Cunha and Lemos *Advances in Difference Equations* (2020) 2020:29
https://doi.org/10.1186/s13662-019-2469-6

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH

Open Access

Infinitely many solutions for a Hénon-type system in hyperbolic space

Patrícia Leal da Cunha[1*] and Flávio Almeida Lemos[2]

*Correspondence:
patcunha80@gmail.com
[1]Departament of Technology and Data Science, Fundação Getulio Vargas, São Paulo, Brazil
Full list of author information is available at the end of the article

Abstract

This paper is devoted to studying the semilinear elliptic system of Hénon type

$$\begin{cases} -\Delta_{\mathbb{B}^N} u = K(d(x))Q_u(u,v), \\ -\Delta_{\mathbb{B}^N} v = K(d(x))Q_v(u,v), \\ \quad u, v \in H_r^1(\mathbb{B}^N), \quad N \geq 3, \end{cases}$$

in the hyperbolic space $\mathbb{B}^N$, where $H_r^1(\mathbb{B}^N) = \{u \in H^1(\mathbb{B}^N) : u \text{ is radial}\}$ and $-\Delta_{\mathbb{B}^N}$ denotes the Laplace–Beltrami operator on $\mathbb{B}^N$, $d(x) = d_{\mathbb{B}^N}(0,x)$, $Q \in C^1(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ is p-homogeneous, and $K \geq 0$ is a continuous function. We prove a compactness result and, together with Clark's theorem, we establish the existence of infinitely many solutions.

MSC: 58J05; 35J60; 58E30; 35J20; 35J75

Keywords: Hyperbolic space; Hénon equation; Variational methods

1 Introduction and the main result

This article concerns the existence of infinitely many solutions for the following semilinear elliptic system of Hénon type in hyperbolic space:

$$\begin{cases} -\Delta_{\mathbb{B}^N} u = K(d(x))Q_u(u,v), \\ -\Delta_{\mathbb{B}^N} v = K(d(x))Q_v(u,v), \\ \quad u, v \in H_r^1(\mathbb{B}^N), \quad N \geq 3, \end{cases} \tag{$\mathcal{H}$}$$

where $\mathbb{B}^N$ is the Poincaré ball model for the hyperbolic space, $H_r^1(\mathbb{B}^N)$ denotes the Sobolev space of a radial $H^1(\mathbb{B}^N)$ function, $r = d(x) = d_{\mathbb{B}^N}(0,x)$, $\Delta_{\mathbb{B}^N}$ is the Laplace–Beltrami type operator on $\mathbb{B}^N$.

We assume the following hypotheses on K and Q:

(K_1) $K \geq 0$ is a continuous function with $K(0) = 0$ and $K(t) \neq 0$ for $t \neq 0$.

(K_2) $K = O(r^\beta)$ as $r \to 0$ and $K = O(r^\beta)$ as $r \to \infty$ for some $\beta > 0$.

(Q_1) $Q \in C^1(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ is such that $Q(-s,t) = Q(s,-t) = Q(s,t)$, $Q(\lambda s, \lambda t) = \lambda^p Q(s,t)$ (Q is p-homogeneous), $\forall \lambda \in \mathbb{R}$ and $p \in (2, \delta)$, where

$$\delta = \begin{cases} \frac{2N+2\beta}{N-2} & \text{if } N-2>0, \\ \infty & \text{if otherwise.} \end{cases}$$

(Q_2) There exist $C, C_1, C_2 > 0$ such that $Q(s,t) \le C(s^p + t^p)$, $Q_s(s,t) \le C_1 s^{p-1}$, and $Q_t(s,t) \le C_2 t^{p-1}$, $\forall s,t \ge 0$.

(Q_3) There exists $C_3 > 0$ such that $C_3(|s|^p + |t|^p) \le Q(s,t)$ with $p \in (2, \delta)$.

In the past few years the prototype problem

$$-\Delta_{\mathbb{B}^N} u = d(x)^\alpha |u|^{p-2} u, \quad u \in H_r^1(\mathbb{B}^N)$$

has attracted attention. Unlike the corresponding problem in the Euclidean space $\mathbb{R}^N$, He in [1] proved the existence of a positive solution to the above problem over the range $p \in (2, \frac{2N+2\alpha}{N-2})$ in the hyperbolic space. More precisely, she explored the Strauss radial estimate for hyperbolic space together with the mountain pass theorem. In a subsequent paper [2], He and Qiu proved the existence of at least one non-trivial positive solution for the critical Hénon equation

$$-\Delta_{\mathbb{B}^N} u = d(x)^\alpha |u|^{2^*-2} u + \lambda u, \quad u \ge 0, u \in H_0^1(\Omega'),$$

provided that $\alpha \to 0^+$ and for a suitable value of λ, where Ω' is a bounded domain in hyperbolic space $\mathbb{B}^N$. Finally, by working in the whole hyperbolic space $\mathbb{H}^N$, He [3] considered the following Hardy–Hénon type system:

$$\begin{cases} -\Delta_{\mathbf{H}^N} u = d_b(x)^\alpha |v|^{p-1} v, \\ -\Delta_{\mathbf{H}^N} v = d_b(x)^\beta |u|^{q-1} u \end{cases}$$

for $\alpha, \beta \in \mathbb{R}$, $N > 4$ and obtained infinitely many non-trivial radial solutions.

