Kasmi et al. *Advances in Difference Equations* (2019) 2019:511
https://doi.org/10.1186/s13662-019-2444-2

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Existence of solution for 2-D time-fractional differential equations with a boundary integral condition

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Abstract
In this article, we prove the existence and uniqueness of a solution for 2-dimensional time-fractional differential equations with classical and integral boundary conditions. We start by writing this problem in the operator form and we choose suitable spaces and norms. Then we establish prior estimates from which we deduce the uniqueness of the strong solution. For the existence of solution for the fractional problem, we prove that the range of the operator generated by the considered problem is dense.

MSC: 35R11; 35B45; 35D35; 35L20

Keywords: Fractional differential equations; A priori estimate method; Boundary integral condition; Existence and uniqueness

1 Introduction

Many physical phenomena bring us back to the study of fractional partial differential equations. We mention, for example, viscoelasticity, signal processing, electro-chemistry, control theory, electrical networks, fluid flow, porous media, rheology, diffusive transport, electromagnetic theory, diffusion phenomena, and a lot of other physical processes are diverse applications of fractional differential equations. For more on this, see [1–6].

Fractional diffusion equations appear widely in natural phenomena; these are suggested as mathematical models of physical problems in many fields, like the inhomogeneous fractional diffusion equations of the form

$$\partial_{0t}^{\beta} u = -Au + F(t), \qquad u(0) = f, \tag{1.1}$$

where ∂_{0t}^{β} is the Caputo fractional derivative, A is a positive self-adjoin operator on a Hilbert space $H, f \in H, F \in C(R^{+}; H)$, and $0 < \beta \leq 1$. The Caputo derivative is more suitable and natural for physical models problems, because it enables us to deal with inhomogeneous initial data easily.

Several methods have been used to investigate the existence and uniqueness of solution for fractional-order initial boundary value problems, such as the Laplace transform method, iteration method, the series method, the Fourier transform technique, the operational calculus method, see for example [7–11], but in general, analytical solutions are hard to obtain for most fractional differential equations especially with nonlocal conditions (integral conditions), i.e. when the data cannot be measured on the boundary or on a part of it. In the absence of a precise solution, we often resort to numerical methods such as finite element methods or spectral methods or backward substitution methods (see [12–15]), which strongly rely on the existence and uniqueness of the solution for a variational problem. The study of existence and uniqueness of a solution for fractional differential equations starts by constructing variational formulation and choosing suitable spaces and norms. Then we choose fixed point theorem or the Lax–Milgram theorem to prove existence results of the solution. For our problem (2.1)–(2.4), we believe that the prior estimate method is the most powerful tool to prove the existence and uniqueness of the solution for fractional differential equation, and is more appropriate with classical and integral boundary conditions. A few papers use the means of the energy inequality method for studying fractional partial differential equations, we cite for example: Alikhanov [16], Akilandeeswari et al. [17], and Mesloub [18]. Our work can be considered as an expansion and generalization of integer order problems such as [19].

We organize our article as follows: In Sect. 2, we give the statement of the problem and the needed functions spaces and the different tools that can be used in other sections. In Sect. 3, we prove an a priori estimate from which we deduce the uniqueness of a strong solution of problem (2.1)–(2.4), and its dependence on the given data. In Sect. 4, we establish the existence of the solution of problem (2.1)–(2.4), by proving that the closure of the range of the operator L generated by the considered problem is dense in the Hilbert space Y.

2 Statement of the problem and associated function spaces

Let $D = \Omega \times [0,T]$ be a bounded domain in $\mathbb{R}^3$ with $\Omega = (0,c) \times (0,d)$. We consider the 2-dimensional time-fractional partial differential equation (PDE)

$$\partial_{0t}^{\beta+1} V - \frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial V}{\partial x}\right) - \frac{1}{x}\frac{\partial^2 V}{\partial y^2} = F(x,y,t), \quad (x,y,t) \in D, \tag{2.1}$$

associated with the initial data

$$\ell_1 V = V(x,y,0) = f(x,y), \qquad \ell_2 V = V_t(x,y,0) = g(x,y). \tag{2.2}$$

We have the Neumann and Dirichlet boundary conditions

$$V(c,y,t) = 0, \qquad V_y(x,d,t) = 0, \tag{2.3}$$

and the integral conditions

$$\int_0^c xV(x,y,t)\,dx = 0, \qquad \int_0^d V(x,y,t)\,dy = 0. \tag{2.4}$$

Here F is a known function, where $F \in C(\overline{D})$.

