) = -\frac{d_1\mu_{2j} + f'(f^{-1}(a))}{d_2\mu_{2j}}a_2(j).$$

Since $b_1 = 0$, we have to solve b_2. We substitute (3.2) and (3.3) into (3.1) and let the higher-order term of ε^3 be equal to 0, then a problem similar to (3.5) is obtained:

$$\begin{cases} d_1u_3'' + (-b_0f'(f^{-1}(a)) - f'(f^{-1}(a)) + \frac{2ab_0}{f^{-1}(a)})u_3 + [f^{-1}(a)]^2v_3 = -F_2, & x \in (0,l), \\ d_2v_3'' + (b_0f'(f^{-1}(a)) - \frac{2ab_0}{f^{-1}(a)})u_3 - [f^{-1}(a)]^2v_3 = F_2, & x \in (0,l), \\ u_3'(0) = u_3'(l) = v_3'(0) = v_3'(l) = 0, \end{cases} \tag{3.8}$$

where

$$F_2 = \left(-b_2f'\left(f^{-1}(a)\right) + \frac{2ab_2}{f^{-1}(a)}\right)u_1 + 2f^{-1}(a)u_1v_2 + 2f^{-1}(a)u_2v_1 + u_1^2v_1 + \frac{2ab_0}{[f^{-1}(a)]^2}u_1u_2.$$

Clearly, (3.6) is also the adjoint system of the homogeneous system related to (3.8), then

$$\int_0^l (z_2 - y_2)F_2\,dx = 0,$$

and, according to values of u_1, u_2, v_1, and v_2, we have

$$\begin{aligned} &\int_0^l (z_2 - y_2)F_2\,dx \\ &= \left(-b_2f'\left(f^{-1}(a)\right) + \frac{2ab_2}{f^{-1}(a)}\right)c_1\int_0^l \cos\left(\frac{j\pi}{l}x\right)dx \\ &\quad + 2f^{-1}(a)\int_0^l c_1\left(a_3 + a_4\cos\left(\frac{2j\pi}{l}x\right)\right)\cdot\cos^2\left(\frac{j\pi}{l}x\right)dx \\ &\quad + 2f^{-1}(a)a_2\int_0^l \cos\left(\frac{2j\pi}{l}x\right)\cdot\cos^2\left(\frac{j\pi}{l}x\right)dx + \int_0^l c_1^2\cos^4\left(\frac{j\pi}{l}x\right)dx \\ &\quad + \frac{2ab_0c_1a_2}{[f^{-1}(a)]^2}\int_0^l \cos^2\left(\frac{j\pi}{l}x\right)\cdot\cos\left(\frac{2j\pi}{l}x\right)dx \\ &= 0. \end{aligned}$$

Thus,

$$b_2 = -\frac{[f^{-1}(a)]^3(2c_1a_3 + c_1a_4 + a_2) + 3c_1^2[f^{-1}(a)]^2 + ab_0^jc_1a_2}{2ac_1f^{-1}(a) - f'(f^{-1}(a))[f^{-1}(a)]^2c_1} := b_2^j \neq 0, \quad j = 1, 2, \ldots.$$

From the above analysis, we obtain the main result of this section.

Theorem 3.1 *Assume that* (H1) *and* (H2) *hold. Then, for any positive integer j,* $(b_0^j, \bar{w})$ *is a bifurcation point of* $P(b,w) = \binom{0}{0}$. *Moreover, there is a nontrivial solution* $\phi(\varepsilon) = (b(\varepsilon), u(\varepsilon), v(\varepsilon))$ *of* (3.1) *if* ε *is small enough, where b, u, and v are continuous with respect* *to ε, and*

$$u(\varepsilon)=f^{-1}(a)+\varepsilon c_1(j)\varphi_j+\varepsilon^2\big(a_1(j)+a_2(j)\varphi_{2j}\big)+o\big(\varepsilon^2\big),$$

$$v(\varepsilon)=\frac{ab}{[f^{-1}(a)]^2}+\varepsilon\varphi_j+\varepsilon^2\big(a_3(j)+a_4(j)\varphi_{2j}\big)+o\big(\varepsilon^2\big),$$

$$b(\varepsilon)=b_0^j+\varepsilon^2 b_2^j+o\big(\varepsilon^2\big).$$

The set of zero-points of P constitutes two curves in a neighborhood of the bifurcation point $(b_0^j,\bar{w})$.

