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Multiple periodic orbits of high-dimensional differential delay systems

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Abstract

In this paper, we consider differential delay systems of the form

$$x'(t) = - \sum_{s=1}^{2k-1} (-1)^{s+1} \nabla F(x(t-s)),$$

in which the coefficients of the nonlinear terms with different hysteresis have different signs. Such systems have not been studied before. The multiplicity of the periodic orbits is related to the eigenvalues of the limit matrix. The results provide a theoretical basis for the study of differential delay systems.

MSC: 34K13; 58E50

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1 Introduction

We consider the asymptotically linear differential delay system

$$x'(t) = - \sum_{s=1}^{2k-1} (-1)^{s+1} \nabla F(x(t-s)), \tag{1.1}$$

where

$$x \in \mathbb{R}^N, \quad F \in C^1(\mathbb{R}^N, \mathbb{R}), \quad \nabla F(-x) = -\nabla F(x), \tag{1.2}$$

and there are real symmetric matrices $A_0, A_\infty \in \mathbb{R}^{N \times N}$ such that

$$\nabla F(x) = A_0 x + o(|x|), \quad |x| \rightarrow 0, \quad \nabla F(x) = A_\infty x + o(|x|), \quad |x| \rightarrow \infty. \tag{1.3}$$

In the past several decades, many papers [1–16] have studied the existence of periodic solutions of delay differential equations. In 1974, Kaplan and Yorke [15] studied the multiple periodic solutions of the equations

$$x'(t) = -f(x(t-1)) \tag{1.4}$$

and

$$x'(t) = -f(x(t-1)) - f(x(t-2)) \tag{1.5}$$

by transforming them respectively into associated systems of ordinary differential equations and then making analysis by qualitative approaches. Meanwhile, they guessed that there should exist $2(n+1)$ -periodic solutions to the equation

$$x'(t) = -\sum_{i=1}^n f(x(t-i)), \tag{1.6}$$

where $f \in C^0(\mathbb{R}, \mathbb{R})$ with $f(-x) = -f(x)$, $xf(x) > 0$, $x \neq 0$. This was proved in [17]. On the basis of this work, Fei [3, 4] studied the multiple periodic solutions of differential delay equations via Hamiltonian systems. Li and He [10–12] studied the multiple solutions by an asymptotically linear Hamiltonian system. Guo and Yu [13, 14] gave some multiple results for periodic solutions via critical point theory.

In this paper, our main purpose is to study system (1.1), in which the coefficients of non-linear terms corresponding to different hysteresis have different signs, which is an extension of [3]. To construct the even functional, the variation structure here is much simpler since we do not transform system (1.1) into a $2kN$ -dimensional system. At the same time, according to the variational method and the method of Kaplan–Yorke coupling system, we get an exact counting method of the number of $4k$ -periodic orbits. Moreover, our results are easier to examine by introducing the eigenvalues and eigenvectors of the matrices A_∞ and A_0 .

Let

$$\alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_N \quad \text{and} \quad \beta_1 \leq \beta_2 \leq \dots \leq \beta_N$$

be the eigenvalues of A_0 and A_∞ , respectively, and let u_1, u_2, \dots, u_N and v_1, v_2, \dots, v_N be the corresponding unit eigenvectors in space. For convenience, we make the following assumptions:

- (f₁) F satisfies (1.2) and (1.3),
- (f₂) there are $M > 0$ and a function $r \in C^0(\mathbb{R}^+, \mathbb{R}^+)$ satisfying $r(s) \rightarrow \infty$ and $r(s)/s \rightarrow 0$ as $s \rightarrow \infty$ such that

$$\left| F(x) - \frac{1}{2}(A_\infty x, x) \right| > r(|x|) - M,$$

- (f₃[±]) $\pm[F(x) - \frac{1}{2}(A_\infty x, x)] > 0$, $|x| \rightarrow \infty$,
- (f₄[±]) $\pm[F(x) - \frac{1}{2}(A_0 x, x)] > 0$, $0 < |x| \ll 1$.

2 Variational structure

Let

$$\begin{aligned} \widehat{X} &= \{x \in L^2 : x(t-2k) = -x(t)\} \\ &= \left\{ \sum_{i=0}^{\infty} \left(a_i \cos \frac{(2i+1)\pi t}{2k} + b_i \sin \frac{(2i+1)\pi t}{2k} \right) : a_i, b_i \in \mathbb{R}^N \right\}, \end{aligned}$$

$$X = \text{cl} \left\{ \sum_{i=0}^{\infty} \left(a_i \cos \frac{(2i+1)\pi t}{2k} + b_i \sin \frac{(2i+1)\pi t}{2k} \right) : a_i, b_i \in R^N, \sum_{i=0}^{\infty} (2i+1)(a_i^2 + b_i^2) < \infty \right\} \subset \widehat{X},$$

and define $P : X \rightarrow L^2$ by

$$\begin{aligned} Px(t) &= P \left(\sum_{i=0}^{\infty} \left(a_i \cos \frac{(2i+1)\pi t}{2k} + b_i \sin \frac{(2i+1)\pi t}{2k} \right) \right) \\ &= \sum_{i=0}^{\infty} (2i+1) \left(a_i \cos \frac{(2i+1)\pi t}{2k} + b_i \sin \frac{(2i+1)\pi t}{2k} \right) \end{aligned}$$

and the inverse of P as

$$P^{-1}x(t) = \sum_{i=0}^{\infty} \frac{1}{2i+1} \left(a_i \cos \frac{(2i+1)\pi t}{2k} + b_i \sin \frac{(2i+1)\pi t}{2k} \right).$$