We would like to mention the paper of Carrião, Faria, and Miyagaki [4] where they extended He's result by considering a general nonlinearity

$$\begin{cases} -\Delta^\alpha_{\mathbb{B}^N} u = K(d(x)) f(u) \\ \quad u \in H_r^1(\mathbb{B}^N). \end{cases} \tag{1}$$

They were able to prove the existence of at least one positive solution through a compact Sobolev embedding with the mountain pass theorem.

In this paper, we investigate the existence of infinitely many solutions by considering a gradient system that generalizes problem (1). We cite [5–11] for related gradient system problems. In order to obtain our result, we applied Clark's theorem [12, 13] and got inspiration on the nonlinearities condition employed by Morais Filho and Souto [14] in a p-Laplacian system defined on a bounded domain in $\mathbb{R}^N$.

Regarding the difficulties, many technical difficulties arise when working on $\mathbb{B}^N$, which is a non-compact manifold. This means that the embedding $H^1(\mathbb{B}^N) \hookrightarrow L^p(\mathbb{B}^N)$ is not compact for $2 \leq p \leq \frac{2N}{N-2}$ and the functional related to the system $\mathcal{H}$ cannot satisfy the $(PS)_c$ condition for all $c > 0$.

We also point out that since the weight function $d(x)$ depends on the Riemannian distance r from a pole 0, we have some difficulties in proving that

$$\int_{\mathbb{B}^N} d(x)^\beta \left(\left|u(x)\right|^p + \left|v(x)\right|^p\right) dV_{\mathbb{B}^N} < \infty, \quad \forall (u,v) \in H^1\left(\mathbb{B}^N\right) \times H^1\left(\mathbb{B}^N\right)$$

leading to a great effort in proving that the associated Euler–Lagrange functional is well defined.

To overcome these difficulties, we restrict ourselves to the radial functions.

Our result is the following.

Theorem 1.1 *Under hypotheses* (K_1)–(K_2) *and* (Q_1)–(Q_3), *problem* $(\mathcal{H})$ *has infinitely many solutions.*

2 Preliminaries

Throughout this paper, C is a positive constant which may change from line to line.

The Poincaré ball for the hyperbolic space is

$$\mathbb{B}^N = \left\{x \in \mathbb{R}^N \,|\, |x| < 1\right\}$$

endowed with Riemannian metric g given by $g_{i,j} = (p(x))^2 \delta_{i,j}$, where $p(x) = \frac{2}{1-|x|^2}$. We denote the hyperbolic volume by $dV_{\mathbb{B}^N} = (p(x))^N\, dx$. The hyperbolic distance from the origin to $x \in \mathbb{B}^N$ is given by

$$d(x) := d_{\mathbb{B}^N}(0,x) = \int_0^{|x|} \frac{2}{1-s^2}\, ds = \log\left(\frac{1+|x|}{1-|x|}\right).$$

The hyperbolic gradient and the Laplace–Beltrami operator are

$$-\Delta_{\mathbb{B}^N} u = -\left(p(x)\right)^{-N} \operatorname{div}\left(p(x)^{N-2} \nabla u\right), \quad \nabla_{\mathbb{B}^N} u = \frac{\nabla u}{p(x)},$$

where $H^1(\mathbb{B}^N)$ denotes the Sobolev space on $\mathbb{B}^N$ with the metric g. ∇ and div denote the Euclidean gradient and divergence in $\mathbb{R}^N$, respectively.

Let $H_r^1(\mathbb{B}^N) = \{u \in H^1(\mathbb{B}^N) : u \text{ is radial}\}$.

We shall find weak solutions of problem $(\mathcal{H})$ in the space

$$H = H_r^1\left(\mathbb{B}^N\right) \times H_r^1\left(\mathbb{B}^N\right)$$

endowed with the norm

$$\left\|(u,v)\right\|^2 = \int_{\mathbb{B}^N} \left(|\nabla_{\mathbb{B}^N} u|_{\mathbb{B}^N}^2 + |\nabla_{\mathbb{B}^N} v|_{\mathbb{B}^N}^2\right) dV_{\mathbb{B}^N}.$$

One can observe that system ($\mathcal{H}$) is formally derived as the Euler–Lagrange equation for the functional

$$I(u,v) = \frac{1}{2}\int_{\mathbb{B}^N} \bigl(|\nabla_{\mathbb{B}^N} u|^2_{\mathbb{B}^N} + |\nabla_{\mathbb{B}^N} v|^2_{\mathbb{B}^N}\bigr)\, dV_{\mathbb{B}^N} - \int_{\mathbb{B}^N} K\bigl(d(x)\bigr)Q(u,v)\, dV_{\mathbb{B}^N}.$$

We endowed the norm for $L^p(\mathbb{B}^N) \times L^p(\mathbb{B}^N)$ as follows:

$$\bigl\|(u,v)\bigr\|_p^p = \int_{\mathbb{B}^N} \bigl(|u|^p + |v|^p\bigr)\, dV_{\mathbb{B}^N}.$$

To solve this problem, we need the following lemmas.