$\partial_{0t}^{\beta+1}$; denotes the Caputo fractional derivative of order $\beta + 1$, defined by

$$\partial_{0t}^{\beta+1} V(x,y,t) = \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{V_{\tau\tau}(x,y,\tau)}{(t-\tau)^{\beta}}\, d\tau, \quad 0 < \beta < 1, \tag{2.5}$$

$\Gamma(.)$ is the Gamma function.

$D_{0t}^{-\beta}$; denotes the Riemann–Liouville integral of order β, defined by

$$D_{0t}^{-\beta} V(x,y,t) = \frac{1}{\Gamma(\beta)} \int_0^t \frac{V(x,y,\tau)}{(t-\tau)^{\beta}}\, d\tau, \quad 0 < \beta < 1. \tag{2.6}$$

The supposed solution $V \in C^{2,2,2}(\overline{D})$, the space of functions together with their partial derivatives of order 2 are continuous on $\overline{D}$ for their three variables (x, y, t).

For more information on fractional differential equations, and applications of fractional calculus in physics, see [1–9].

We often use the following two lemmas.

Lemma 2.1 ([16]) *For any absolutely continuous function $v(t)$ on the interval $[0,T]$, the following inequality holds*:

$$v(t)\partial_{0t}^{\beta} v(t) \geq \frac{1}{2}\partial_{0t}^{\beta} v^2(t), \quad 0 < \beta < 1. \tag{2.7}$$

Lemma 2.2 ([16]) *Let a nonnegative absolutely continuous function $y(t)$ satisfy the inequality*

$$\partial_{0t}^{\beta} y(t) \leq cy(t) + k(t), \quad 0 < \beta < 1, \tag{2.8}$$

for almost all $t \in [0,T]$, where c is positive and $k(t)$ is an integrable nonnegative function on $[0,T]$. Then

$$y(t) \leq y(0)E_{\beta}\bigl(ct^{\beta}\bigr) + \Gamma(\beta)E_{\beta,\beta}\bigl(ct^{\beta}\bigr)D_{0t}^{-\beta} k(t), \tag{2.9}$$

where

$$E_{\beta}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\beta n + 1)} \quad \textit{and} \quad E_{\beta,\alpha}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\beta n + \alpha)}, \tag{2.10}$$

are the Mittag-Leffler functions.

The Cauchy ε-inequality:

$$AB \leq \frac{\varepsilon}{2}A^2 + \frac{1}{2\varepsilon}B^2, \quad \varepsilon > 0, \tag{2.11}$$

where A and B are positive numbers.

A Poincaré type inequalities: [20]

$$\begin{cases} (1)\|\Im_y(V)\|^2_{L_p^2(\Omega)} \leq \frac{d^2}{2}\|V\|^2_{L_p^2(\Omega)}; \\ (2)\|\Im_y^2(V)\|^2_{L_p^2(\Omega)} \leq \frac{d^2}{2}\|\Im_y(V)\|^2_{L_p^2(\Omega)}, \end{cases} \tag{2.12}$$

where

$$\Im_y(V) = \int_0^y V(x,\xi,t)\,d\xi, \qquad \Im_y^2(V) = \int_0^y \int_0^\xi V(x,\eta,t)\,d\eta\,d\xi. \tag{2.13}$$

We now introduce the appropriate function spaces needed in our posed problem. Let $L_p^2(D)$ the weighted L^2-space with finite norm

$$\|V\|^2_{L_p^2(D)} = \int_D xU^2\,dx\,dy\,dt, \tag{2.14}$$

from the inner product

$$(U,V)_{L_p^2(D)} = (xU,V)_{L^2(D)} \tag{2.15}$$

Let $V_p^{1,y}(D)$,and $V^1(D)$ be the weighted Sobolev spaces with finite norms

$$\|V\|^2_{V_p^{1,y}(D)} = \left\|\Im_y(V)\right\|^2_{L_p^2(D)} + \left\|\Im_y(V_x)\right\|^2_{L_p^2(D)}, \tag{2.16}$$

$$\|V\|^2_{V^1(D)} = \|V\|^2_{L^2(D)} + \|V_x\|^2_{L^2(D)}. \tag{2.17}$$

Problem (2.1)–(2.4) can be formulated in operational form:

$$LV = W, \quad \forall V \in D(L), \tag{2.18}$$

where $W = (F,f,g)$, and $L = (\mathcal{L},\ell_1,\ell_2)$ is the operator $L : X \longrightarrow Y$ with domain of definition

$$D(L) = \begin{cases} V \in L^2(D), \partial_{0t}^{\beta+1}V, V_t, V_x, V_y, V_{xx}, V_{yy} \in L^2(D), \\ V(c,y,t) = 0, \qquad V_y(x,d,t) = 0, \\ \int_0^c xV(x,y,t)\,dx = 0, \qquad \int_0^d V(x,y,t)\,dy = 0, \quad t \in [0,T]. \end{cases} \tag{2.19}$$