Let $\mathbb{C}$ be the closure of the nonconstant solution set of $P(b,w)=\binom{0}{0}$, Γ_j be a connected component of $\mathbb{C}\cup\{(b_0^j,\bar{w})\}$, and $(b_0^j,\bar{w})\in\Gamma_j$. In a small neighborhood of bifurcation point $(b_0^j,\bar{w})$, the curve Γ_j is determined by the eigenfunction φ_j, where φ_j has j zeros in the interval $[0,l]$.

4 Global bifurcation

Theorem 4.1 *Let* (H1), (H2), *and* (H3) *hold. If $\mu_j\neq\frac{[f^{-1}(a)]^2}{d_2}$, $j=1,2,\dots$, then the projection of continuum Γ_j is unbounded on the b-axis.*

Proof (3.1) can be written as follows:

$$\begin{cases}-u''(x)=g(u,v), & x\in(0,l),\\ -v''(x)=h(u,v), & x\in(0,l),\\ u'(0)=u'(l)=v'(0)=v'(l)=0,\end{cases}\tag{4.1}$$

where

$$g(u,v)=\frac{1}{d_1}\big(a-(b+1)f(u)+u^2v\big),h(u,v)=\frac{1}{d_2}\big(bf(u)-u^2v\big).$$

Let $\tilde{u}=u-f^{-1}(a)$, $\tilde{v}=v-\frac{ab}{[f^{-1}(a)]^2}$. Then (4.1) is equivalent to the following problem:

$$\begin{cases}-\tilde{u}''=g_0\tilde{u}+g_1\tilde{v}+\tilde{g}(\tilde{u},\tilde{v}), & x\in(0,l),\\ -\tilde{v}''=h_0\tilde{u}+h_1\tilde{v}+\tilde{h}(\tilde{u},\tilde{v}), & x\in(0,l),\\ \tilde{u}'(0)=\tilde{u}'(l)=\tilde{v}'(0)=\tilde{v}'(l)=0,\end{cases}\tag{4.2}$$

where $\tilde{g}$ and $\tilde{h}$ are higher-order terms of $\tilde{u}$, $\tilde{v}$, and

$$g_0=g_u(u,v)|_{(f^{-1}(a),\frac{ab}{[f^{-1}(a)]^2})}=\frac{1}{d_1}\left(-(b+1)f'\big(f^{-1}(a)\big)+\frac{2ab}{f^{-1}(a)}\right),$$

$$g_1=g_v(u,v)|_{(f^{-1}(a),\frac{ab}{[f^{-1}(a)]^2})}=\frac{[f^{-1}(a)]^2}{d_1},$$

$$h_0=h_u(u,v)|_{(f^{-1}(a),\frac{ab}{[f^{-1}(a)]^2})}=\frac{1}{d_2}\left(bf'\big(f^{-1}(a)\big)-\frac{2ab}{f^{-1}(a)}\right),$$

$$h_1=h_v(u,v)|_{(f^{-1}(a),\frac{ab}{[f^{-1}(a)]^2})}=-\frac{[f^{-1}(a)]^2}{d_2}.$$

In this way, we convert the constant solution $\bar{w} = (f^{-1}(a), \frac{ab}{[f^{-1}(a)]^2})$ of (3.1) to the trivial solution $\theta = (0,0)$ of (4.2).

Let $H_1 : Y \to X$ and $H_2 : Y \to X$ be the inverse of operators $\frac{f'(f^{-1}(a))}{d_1} I - \frac{d^2}{dx^2}$ and $\frac{[f^{-1}(a)]^2}{d_2} I - \frac{d^2}{dx^2}$ with Neumann boundary conditions, respectively. Set $U = (\tilde{u}, \tilde{v})$,

$$\begin{aligned} K(b)U = {} & \biggl(\frac{1}{d_1}\biggl[-bf'\bigl(f^{-1}(a)\bigr) + \frac{2ab}{f^{-1}(a)}\biggr]H_1(\tilde{u}) \\ & + \frac{[f^{-1}(a)]^2}{d_1}H_1(\tilde{v}), \frac{1}{d_2}\biggl[bf'\bigl(f^{-1}(a)\bigr) - \frac{2ab}{f^{-1}(a)}\biggr]H_2(\tilde{u})\biggr), \\ W(U) = {} & \bigl(H_1\bigl(\tilde{g}(\tilde{u},\tilde{v})\bigr), H_2\bigl(\tilde{h}(\tilde{u},\tilde{v})\bigr)\bigr). \end{aligned}$$

It can be verified that (4.2) is equivalent to

$$U = K(b)U + W(U) \tag{4.3}$$

in X. For any fixed number $b > 0$, $K(b)$ and $W(U)$ are linear compact operators in X and $W(U) = o(\|U\|)$. By the Rabinowitz global bifurcation theorem [16], we need to verify

(i) 1 is an eigenvalue of $K(b_0^j)$ and its algebraic multiplicity is 1;

(ii) the index of $I - K(b) - W$ changes when b crosses b_0^j.