Lemma 2.1 *The map* $(u,v) \longmapsto (d(x)^m u, d(x)^m v)$ *from* $H = H_r^1(\mathbb{B}^N) \times H_r^1(\mathbb{B}^N)$ *to* $L^p(\mathbb{B}^N) \times L^p(\mathbb{B}^N)$ *is continuous for* $p \in (2,\tilde{m})$, *where* $m > 0$ *and*

$$\tilde{m} = \begin{cases} \frac{2N}{N-2-2m} & \textit{if } m < \frac{N-2}{2}, \\ \infty & \textit{if otherwise.} \end{cases}$$

Proof In [1, Lemma 2.2] it has been proved that the map $u \longmapsto d(x)^m u$ from $H_r^1(\mathbb{B}^N)$ to $L^p(\mathbb{B}^N)$ is continuous for $p \in (2,\tilde{m})$. Therefore $\|d(x)^m u\|_p \le C\|u\|_{H_r^1}$ and $\|d(x)^m v\|_p \le C\|v\|_{H_r^1}$. Hence,

$$\bigl(\bigl\|d(x)^m u\bigr\|_p^2 + \bigl\|d(x)^m v\bigr\|_p^2\bigr)^{\frac{1}{2}} \le C\bigl\|(u,v)\bigr\|.$$

Now observe that

$$\begin{aligned} \bigl\|d(x)^m u\bigr\|_p + \bigl\|d(x)^m v\bigr\|_p &= \bigl[\bigl(\bigl\|d(x)^m u\bigr\|_p + \bigl\|d(x)^m v\bigr\|_p\bigr)^2\bigr]^{\frac{1}{2}} \\ &= \bigl(\bigl\|d(x)^m u\bigr\|_p^2 + 2\bigl\|d(x)^m u\bigr\|_p\bigl\|d(x)^m v\bigr\|_p + \bigl\|d(x)^m v\bigr\|_p\bigr)^{\frac{1}{2}}. \end{aligned}$$

Applying Cauchy's inequality $ab \le \frac{a^2+b^2}{2}$, we get

$$\bigl\|d(x)^m u\bigr\|_p + \bigl\|d(x)^m v\bigr\|_p \le \sqrt{2}\bigl(\bigl\|d(x)^m u\bigr\|_p^2 + \bigl\|d(x)^m v\bigr\|_p^2\bigr)^{\frac{1}{2}}.$$

By the subadditivity, we get

$$\bigl(\bigl\|d(x)^m u\bigr\|_p^p + \bigl\|d(x)^m v\bigr\|_p^p\bigr)^{\frac{1}{p}} \le \bigl\|d(x)^m u\bigr\|_p + \bigl\|d(x)^m v\bigr\|_p.$$

Therefore,

$$\bigl\|\bigl(d(x)^m u, d(x)^m v\bigr)\bigr\|_p \le C\bigl\|(u,v)\bigr\|,$$

and the lemma holds. □

Remark 2.1 From the previous lemma, there exists a positive constant $C > 0$ such that

$$\begin{aligned}
&\int_{\mathbb{B}^N} d(x)^\beta \left|u(x)\right|^p dV_{\mathbb{B}^N} + \int_{\mathbb{B}^N} d(x)^\beta \left|v(x)\right|^p dV_{\mathbb{B}^N} \\
&\quad \le C\left(\int_{\mathbb{B}^N} |\nabla_{\mathbb{B}^N} u|^2_{\mathbb{B}^N} \, dV_{\mathbb{B}^N} + \int_{\mathbb{B}^N} |\nabla_{\mathbb{B}^N} v|^2_{\mathbb{B}^N} \, dV_{\mathbb{B}^N}\right)^{\frac{p}{2}},
\end{aligned}$$

where $m = \frac{\beta}{p}$ and $2 < p < \frac{2N}{N-2-2(\frac{\beta}{p})}$, that is, $2 < p < \delta$.

Lemma 2.2 *The map $(u,v) \longmapsto (d(x)^m u, d(x)^m v)$ from $H = H_r^1(\mathbb{B}^N) \times H_r^1(\mathbb{B}^N)$ to $L^p(\mathbb{B}^N) \times L^p(\mathbb{B}^N)$ is compact for $p \in (2, \tilde{m})$, where $m > 0$ and*

$$\tilde{m} = \begin{cases} \frac{2N}{N-2-2m} & \textit{if } m < \frac{N-2}{2}, \\ \infty & \textit{if otherwise.} \end{cases}$$

Proof Let $(u_n, v_n) \in H$ be a bounded sequence. Then, up to a subsequence, if necessary, we may assume that

$$(u_n, v_n) \rightharpoonup (u, v).$$

It is easy to see that $u_n \rightharpoonup u$ and $v_n \rightharpoonup v$ in $H_r^1(\mathbb{B}^N)$.