Here X is a Banach space of functions V obtained by enclosing $D(L)$ with respect to the finite norm

$$\|V\|^2_X = \sup_{0\le t\le T}\left(D_{0t}^{\beta-1}\left\|\Im_y(V_t)\right\|^2_{L_p^2(\Omega)} + \|V\|^2_{V_p^{1,y}(\Omega)}\right), \tag{2.20}$$

and Y is the Hilbert space associated with the finite norm

$$\|W\|^2_Y = \|F\|^2_{L_p^2(D)} + \|f\|^2_{V^1(\Omega)} + \|g\|^2_{L_p^2(\Omega)} \tag{2.21}$$

$L_p^2(\Omega)$, $V_p^{1,y}(\Omega)$ and $V^1(\Omega)$ the weighted Sobolev spaces on Ω are defined analogously to that on D.

3 Uniqueness of the solution

In this section, we prove a uniqueness result for problem (2.1)–(2.4), that is, we establish an a priori estimate from which we deduce the uniqueness of the consequences of the solution.

Theorem 3.1 *For any $V \in D(L)$, there exists a positive constant M_5 independent of V such that*

$$\sup_{0\le t\le T}\left(D_{0t}^{\beta-1}\left\|\Im_y(V_t)\right\|^2_{L^2_p(\Omega)} + \|V\|^2_{V_p^{1,y}(\Omega)}\right) \le M_5\left(\int_0^T \|F\|^2_{L^2_p(\Omega)}\,dt + \|f\|^2_{V^1(\Omega)} + \|g\|^2_{L^2_p(\Omega)}\right), \tag{3.1}$$

where

$$M_1 = \max\left(1, \frac{d^2}{2}\right), \tag{3.2}$$

$$M_2 = \frac{M_1 \max\{1, \frac{cd^2}{2}, \frac{d^2}{2}\frac{T^{1-\beta}}{\Gamma(1-\beta)}\}}{\min\{1, \frac{2}{cd^2}\}}, \tag{3.3}$$

$$M_3 = \Gamma(\beta)E_{\beta,\beta}\left(M_2 T^\beta\right)\max\left(M_2, \frac{M_2 T^{\beta+1}}{(\beta+1)\Gamma(\beta+1)}\right), \tag{3.4}$$

$$M_4 = M_2 M_3 + M_2, \tag{3.5}$$

$$M_5 = M_4\left(1 + \frac{T^\beta}{\Gamma(\beta+1)}\right). \tag{3.6}$$

Proof Taking the inner product in $L^2_p(\Omega)$ of Eq. (2.1) and the integro-differential operator

$$MV = -\Im_y^2(V_t) = -\int_0^y\int_0^\xi V_t(x,\eta,t)\,d\eta\,d\xi. \tag{3.7}$$

We have

$$\left(\partial_{0t}^{\beta+1}V - \frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial V}{\partial x}\right) - \frac{1}{x}\frac{\partial^2 V}{\partial y^2}, -\Im_y^2(V_t)\right)_{L^2_p(\Omega)} = \left(F, -\Im_y^2(V_t)\right)_{L^2_p(\Omega)}. \tag{3.8}$$

Using conditions (2.3)–(2.4), then standard integration by parts of each term of the left-hand side in (3.8), leads to

$$-\left(\partial_{0t}^{\beta+1}V, \int_0^y\int_0^\xi V_t(x,\eta,t)\,d\eta\,d\xi\right)_{L^2_p(\Omega)} = \left(\partial_{0t}^{\beta}\left(\int_0^y V_t(x,\xi,t)\,d\xi\right), \left(\int_0^y V_t(x,\xi,t)\,d\xi\right)\right)_{L^2_p(\Omega)}, \tag{3.9}$$

$$\left(\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial V}{\partial x}\right), \int_0^y\int_0^\xi V_t(x,\eta,t)\,d\eta\,d\xi\right)_{L^2_p(\Omega)} = \left(\int_0^y V_x(x,\xi,t)\,d\xi, \int_0^y V_{xt}(x,\xi,t)\,d\xi\right)_{L^2_p(\Omega)}, \tag{3.10}$$

$$\left(\frac{1}{x}\frac{\partial^2 V}{\partial y^2}, \int_0^y\int_0^\xi V_t(x,\eta,t)\,d\eta\,d\xi\right)_{L^2_p(\Omega)} = (V, V_t)_{L^2(\Omega)}. \tag{3.11}$$

