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Multiple positive solutions to singular fractional differential equations with integral boundary conditions involving $p-q$-order derivatives

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Abstract

In this paper, we investigate the existence for a class of higher-order fractional differential equations with integral boundary value conditions involving $p-q$-order derivatives. As an application of the height functions on some special bounded sets, we obtain the existence of two positive solutions by means of the Leray–Schauder nonlinear alternative and cone expansion and cone compression fixed point theory. The nonlinearity may take negative infinity, and there may appear a singular phenomenon on both time and space variables.

MSC: 26A33; 34A08; 34B18

Keywords: Nonlocal fractional differential equations; Integral conditions; Semipositone; Singularity on space variable; Positive solutions

1 Introduction

The purpose of this paper is to obtain the existence of multiple positive solutions of the following singular fractional differential equations (FDEs for short) with $p-q$-order derivatives:

$$\begin{cases} D_{0^+}^{\alpha}x(t)+f(t,x(t))=0, \quad t\in(0,1),\\ x(0)=x'(0)=x''(0)=\cdots=x^{(n-2)}(0)=0,\\ D_{0^+}^{p}x(1)=\lambda\int_0^{\eta}h(t)D_{0^+}^{q}x(t)\,dt, \end{cases} \tag{1.1}$$

where $D_{0^+}^{\alpha}$ is the standard Riemann–Liouville derivative of order α, $n-1<\alpha\le n$, $n\ge 3$, $h\in L^1[0,1]$ is nonnegative and may be singular at $t=0$ and $t=1$, $p,q\in\mathbb{R}$, $1\le p\le n-2$, $0\le q\le p$, $\Delta=\Gamma(\alpha)/\Gamma(\alpha-p)(1-\lambda\int_0^{\eta}h(t)t^{\alpha-q-1}\,dt)>0$, $f(t,x)$ permits sign-changing and singularities at $t=0,1$ and/or $x=0$.

Nowadays FDEs nonlocal problems are of great interest because of their abilities to modeling complex phenomena in almost every field of science and technology. Many excellent works about FDEs nonlocal problems can be found in the literature (see [1, 3, 9–11, 13, 16, 17, 20, 22, 27, 31] for instance). The existence of positive solutions for FDEs integral boundary problems (BVPs) with one or two parameters in boundary conditions is obtained by means of fixed point theory [5, 14, 22, 28]. The existence of positive solutions for integer-order differential equations with integral boundary conditions can be found in the literature (see [8, 13, 15, 18, 24] for instance). As is well known, semipositone problems arise in bulking of mechanical systems, chemical reactions, astrophysics, combustion, management of natural resources, etc. Details are available in the works [2, 6, 15, 21, 23, 24, 26, 29, 32, 34, 35, 37, 39]. Studying positive solutions for semipositone problems is more difficult than that for positive problems. Many methods used to deal with semipositone problems are, for example, variational methods, fixed point theory, subsuper solution methods, and degree theory.

Xu et al. [30] studied the existence and uniqueness of positive solutions for the fractional boundary value problem

$$\begin{cases} D^{\alpha}_{0^+}x(t) + h(t)f(t,x(t)) = 0, & t \in (0,1), n-1 < \alpha \leq n, \\ x^{(k)}(0) = 0, \quad 0 \leq k \leq n-2, & [D^{\beta}_{0^+}x]_{t=1} = 0, \quad 1 \leq \beta \leq n-2, \end{cases}$$

where $n > 3$, $D^{\alpha}_{0^+}$ is the standard Riemann–Liouville derivative, $f \in C([0,1] \times [0,\infty), [0,\infty))$ and $h \in C(0,1) \cap L(0,1)$ is nonnegative and may be singular at $t = 0$ and/or $t = 1$. In the case that f is growing sublinearly, by means of fixed index theory and some spectral properties of associated linear integral operators the existence and uniqueness of positive solutions are obtained.

Salem [25] obtained the existence of pseudosolutions for the nonlinear multipoint boundary value problem of fractional order

$$\begin{cases} D^{\alpha}_{0^+}x(t) + q(t)f(t,x(t)) = 0, & \text{a.e. on } [0,1], \alpha \in (n-1,n], n \geq 2, \\ x(0) = x'(0) = x''(0) = \cdots = x^{(n-1)}(0) = 0, & x(1) = \sum_{i=1}^{m-2} \zeta_i x(\eta_i), \end{cases}$$

where $0 < \eta_1 < \eta_2 < \cdots < \eta_{m-2} < 1$, and $\zeta_i > 0$ with $\sum_{i=1}^{m-2} \zeta_i \eta_i^{\alpha-1} < 1$. It was assumed that q is a real-valued continuous function and f is a nonlinear Pettis-integrable function. Other relative papers on multipoint boundary value problem can be found in [4, 19, 33, 37].