We will use the same calculus used by Haiyang He [1] (page 26). We want to show that

$$\begin{aligned}
\lim_{n\to\infty} \int_{\mathbb{B}^N} d(x)^{mp} \left|u_n(x)\right|^p dV_{\mathbb{B}^N} &= \int_{\mathbb{B}^N} d(x)^{mp} \left|u(x)\right|^p dV_{\mathbb{B}^N}, \\
\lim_{n\to\infty} \int_{\mathbb{B}^N} d(x)^{mp} \left|v_n(x)\right|^p dV_{\mathbb{B}^N} &= \int_{\mathbb{B}^N} d(x)^{mp} \left|v(x)\right|^p dV_{\mathbb{B}^N}.
\end{aligned}$$

Let $u \in H_r^1(\mathbb{B}^N)$, then by Haiyang He [1] we have

$$\begin{aligned}
\left|u_n(x)\right| &\le \frac{1}{\sqrt{\omega_{n-1}(N-2)}} \left(\frac{1-|x|^2}{2}\right)^{\frac{N-2}{2}} \frac{1}{|x|^{\frac{N-2}{2}}} \|u_n\|_{H_r^1(\mathbb{B}^N)}, \\
\left|u_n(x)\right| &\le \frac{1}{\sqrt{\omega_{n-1}}} \left(\frac{1-|x|^2}{2}\right)^{\frac{N-1}{2}} \frac{1}{|x|^{\frac{N}{2}}} \|u_n\|_{H_r^1(\mathbb{B}^N)}.
\end{aligned}$$

Since $\{|x| \le \frac{1}{2}\}$, $\ln \frac{1+|x|}{1-|x|} \le \frac{2r}{1-r^2}$, and $2 < p < \tilde{m}$, we have

$$\begin{aligned}
d(x)^{mp}|u|^p &\le C\left(\ln \frac{1+|x|}{1-|x|}\right)^{mp} \left(\frac{1-|x|^2}{2}\right)^{p\frac{N-2}{2}} \left(\frac{1}{|x|^{\frac{N-2}{2}}}\right)^p \\
&\le C\left(\frac{2|x|}{1-|x|^2}\right)^{mp} \left(\frac{1-|x|^2}{2}\right)^{p\frac{N-2}{2}} \left(\frac{1}{|x|^{\frac{N-2}{2}}}\right)^p \equiv h_1.
\end{aligned}$$

Set

$$g_1(x) = \begin{cases} h_1(x) & \text{if } 0 \le |x| < \frac{1}{2}, \\ 0 & \text{if } \frac{1}{2} \le |x| < 1, \end{cases}$$

then

$$\begin{aligned}\int_{\mathbb{B}^N} g_1\,dV_{\mathbb{B}^N} &= \int_0^{\frac{1}{2}} \left(\frac{2r}{1-r^2}\right)^{mp} \left(\frac{1-r^2}{2}\right)^{p\frac{N-2}{2}} \left(\frac{1}{r^{\frac{N-2}{2}}}\right)^{p} r^{N-1} \left(\frac{2}{1-r^2}\right)^{N} dr \\ &\le C\int_0^{\frac{1}{2}} \left(\frac{2r}{1-r^2}\right)^{mp} \left(\frac{1-r^2}{2}\right)^{(p\frac{N-2}{2}-N)} r^{N-1-p\frac{N-2}{2}}\,dr \\ &\le C\int_0^{\frac{1}{2}} r^{mp+N-1-p\frac{N-2}{2}}\,dr < \infty.\end{aligned}$$

Since $\{|x| > \frac{1}{2}\}$ and $2 < p < \tilde{m}$, we have

$$\begin{aligned}d(x)^{mp}|u|^p &\le C\left(\ln\frac{1+|x|}{1-|x|}\right)^{mp} \left(\frac{1-|x|^2}{2}\right)^{p\frac{N-1}{2}} \left(\frac{1}{|x|^{\frac{N}{2}}}\right)^{p} \\ &\le C\left(\ln\frac{1+|x|}{1-|x|}\right)^{mp} \left(\frac{1-|x|^2}{2}\right)^{p\frac{N-1}{2}} \left(\frac{1}{|x|^{\frac{N}{2}}}\right)^{p} \equiv h_2.\end{aligned}$$

Set

$$g_2(x) = \begin{cases} 0 & \text{if } 0 \le |x| < \frac{1}{2} \\ h_2(x) & \text{if } \frac{1}{2} \le |x| < 1, \end{cases}$$

then

$$\begin{aligned}\int_{\mathbb{B}^N} g_2\,dV_{\mathbb{B}^N} &= \int_{\frac{1}{2}}^{1} \left(\ln\frac{1+r}{1-r}\right)^{mp} \left(\frac{1-r^2}{2}\right)^{p\frac{N-1}{2}} \left(\frac{1}{r^{\frac{N}{2}}}\right)^{p} r^{N-1} \left(\frac{2}{1-r^2}\right)^{N} dr \\ &\le C\int_{\frac{1}{2}}^{1} \left(\ln\frac{1+r}{1-r}\right)^{mp} \left(\frac{1-r^2}{2}\right)^{(p\frac{N-1}{2}-N)} r^{N-1-\frac{N}{2}p}\,dr \\ &\le \int_{\ln 3}^{\infty} s^{mp} \left(\frac{2e^s}{(e^s+1)^2}\right)^{\frac{N-1}{2}p-N+1} ds < \infty.\end{aligned}$$

Hence, we have

$$\left|d(x)^{mq} u_n(x)^q\right| \le g_1(x) + g_2(x).$$

By the dominated convergence theorem, we obtain

$$\lim_{n\to\infty} \int_{\mathbb{B}^N} d(x)^{mq} u_n^p(x)\,dV_{\mathbb{B}^N} = \int_{\mathbb{B}^N} d(x)^{mp} u^p(x)\,dV_{\mathbb{B}^N}.$$

In the same way we conclude that

$$\lim_{n\to\infty} \int_{\mathbb{B}^N} d(x)^{mq} v_n^p(x)\,dV_{\mathbb{B}^N} = \int_{\mathbb{B}^N} d(x)^{mp} v^p(x)\,dV_{\mathbb{B}^N},$$

and the lemma holds. □

3 Proof of Theorem 1.1

Clark's theorem is one of the most important results in critical point theory (see [12]). It was successfully applied to sublinear elliptic problems with symmetry and the existence of infinitely many solutions around was shown.