By substitution of (3.9)–(3.11) into (3.8) we obtain

$$\begin{aligned} &2\big(\partial_{0t}^{\beta}\big(\Im_y(V_t)\big),\Im_y(V_t)\big)_{L_p^2(\Omega)}+\frac{\partial}{\partial t}\int_\Omega x\big(\Im_y(V_x)\big)^2\,dx\,dy+\frac{\partial}{\partial t}\int_\Omega V^2\,dx\,dy \\ &\quad=-2\left(F,\int_0^y\int_0^\xi V_t(x,\eta,t)\,d\eta\,d\xi\right)_{L_p^2(\Omega)}. \end{aligned}\tag{3.12}$$

In the light of Lemma (2.1), the first term on the LHS of (3.12) is estimated as follows:

$$2\big(\partial_{0t}^{\beta}\big(\Im_y(V_t)\big),\Im_y(V_t)\big)_{L_p^2(\Omega)}\ge\partial_{0t}^{\beta}\big\|\Im_y(V_t)\big\|_{L_p^2(\Omega)}^2.\tag{3.13}$$

Combination of inequality (3.13) and equality (3.12) gives

$$\begin{aligned} &\partial_{0t}^{\beta}\big\|\Im_y(V_t)\big\|_{L_p^2(\Omega)}^2+\frac{\partial}{\partial t}\big\|\Im_y(V_x)\big\|_{L_p^2(\Omega)}^2+\frac{\partial}{\partial t}\|V\|_{L^2(\Omega)}^2 \\ &\quad\le-2\left(F,\int_0^y\int_0^\xi V_t(x,\eta,t)\,d\eta\,d\xi\right)_{L_p^2(\Omega)}. \end{aligned}\tag{3.14}$$

By using the Cauchy-ε inequality and Poincaré inequality 2), we infer from (3.14) that

$$\begin{aligned} &\partial_{0t}^{\beta}\big\|\Im_y(V_t)\big\|_{L_p^2(\Omega)}^2+\frac{\partial}{\partial t}\big\|\Im_y(V_x)\big\|_{L_p^2(\Omega)}^2+\frac{\partial}{\partial t}\|V\|_{L^2(\Omega)}^2 \\ &\quad\le M_1\big(\|F\|_{L_p^2(\Omega)}^2+\big\|\Im_y(V_t)\big\|_{L_p^2(\Omega)}^2\big), \end{aligned}\tag{3.15}$$

where

$$M_1=\max\left(1,\frac{d^2}{2}\right).\tag{3.16}$$

Note that

$$\int_0^t\partial_{0t}^{\beta}\big\|\Im_y(V_t)\big\|_{L_p^2(\Omega)}^2\,d\tau=D_{0t}^{\beta-1}\big\|\Im_y(V_t)\big\|_{L_p^2(\Omega)}^2-\frac{t^{1-\beta}}{\Gamma(1-\beta)}\big\|\Im_y\big(V_t(x,y,0)\big)\big\|_{L_p^2(\Omega)}^2.\tag{3.17}$$

By changing t by τ, and integrating both sides of (3.15) with respect to τ on $[0;t]$ we find

$$\begin{aligned} &D_{0t}^{\beta-1}\big\|\Im_y(V_t)\big\|_{L_p^2(\Omega)}^2+\big\|\Im_y(V_x)\big\|_{L_p^2(\Omega)}^2+\|V\|_{L^2(\Omega)}^2 \\ &\quad\le M_1\left(\int_0^t\|F\|_{L_p^2(\Omega)}^2\,d\tau+\int_0^t\big\|\Im_y\big(V_\tau(x,y,\tau)\big)\big\|_{L_p^2(\Omega)}^2\,d\tau\right. \\ &\qquad\left.+\frac{T^{1-\beta}}{\Gamma(1-\beta)}\big\|\Im_y(g)\big\|_{L_p^2(\Omega)}^2+\big\|\Im_y(f_x)\big\|_{L_p^2(\Omega)}^2+\|f\|_{L^2(\Omega)}^2\right). \end{aligned}\tag{3.18}$$