In 2018, Zhang et al. [37] investigated the following singular differential equation with fractional derivative:

$$\begin{cases} D^{\alpha}_{0^+}x(t) + f(t,x(t)) = 0, & t \in (0,1), \\ x^{(k)}(0) = 0, \quad 0 \leq k \leq n-2, & D^{p}_{0^+}x(1) = \sum_{i=1}^{m} a_i D^{q}_{0^+}x(\xi_i), \end{cases}$$

where $D^{\alpha}_{0^+}$ is the standard Riemann–Liouville derivative of order α, $n-1 < \alpha \leq n$, $n \geq 3$, $a_i \geq 0$, $i = 1,2,\ldots,m$ ($m \in \mathbb{N}^+$), $0 < \xi_1 < \xi_2 < \cdots < \xi_m < 1$, $p,q \in \mathbb{R}$, $1 \leq p \leq n-2$, and $0 \leq q \leq p$ with $\Delta = \Gamma(\alpha)/\Gamma(\alpha-p)(1 - \sum_{i=1}^{m} a_i \xi_i^{\alpha-q-1}) > 0$. The authors obtained the existence of triple positive solutions for fractional differential equations subject to multipoint boundary conditions by virtue of height functions on some special bounded sets.

Zhang and Han [36] investigated the higher-order nonlocal fractional differential equations

$$\begin{cases} D_{0^+}^{\alpha}x(t) + f(t, x(t)) = 0, & t \in (0,1), n-1 < \alpha \le n, \\ x^{(k)}(0) = 0, \quad 0 \le k \le n-2, & x(1) = \int_0^1 x(s)\,dA(s), \end{cases}$$

where $\alpha \ge 2$, $D_{0^+}^{\alpha}$ is the standard Riemann–Liouville derivative, A is a function of bounded variation, and $\int_0^1 x(s)\,dA(s)$ denotes the Riemann–Stieltjes integral. By applying the monotone iterative technique the existence and uniqueness of positive solutions were obtained, provided that $f(t,x)$ satisfies some growth conditions.

Inspired by the achievements mentioned, we consider the existence of multiple positive solutions of FDEs (1.1). In comparison with known results, this paper has some new features. Firstly, the nonlinearity f may take negative infinity and change its sign. Secondly, the function $f(t,x)$ may be singular with respect to the time variable t and/or the space variable x. Thirdly, the boundary conditions include an integral boundary condition involving $p-q$-order derivatives, quite different from those in [25, 30, 36]. Finally, the method in this paper is different from that in [38]. We devote ourselves to obtaining two existence results for BVP (1.1) by fixed point theory and the Leray–Schauder nonlinear alternative. Integration of height functions on special bounded sets is utilized to obtain the existence of positive solutions.

2 Basic definitions and preliminaries

This paper involves Banach spaces $X = C[0,1]$ and $L^1[0,1]$, the spaces of continuous functions and Lebesgue-integrable functions equipped with the norms $\|x\| = \max_{0\le t\le 1}|x(t)|$ and $\|x\|_1 = \int_0^1 |x(t)|\,dt$.

Lemma 2.1 ([38], p. 13) *Assume that $\Delta \neq 0$. Then for any $z \in C[0,1] \cap L^1[0,1]$, the solution of the boundary value problem*

$$\begin{cases} D_{0^+}^{\alpha}x(t) + z(t) = 0, & t \in (0,1), \\ x(0) = x'(0) = x''(0) = \cdots = x^{(n-2)}(0) = 0, \\ D_{0^+}^{p}x(1) = \lambda \int_0^{\eta} h(t) D_{0^+}^{q}x(t)\,dt, \end{cases}$$

satisfies

$$x(t) = \int_0^1 G(t,s)z(s)\,ds, \quad t \in (0,1),$$

where

$$G(t,s) = G_1(t,s) + G_2(t,s)$$

with

$$G_1(t,s) = \begin{cases} \frac{t^{\alpha-1}(1-s)^{\alpha-p-1} - (t-s)^{\alpha-p-1}}{\Gamma(\alpha)}, & 0 \le s \le t \le 1, \\ \frac{t^{\alpha-1}(1-s)^{\alpha-p-1}}{\Gamma(\alpha)}, & 0 \le t \le s \le 1, \end{cases}$$

$$G_2(t,s) = \frac{\lambda t^{\alpha-1}}{\Delta}\int_0^{\eta} h(t)H(t,s)\,dt,$$

$$H(t,s) = \begin{cases} \frac{t^{\alpha-q-1}(1-s)^{\alpha-p-1}-(t-s)^{\alpha-p-1}}{\Gamma(\alpha-q)}, & 0 \le s \le t \le 1, \\ \frac{t^{\alpha-q-1}(1-s)^{\alpha-p-1}}{\Gamma(\alpha-q)}, & 0 \le t \le s \le 1. \end{cases}$$