Define

$$\begin{aligned} \langle x, y \rangle &= \int_0^{4k} (Px(t), y(t)) dt, \quad \|x\| = \sqrt{\langle x, x \rangle}, \\ \langle x, y \rangle_2 &= \int_0^{4k} (x(t), y(t)) dt, \quad \|x\|_2 = \sqrt{\langle x, x \rangle_2}. \end{aligned}$$

Then $(X, \|\cdot\|)$ is an $H_{4k}^{\frac{1}{2}}([0, 4k], R^N)$ space.

For system (1.1), define the following functional $\Phi : X \rightarrow R$:

$$\Phi(x) = \frac{1}{2} \langle Lx, x \rangle + \int_0^{4k} F(x(t)) dt, \tag{2.1}$$

where

$$Lx = -P^{-1} \sum_{s=1}^{2k-1} x'(t-s). \tag{2.2}$$

Let

$$X(i) = \left\{ x(t) = a_i \cos \frac{(2i+1)\pi t}{2k} + b_i \sin \frac{(2i+1)\pi t}{2k} : a_i, b_i \in R^N \right\}.$$

Then we have

$$X = \sum_{h=0}^{\infty} \left[\sum_{i=0}^{k-1} (X(2hk+i) + X(2hk+2k-i-1)) \right]. \tag{2.3}$$

On the basis of Theorem 1.4 in [18], we can get that the differential of Φ satisfies

$$\Phi'(x) = Lx + K(x), \tag{2.4}$$

where $K(x) = P^{-1} \nabla F(x)$.

For convenience of further calculations, we can also make a more detailed division of space X by introducing the eigenvalues and eigenvectors mentioned before.

Suppose that

$$X_{0j}(i) = \left\{ x(t) = \left(a_j \cos \frac{(2i+1)\pi t}{2k} + b_j \sin \frac{(2i+1)\pi t}{2k} \right) u_j : a_j, b_j \in R, j = 1, 2, \dots, N \right\},$$

$$X_{\infty j}(i) = \left\{ x(t) = \left(a_j \cos \frac{(2i+1)\pi t}{2k} + b_j \sin \frac{(2i+1)\pi t}{2k} \right) v_j : a_j, b_j \in R, j = 1, 2, \dots, N \right\}.$$

Then we have

$$X(i) = \sum_{j=1}^N X_{\infty j}(i) = \sum_{j=1}^N X_{0j}(i)$$

and

$$X = \sum_{i=0}^{\infty} X(i) = \sum_{i=0}^{\infty} \sum_{j=1}^N X_{\infty j}(i) = \sum_{i=0}^{\infty} \sum_{j=1}^N X_{0j}(i).$$

Therefore from (2.2) we find that if

$$x(t) = \sum_{i=0}^{\infty} \left(a_i \cos \frac{(2i+1)\pi t}{2k} + b_i \sin \frac{(2i+1)\pi t}{2k} \right),$$

then

$$\begin{aligned} \langle Lx, x \rangle &= - \sum_{i=0}^{\infty} (2i+1)\pi (a_i^2 + b_i^2) \cot \frac{(2i+1)\pi}{4k} \\ &= \sum_{h=0}^{\infty} \left[- \sum_{i=0}^{k-1} (4hk+2i+1)\pi (a_{2hk+i}^2 + b_{2hk+i}^2) \cot \frac{(2i+1)\pi}{4k} \right. \\ &\quad \left. + \sum_{i=0}^{k-1} (4hk+4k-2i-1)\pi (a_{2hk+2k-i-1}^2 + b_{2hk+2k-i-1}^2) \cot \frac{(2i+1)\pi}{4k} \right]. \end{aligned}$$

On the other hand, when

$$\begin{aligned} x \in X_{\infty j} &= \sum_{i=0}^{\infty} X_{\infty j}(i) \\ &= \sum_{i=0}^{\infty} \left\{ x(t) = \left(a_j \cos \frac{(2i+1)\pi t}{2k} + b_j \sin \frac{(2i+1)\pi t}{2k} \right) v_j : a_j, b_j \in R, j = 1, 2, \dots, N \right\}, \end{aligned}$$

we have

$$\begin{aligned} \langle P^{-1}A_{\infty}x, x \rangle &= \sum_{i=0}^{\infty} 2k\beta_j (a_i^2 + b_i^2) \\ &= \sum_{h=0}^{\infty} \left[\sum_{i=0}^{k-1} 2k\beta_j (a_{2hk+i}^2 + b_{2hk+i}^2) + \sum_{i=0}^{k-1} 2k\beta_j (a_{2hk+2k-i-1}^2 + b_{2hk+2k-i-1}^2) \right]. \end{aligned}$$