We consider the following two elementary inequalities:

$$\big\|\Im_y(V)\big\|_{L_p^2(\Omega)}^2\le\frac{d^2}{2}\|V\|_{L_p^2(\Omega)}^2\le\frac{cd^2}{2}\|V\|_{L^2(\Omega)}^2,\tag{3.19}$$

$$\big\|\Im_y(f_x)\big\|_{L_p^2(\Omega)}^2\le\frac{d^2}{2}\|f_x\|_{L_p^2(\Omega)}^2\le\frac{cd^2}{2}\|f_x\|_{L^2(\Omega)}^2.\tag{3.20}$$

Inequality (3.18) takes the form

$$\begin{aligned} &D_{0t}^{\beta-1}\left\|\Im_y(V_t)\right\|^2_{L_p^2(\Omega)}+\left\|\Im_y(V_x)\right\|^2_{L_p^2(\Omega)}+\left\|\Im_y(V)\right\|^2_{L_p^2(\Omega)} \\ &\quad\leq M_2\biggl(\int_0^t\|F\|^2_{L_p^2(\Omega)}\,d\tau+\int_0^t\left\|\Im_y\bigl(V_\tau(x,y,\tau)\bigr)\right\|^2_{L_p^2(\Omega)}\,d\tau \\ &\qquad+\|f\|^2_{V^1(\Omega)}+\|g\|^2_{L_p^2(\Omega)}\biggr), \end{aligned} \tag{3.21}$$

where

$$M_2=\frac{M_1\max\{1,\frac{cd^2}{2},\frac{d^2}{2}\frac{T^{1-\beta}}{\Gamma(1-\beta)}\}}{\min\{1,\frac{2}{cd^2}\}}. \tag{3.22}$$

Now by omitting the last two terms on the left-hand side of (3.21) we get

$$\begin{aligned} &D_{0t}^{\beta-1}\left\|\Im_y(V_t)\right\|^2_{L_p^2(\Omega)} \\ &\quad\leq M_2\biggl(\int_0^t\|F\|^2_{L_p^2(\Omega)}\,d\tau+\int_0^t\left\|\Im_y\bigl(V_\tau(x,y,\tau)\bigr)\right\|^2_{L_p^2(\Omega)}\,d\tau \\ &\qquad+\|f\|^2_{V^1(\Omega)}+\|g\|^2_{L_p^2(\Omega)}\biggr). \end{aligned} \tag{3.23}$$

We apply Lemma (2.2) as follows:

$$y(t)=\int_0^t\left\|\Im_y\bigl(V_\tau(x,y,\tau)\bigr)\right\|^2_{L_p^2(\Omega)}\,d\tau,\qquad y(0)=0, \tag{3.24}$$

$$\partial_{0t}^\beta y(t)=D_{0t}^{\beta-1}\left\|\Im_y(V_t)\right\|^2_{L_p^2(\Omega)}. \tag{3.25}$$

Thence (3.23) becomes

$$\int_0^t\left\|\Im_y\bigl(V_\tau(x,y,\tau)\bigr)\right\|^2_{L_p^2(\Omega)}\,d\tau\leq M_3\bigl(D_{0t}^{-\beta-1}\|F\|^2_{L_p^2(\Omega)}+\|f\|^2_{V^1(\Omega)}+\|g\|^2_{L_p^2(\Omega)}\bigr), \tag{3.26}$$

where

$$M_3=\Gamma(\beta)E_{\beta,\beta}\bigl(M_2T^\beta\bigr)\max\biggl(M_2,\frac{M_2T^{\beta+1}}{(\beta+1)\Gamma(\beta+1)}\biggr). \tag{3.27}$$

Combination of inequalities (3.21) and (3.26) leads to

$$\begin{aligned} &D_{0t}^{\beta-1}\left\|\Im_y(V_t)\right\|^2_{L_p^2(\Omega)}+\|V\|^2_{V_p^{1,y}(\Omega)} \\ &\quad\leq M_4\biggl(D_{0t}^{-\beta-1}\|F\|^2_{L_p^2(\Omega)}+\int_0^t\|F\|^2_{L_p^2(\Omega)}\,d\tau+\|f\|^2_{V^1(\Omega)}+\|g\|^2_{L_p^2(\Omega)}\biggr), \end{aligned} \tag{3.28}$$

where

$$M_4=M_2M_3+M_2. \tag{3.29}$$

We make use of the inequality

$$D_{0t}^{-\beta-1}\|F\|^2_{L^2_p(\Omega)} \leq \frac{T^\beta}{\Gamma(\beta+1)}\int_0^t \|F\|^2_{L^2_p(\Omega)}\,d\tau. \tag{3.30}$$