Now, we will prove (i). Suppose $\Psi = \binom{\xi}{\psi}$, $\xi = \sum a_j\varphi_j$, $\psi = \sum c_j\varphi_j$. Let

$$\bigl(K(b) - I\bigr)\Psi = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

i.e.,

$$\begin{pmatrix} \frac{1}{d_1}[-(b+1)f'(f^{-1}(a)) + \frac{2ab}{f^{-1}(a)}] + \frac{d^2}{dx^2} & \frac{[f^{-1}(a)]^2}{d_1} \\ \frac{1}{d_2}[bf'(f^{-1}(a)) - \frac{2ab}{f^{-1}(a)}] & -\frac{[f^{-1}(a)]^2}{d_2} + \frac{d^2}{dx^2} \end{pmatrix} \Psi = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

thus,

$$\sum_{j=0}^{\infty} L_j \begin{pmatrix} a_j \\ c_j \end{pmatrix} \varphi_j = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where

$$L_j = \begin{pmatrix} \frac{1}{d_1}[-(b+1)f'(f^{-1}(a)) + \frac{2ab}{f^{-1}(a)}] - \mu_j & \frac{[f^{-1}(a)]^2}{d_1} \\ \frac{1}{d_2}[bf'(f^{-1}(a)) - \frac{2ab}{f^{-1}(a)}] & -\frac{[f^{-1}(a)]^2}{d_2} - \mu_j \end{pmatrix}.$$

By computation, $\det L_j = 0$ if and only if $b = b_0^j$, taking $b = b_0^j$ leads to

$$\det L_j = \det \begin{pmatrix} \frac{[f^{-1}(a)]^2}{d_1 d_2 \mu_j}(f'(f^{-1}(a)) + d_1\mu_j) & \frac{[f^{-1}(a)]^2}{d_1 d_2 \mu_j} \cdot d_2\mu_j \\ -\frac{1}{d_2}f'(f^{-1}(a)) - \frac{d_1}{d_2}\mu_j & -\mu_j \end{pmatrix} = 0$$

and

$$L_j\begin{pmatrix}a_j\\c_j\end{pmatrix}=\begin{pmatrix}0&0\\f'(f^{-1}(a))+d_1\mu_j&d_2\mu_j\end{pmatrix}\begin{pmatrix}a_j\\c_j\end{pmatrix}.$$

Then $\ker(K(b_0^j)-I)=\operatorname{span}(\Psi)$, $\Psi=\binom{-d_2\mu_j}{f'(f^{-1}(a))+d_1\mu_j}\varphi_j)$. This implies that 1 is the eigenvalue of $K=K(b_0^j)$ and $\dim\ker(K-I)=1$. The algebraic multiplicity of the eigenvalue 1 is the dimension of the generalized null space $\bigcup_{i=1}^{\infty}\ker(K-I)^i$, therefore, $\ker(K-I)\cap\operatorname{Im}(K-I)=\{\theta^T\}$.

Let K^T be the transposed matrix of K,

$$K^T=\begin{pmatrix}\frac{1}{d_1}[-bf'(f^{-1}(a))+\frac{2ab}{f^{-1}(a)}]H_1 & \frac{1}{d_2}[bf'(f^{-1}(a))-\frac{2ab}{f^{-1}(a)}]H_2\\ \frac{[f^{-1}(a)]^2}{d_1}H_1 & 0\end{pmatrix},$$

and $\Psi^*=\binom{\xi^*}{\psi^*}$, $\xi^*=\sum a_j^*\varphi_j$, $\psi^*=\sum c_j^*\varphi_j$. Suppose $\Psi^*\in\ker(K^T-I)$. Then

$$\begin{cases}\frac{1}{d_1}[-bf'(f^{-1}(a))+\frac{2ab}{f^{-1}(a)}]H_1(\xi^*)+\frac{1}{d_2}[bf'(f^{-1}(a))-\frac{2ab}{f^{-1}(a)}]H_2(\psi^*)=\xi^*,\\ \frac{[f^{-1}(a)]^2}{d_1}H_1(\xi^*)=\psi^*.\end{cases}\tag{4.4}$$