Then we can get

$$\begin{aligned} & \langle (L + P^{-1}A_\infty)x, x \rangle \\ &= 2k \sum_{j=1}^N \sum_{h=0}^\infty \left[\sum_{i=0}^{k-1} \left(-\frac{(4hk + 2i + 1)\pi}{2k} \cot \frac{(2i + 1)\pi}{4k} + \beta_j \right) (a_{2lk+i}^2 + b_{2hk+i}^2) \right. \\ & \quad \left. + \sum_{i=0}^{k-1} \left(\frac{(4hk + 4k - 2i - 1)\pi}{2k} \cot \frac{(2i + 1)\pi}{4k} + \beta_j \right) (a_{2hk+2k-i-1}^2 + b_{2hk+2k-i-1}^2) \right]. \end{aligned}$$

Similarly,

$$\begin{aligned} & \langle (L + P^{-1}A_0)x, x \rangle \\ &= 2k \sum_{j=1}^N \sum_{h=0}^\infty \left[\sum_{i=0}^{k-1} \left(-\frac{(4hk + 2i + 1)\pi}{2k} \cot \frac{(2i + 1)\pi}{4k} + \alpha_j \right) (a_{2hk+i}^2 + b_{2hk+i}^2) \right. \\ & \quad \left. + \sum_{i=0}^{k-1} \left(\frac{(4hk + 4k - 2i - 1)\pi}{2k} \cot \frac{(2i + 1)\pi}{4k} + \alpha_j \right) (a_{2hk+2k-i-1}^2 + b_{2hk+2k-i-1}^2) \right]. \end{aligned}$$

3 Division of space X and lemmas

Let

$$\begin{aligned} X_\infty^+ &= \sum_{j=1}^N \left\{ X_{\infty j}(2hk + i) : h \geq 0, 0 \leq i \leq k - 1, -\frac{(4hk + 2i + 1)\pi}{2k} \cot \frac{(2i + 1)\pi}{4k} + \beta_j > 0 \right\} \\ & \quad \cup \sum_{j=1}^N \left\{ X_{\infty j}(2hk + 2k - i - 1) : \right. \\ & \quad \left. h \geq 0, 0 \leq i \leq k - 1, \frac{(4hk + 4k - 2i - 1)\pi}{2k} \cot \frac{(2i + 1)\pi}{4k} + \beta_j > 0 \right\}, \\ X_\infty^- &= \sum_{j=1}^N \left\{ X_{\infty j}(2hk + i) : h \geq 0, 0 \leq i \leq k - 1, -\frac{(4hk + 2i + 1)\pi}{2k} \cot \frac{(2i + 1)\pi}{4k} + \beta_j < 0 \right\} \\ & \quad \cup \sum_{j=1}^N \left\{ X_{\infty j}(2hk + 2k - i - 1) : \right. \\ & \quad \left. h \geq 0, 0 \leq i \leq k - 1, \frac{(4hk + 4k - 2i - 1)\pi}{2k} \cot \frac{(2i + 1)\pi}{4k} + \beta_j < 0 \right\}, \\ X_0^+ &= \sum_{j=1}^N \left\{ X_{0j}(2hk + i) : h \geq 0, 0 \leq i \leq k - 1, -\frac{(4hk + 2i + 1)\pi}{2k} \cot \frac{(2i + 1)\pi}{4k} + \alpha_j > 0 \right\} \\ & \quad \cup \sum_{j=1}^N \left\{ X_{0j}(2hk + 2k - i - 1) : \right. \\ & \quad \left. h \geq 0, 0 \leq i \leq k - 1, \frac{(4hk + 4k - 2i - 1)\pi}{2k} \cot \frac{(2i + 1)\pi}{4k} + \alpha_j > 0 \right\}, \end{aligned}$$

$$\begin{aligned}
 X_0^- &= \sum_{j=1}^N \left\{ X_{0j}(2hk+i) : h \geq 0, 0 \leq i \leq k-1, -\frac{(4hk+2i+1)\pi}{2k} \cot \frac{(2i+1)\pi}{4k} + \alpha_j < 0 \right\} \\
 &\cup \sum_{j=1}^N \left\{ X_{0j}(2hk+2k-i-1) : \right. \\
 &\quad \left. h \geq 0, 0 \leq i \leq k-1, \frac{(4hk+4k-2i-1)\pi}{2k} \cot \frac{(2i+1)\pi}{4k} + \alpha_j < 0 \right\}, \\
 X_\infty^0 &= \sum_{j=1}^N \left\{ X_{\infty j}(2hk+i) : h \geq 0, 0 \leq i \leq k-1, -\frac{(4hk+2i+1)\pi}{2k} \cot \frac{(2i+1)\pi}{4k} + \beta_j = 0 \right\} \\
 &\cup \sum_{j=1}^N \left\{ X_{\infty j}(2hk+2k-i-1) : \right. \\
 &\quad \left. h \geq 0, 0 \leq i \leq k-1, \frac{(4hk+4k-2i-1)\pi}{2k} \cot \frac{(2i+1)\pi}{4k} + \beta_j = 0 \right\}, \\
 X_0^0 &= \sum_{j=1}^N \left\{ X_{0j}(2hk+i) : h \geq 0, 0 \leq i \leq k-1, -\frac{(4hk+2i+1)\pi}{2k} \cot \frac{(2i+1)\pi}{4k} + \alpha_j = 0 \right\} \\
 &\cup \sum_{j=1}^N \left\{ X_{0j}(2hk+2k-i-1) : \right. \\
 &\quad \left. h \geq 0, 0 \leq i \leq k-1, \frac{(4hk+4k-2i-1)\pi}{2k} \cot \frac{(2i+1)\pi}{4k} + \alpha_j = 0 \right\}.
 \end{aligned}$$

It is easy to see that $\dim X_\infty^0 < \infty$ and $\dim X_0^0 < \infty$.