Definition 2.1 ([38], p. 13) A function $\zeta : K \longrightarrow [0,+\infty)$ is called a concave positive functional on a cone K if

$$\zeta\bigl(tx + (1-t)y\bigr) \ge t\zeta(x) + (1-t)\zeta(y), \quad x, y \in K, 0 \le t \le 1.$$

Lemma 2.2 ([37], p. 891) *The Green function defined in Lemma 2.1 is a continuous function on* $[0,1] \times [0,1]$ *and satisfies the following conditions*:

(i) $G(t,s) \le J(s)$, $t,s \in [0,1] \times [0,1]$, *where*

$$J(s) = h_1(s) + \frac{\lambda}{\Delta}\int_0^{\eta} h(t)H(t,s)\,dt, \quad s \in [0,1],$$

$$h_1(s) = (1-s)^{\alpha-p-1}\bigl(1-(1-s)^p\bigr)/\Gamma(\alpha), \quad s \in [0,1];$$

(ii) $t^{\alpha-1}J(s) \le G(t,s) \le \sigma t^{\alpha-1}$, *where*

$$\sigma = 1/\Gamma(\alpha) + \lambda\int_0^{\eta} h(t)t^{\alpha-q-1}\,dt/\bigl(\Delta\Gamma(\alpha-q)\bigr), \quad t,s \in [0,1].$$

Lemma 2.3 *Let* $w \in C[0,1]$ *be a solution of*

$$\begin{cases} D_{0^+}^{\alpha}x(t) + \phi(t) = 0, \quad t \in (0,1), \\ x(0) = x'(0) = x''(0) = \cdots = x^{(n-2)}(0) = 0, \\ D_{0^+}^{p}x(1) = \lambda\int_0^{\eta} h(t)D_{0^+}^{q}x(t)\,dt, \end{cases}$$

where $\phi \in L^1[0,1]$, $\phi(t) > 0$. *Then* $w(t) \le \sigma\|\phi\|_1 t^{\alpha-1}$, $0 \le t \le 1$.

Lemma 2.4 ([12], p. 94) *Let* Ω_1 *and* Ω_2 *be two bounded open sets in a Banach space* X *such* $\theta \in \Omega_1$ *and* $\overline{\Omega}_1 \subset \Omega_2$, *and let* $A : P \cap (\overline{\Omega}_2 \backslash \Omega_1) \to P$ *be a completely continuous operator, where* θ *denotes the zero element of* X, *and* P *is a cone in* X. *Suppose that one of the two conditions holds*:

(i) $\|Ax\| \le \|x\|$, $x \in P \cap \partial\Omega_1$; $\|Ax\| \ge \|x\|$, $x \in P \cap \partial\Omega_2$,

(ii) $\|Ax\| \ge \|x\|$, $x \in P \cap \partial\Omega_1$; $\|Ax\| \le \|x\|$, $x \in P \cap \partial\Omega_2$.

Then A *has a fixed point in* $P \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

Lemma 2.5 ([7], P3) *If* X *is a Banach space,* $D \subset X$ *is convex with* $\theta \subset D$, *and* $A : D \to D$ *is a completely continuous operator, then either*

(i) *the set* $B = \{x \in D : x = \lambda A(x), 0 < \lambda < 1\}$ *is unbounded, or*

(ii) A *has a fixed point.*

3 Main results

Let $K = \{x \in X : x(t) \ge t^{\alpha-1}\|x\| \text{ for } t \in [0,1]\}$. Obviously, K is a cone in X, and (X,K) is a partially ordered Banach space.

The following hypotheses will be used throughout this paper:

(H_1) $f \in C((0,1) \times (0,+\infty), (-\infty,+\infty))$, and there exists a function $\phi \in L^1[0,1]$, $\phi(t) > 0$, such that $f(t,x) \geq -\phi(t)$ for all $t \in (0,1)$ and $x > 0$;

(H_2) $f^*(t,x) = f(t,x) + \phi(t)$ and $f^*(t,x) \leq q(t)[g(x) + h(x)]$ on $(0,1] \times (0,+\infty)$ with $g > 0$ continuous and nonincreasing on $(0,+\infty)$, $h > 0$ continuous on $[0,+\infty)$, $\frac{h}{g}$ nondecreasing on $(0,+\infty)$, and $q \in L^1[0,1]$ such that $q > 0$ on $(0,1)$;

(H_3) There exists $K_0 > 0$ such that $g(ab) \leq K_0 g(a)g(b)$ for $a > 0$, $b > 0$;

(H_4) $a_0 = \int_0^1 q(s)g(s^{\alpha-1})\,ds < +\infty$, and there exists $r > \sigma\|\phi\|_1$ such that

$$\frac{r}{g(r-\sigma\|\phi\|_1)\{1+\frac{h(r)}{g(r)}\}} > \sigma a_0 K_0;$$

(H_5) For each $L > 0$, there exists a positive function $\tau_L \in C[0,1]$ such that $f^*(t,x) \geq \tau_L(t)$ for $(t,x) \in [0,1] \times (0,L)$ and $\tau_r(t) > \phi(t)$, where r is as in (H_4);