In order to state Clark's theorem, we need some terminologies.

Let $(X, \|\cdot\|_X)$ be a Banach space and $\mathcal{I} \in C^1(X, \mathbb{R})$.

(i) For $c \in \mathbb{R}$, we say that $\mathcal{I}(u)$ satisfies the $(PS)_c$ condition if any sequence $(u_j)_{j=1}^{\infty} \subset X$ such that $\mathcal{I}(u_j) \to c$ and $\|\mathcal{I}'(u_j)\| \to 0$ has a convergent subsequence.

(ii) Let S be a symmetric and closed set family in $X \diagdown \{0\}$. For $A \in S$, the genus $\gamma(A) = \min\{n \in \mathbb{N} : \phi \in C(A, \mathbb{R}^n\{0\}) \text{ is odd}\}$. If there is no such natural, we set $\gamma(A) = \infty$.

(iii) Let Ω be an open and bounded set, $0 \in \Omega$ in $\mathbb{R}^n$. If $A \in S$ is such that there exists an odd homeomorphism function from A to $\partial\Omega$, then $\gamma(A) = n$.

Theorem 3.1 (Clark's theorem) *Let $\mathcal{I} \in C(X, \mathbb{R})$ be an even function bounded from below with $\mathcal{I}(0) = 0$, and there exists a compact, symmetric set $K \in X$ such that $\gamma(K) = k$ and $\sup_K \mathcal{I} < 0$. Then I has at least k distinct pairs of critical points.*

The proof of Theorem 1.1 is made by using Theorem 3.1.

The $(\mathcal{H})$ system is the Euler–Lagrange equations related to the functional

$$I(u,v) = \frac{1}{2}\int_{\mathbb{B}^N} \left(|\nabla_{\mathbb{B}^N} u|_{\mathbb{B}^N}^2 + |\nabla_{\mathbb{B}^N} v|_{\mathbb{B}^N}^2\right) dV_{\mathbb{B}^N} - \int_{\mathbb{B}^N} K\big(d(x)\big) Q(u,v)\, dV_{\mathbb{B}^N}, \tag{2}$$

which is C^1 on H.

The functional I is not bounded from below, therefore, we cannot apply Clark's technique for this functional.

In order to overcome this difficulty, we consider the auxiliary functional

$$J(u,v) = \left(\int_{\mathbb{B}^N} \left(|\nabla_{\mathbb{B}^N} u|_{\mathbb{B}^N}^2 + |\nabla_{\mathbb{B}^N} v|_{\mathbb{B}^N}^2\right) dV_{\mathbb{B}^N}\right)^{p-1} - \int_{\mathbb{B}^N} K\big(d(x)\big) Q(u,v)\, dV_{\mathbb{B}^N}, \tag{3}$$

where $p \in (2, \delta)$, while for J' we have $\forall(\phi, \psi) \in H$

$$\begin{aligned} J'(u,v)(\phi,\psi) = {} & (2p-2)\big\|(u,v)\big\|^{2p-4} \int_{\mathbb{B}^N} \left(\langle \nabla_{\mathbb{B}^N} u, \nabla_{\mathbb{B}^N} \phi\rangle_{\mathbb{B}^N} + \langle \nabla_{\mathbb{B}^N} v, \nabla_{\mathbb{B}^N} \psi\rangle_{\mathbb{B}^N}\right) dV_{\mathbb{B}^N} \\ & - \int_{\mathbb{B}^N} K\big(d(x)\big)\big(\phi Q_u(u,v) + \psi Q_v(u,v)\big)\, dV_{\mathbb{B}^N}. \end{aligned} \tag{4}$$

We will show that the set of critical points of J is related to a set of critical points of I and J satisfies the conditions of Theorem 3.1.

The proof of Theorem 1.1 is divided into several lemmas.

Lemma 3.1 *If $(u,v) \in H$, $(u,v) \neq (0,0)$ is a critical point for J, then*

$$(w,z) = \left(\frac{u}{[(2p-2)\|(u,v)\|^{2p-4}]^{\frac{1}{p-2}}}, \frac{v}{[(2p-2)\|(u,v)\|^{2p-4}]^{\frac{1}{p-2}}}\right)$$

is a critical point for I.