From the definition of H_1, H_2, (4.4) can also be written as

$$\begin{cases}-d_1d_2\xi^{*\prime\prime}=(d_2[-bf'(f^{-1}(a))+\frac{2ab}{f^{-1}(a)}]-[f^{-1}(a)]^2d_1)\xi^*\\ \qquad\qquad -\frac{d_2f'(f^{-1}(a))}{[f^{-1}(a)]^2}[-bf'(f^{-1}(a))+\frac{2ab}{f^{-1}(a)}]\psi^*,\\ -d_1\psi^{*\prime\prime}=[f^{-1}(a)^2]\xi^*-f'(f^{-1}(a))\psi^*.\end{cases}$$

That is to say,

$$\sum_{j=0}^{\infty}B_j\begin{pmatrix}a_j^*\\c_j^*\end{pmatrix}\varphi_j=\begin{pmatrix}0\\0\end{pmatrix},$$

where

$$B_j=\begin{pmatrix}d_2[-bf'(f^{-1}(a))+\frac{2ab}{f^{-1}(a)}]-[f^{-1}(a)]^2d_1-d_1d_2\mu_j & -\frac{d_2f'(f^{-1}(a))}{[f^{-1}(a)]^2}[-bf'(f^{-1}(a))+\frac{2ab}{f^{-1}(a)}]\\ [f^{-1}(a)]^2 & -d_1\mu_j-f'(f^{-1}(a))\end{pmatrix}.$$

Similarly, $\det B_j=0$ if and only if $b=b_0^j$, taking $b=b_0^j$ leads to

$$\det B_j=\det\begin{pmatrix}\frac{[f^{-1}(a)]^2+d_2\mu_j}{[f^{-1}(a)]^2\mu_j}[f^{-1}(a)]^2 & \frac{[f^{-1}(a)]^2+d_2\mu_j}{[f^{-1}(a)]^2\mu_j}(-d_1\mu_j-f'(f^{-1}(a)))\\ [f^{-1}(a)]^2 & -d_1\mu_j-f'(f^{-1}(a))\end{pmatrix}=0$$

and

$$B_j\begin{pmatrix}a_j^*\\c_j^*\end{pmatrix}=\begin{pmatrix}0&0\\ [f^{-1}(a)]^2 & -d_1\mu_j-f'(f^{-1}(a))\end{pmatrix}\begin{pmatrix}a_j^*\\c_j^*\end{pmatrix}.$$

Then $\ker(K^*-I)=\operatorname{span}\binom{d_1\mu_j+f'(f^{-1}(a))}{[f^{-1}(a)]^2}\varphi_j$. According to $\mu_j\neq\frac{[f^{-1}(a)]^2}{d_2}$, we obtain

$$\begin{aligned}(\Psi,\Psi^*)_Y&=\bigl(-d_2\mu_j\varphi_j,\bigl(d_1\mu_j+f'\bigl(f^{-1}(a)\bigr)\varphi_j\bigr)\bigr)_{L^2[0,l]}\\&\quad+\bigl(\bigl(f'\bigl(f^{-1}(a)+d_1\mu_j\bigr)\bigr)\varphi_j,\bigl[f^{-1}(a)\bigr]^2\varphi_j\bigr)_{L^2[0,l]}\\&=\bigl(d_1\mu_j+f'\bigl(f^{-1}(a)\bigr)\bigr)\bigl(\bigl[f^{-1}(a)\bigr]^2-d_2\mu_j\bigr)\int_0^l\cos^2\left(\frac{j\pi}{l}x\right)dx\\&=\frac{l}{2}\bigl(d_1\mu_j+f'\bigl(f^{-1}(a)\bigr)\bigr)\bigl(\bigl[f^{-1}(a)\bigr]^2-d_2\mu_j\bigr)\neq0.\end{aligned}$$