(H_6) There exist $R > r > \sigma\|\phi\|_1$ such that

$$\int_0^1 J(s)\varphi(s,r)\,ds < r \tag{3.1}$$

and

$$\int_0^1 J(s)\psi(s,R)\,ds > R, \tag{3.2}$$

where

$$\varphi(t,r) = \max\bigl\{f^*(t,x) : \bigl(r-\sigma\|\phi\|_1\bigr)t^{\alpha-1} \leq x \leq r\bigr\},$$
$$\psi(t,R) = \min\bigl\{f^*(t,x) : \bigl(R-\sigma\|\phi\|_1\bigr)t^{\alpha-1} \leq x \leq R\bigr\}.$$

To establish the existence of a positive solution for BVP (1.1), we will concentrate on the following modified approximating BVP to overcome difficulties caused by singularities:

$$\begin{cases} D_{0^+}^{\alpha}x(t) + f^*(t,x(t)-w(t)) = 0, \quad t \in (0,1), \\ x(0) = x'(0) = x''(0) = \cdots = x^{(n-2)}(0) = 0, \\ D_{0^+}^{p}x(1) = \lambda\int_0^{\eta} h(t)D_{0^+}^{q}x(t)\,dt. \end{cases} \tag{3.3}$$

Define the operator T by

$$(Tx)(t) = \int_0^1 G(t,s)f^*\bigl(s,x(s)-w(s)\bigr)\,ds, \quad 0 \leq t \leq 1. \tag{3.4}$$

Lemma 3.1 *Suppose that (H_1)–(H_4) hold. Then for any $\sigma\|\phi\|_1 < r < R$, the operator $T: K \cap (\overline{\Omega}_R \backslash \Omega_r) \longrightarrow K$ is completely continuous.*

Proof For any $0 < t < 1$ and $\sigma\|\phi\|_1 < r < R$, since $x(t) - w(t) \geq (r-\sigma\|\phi\|_1)t^{\alpha-1} > 0$, we have

$$\bigl(r-\sigma\|\phi\|_1\bigr)t^{\alpha-1} \leq x(t) - w(t) \leq R. \tag{3.5}$$

From this we get that

$$
\begin{aligned}
(Tx)t &= \int_0^1 G(t,s)f^*\bigl(s,x(s)-w(s)\bigr)\,ds \\
&\le \sigma t^{\alpha-1}\int_0^1 f^*\bigl(s,x(s)-w(s)\bigr)\,ds \\
&\le \sigma \int_0^1 q(s)\bigl[g\bigl(x(s)-w(s)\bigr)+h\bigl(x(s)-w(s)\bigr)\bigr]\,ds \\
&\le \sigma \int_0^1 q(s)g\bigl(\bigl(r-\sigma\|\phi\|_1\bigr)s^{\alpha-1}\bigr)\left\{1+\frac{h(R)}{g(R)}\right\}ds \\
&\le \sigma a_0K_0g\bigl(r-\sigma\|\phi\|_1\bigr)\left\{1+\frac{h(R)}{g(R)}\right\}<+\infty.
\end{aligned}
\tag{3.6}
$$

Therefore the operator $T:K\cap(\overline{\Omega}_R\backslash\Omega_r)\longrightarrow X$ is well defined. At the same time, we have the operator decomposition $T=N_1\circ N_2$, where

$$
\begin{aligned}
&(N_1x)(t)=\int_0^1 G(t,s)x(s)\,ds, \quad t\in(0,1), x\in L_1[0,1], \\
&(N_2x)(t)=f^*\bigl(t,x(t)-w(t)\bigr), \quad \forall x\in K\cap(\overline{\Omega}_R\backslash\Omega_r).
\end{aligned}
$$

By Lemma 2.2 we have

$$
(N_1x)(t)=\int_0^1 G(t,s)x(s)\,ds\ge t^{\alpha-1}\int_0^1 J(s)x(s)\,ds\ge t^{\alpha-1}\|N_1x\|.
$$

This shows that $N_1:L_1[0,1]\to K$. Furthermore, the operator $N_1:L_1[0,1]\longrightarrow K$ is completely continuous by the Arzelà–Ascoli theorem. To prove that the operator $T:K\cap(\overline{\Omega}_R\backslash\Omega_r)\longrightarrow K$ is completely continuous, we only need to prove that $N_2:K\cap(\overline{\Omega}_R\backslash\Omega_r)\to L_1[0,1]$ is bounded and continuous. By assumptions (H_2)–(H_4) we have

$$
\begin{aligned}
&\bigl|f^*\bigl(t,x(t)-w(t)\bigr)\bigr|\le K_0q(t)g\bigl(t^{\alpha-1}\bigr)g\bigl(r-\sigma\|\phi\|_1\bigr)\left\{1+\frac{h(R)}{g(R)}\right\}, \quad t\in[0,1], \\
&\bigl(r-\sigma\|\phi\|_1\bigr)t^{\alpha-1}\le x(t)-w(t)\le R.
\end{aligned}
$$