Proof Note that $(u,v) \neq (0,0)$ is a critical point for J if, and only if, (u,v) is a weak solution to the problem

$$\begin{cases} -(2p-2)\|(u,v)\|^{2p-4}\Delta_{\mathbb{B}^N} u = K(d(x))Q_u(u,v), \\ -(2p-2)\|(u,v)\|^{2p-4}\Delta_{\mathbb{B}^N} v = K(d(x))Q_v(u,v), \\ \quad u,v \in H_r^1(\mathbb{B}^N), \quad N \geq 3. \end{cases} \qquad (\mathcal{S})$$

Define $\lambda(\|(u,v)\|) = [(2p-2)\|(u,v)\|^{2p-4}]^{\frac{-1}{p-2}}$, then $(w,z) = \lambda(\|(u,v)\|)(u,v)$. Using the $p-1$-homogeneity condition of $Q_u(u,v)$ and $Q_v(u,v)$, observe that

$$\begin{aligned} &-\Delta_{\mathbb{B}^N} w - K\big(d(x)\big)Q_u(w,z) \\ &\quad = -\lambda\big(\|(u,v)\|\big)\Delta_{\mathbb{B}^N} u - \big(\lambda\big(\|(u,v)\|\big)\big)^{p-1}K\big(d(x)\big)Q_u(u,v) \\ &\quad = -\lambda\big(\|(u,v)\|\big)K\big(d(x)\big)Q_u(u,v)\left(\big(\lambda\big(\|(u,v)\|\big)\big)^{p-2} - \frac{1}{(2p-2)\|(u,v)\|^{2p-4}}\right) \end{aligned}$$

and

$$\begin{aligned} &-\Delta_{\mathbb{B}^N} z - K\big(d(x)\big)Q_v(w,z) \\ &\quad = -\lambda\big(\|(u,v)\|\big)\Delta_{\mathbb{B}^N} v - \big(\lambda\big(\|(u,v)\|\big)\big)^{p-1}K\big(d(x)\big)Q_v(u,v) \\ &\quad = -\lambda\big(\|(u,v)\|\big)K\big(d(x)\big)Q_v(u,v)\left(\big(\lambda\big(\|(u,v)\|\big)\big)^{p-2} - \frac{1}{(2p-2)\|(u,v)\|^{2p-4}}\right). \end{aligned}$$

Hence (w,z) is a weak solution for problem $(\mathcal{H})$ and so, a critical point for I. □

Lemma 3.2 *$J(u,v)$ is bounded from below and satisfies the $(PS)_c$ condition.*

Proof From (K_1)–(K_2), (Q_2)–(Q_3), and Remark 2.1

$$\begin{aligned} J(u,v) &= \left(\int_{\mathbb{B}^N} |\nabla_{\mathbb{B}^N} u|_{\mathbb{B}^N}^2 + |\nabla_{\mathbb{B}^N} v|_{\mathbb{B}^N}^2 \, dV_{\mathbb{B}^N}\right)^{p-1} - \int_{\mathbb{B}^N} K\big(d(x)\big)Q(u,v)\, dV_{\mathbb{B}^N} \\ &\geq \big\|(u,v)\big\|^{2p-2} - C\int_{\mathbb{B}^N} d(x)^\beta \big(|u|^p + |v|^p\big)\, dV_{\mathbb{B}^N} \\ &\geq \big\|(u,v)\big\|^{2p-2} - \big\|(u,v)\big\|^p, \end{aligned}$$

so that $J(u,v)$ is bounded from below.

Let $(u_n,v_n) \in H$ be such that $|J(u_n,v_n)| \leq C$ with $C \in \mathbb{R}^+$, $J'(u_n,v_n) \to 0$. Since

$$C \geq J(u_n,v_n) \geq \big\|(u_n,v_n)\big\|^{2p-2} - \big\|(u_n,v_n)\big\|^p,$$

we conclude that $\|(u_n,v_n)\|$ is bounded. So, there exists $(u,v) \in H$ such that, passing to a subsequence if necessary,

$$(u_n,v_n) \rightharpoonup (u,v), \quad \text{as } n \to \infty.$$

From the embedding Lemma 2.2, we have

$$\int_{\mathbb{B}^N} (d(x))^\beta \big(|u_n|^p + |v_n|^p\big)\, dV_{\mathbb{B}^N} \longrightarrow \int_{\mathbb{B}^N} (d(x))^\beta \big(|u|^p + |v|^p\big)\, dV_{\mathbb{B}^N},$$

and by $(K_2)-(Q_2)$, we infer that

$$|K(d(x)\big(u_n Q_u(u_n, v_n) + v_n Q_v(u_n, v_n)\big)| \le C\big(d(x)\big)^\beta \big(|u_n|^p + |v_n|^p\big).$$

Therefore, by the Lebesgue dominated convergence theorem,

$$\begin{aligned} &\int_{\mathbb{B}^N} K\big(d(x)\big)\big(Q_u(u_n, v_n)u_n + Q_v(u_n, v_n)v_n\big)\, dV_{\mathbb{B}^N} \\ &\quad \to \int_{\mathbb{B}^N} K\big(d(x)\big)\big(Q_u(u, v)u + Q_v(u, v)v\big)\, dV_{\mathbb{B}^N}. \end{aligned}$$

Since $J'(u,v)(u,v) = 0$ and $J'(u_n, v_n)(u_n, v_n) = o_n(1)$ as $n \to \infty$, we have

$$\begin{aligned} &(2p-2)\big(\big\|(u_n, v_n)\big\|^{2p-2} - \big\|(u,v)\big\|^{2p-2}\big) \\ &\quad = J'(u_n, v_n)(u_n, v_n) - J'(u,v)(u,v) \\ &\qquad + \int_{\mathbb{B}^N} K\big(d(x)\big)\big(Q_u(u_n, v_n)u_n + Q_v(u_n, v_n)v_n\big)\, dV_{\mathbb{B}^N} \\ &\qquad - \int_{\mathbb{B}^N} K\big(d(x)\big)\big(Q_u(u,v)u + Q_v(u,v)v\big)\, dV_{\mathbb{B}^N} = o_n(1), \end{aligned}$$

then $\|(u_n, v_n)\| \to \|(u,v)\|$. Therefore,

$$(u_n, v_n) \to (u,v), \quad \text{as } n \to \infty, \text{ in } H. \qquad \square$$

The next lemma ends the proof of Theorem 1.1.