This suggests that $\Psi\notin(\ker(K^*-I))^\perp=\operatorname{Im}(K-I)$, and so (i) is proved. Now, we will prove (ii). From (i), for any $b>0$, $b\neq b_0^j$ and b belongs to a small neighborhood of b_0^j, $K(b)-I: X\to X$ is a bijection. Fix $b>0$, then θ is a solution of (4.3) and θ is isolated. From the Leray–Schauder fixed point theory, we can get

$$\operatorname{index}\bigl(I-K(b)-W,(b,\theta)\bigr)=\deg\bigl(I-K(b),B,\theta\bigr)=(-1)^\gamma,$$

where B is a sufficiently small ball centered at θ, γ is the sum of the algebraic multiplicity of the eigenvalues of $K(b)$ and $\gamma>1$. We are going to verify that, for $\varepsilon>0$ is small enough,

$$\operatorname{index}\bigl(I-K\bigl(b_0^j-\varepsilon\bigr)-W,\bigl(b_0^j-\varepsilon,\theta\bigr)\bigr)\neq\operatorname{index}\bigl(I-K\bigl(b_0^j+\varepsilon\bigr)-W,\bigl(b_0^j+\varepsilon,\theta\bigr)\bigr).\tag{4.5}$$

If τ is an eigenvalue of $K(b)$ and $\Psi=\binom{\xi}{\psi}$ is the corresponding eigenfunction, then

$$\bigl(K(b)-I\bigr)\Psi=\begin{pmatrix}0\\0\end{pmatrix},$$

i.e.,

$$\begin{cases}-\tau d_1\xi''=(-bf'(f^{-1}(a))+\frac{2ab}{f^{-1}(a)}-\tau f'(f^{-1}(a)))\xi+[f^{-1}(a)]^2\psi,\\-\tau d_2\psi''=(bf'(f^{-1}(a))+\frac{2ab}{f^{-1}(a)})\xi-[f^{-1}(a)]^2\tau\psi.\end{cases}$$

By virtue of $\xi=\sum a_j\varphi_j$ and $\psi=\sum c_j\varphi_j$, we can get

$$\sum_{j=0}^\infty\begin{pmatrix}\tau\mu_jd_1+bf'(f^{-1}(a))-\frac{2ab}{f^{-1}(a)}+\tau f'(f^{-1}(a))&-[f^{-1}(a)]^2\\-bf'(f^{-1}(a))+\frac{2ab}{f^{-1}(a)}&\tau\mu_jd_2+\tau[f^{-1}(a)]^2\end{pmatrix}\begin{pmatrix}a_j\\c_j\end{pmatrix}\varphi_j=\begin{pmatrix}0\\0\end{pmatrix}.$$

Then the characteristic equation is

$$\begin{aligned}&\bigl(d_1d_2\mu_j^2+d_2\mu_jf'\bigl(f^{-1}(a)\bigr)+d_1\mu_j\bigl[f^{-1}(a)\bigr]^2+f'\bigl(f^{-1}(a)\bigr)\bigl[f^{-1}(a)\bigr]^2\bigr)\tau^2\\&\quad+b\left(d_2f'\bigl(f^{-1}(a)\bigr)\mu_j-\frac{2ad_2\mu_j}{f^{-1}(a)}+f'\bigl(f^{-1}(a)\bigr)\bigl[f^{-1}(a)\bigr]^2-2af^{-1}(a)\right)\tau\\&\quad+\bigl[f^{-1}(a)\bigr]^2\left(-bf'\bigl(f^{-1}(a)\bigr)+\frac{2ab}{f^{-1}(a)}\right)=0,\quad j=0,1,2,\ldots.\end{aligned}\tag{4.6}$$

If $\tau = 1$, b can be solved from (4.6):

$$b = \frac{f^{-1}(a)\{d_1 d_2 \mu_j^2 + d_2 f'(f^{-1}(a))\mu_j + d_1 [f^{-1}(a)]^2 \mu_j + [f^{-1}(a)]^2 f'(f^{-1}(a))\}}{2ad_2\mu_j - d_2 f^{-1}(a) f'(f^{-1}(a))\mu_j} = b_0^j. \quad (4.7)$$

Therefore, by calculating the corresponding eigenvalues of (4.6), we can obtain that when b passes through b_0^j, the number of eigenvalues of $K(b)$, which is greater than 1, is the same and their algebraic multiplicities are equal. By plugging (4.7) into (4.6), we have

$$\begin{aligned} &\frac{2ad_2\mu_j - d_2 f^{-1}(a) f'(f^{-1}(a))\mu_j}{f^{-1}(a)}\tau^2 + \Big(f'\big(f^{-1}(a)\big)\big(d_2\mu_j + [f^{-1}(a)]^2\big) \\ &\quad - \frac{2ad_2\mu_j}{f^{-1}(a)} - 2af^{-1}(a)\Big)\tau + [f^{-1}(a)]^2 \Big(-f'\big(f^{-1}(a)\big) + \frac{2a}{f^{-1}(a)}\Big) = 0. \end{aligned} \quad (4.8)$$