This shows that for $x\in K\cap(\overline{\Omega}_R\backslash\Omega_r)$,

$$
\begin{aligned}
\int_0^1 (N_2x)(t)\,dt &= \int_0^1 f^*\bigl(t,x(t)-w(t)\bigr)\,dt \\
&\le a_0K_0g\bigl(r-\sigma\|\phi\|_1\bigr)\left\{1+\frac{h(R)}{g(R)}\right\}<+\infty.
\end{aligned}
\tag{3.7}
$$

Consequently, $N_2(K\cap(\overline{\Omega}_R\backslash\Omega_r))\subset L_1[0,1]$ is bounded. Let

$$
x_m,x_0\in K\cap(\overline{\Omega}_R\backslash\Omega_r), \quad m=1,2,\dots,
$$

and $\|x_m-x_0\|\to 0$. Then we have

$$
x_m(t)-x_0(t)\to 0, \quad t\in[0,1].
$$

By (H_1) we get

$$f^*\bigl(t, x_m(t) - w(t)\bigr) - f^*\bigl(t, x_0(t) - w(t)\bigr) \longrightarrow 0, \quad \text{a.e. } t \in [0,1].$$

By the Lebesgue dominated convergence theorem we have that $N_2 : K \cap (\overline{\Omega}_R \backslash \Omega_1) \to L_1[0,1]$ is continuous. Thus $T : K \cap (\overline{\Omega}_R \backslash \Omega_r) \longrightarrow K$ is completely continuous. □

Theorem 3.1 *Suppose that* (H_1)–(H_6) *hold. Then BVP* (1.1) *has at least two positive solutions.*

Proof For $x \in K \cap \partial\Omega_r$, we have $x(t) \geq rt^{\alpha-1}$, $t \in [0,1]$. Thus we get

$$\bigl(r - \sigma\|\phi\|_1\bigr)t^{\alpha-1} \leq x(t) - w(t) \leq r, \quad 0 < t < 1.$$

By (H_6) and Lemma 2.2 we have

$$\begin{aligned}\|Tx\| &= \max_{0\leq t\leq 1} \int_0^1 G(t,s)f^*\bigl(s, x(s) - w(s)\bigr)\,ds \\ &\leq \int_0^1 J(s)\varphi(s,r)\,ds < r.\end{aligned} \tag{3.8}$$

Obviously, $\|Tx\| \leq \|x\|$ for $x \in K \cap \partial\Omega_r$. For $x \in K \cap \partial\Omega_R$, $x(t) \geq Rt^{\alpha-1}$, $t \in [0,1]$. Thus we have

$$R \geq x(t) - w(t) \geq \bigl(R - \sigma\|\phi\|_1\bigr)t^{\alpha-1}.$$

By (H_6) and Lemma 2.2 we also obtain

$$\begin{aligned}\|Tx\| &= \max_{0\leq t\leq 1} \int_0^1 G(t,s)f^*\bigl(s, x(s) - w(s)\bigr)\,ds \\ &\geq \max_{0\leq t\leq 1} \int_0^1 J(s)f^*\bigl(s, x(s) - w(s)\bigr)\,ds \\ &\geq \int_0^1 J(s)\psi(s,R)\,ds \\ &> R,\end{aligned} \tag{3.9}$$

that is, $\|Tx\| \geq \|x\|$ for all $x \in K \cap \partial\Omega_R$. By Lemma 2.4 we have that T has one fixed point $x_1 \in K \cap (\overline{\Omega}_R \backslash \Omega_r)$. Therefore

$$\begin{aligned}&x_1(t) \geq \|x_1\|t^{\alpha-1} \geq rt^{\alpha-1} > \sigma\|\phi\|_1 t^{\alpha-1} \geq w(t), \quad t \in [0,1]. \\ &x_1(t) = \int_0^1 G(t,s)f^*\bigl(s, x_1(s) - w(s)\bigr)\,ds, \quad 0 < t < 1.\end{aligned}$$

Consider the family of equations

$$(T_n x)(t) = \int_0^1 G(t,s)f_n^*\bigl(s, x(s) - w(s)\bigr)\,ds + \frac{1}{n}, \quad 0 \leq t \leq 1, n \in \mathbb{N}, \tag{3.10}$$

where

$$f_n^*(t,x) = \begin{cases} f^*(t,x), & x \geq \frac{1}{n}, \\ f^*(t,\frac{1}{n}), & x < \frac{1}{n}. \end{cases} \tag{3.11}$$

Similar to the proof of Lemma 3.1, we can show that the operator $T_n : K \cap \Omega_r \longrightarrow K$ is completely continuous.