Lemma 3.1 ([3], Lemma 2.4) *Suppose X is a Hilbert space, $\Phi : X \rightarrow R$ is a differentiable functional, and $L : X \rightarrow X$ is a linear operator. Then there are two closed S^1 -invariant linear subspaces X^+ and X^- such that*

- (a) $X^+ \cup X^-$ is closed and of finite codimension in X ,
- (b) $\widehat{L}(X^-) \subset X^-$, $\widehat{L} = L + P^{-1}A_0$ or $\widehat{L} = L + P^{-1}A_\infty$,
- (c) there exists $c_0 \in R$ such that

$$\inf_{x \in X^+} \Phi(x) \geq c_0,$$

- (d) there is $c_\infty \in R$ such that

$$\Phi(x) \leq c_\infty < \Phi(0) = 0, \quad \forall x \in X^- \cap S_r = \{x \in X^- : \|x\| = r\},$$

- (e) Φ satisfies the $(P.S)_c$ -condition for $c_0 < c < c_\infty$, that is, every $\{x_n\} \subseteq X$ satisfying $\Phi(x_n) \rightarrow c$ and $\Phi'(x_n) \rightarrow 0$ has a convergent subsequence. Then Φ has at least $\frac{1}{2}[\dim(X^+ \cap X^-) - \text{codim}_X(X^+ \cup X^-)]$ generally different critical orbits in $\Phi^{-1}([c_0, c_\infty])$ if

$$[\dim(X^+ \cap X^-) - \text{codim}_X(X^+ \cup X^-)] > 0.$$

Lemma 3.2 *There exists $\sigma > 0$ such that*

$$\langle (L + P^{-1}A_\infty)x, x \rangle > \sigma \|x\|^2, \quad x \in X_\infty^+, \tag{3.1}$$

and

$$\langle (L + P^{-1}A_\infty)x, x \rangle < -\sigma \|x\|^2, \quad x \in X_\infty^-. \tag{3.2}$$

Proof Let

$$\begin{aligned} X_{\infty j} &= \sum_{i=0}^\infty X_{\infty j}(i) \\ &= \left\{ x(t) = \sum_{i=0}^\infty \left(a_j \cos \frac{(2i+1)\pi t}{2k} + b_j \sin \frac{(2i+1)\pi t}{2k} \right) v_j : a_j, b_j \in R, j = 1, 2, \dots, N \right\}. \end{aligned}$$

Then $X = \sum_{j=1}^N X_{\infty j}$. We need to consider two cases, $\beta_j \geq 0$ and $\beta_j < 0$. In the following part, we just give the proof for $\beta_j \geq 0$, as the other case is similar.

For $\beta_j \geq 0, i \in \{0, 1, \dots, k-1\}$, and $x \in X_{\infty j}$,

$$-\frac{(4hk + 2i + 1)\pi}{2k} \cot \frac{(2i + 1)\pi}{4k} + \beta_j > -\frac{(4h^+(i)k + 2i + 1)\pi}{2k} \cot \frac{(2i + 1)\pi}{4k} + \beta_j > 0,$$

where $h^+(i) = \max\{h \in N : -\frac{(4hk+2i+1)\pi}{2k} \cot \frac{(2i+1)\pi}{4k} + \beta_j > 0\}$, and

$$-\frac{(4hk + 2i + 1)\pi}{2k} \cot \frac{(2i + 1)\pi}{4k} + \beta_j < -\frac{(4h^-(i)k + 2i + 1)\pi}{2k} \cot \frac{(2i + 1)\pi}{4k} + \beta_j < 0,$$

where $h^-(i) = \min\{h \in N : -\frac{(4hk+2i+1)\pi}{2k} \cot \frac{(2i+1)\pi}{4k} + \beta_j < 0\}$.

Then we can choose

$$\begin{aligned} \sigma_i &= \min \left\{ -\frac{\pi}{2k} \cot \frac{(2i + 1)\pi}{4k} + \frac{\beta_j}{4h^+(i)k + 2i + 1}, \right. \\ &\quad \left. \frac{\pi}{2k} \cot \frac{(2i + 1)\pi}{4k} - \frac{\beta_j}{4h^-(i)k + 2i + 1} \right\} \\ &> 0, \end{aligned}$$

and let $\sigma_j = \min\{\sigma_0, \sigma_1, \dots, \sigma_{k-1}\} > 0$, and then let $\sigma = \min\{\sigma_j : j = 1, 2, \dots, N\}$. The proof is over. □

Lemma 3.3 *If (f_1) and (f_2) hold, then the functional Φ given by (2.1) satisfies the (P.S)-condition.*

Proof Let Π, Λ , and Γ be the orthogonal mappings from X to X_∞^+, X_∞^- , and X_∞^0 , respectively. From (1.3) we get

$$|\langle P^{-1}(\nabla F(x) - A_\infty x), x \rangle| < \frac{\sigma}{2} \|x\|^2 + \tilde{M}, \quad x \in X,$$

for some $\tilde{M} > 0$.