Lemma 3.3 *Given $k \in \mathbb{N}$, there exists a compact and symmetric set $K \in H$ such that $\gamma(K) = k$ and $\sup_K J < 0$.*

Proof Let $X_k \subset H$ be a subspace of dimension k. Consider the following norm in X_k:

$$\big\|(u,v)\big\|_{X_k} = \left(\int_{\mathbb{B}^N} K\big(d(x)\big)\big(|u|^p + |v|^p\big)\, dV_{\mathbb{B}^N}\right)^{\frac{1}{p}}.$$

Since $X_k \subset H$ has finite dimension, there exists $a > 0$ such that

$$a\big\|(u,v)\big\|_{X_k} \le \big\|(u,v)\big\| \le \frac{1}{a}\big\|(u,v)\big\|_{X_k}, \quad \forall (u,v) \in X_k.$$

Therefore, we obtain from (Q_3) that

$$J(u,v) \le \big\|(u,v)\big\|^{2p-2} - C\int_{\mathbb{B}^N} K\big(d(x)\big)\big(|u|^p + |v|^p\big)\, dV_{\mathbb{B}^N} = \big\|(u,v)\big\|^{2p-2} - C\big\|(u,v)\big\|_{X_k}^p,$$

where $C \in \mathbb{R}$ is a positive constant. We then conclude that

$$J(u,v) \leq \|(u,v)\|_{X_k}^p \left(\frac{\|(u,v)\|_{X_k}^{p-2}}{a^{2p-2}} - C \right).$$

Let $A = a^{\frac{2p-2}{p-2}}$ and consider the set $K = \{(u,v) \in X_k : \|(u,v)\|_{X_k} = \frac{A}{2} C^{\frac{1}{p-2}}\}$, then

$$J(u,v) \leq C \frac{A^p}{2^p} \left(\frac{1}{2^{q-2}} - 1 \right) < 0, \quad \forall (u,v) \in K.$$

We get that $\sup_K J < 0$, where $K \subset H$ is a compact and symmetric set such that $\gamma(K) = k$. □

Finally, from Lemmas 3.2 and 3.3, Theorem 3.1 implies the existence of at least k distinct pairs of critical points for the functional J. Since k is arbitrary, we obtain infinitely many critical points in H.

In view of Lemma 3.1, we conclude that the functional J possesses, together with I, infinitely many critical points in H.

Finally, we point out that since H is a closed subspace of the Hilbert space $H^1(\mathbb{B}^N) \times H^1(\mathbb{B}^N)$, following some ideas in [4, 15], we can conclude that (u,v) is a critical point in $H^1(\mathbb{B}^N) \times H^1(\mathbb{B}^N)$.

4 Further result

We can apply the same method used in the proof of Theorem 1.1 to establish the existence of infinitely many solutions for the following semilinear elliptic equation:

$$\begin{cases} -\Delta_{\mathbb{B}^N}^{\alpha} u = K(d(x))|u|^{p-2}u, \\ u \in \mathbb{E} \subset H_r^1(\mathbb{B}^N), \quad N \geq 3, \end{cases} \tag{$\mathcal{H}*$}$$

where K satisfies $(K_1) - (K_2)$, $-\Delta_{\mathbb{B}^N}^{\alpha}$ is the Laplace–Beltrami type operator

$$-\Delta_{\mathbb{B}^N}^{\alpha} u = -\big(p(x)\big)^{-N} \operatorname{div}\big(p(x)^{N-2}\big(d(x)\big)^{\alpha} \nabla u\big)$$

and

$$\mathbb{E} = \left\{ u \in H_r^1(\mathbb{B}^N) : \|u\|_{\mathbb{E}} = \left(\int_{\mathbb{B}^N} \big(d(x)\big)^{\alpha} |\nabla_{\mathbb{B}^N} u|_{\mathbb{B}^N}^2 \, dV_{\mathbb{B}^N} \right)^{\frac{1}{2}} < \infty \right\}.$$

We obtain the following result.

Theorem 4.1 *Under hypotheses* (K_1)–(K_2), $(\mathcal{H}*)$ *equation has infinitely many solutions.*

The energy functional corresponding to $(\mathcal{H}*)$ is

$$I(u) = \frac{1}{2} \int_{\mathbb{B}^N} \big(d(x)\big)^{\alpha} |\nabla_{\mathbb{B}^N} u|_{\mathbb{B}^N}^2 \, dV_{\mathbb{B}^N} - \frac{1}{q} \int_{\mathbb{B}^N} K\big(d(x)\big)|u|^p \, dV_{\mathbb{B}^N} \tag{5}$$

defined on E.