Inequality (3.28) becomes

$$D_{0t}^{\beta-1}\left\|\Im_y(V_t)\right\|^2_{L^2_p(\Omega)} + \|V\|^2_{V_p^{1,y}(\Omega)} \tag{3.31}$$

$$\leq M_5\left(\int_0^t \|F\|^2_{L^2_p(\Omega)}\,d\tau + \|f\|^2_{V^1(\Omega)} + \|g\|^2_{L^2_p(\Omega)}\right) \tag{3.32}$$

$$\leq M_5\left(\int_0^T \|F\|^2_{L^2_p(\Omega)}\,dt + \|f\|^2_{V^1(\Omega)} + \|g\|^2_{L^2_p(\Omega)}\right). \tag{3.33}$$

Then we take the supremum of LHS in (3.31) with respect to t over $[0;T]$, we get the desired estimate (3.1) with

$$M_5 = M_4\left(1+\frac{T^\beta}{\Gamma(\beta+1)}\right). \tag{3.34}$$

□

Proposition 3.2 *The operator $L : X \longrightarrow Y$ is closable.*

Proof The proof is analogous to that in [21]. □

Definition A solution of the operator equation

$$\overline{L}V = W = (F,f,g) \tag{3.35}$$

is called a strong solution of problem (2.1)–(2.4) where $\overline{L}$ is the closure of the operator L and $D(\overline{L})$ its domain of definition.

The points of the graph of $\overline{L}$ are limits of sequences of points of the graph of L, by passing to the limit, the estimate (3.1) can be extended to

$$\|V\|_X \leq \sqrt{M_5}\|\overline{L}V\|_Y, \quad \forall V \in D(\overline{L}). \tag{3.36}$$

From this we deduce the *following results.*

Corollary 3.3 *If a strong solution of problem* (2.1)–(2.4) *exists, it is unique and depends continuously on elements $W = (F,f,g) \in Y$.*

Corollary 3.4 *The range of $R(\overline{L})$ of $\overline{L}$ is closed in Y, and $R(\overline{L}) = \overline{R(L)}$.*

4 Existence of the solution

To prove the existence of a strong solution $V = \overline{L}^{-1}W = \overline{L^{-1}}W$ of the problem (2.1)–(2.4) $\forall W = (F,f,g) \in Y$, it suffices to prove that $\overline{R(L)} = Y$, the density of the range $R(L)$ in Y is equivalent to the orthogonality of a vector $W = (F,f,g) \in Y$ to the set $R(L)$. For this purpose, we begin by the following theorem (the proof of the density in a special case).

Theorem 4.1 *For some function $G \in L_p^2(D)$, and for all $U \in D_0(L) = \{U/U \in D(L), \ell_1 U = 0, \ell_2 U = 0\}$, we have*

$$(\mathcal{L}U, G)_{L_p^2(D)} = 0. \tag{4.1}$$

Then $G = 0$ almost everywhere in the domain D.

Proof Equation (4.1) can be written

$$\left(\partial_{0t}^{\beta+1} U - \frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial U}{\partial x}\right) - \frac{1}{x}\frac{\partial^2 U}{\partial y^2}, G\right)_{L_p^2(D)} = 0. \tag{4.2}$$

Let $h(x,y,t)$ be a function that satisfies the boundary conditions (2.2)–(2.4) and

$$h, h_x, h_y, \int_0^t h(x,y,s)\,ds, x\int_0^t h_x(x,y,s)\,ds, x\int_0^t h_y(x,y,s)\,ds, \partial_{0t}^{\beta+1} h \in L^2(D). \tag{4.3}$$

Then we suppose

$$U(x,y,t) = \int_0^t \int_0^s h(x,y,z)\,dz\,ds. \tag{4.4}$$

By replacing $U(x,y,t)$ in (4.2) we have

$$\begin{aligned}&\left(\partial_{0t}^{\beta+1}\left(x\int_0^t \int_0^s h(x,y,z)\,dz\,ds\right) - \frac{\partial}{\partial x}\left(x\int_0^t \int_0^s h_x(x,y,z)\,dz\,ds\right)\right.\\ &\quad\left. - \frac{\partial}{\partial y}\left(\int_0^t \int_0^s h_y(x,y,z)\,dz\,ds\right), G\right)_{L^2(D)} = 0.\end{aligned} \tag{4.5}$$