Then

$$\begin{aligned} \Delta &:= \Big(d_2 f'\big(f^{-1}(a)\big)\mu_j - \frac{2ad_2\mu_j}{f^{-1}(a)} + f'\big(f^{-1}(a)\big)[f^{-1}(a)]^2 - 2af^{-1}(a)\Big)^2 \\ &\quad - 4\big(2ad_2\mu_j - d_2 f^{-1}(a) f'\big(f^{-1}(a)\big)\mu_j\big) \cdot \big(-f'\big(f^{-1}(a)\big) f^{-1}(a) + 2a\big) \\ &= \Big[\Big(d_2 f'\big(f^{-1}(a)\big)\mu_j - \frac{2ad_2\mu_j}{f^{-1}(a)}\Big) - \big(f'\big(f^{-1}(a)\big)[f^{-1}(a)]^2 - 2af^{-1}(a)\big)\Big]^2 > 0, \end{aligned}$$

and so (4.8) has two different roots $\tau_1 = 1$, $\tau_2 = \frac{[f^{-1}(a)]^2}{d_2\mu_j}$. Thus two things will happen:

(a) if $\mu_j > \frac{[f^{-1}(a)]^2}{d_2}$, then $\tau_1(b_0^j) = 1$, $\tau_2(b_0^j) < 1$;

(b) if $\mu_j < \frac{[f^{-1}(a)]^2}{d_2}$, then $\tau_1(b_0^j) = 1$, $\tau_2(b_0^j) > 1$.

When scenario (a) occurs, b passes through b_0^j and $\tau_2(b) < 1$. From (4.6), $\tau_1(b_0^j + \varepsilon) > 1$, $\tau_1(b_0^j - \varepsilon) < 1$. Therefore, the matrix $K(b_0^j + \varepsilon)$ has exactly one more eigenvalue, that is, > 1, than $K(b_0^j - \varepsilon)$ does, and its algebraic multiplicity is 1. Then (4.5) holds. That is to say, the index jumps as b goes through b_0^j.

When scenario (b) occurs, b passes through b_0^j and $\tau_2(b) > 1$. From (4.6), $\tau_1(b_0^j + \varepsilon) > 1$, $\tau_1(b_0^j - \varepsilon) < 1$. Similarly, the index jumps as b goes through b_0^j. Therefore, (ii) is true regardless of (a) or (b). Thus, by the index jump principle and [16, Theorem 1.3], it follows that there exists a connected component $\hat{\Gamma}_j$ of nontrivial solutions of (4.3) and $\hat{\Gamma}_j$ comes from the bifurcation point (b_0^j, θ). We know that $\hat{\Gamma}_j$ is also the connected component Γ_j of the nonconstant solution of (3.1) from $(b_0^j, \bar{w})$. $\hat{\Gamma}_j$ and Γ_j are both in $\mathbb{R} \times X$. By the Rabinowitz global bifurcation theorem, the connected component Γ_j joins $(b_0^j, \bar{w})$ to either ∞ or $(b_0^k, \bar{w})$ in $\mathbb{R} \times X$, where $k \neq j$. We first prove that the latter situation will not happen. According to Theorem 3.1, the solution on the connected component sent from $(b_0^j, \bar{w})$ is related to φ_j, and φ_j has j zeros in the interval $[0, l]$. In the same way, the solution on the connected component sent from $(b_0^k, \bar{w})$ is related to φ_k, and φ_k has k zeros in the interval $[0, l]$. If the connected component sent Γ_j joining $(b_0^j, \bar{w})$ to $(b_0^k, \bar{w})$, the solution $(b, w) \in \Gamma_j$ is related to both φ_j and φ_k, which is impossible. On the other hand, Lemma 2.2 shows that, if $b = b_c \in (0, \infty)$, then the solutions u and v of (3.1) are both bounded. So the connected component Γ_j will not join $(b_0^j, \bar{w})$ to (b_c, ∞). Therefore, the connected component Γ_j can only join $(b_0^j, \bar{w})$ to either (∞, ∞) or (∞, m), where $m \in (0, \infty)$. But in any case, the projection of continuum Γ_j is unbounded on the b-axis. □

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