Suppose that $\{x_n\} \subset X$ is a subsequence such that $\Phi'(x_n) \rightarrow 0$ and $\Phi(x_n)$ is bounded. Let $w_n = \Pi x_n$, $y_n = \Lambda x_n$, and $z_n = \Gamma x_n$. Then

$$\Pi(L + P^{-1}A_\infty) = (L + P^{-1}A_\infty)\Pi, \Lambda(L + P^{-1}A_\infty) = (L + P^{-1}A_\infty)\Lambda. \tag{3.3}$$

From

$$\langle \Phi'(x_n), x_n \rangle = \langle Lx_n + P^{-1}\nabla F(x_n), x_n \rangle = \langle (L + P^{-1}A_\infty)x_n, x_n \rangle + \langle P^{-1}(\nabla F(x_n) - A_\infty x_n), x_n \rangle$$

and (3.1) we have

$$\begin{aligned} \langle \Pi \Phi'(x_n), x_n \rangle &= \langle \Pi(L + P^{-1}A_\infty)x_n, x_n \rangle + \langle \Pi P^{-1}(\nabla F(x_n) - A_\infty x_n), x_n \rangle \\ &= \langle (L + P^{-1}A_\infty)w_n, w_n \rangle + \langle \Pi P^{-1}(\nabla F(x_n) - A_\infty x_n), w_n \rangle \\ &> \frac{\sigma}{2} \|w_n\|^2 - \tilde{M}. \end{aligned}$$

Then we get that w_n is bounded. Similarly, y_n is bounded. Meanwhile, from (f_2) we get

$$\begin{aligned} \Phi(x_n) &= \frac{1}{2} \langle (L + P^{-1}A_\infty)x_n, x_n \rangle + \int_0^{4k} \left(F(x_n) - \frac{1}{2} \langle A_\infty x_n, x_n \rangle \right) dt \\ &\geq \frac{1}{2} \langle (L + P^{-1}A_\infty)w_n, w_n \rangle + \frac{1}{2} \langle (L + P^{-1}A_\infty)y_n, y_n \rangle \\ &\quad + \int_0^{4k} r(|x_n(t)|) dt - 4kM. \end{aligned}$$

Then $\|z_n\|$ is bounded since $\Phi(x_n)$ is bounded. So, $\|x_n\|$ is bounded.

Furthermore, from (2.4) we have

$$\begin{aligned} (\Pi + N)\Phi'(x_n) &= (\Pi + \Lambda)Lx_n + (\Pi + N)K(x_n) \\ &= L(w_n + y_n) + (\Pi + N)K(x_n). \end{aligned}$$

Then we can suppose without loss of generality that $K(x_n) \rightarrow \eta$ because K is compact and x_n is bounded. Then

$$L|_{x_\infty^+ + x_\infty^-}(w_n + y_n) \rightarrow -(\Pi + N)\eta. \tag{3.4}$$

Meanwhile, we can easily see that the dimension of X_∞^0 is finite, so we can suppose that $z_n \rightarrow \varphi$ as z_n is bounded. Hence

$$x_n = z_n + w_n + y_n \rightarrow \varphi - (L|_{x_\infty^+ + x_\infty^-})^{-1}(\Pi + \Lambda)\eta,$$

and the (P.S)-condition is proved. □

Lemma 3.4 *If x is a critical point of Φ , then it is a solution to system (1.4).*

Proof Suppose x is a critical point of Φ given by (2.1). Then $x(t)$ satisfies

$$-\sum_{s=1}^{2k-1} x'(t-s) + \nabla F(x(t)) = 0, \quad \text{a.e. } t \in [0, 4k]. \tag{3.5}$$

Consequently,

$$-\sum_{s=1}^{2k-1} x'(t-s-1) + \nabla F(x(t-1)) = 0, \tag{3.5_1}$$

$$-\sum_{s=1}^{2k-1} x'(t-s-2) + \nabla F(x(t-2)) = 0, \tag{3.5_2}$$

$$-\sum_{s=1}^{2k-1} x'(t-s-3) + \nabla F(x(t-3)) = 0, \tag{3.5_3}$$

⋮

$$-\sum_{s=0}^{2k-1} x'(t-s-(2k-1)) + \nabla F(x(t-(2k-1))) = 0. \tag{3.5_{(2k-1)}}$$

Calculating (3.5_1) − (3.5_2) + (3.5_3) − ⋯ + (3.5_{(2k-1)}), we have

$$x'(t) + \sum_{s=1}^{2k-1} (-1)^{s+1} \nabla F(x(t-s)) = 0, \quad \text{a.e. } t \in [0, 4k],$$

that is,

$$x'(t) = -\sum_{s=1}^{2k-1} (-1)^{s+1} \nabla F(x(t-s)), \quad \text{a.e. } t \in [0, 4k],$$

and hence x is a solution of (1.1). □

4 Main results

Denote

$$N(\alpha_j) = \begin{cases} -\sum_{i=0}^{k-1} \# \{h \geq 0 : 0 < \frac{(4hk+4k-2i-1)\pi}{2k} \cot \frac{(2i+1)\pi}{4k} < -\alpha_j\}, & \alpha_j < 0, \\ \sum_{i=0}^{k-1} \# \{h \geq 0 : 0 < \frac{(4hk+2i+1)\pi}{2k} \cot \frac{(2i+1)\pi}{4k} < \alpha_j\}, & \alpha_j \geq 0, \end{cases}$$