Now, we assume the function

$$G(x,y,t) = -\int_0^y \int_0^\xi \int_0^t h(x,\eta,s)\,ds\,d\eta\,d\xi. \tag{4.6}$$

Then Eq. (4.5) becomes

$$\begin{aligned}&-\left(\partial_{0t}^{\beta+1}\left(\int_0^t \int_0^s h(x,y,z)\,dz\,ds\right), \int_0^y \int_0^\xi \int_0^t h(x,\eta,s)\,ds\,d\eta\,d\xi\right)_{L_p^2(D)}\\ &\quad + \left(\frac{\partial}{\partial x}\left(x\int_0^t \int_0^s h_x(x,y,z)\,dz\,ds\right), \int_0^y \int_0^\xi \int_0^t h(x,\eta,s)\,ds\,d\eta\,d\xi\right)_{L^2(D)}\\ &\quad + \left(\frac{\partial}{\partial y}\left(\int_0^t \int_0^s h_y(x,y,z)\,dz\,ds\right), \int_0^y \int_0^\xi \int_0^t h(x,\eta,s)\,ds\,d\eta\,d\xi\right)_{L^2(D)} = 0.\end{aligned} \tag{4.7}$$

Taking into account that the function h verifies the conditions (2.2)–(2.4), then integrating by parts each term of (4.7) we have

$$-\left(\partial_{0t}^{\beta+1}\left(\int_0^t \int_0^s h(x,y,z)\,dz\,ds\right), \int_0^y \int_0^\xi \int_0^t h(x,\eta,s)\,ds\,d\eta\,d\xi\right)_{L_p^2(\Omega)}$$

$$= \left(\partial_{0t}^{\beta}\left(\int_0^y\int_0^t h(x,\xi,s)\,ds\,d\xi\right), \int_0^y\int_0^t h(x,\xi,s)\,ds\,d\xi\right)_{L_p^2(\Omega)}, \tag{4.8}$$

$$\left(\frac{\partial}{\partial x}\left(x\int_0^t\int_0^s h_x(x,y,z)\,dz\,ds\right), \int_0^y\int_0^\xi\int_0^t h(x,\eta,s)\,ds\,d\eta\,d\xi\right)_{L^2(\Omega)}$$
$$= \left(\int_0^y\int_0^t\int_0^s h_x(x,\xi,z)\,dz\,ds\,d\xi, \int_0^y\int_0^t h_x(x,\xi,s)\,ds\,d\xi\right)_{L_p^2(\Omega)}$$
$$= \frac{1}{2}\frac{\partial}{\partial t}\left\|\int_0^y\int_0^t\int_0^s h_x(x,\xi,z)\,dz\,ds\,d\xi\right\|^2_{L_p^2(\Omega)}, \tag{4.9}$$

$$\left(\frac{\partial}{\partial y}\left(\int_0^t\int_0^s h_y(x,y,z)\,dz\,ds\right), \int_0^y\int_0^\xi\int_0^t h(x,\eta,s)\,ds\,d\eta\,d\xi\right)_{L^2(\Omega)}$$
$$= \left(\int_0^t\int_0^s h(x,y,z)\,dz\,ds, \int_0^t h(x,y,s)\,ds\right)_{L^2(\Omega)}$$
$$= \frac{1}{2}\frac{\partial}{\partial t}\left\|\int_0^t\int_0^s h(x,y,z)\,dz\,ds\right\|^2_{L^2(\Omega)}. \tag{4.10}$$

Substituting (4.8), (4.9), and (4.10) into (4.7) we get

$$\begin{aligned}
&2\left(\partial_{0t}^{\beta}\left(\int_0^y\int_0^t h(x,\xi,s)\,ds\,d\xi\right), \int_0^y\int_0^t h(x,\xi,s)\,ds\,d\xi\right)_{L_p^2(\Omega)} \\
&\quad + \frac{\partial}{\partial t}\left\|\int_0^y\int_0^t\int_0^s h_x(x,\xi,z)\,dz\,ds\,d\xi\right\|^2_{L_p^2(\Omega)} \\
&\quad + \frac{\partial}{\partial t}\left\|\int_0^t\int_0^s h(x,y,z)\,dz\,ds\right\|^2_{L^2(\Omega)} = 0.
\end{aligned} \tag{4.11}$$