We consider

$$x = \lambda T_n x + (1-\lambda)\frac{1}{n},$$

that is,

$$x(t) = \lambda \int_0^1 G(t,s) f_n^*\bigl(s, x(s) - w(s)\bigr)\, ds + \frac{1}{n}, \quad t \in [0,1], \tag{3.12}$$

where $\lambda \in [0,1]$. We claim that any fixed point of (3.12) for any $\lambda \in [0,1]$ must satisfy $\|x\| \neq r$. Otherwise, assume that x is a fixed point of (3.12) for some $\lambda \in [0,1]$ such that $\|x\| = r$. Note that

$$\begin{aligned} x(t) - \frac{1}{n} &= \lambda \int_0^1 G(t,s) f_n^*\bigl(s, x(s) - w(s)\bigr)\, ds \\ &\leq \lambda \int_0^1 J(s) f_n^*\bigl(s, x(s) - w(s)\bigr)\, ds. \end{aligned}$$

Then we have

$$\left\| x - \frac{1}{n} \right\| \leq \lambda \int_0^1 J(s) f_n^*\bigl(s, x(s) - w(s)\bigr)\, ds.$$

On the other hand, we get

$$\begin{aligned} x(t) - \frac{1}{n} &= \lambda \int_0^1 G(t,s) f_n^*\bigl(s, x(s) - w(s)\bigr)\, ds \\ &\geq \lambda t^{\alpha-1} \int_0^1 J(s) f_n^*\bigl(s, x(s) - w(s)\bigr)\, ds \\ &\geq t^{\alpha-1} \left\| x - \frac{1}{n} \right\|. \end{aligned}$$

By the choice of n_0, $\frac{1}{n} \leq \frac{1}{n_0} < r - \sigma \|\phi\|_1$. Hence we have

$$x(t) \geq t^{\alpha-1} \left\| x - \frac{1}{n} \right\| + \frac{1}{n} \quad \geq t^{\alpha-1}\left(\|x\| - \frac{1}{n} \right) + \frac{1}{n} \geq r t^{\alpha-1} + \bigl(1 - t^{\alpha-1}\bigr)\frac{1}{n}.$$

Therefore

$$\begin{aligned} x(t) - w(t) &\geq r t^{\alpha-1} + \bigl(1 - t^{\alpha-1}\bigr)\frac{1}{n} - \sigma \|\phi\|_1 t^{\alpha-1} \\ &\geq \left(r - \sigma \|\phi\|_1 - \frac{1}{n} \right) t^{\alpha-1} + \frac{1}{n} > \frac{1}{n} \end{aligned}$$

and

$$\begin{aligned} x(t)-w(t) &\geq rt^{\alpha-1}+\left(1-t^{\alpha-1}\right)\frac{1}{n}-\sigma\|\phi\|_1 t^{\alpha-1} \\ &\geq \left(r-\sigma\|\phi\|_1\right)t^{\alpha-1}+\left(1-t^{\alpha-1}\right)\frac{1}{n} \\ &\geq \left(r-\sigma\|\phi\|_1\right)t^{\alpha-1}. \end{aligned}$$

For all $t \in [0,1]$, from condition (H_4) we have that

$$\begin{aligned} x(t) &= \lambda\int_0^1 G(t,s)f_n^*\big(s,x(s)-w(s)\big)\,ds+\frac{1}{n} \\ &= \lambda\int_0^1 G(t,s)f^*\big(s,x(s)-w(s)\big)\,ds+\frac{1}{n} \\ &\leq \sigma t^{\alpha-1}\int_0^1 f^*\big(s,x(s)-w(s)\big)\,ds+\frac{1}{n} \\ &\leq \sigma a_0K_0 g\big(r-\sigma\|\phi\|_1\big)\left\{1+\frac{h(r)}{g(r)}\right\}+\frac{1}{n}. \end{aligned}$$

Therefore

$$r=\|x\|\leq \sigma a_0K_0 g\big(r-\sigma\|\phi\|_1\big)\left\{1+\frac{h(r)}{g(r)}\right\}+\frac{1}{n}.$$

This is a contradiction to the choice of n_0, and the claim is proved. Now the Leray–Schauder alternative principle guarantees that

$$x(t)=\lambda\int_0^1 G(t,s)f_n^*\big(s,x(s)-w(s)\big)\,ds+\frac{1}{n}$$

has a fixed point $\bar{x}_n$ in $K\cap\Omega_r$.

Next, we claim that $\bar{x}_n(t)-w(t)$ has a uniform positive lower bound: there exists a constant $\delta>0$ such that

$$\min_{t\in[0,1]}\big\{\bar{x}_n(t)-w(t)\big\}\geq \delta t^{\alpha-1}$$

for all $n\in\mathbb{N}$. Since (H_6) holds, there exists a continuous function $\tau_r(t)>0$ such that $f^*(t,x)>\tau_r(t)>\phi(t)$ for all $(t,x)\in[0,1]\times(0,r]$.