$$N(\beta_j) = \begin{cases} -\sum_{i=0}^{k-1} \# \{h \geq 0 : 0 < \frac{(4hk+4k-2i-1)\pi}{2k} \cot \frac{(2i+1)\pi}{4k} < -\beta_j\}, & \beta_j < 0, \\ \sum_{i=0}^{k-1} \# \{h \geq 0 : 0 < \frac{(4hk+2i+1)\pi}{2k} \cot \frac{(2i+1)\pi}{4k} < \beta_j\}, & \beta_j \geq 0, \end{cases}$$

$$N^0(\alpha_{j-}) = \sum_{i=0}^{k-1} \# \left\{ h \geq 0 : 0 < \frac{(4hk+4k-2i-1)\pi}{2k} \cot \frac{(2i+1)\pi}{4k} = -\alpha_j \right\}, \quad \alpha_j < 0,$$

$$N^0(\alpha_{j+}) = \sum_{i=0}^{k-1} \# \left\{ h \geq 0 : 0 < \frac{(4hk+2i+1)\pi}{2k} \cot \frac{(2i+1)\pi}{4k} = \alpha_j \right\}, \quad \alpha_j \geq 0,$$

$$N^0(\beta_{j-}) = \sum_{i=0}^{k-1} \#\left\{h \geq 0 : 0 < \frac{(4hk + 4k - 2i - 1)\pi}{2k} \cot \frac{(2i + 1)\pi}{4k} = -\beta_j\right\}, \quad \beta_j < 0,$$

$$N^0(\beta_{j+}) = \sum_{i=0}^{k-1} \#\left\{h \geq 0 : 0 < \frac{(4hk + 2i + 1)\pi}{2k} \cot \frac{(2i + 1)\pi}{4k} = \beta_j\right\}, \quad \beta_j \geq 0,$$

and

$$N(A_\infty) = \sum_{j=1}^N N(\beta_j), \quad N(A_0) = \sum_{j=1}^N N(\alpha_j),$$

$$N^0(A_{\infty-}) = \sum_{j=1}^N N^0(\beta_{j-}), \quad N^0(A_{\infty+}) = \sum_{j=1}^N N^0(\beta_{j+}),$$

$$N^0(A_{0-}) = \sum_{j=1}^N N^0(\alpha_{j-}), \quad N^0(A_{0+}) = \sum_{j=1}^N N^0(\alpha_{j+}).$$

Theorem 4.1 *System (1.1) has at least*

$$n = \max\{N(A_\infty) - N(A_0) - N^0(A_{\infty-}) - N^0(A_{0+}),$$

$$N(A_0) - N(A_\infty) - N^0(A_{0-}) - N^0(A_{\infty+})\}$$

$$> 0$$

4k-periodic orbits when (f₁) and (f₂) hold.

Proof Without loss of generality, we suppose

$$n = N(A_\infty) - N(A_0) - N^0(A_{\infty-}) - N^0(A_{0+}).$$

Then letting $X^+ = X_\infty^+$ and $X^- = X_0^-$, we get

$$X \setminus (X^+ \cup X^-) = X \setminus (X_\infty^+ \cup X_0^-) \subseteq X_\infty^0 \cup X_0^0 \cup (X_\infty^+ \cap X_0^-).$$

Obviously,

$$\text{codim}_X(X^+ + X^-) \leq \dim X_\infty^0 + \dim X_0^0 + \dim(X_\infty^+ \cap X_0^-) < \infty,$$

which means that the codimension of $(X^+ \cup X^-)$ is finite. For each $x \in X(i)$, we have $(L + P^{-1}A_\infty)x \in X(i)$. The (PS)-condition is satisfied by Lemma 3.3. Moreover, from (1.3) we get $|F(x) - \frac{1}{2}(A_\infty x, x)| < \frac{1}{4}\sigma \|x\|^2 + M_1$, $x \in R^N$, for some $M_1 > 0$, and from Lemma 3.2 we know that there exists $\sigma > 0$ such that $\langle (L + P^{-1}A_\infty)x, x \rangle > \sigma \|x\|^2$, $x \in X_\infty^+$. Then

$$\Phi(x) = \frac{1}{2} \langle (L + P^{-1}A_\infty)x, x \rangle + \int_0^{4k} \left[F(x(t)) - \frac{1}{2}(A_\infty x, x) \right] dt$$

$$\geq \frac{1}{2}\sigma \|x\|^2 - \frac{1}{4}\sigma \|x\|^2 - 4kM_1$$

$$\geq \frac{1}{4}\sigma \|x\|^2 - 4kM_1$$

for $x \in X^+$. Therefore there exists $c_0 \in R$ such that

$$\inf_{x \in X^+} \Phi(x) \geq c_0.$$

Similarly, we get that there exist $r, \sigma > 0$ such that $|F(x) - \frac{1}{2}(A_0x, x)| < \frac{1}{4}\sigma \|x\|^2, \|x\| = r$. Then

$$\begin{aligned} \Phi(x) &= \frac{1}{2} \langle (L + P^{-1}A_0)x, x \rangle + \int_0^{4k} \left[F(x(t)) - \frac{1}{2}(A_0x, x) \right] dt \\ &\leq -\frac{1}{2}\sigma \|x\|^2 + \frac{1}{4}\sigma \|x\|^2 \\ &\leq -\frac{1}{4}\sigma \|x\|^2 \end{aligned}$$

for $x \in X^-$. This means that there exist $r > 0$ and $c_\infty < 0$, such that

$$\Phi(x) \leq c_\infty < 0 = \Phi(0), \quad \forall x \in X^- \cap S_r = \{x \in X : \|x\| = r\}.$$