According to Lemma (2.1) the first term on the LHS of (4.11) can be estimated as follows:

$$\begin{aligned}
&2\left(\partial_{0t}^{\beta}\left(\int_0^y\int_0^t h(x,\xi,s)\,ds\,d\xi\right), \int_0^y\int_0^t h(x,\xi,s)\,ds\,d\xi\right)_{L_p^2(\Omega)} \\
&\quad \geq \partial_{0t}^{\beta}\left\|\int_0^y\int_0^t h(x,\xi,s)\,ds\,d\xi\right\|^2_{L_p^2(\Omega)}.
\end{aligned} \tag{4.12}$$

Equation (4.11) can be written

$$\begin{aligned}
&\partial_{0t}^{\beta}\left\|\int_0^y\int_0^t h(x,\xi,s)\,ds\,d\xi\right\|^2_{L_p^2(\Omega)} + \frac{\partial}{\partial t}\left\|\int_0^y\int_0^t\int_0^s h_x(x,\xi,z)\,dz\,ds\,d\xi\right\|^2_{L_p^2(\Omega)} \\
&\quad + \frac{\partial}{\partial t}\left\|\int_0^t\int_0^s h(x,y,z)\,dz\,ds\right\|^2_{L^2(\Omega)} \leq 0.
\end{aligned} \tag{4.13}$$

By replacing t by τ and integrating of (4.13) with respect to τ over $[0;t]$ gives

$$\begin{aligned}
&D_{0t}^{\beta-1}\left\|\int_0^y\int_0^t h(x,\xi,s)\,ds\,d\xi\right\|^2_{L_p^2(\Omega)} \\
&\quad + \left\|\int_0^y\int_0^t\int_0^s h_x(x,\xi,z)\,dz\,ds\,d\xi\right\|^2_{L_p^2(\Omega)}
\end{aligned}$$

$$+\left\| \int_0^t \int_0^s h(x,y,z)\,dz\,ds \right\|^2_{L^2(\Omega)} \le 0. \tag{4.14}$$

We find from inequality (4.14) that $G \equiv 0$ almost everywhere in D. □

Theorem 4.2 *The range $R(L)$ of the operator L, coincides with the whole space Y.*

Proof Let $W = (\varphi, g_1, g_2) \in R(L)^{\perp}$ such that

$$(\mathcal{L}u, \varphi)_{L^2_p(D)} + (\ell_1 u, g_1)_{V^1(\Omega)} + (\ell_2 u, g_2)_{L^2_p(\Omega)} = 0. \tag{4.15}$$

If we put $u \in D_0(L)$ into (4.15) we get

$$(\mathcal{L}u, \varphi)_{L^2_p(D)} = 0, \quad \forall u \in D_0(L). \tag{4.16}$$

By virtue of Theorem 4.1 we deduce that $\varphi \equiv 0$, thus (4.15) becomes

$$(\ell_1 u, g_1)_{V^1(\Omega)} + (\ell_2 u, g_2)_{L^2_p(\Omega)} = 0, \quad \forall u \in D(L). \tag{4.17}$$

The trace operators ℓ_1 and ℓ_2 are independent, and $R(\ell_1)$ and $R(\ell_2)$ are everywhere dense in the spaces $V^1(\Omega)$ and $L^2_p(\Omega)$, respectively. Then $g_1 = 0$, $g_2 = 0$. Consequently $W = 0$. Hence $R(L)^{\perp} = 0$ i.e. $\overline{R(L)} = Y$. □

5 Conclusion

The well posedness of 2-D time-fractional differential equations with boundary integral conditions is proved. The functional analysis method was successfully applied to a fractional-order initial boundary value problem.

Our results develop the traditional functional analysis method which relies on some a priori estimates and some density arguments for a fractional hyperbolic equation with fractional Caputo derivative.

Acknowledgements
The authors wish to thank the anonymous referees for their comments and valuable suggestions.

Funding
The authors stress that there are no sources of funding.

Availability of data and materials
The authors emphasize the existence of all data and materials in their manuscript and everything that supports their findings.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors contributed equally to the writing of this paper. Both authors read and approved the final manuscript.

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Publisher's Note

Received: 16 August 2019 Accepted: 3 December 2019 Published online: 11 December 2019

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