Since $\bar{x}_n(t)-w(t)<r$, we have

$$\begin{aligned} \bar{x}_n(t)-w(t) &= \int_0^1 G(t,s)f_n^*\big(s,\bar{x}_n(s)-w(s)\big)\,ds+\frac{1}{n}-\int_0^1 G(t,s)\phi(s)\,ds \\ &\geq \int_0^1 G(t,s)\big(\tau_r(s)-\phi(s)\big)\,ds \\ &\geq t^{\alpha-1}\int_0^1 J(s)\big(\tau_r(s)-\phi(s)\big)\,ds=\delta t^{\alpha-1}, \end{aligned}$$

where $\delta=\int_0^1 J(s)(\tau_r(s)-\phi(s))\,ds$.

By (H_1) we have that $\bar{x}_n$ are bounded and equicontinuous on $[0,1]$. The Arzelà–Ascoli theorem implies that there exist a subsequence N_0 of N and a function x_2 such that $\bar{x}_n$ converges to x_2 uniformly on $[0,1]$ as $n\to\infty$ through N_0. Since $\min_{t\in[0,1]}\{\bar{x}_n(t)-w(t)\}\ge\delta t^{\alpha-1}$ and $\|\bar{x}_n\|<r$, it follow that x_2 satisfies $\delta t^{\alpha-1}\le x_2(t)-w(t)<r$ for all $t\in[0,1]$. Therefore x_2 is a positive solution of (3.4). Letting $n\to\infty$ on both sides, we have

$$x_i(t)=\int_0^1 G(t,s)f^*\bigl(s,x_i(s)-w(s)\bigr)\,ds,\quad i=1,2,t\in[0,1]. \tag{3.13}$$

Let $x_i^\star(t)=x_i(t)-w(t)$. Then from (3.13) it follows that $x_i^\star(t)$ $(i=1,2)$ are positive solutions of BVP (1.1). □

Remark 3.1 Suppose that (H_1)–(H_5) and the following conditions are satisfied:

$$\int_0^1 J(s)\varphi(s,r_i)\,ds<r_i,\qquad \int_0^1 J(s)\varphi(s,R_i)\,ds>R_i,$$

where $R_i>r_i>\sigma\|\phi\|_1$ $(i=1,2,\ldots,m)$. Then BVP (1.1) has at least $m+1$ positive solutions.

4 An example

Example 4.1 Consider the fractional differential equation

$$\begin{cases} D_{0^+}^{\frac{5}{2}}x(t)+f(t,x(t))=0, & t\in(0,1),\\ x(0)=x'(0)=0, & x'(1)=\frac{1}{3}\int_0^{\frac{1}{3}}t^{-\frac{1}{4}}D_{0^+}^{\frac{1}{2}}x(t)\,dt, \end{cases} \tag{4.1}$$

where $f(t,x)=\frac{1}{10}t^{10}(\frac{1}{\sqrt{u}}+u^{\frac{7}{2}})-\frac{1}{30}t^{10}$. It is clear that $\alpha=\frac{5}{2}$, $n=3$, $p=1$, $q=\frac{1}{2}$, $\lambda=\frac{1}{3}$, $\eta=\frac{1}{3}$, $h(t)=t^{-\frac{1}{4}}$.

By a simple computation $\Gamma(\alpha)=1.3294$ and $\lambda\int_0^\eta h(t)t^{\alpha-q-1}\,dt=0.0279<1$.

Clearly, (H_1) holds for $\phi(t)=\frac{1}{30}t^{10}$.

After direct calculation, we have $\sigma=0.7714$, $\|\phi\|_1=0.0031$, and $\sigma\|\phi\|_1=0.0024$. Let $q(t)=\frac{1}{10}t^{10}$, $g(x)=\frac{1}{\sqrt{x}}$, $h(x)=x^{\frac{7}{2}}$, $K_0=0.1$. Then (H_2) and (H_3) are satisfied.

We take $r=1>0.0024=\sigma\|\phi\|_1$, and by direct calculation we have

$$a_0=\int_0^1 q(s)g\bigl(s^{\alpha-1}\bigr)\,ds=0.9978<+\infty$$

and

$$\frac{r}{g(r-\sigma\|\phi\|_1)\{1+\frac{h(r)}{g(r)}\}}=0.4994>0.0769=\sigma a_0K_0.$$