On the other hand, for $i \in \{0, 1, \dots, k - 1\}$,

$$\begin{aligned} \langle (L + P^{-1}A_\infty)x, x \rangle &= \left(-\frac{\pi}{2k} \cot \frac{(2i + 1)\pi}{4k} + \frac{\beta_j}{4hk + 2i + 1} \right) \|x\|^2, \quad x \in X(2hk + i), \\ \langle (L + P^{-1}A_\infty)x, x \rangle &= \left(\frac{\pi}{2k} \cot \frac{(2i + 1)\pi}{4k} + \frac{\beta_j}{4hk + 4k - 2i - 1} \right) \|x\|^2, \\ &\quad x \in X(2hk + 2k - i - 1), \end{aligned}$$

and

$$\begin{aligned} \langle (L + P^{-1}A_0)x, x \rangle &= \left(-\frac{\pi}{2k} \cot \frac{(2i + 1)\pi}{4k} + \frac{\alpha_j}{4hk + 2i + 1} \right) \|x\|^2, \quad x \in X(2hk + i), \\ \langle (L + P^{-1}A_0)x, x \rangle &= \left(\frac{\pi}{2k} \cot \frac{(2i + 1)\pi}{4k} + \frac{\alpha_j}{4hk + 4k - 2i - 1} \right) \|x\|^2, \\ &\quad x \in X(2hk + 2k - i - 1). \end{aligned}$$

Hence we have that

$$\begin{aligned} X_\infty^+(2hk + i) &= X_\infty^+ \cap X(2hk + i) = \emptyset, \\ X_0^-(2hk + 2k - i - 1) &= X_0^- \cap X(2hk + 2k - i - 1) = \emptyset, \\ X_0^-(2hk + i) &= X_0^- \cap X(2hk + i) = X(2hk + i), \\ X_\infty^+(2hk + 2k - i - 1) &= X_\infty^+ \cap X(2hk + 2k - i - 1) = X(2hk + 2k - i - 1), \end{aligned}$$

when $h \geq 0$ is large enough. So there is $M > 0$ such that

$$\dim(X_\infty^+(s) \cap X_0^-(r)) - \text{codim}_X(X_\infty^+(s) + X_0^-(s)) = 0, \quad s > M.$$

Then

$$\begin{aligned}
 n &= \frac{1}{2} [\dim(X^+ \cap X^-) - \text{codim}_X(X^+ + X^-)] \\
 &= \frac{1}{2} [\dim(X_\infty^+ \cap X_0^-) - \text{codim}_X(X_\infty^+ + X_0^-)] \\
 &= \frac{1}{2} \sum_{s=0}^M [\dim(X_\infty^+(s) \cap X_0^-(s)) - \text{codim}_{X(s)}(X_\infty^+(s) + X_0^-(s))] \\
 &= \frac{1}{2} \sum_{s=0}^M [\dim X_\infty^+(s) + \dim X_0^-(s) - 2N] \\
 &= \frac{1}{2} \sum_{s=0}^M [\dim X_\infty^+(s) + \dim X_0^-(s)] - N(M + 1).
 \end{aligned}$$

Then we have

$$\begin{aligned}
 \sum_{s=0}^M \dim(X_\infty^+(s)) &= 2 \begin{cases} N(A_\infty) & \\ + N \times \#\{2hk + 2k - i - 1 : 0 \leq 2hk + 2k - i - 1 \leq M\}, & \beta_j \geq 0, \\ N(A_\infty) - N^0(A_{\infty-}) & \\ + N \times \#\{2hk + 2k - i - 1 : 0 \leq 2hk + 2k - i - 1 \leq M\}, & \beta_j < 0, \end{cases} \\
 \sum_{s=0}^M \dim(X_0^-(s)) &= 2 \begin{cases} -N(A_0) - N^0(A_{0+}) + N \times \#\{2hk + i : 0 \leq 2hk + i \leq M\}, & \alpha_j \geq 0, \\ -N(A_0) + N \times \#\{2hk + i : 0 \leq 2hk + i \leq M\}, & \alpha_j < 0, \end{cases}
 \end{aligned}$$

and

$$\begin{aligned}
 \sum_{s=0}^M [\dim X_\infty^+(s) + \dim X_0^-(s)] &= 2[N(A_\infty) - N(A_0) - N^0(A_{\infty-}) - N^0(A_{0+})] \\
 &\quad + 2N(M + 1). \tag{4.1}
 \end{aligned}$$

Therefore

$$n = N(A_\infty) - N(A_0) - N^0(A_{\infty-}) - N^0(A_{0+}). \quad \square$$

Theorem 4.2 *System (1.1) possesses at least*

$$n = N(A_\infty) - N(A_0) + N^0(A_{\infty+}) + N^0(A_{0-}) > 0$$

4k-periodic orbits when (f_1) , (f_2) , (f_3^+) , and (f_4^-) hold.

Proof Let $X^+ = X_\infty^+ + X_0^0$ and $X^- = X_0^- + X_0^0$. The verification of conditions (a), (b), (c), (d), and (e) is similar to Theorem 4.1, so we can assume that (4.1) still holds. Let $X_\infty^0(i) =$

$X_\infty^0 \cap X(i)$ and $X_0^0(i) = X_0^0 \cap X(i)$. Then

$$\begin{aligned} n &= \frac{1}{2} \sum_{i=0}^M [\dim(X_\infty^+(i) \cap X_0^-(i)) - \text{codim}_{X(i)}(X_\infty^+(i) + X_0^-(i))] + (\dim X_\infty^0 + \dim X_0^0) \\ &= \frac{1}{2} \sum_{i=0}^M [\dim X_\infty^+(i) + \dim X_0^-(i) - 2N] + (\dim X_\infty^0 + \dim X_0^0) \\ &= \frac{1}{2} \sum_{i=0}^M [\dim X_\infty^+(i) + \dim X_0^-(i)] - N(M + 1) + (\dim X_\infty^0 + \dim X_0^0) \\ &= N(A_\infty) - N(A_0) - N^0(A_{\infty-}) - N^0(A_{0+}) \\ &\quad + (N^0(A_{\infty+}) + N^0(A_{0-}) + N^0(A_{0+}) + N^0(A_{0-})) \\ &= N(A_\infty) - N(A_0) + N^0(A_{\infty+}) + N^0(A_{0-}). \end{aligned}$$

□

Theorem 4.3 *System (1.1) possesses at least*

$$n = N(A_0) - N(A_\infty) + N^0(A_{0+}) + N^0(A_{\infty-}) > 0$$

4k-periodic orbits when (f_1) , (f_2) , (f_3^-) , and (f_4^+) hold.

The proof is almost the same as that of Theorem 4.2, and we omit it.

5 Example

Assume that $F \in C^1(R^2, R)$ satisfies

$$F(x) = \begin{cases} \frac{3\pi}{2}x_1^2 + \frac{\pi}{2}x_2^2 + (2x_1^2 + x_2^2)^{\frac{2}{3}}, & |x| \gg 1, \\ \frac{\pi}{2}x_1^2 - \frac{3\pi}{2}x_2^2 - 3x_1^{\frac{12}{5}} - x_2^{\frac{8}{3}}, & |x| \ll 1. \end{cases}$$

We are to discuss the multiplicity of 12-periodic solutions of the equation

$$x'(t) = - \sum_{s=1}^5 (-1)^{s+1} \nabla F(x(t-s)). \tag{5.1}$$

In this case, $k = 3$, $\alpha_1 = \pi$, $\alpha_2 = -3\pi$, $\beta_1 = 3\pi$, $\beta_2 = \pi$. Then

$$\begin{aligned} N(\alpha_1) &= \# \left\{ h \geq 0 : 0 < \frac{(12h+1)\pi}{6} \cot \frac{\pi}{12} < \pi \right\} \\ &\quad + \# \left\{ h \geq 0 : 0 < \frac{(12h+3)\pi}{6} \cot \frac{3\pi}{12} < \pi \right\} \\ &\quad + \# \left\{ h \geq 0 : 0 < \frac{(12h+5)\pi}{6} \cot \frac{5\pi}{12} < \pi \right\} \\ &= 4, \end{aligned}$$

$$\begin{aligned}
N(\alpha_2) &= -\#\left\{h \geq 0 : 0 < \frac{(12h+11)\pi}{6} \cot \frac{\pi}{12} < 3\pi\right\} \\
&\quad -\#\left\{h \geq 0 : 0 < \frac{(12h+9)\pi}{6} \cot \frac{3\pi}{12} < 3\pi\right\} \\
&\quad -\#\left\{h \geq 0 : 0 < \frac{(12h+7)\pi}{6} \cot \frac{5\pi}{12} < 3\pi\right\} \\
&= -7,
\end{aligned}$$

$$\begin{aligned}
N(\beta_1) &= \#\left\{h \geq 0 : 0 < \frac{(12h+1)\pi}{6} \cot \frac{\pi}{12} < 3\pi\right\} \\
&\quad +\#\left\{h \geq 0 : 0 < \frac{(12h+3)\pi}{6} \cot \frac{3\pi}{12} < 3\pi\right\} \\
&\quad +\#\left\{h \geq 0 : 0 < \frac{(12h+5)\pi}{6} \cot \frac{5\pi}{12} < 3\pi\right\} \\
&= 9,
\end{aligned}$$

$$\begin{aligned}
N(\beta_2) &= \#\left\{h \geq 0 : 0 < \frac{(12h+1)\pi}{6} \cot \frac{\pi}{12} < \pi\right\} \\
&\quad +\#\left\{h \geq 0 : 0 < \frac{(12h+3)\pi}{6} \cot \frac{3\pi}{12} < \pi\right\} \\
&\quad +\#\left\{h \geq 0 : 0 < \frac{(12h+5)\pi}{6} \cot \frac{5\pi}{12} < \pi\right\} \\
&= 4,
\end{aligned}$$

$$N(A_0) = N(\alpha_1) + N(\alpha_2) = -3, \quad N(A_\infty) = N(\beta_1) + N(\beta_2) = 13,$$

$$N^0(\alpha_+) = N^0(\beta_-) = N^0(\alpha_-) = N^0(\beta_+) = 0.$$

According to Theorem 4.2, we get that Eq. (5.1) has at least 16 different 12-periodic orbits satisfying $x(t-6) = -x(t)$.

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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