Thus (H_4) is verified. Then by Lemma 2.2 we get

$$\begin{aligned}\int_0^1 J(s)\varphi(s,1)\,ds\le{}&\frac{1}{10}\int_0^1\biggl[\frac{(1-s)^{\alpha-p-1}(1-(1-s)^p)}{\Gamma(\alpha)}\\ &+\frac{\lambda(1-s)^{\alpha-p-1}\int_0^\eta h(t)t^{\alpha-q-1}\,dt}{\Delta\Gamma(\alpha-p)}\biggr]\end{aligned}$$

$$
\begin{aligned}
&\cdot s^{10}\max\left\{\left(\frac{1}{\sqrt{x}}+x^{\frac{7}{2}}\right):0.9766s^{\alpha-1}\le x\le 1\right\}ds\\
&=\frac{1}{10}\int_0^1\left[\frac{(1-s)^{\frac{1}{2}}(1-(1-s)^1)}{\Gamma(\frac{5}{2})}+\frac{\frac{1}{3}(1-s)^{\frac{1}{2}}\int_0^{\frac{1}{3}}t^{\frac{3}{4}}\,dt}{\Delta\Gamma(\frac{3}{2})}\right]\\
&\quad\cdot s^{10}\max\left\{\left(\frac{1}{\sqrt{x}}+x^{\frac{7}{2}}\right):0.9766s^{\frac{3}{2}}\le x\le 1\right\}ds\\
&\le\frac{1}{10}\int_0^1\left[0.7523(1-s)^{\frac{1}{2}}-0.7523(1-s)^{\frac{3}{2}}+0.215(1-s)^{\frac{1}{2}}\right]\\
&\quad\cdot s^{10}\left(\frac{1}{\sqrt{0.9766}}s^{-\frac{3}{4}}+1\right)ds\\
&\le\frac{1}{10}\times(0.5016-0.3009+0.1434)\left(\frac{1}{\sqrt{0.9766}}+1\right)\\
&\le\frac{1}{10}\times 0.3441\times 1.9883\\
&=0.0685<1,
\end{aligned}
$$

which means that $\int_0^1 J(s)\varphi(s,1)\,ds<1$. Then by taking $\tau_r(t)=\frac{1}{15}t^{10}$ (H_5) is verified.

Since $\Gamma(\frac{5}{4})=0.9064$ and $\Gamma(\frac{3}{4})=1.2255$, taking $R=50$, we have

$$
\begin{aligned}
\int_0^1 J(s)\psi(s,R)\,ds&\ge\frac{1}{10}\int_0^1\left[\frac{(1-s)^{\alpha-p-1}(1-(1-s)^p)}{\Gamma(\alpha)}\right.\\
&\quad\left.+\frac{\lambda\int_0^{\eta}h(t)[t^{\alpha-q-1}(1-s)^{\alpha-p-1}-(t-s)^{\alpha-p-1}]\,dt}{\Delta\Gamma(\alpha-p)}\right]\\
&\quad\cdot s^{10}\min\left\{\left(\frac{1}{\sqrt{x}}+x^{\frac{7}{2}}\right):49.9976s^{\alpha-1}\le x\le 50\right\}ds\\
&\ge\frac{1}{10}\int_0^1\frac{(1-s)^{\alpha-p-1}(1-(1-s)^p)}{\Gamma(\alpha)}\\
&\quad\cdot s^{10}\min\left\{\left(\frac{1}{\sqrt{x}}+x^{\frac{7}{2}}\right):49.9976s^{\alpha-1}\le x\le 50\right\}ds\\
&=\frac{1}{10}\int_0^1\frac{(1-s)^{\frac{1}{2}}(1-(1-s)^1)}{\Gamma(\frac{5}{2})}\\
&\quad\cdot s^{10}\min\left\{\left(\frac{1}{\sqrt{x}}+x^{\frac{7}{2}}\right):49.9976s^{\frac{3}{2}}\le x\le 50\right\}ds\\
&\ge\frac{1}{10}\int_0^1\frac{(1-s)^{\frac{1}{2}}}{\Gamma(\frac{5}{2})}\cdot s^{11}\left(\frac{1}{\sqrt{50}}+(49.9976)^{\frac{7}{2}}s^{\frac{21}{4}}\right)ds\\
&\ge\frac{(49.9976)^{\frac{7}{2}}}{10\Gamma(\frac{5}{2})}\int_0^1(1-s)^{\frac{1}{2}}s^{\frac{65}{4}}\,ds\\
&\ge\frac{(49.9976)^{\frac{7}{2}}}{10\Gamma(\frac{5}{2})}\times\frac{\Gamma(\frac{69}{4})\Gamma(\frac{3}{2})}{\Gamma(\frac{75}{4})}\\
&\ge 0.0001\times(49.9976)^{\frac{7}{2}}=88.8734>50.
\end{aligned}
$$

Hence (H_6) is checked. Therefore all conditions of Theorem 3.1 are satisfied, and by Theorem 3.1 the BVP (4.1) has at least two solutions.

5 Conclusion

In this paper, we obtained several sufficient conditions for the existence of positive solutions for nonlinear fractional differential equation involving integral boundary conditions. Our results will be a useful contribution to the existing literature on fractional-order nonlocal differential equations.

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Abbreviations
BVP, Boundary value problems.

Availability of data and materials
Data sharing not applicable to this paper as no data sets were generated or analyzed during the current study.

Competing interests
The authors declare that they have no competing interests.

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