Problem ($\mathcal{H}*$) is closely related to the one studied by Carrião, Faria, and Miyagaki [4]. In [4], they proved that the map $u \longmapsto d(x)^m u$ from $\mathbb{E}$ to $L^q(\mathbb{B}^N)$ is compact for $q \in (2, \tilde{m})$, where

$$\tilde{m} = \begin{cases} \frac{2N}{N-2-2m+\alpha} & \text{if } m < \frac{N-2+\alpha}{2}, \\ \infty & \text{if otherwise} \end{cases}$$

and then there exists a positive constant $C > 0$ such that

$$\int_{\mathbb{B}^N} d(x)^\beta \left|u(x)\right|^q dV_{\mathbb{B}^N} \le C\left(\int_{\mathbb{B}^N} \left(d(x)\right)^\alpha |\nabla_{\mathbb{B}^N} u|_{\mathbb{B}^N} \, dV_{\mathbb{B}^N}\right)^{\frac{q}{2}}, \tag{6}$$

by taking $m = \frac{\beta}{q}$ with $2 < q < \frac{2N}{N-2-2\frac{\beta}{q}+\alpha}$.

Using (K_1)–(K_2) together with inequality (6), we get that the functional I is well defined. This functional is not bounded from below, hence we cannot apply Clark's technique [12].

In order to overcome this difficulty, we consider the auxiliary functional

$$\psi(u) = \left(\int_{\mathbb{B}^N} \left(d(x)\right)^\alpha |\nabla_{\mathbb{B}^N} u|^2_{\mathbb{B}^N} \, dV_{\mathbb{B}^N}\right)^{p-1} - \int_{\mathbb{B}^N} K\left(d(x)\right)|u|^p \, dV_{\mathbb{B}^N}, \tag{7}$$

where $p \in (2, 2^\beta_\alpha)$, $u \in \mathbb{E}$, and

$$\begin{aligned} \psi'(u)v = (2p-2)\|u\|^{2p-4}_{\mathbb{E}} \int_{\mathbb{B}^N} \left(d(x)\right)^\alpha \langle \nabla_{\mathbb{B}^N} u, \nabla_{\mathbb{B}^N} v\rangle_{\mathbb{B}^N} \, dV_{\mathbb{B}^N} \\ - \int_{\mathbb{B}^N} K\left(d(x)\right)|u|^{p-2} uv \, dV_{\mathbb{B}^N}. \end{aligned} \tag{8}$$

We have the corresponding results of Lemmas 3.1, 3.2, and 3.3 for problem ($\mathcal{H}*$). The set of critical points of ψ is related to a set of critical points of I and ψ satisfies the conditions of Theorem 3.1.

Lemma 4.1 *If $u \in \mathbb{E}$, $u \neq 0$ is a critical point for ψ, then $v = \frac{u}{[(2p-2)\|u\|^{2p-4}_{\mathbb{E}}]^{\frac{1}{p-2}}}$ is a critical point for I.*

Lemma 4.2 *$\psi(u)$ is bounded from below and satisfies the Palais–Smale condition (PS).*

Lemma 4.3 *Given $k \in \mathbb{N}$, there exists a compact and symmetric set $K \in \mathbb{E}$ such that $\gamma(K) = k$ and $\sup_K \psi < 0$.*

From Lemmas 4.2 and 4.3 and Theorem 3.1, we conclude that the functional I possesses infinitely many critical points.

Acknowledgements

This paper was carried out while the first author was visiting the Mathematics Department of UFOP, and she would like to thank the members of DEMAT-UFOP for their hospitality. The authors are grateful to professor Giovany Figueiredo for helpful comments.

Funding

Not applicable.

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors worked together to produce the results, read and approved the final manuscript.

Author details
[1]Departament of Technology and Data Science, Fundação Getulio Vargas, São Paulo, Brazil. [2]Departamento de Matemática, Universidade Federal de Ouro Preto, Minas Gerais, Brazil.

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 29 June 2019 Accepted: 16 December 2019 Published online: 15 January 2020

References

1. He, H.: The existence of solutions for Hénon equation in hyperbolic space. Proc. Jpn. Acad., Ser. A, Math. Sci. **89**, 24–28 (2013)
2. He, H., Qiu, J.: Existence of solutions for critical Hénon equations in hyperbolic spaces. Electron. J. Differ. Equ. **13**, 5 (2013)
3. He, H.: Infinitely many solutions for Hardy–Hénon type elliptic system in hyperbolic space. Ann. Acad. Sci. Fenn., Math. **40**, 969–983 (2015)
4. Carrião, P.C., Faria, L.F.O., Miyagaki, O.H.: Semilinear elliptic equations of the Hénon-type in hyperbolic space. Commun. Contemp. Math. **18**(2), 13 pages (2016)
5. Boccardo, L., de Figueiredo, D.G.: Some remarks on a system of quasilinear elliptic equations. Nonlinear Differ. Equ. Appl. **9**, 309–323 (2002)
6. Chipot, M.: Handbook of Differential Equations—Stationary Partial Differential Equations. Elsevier, Amsterdam (2008)
7. de Figueiredo, D.G.: Nonlinear elliptic systems. Anais da Academia Brasileira de Ciências **72** (2000)
8. de Figueiredo, D.G., Ding, Y.H.: Strongly indefinite functionals and multiple solutions of elliptic systems. Trans. Am. Math. Soc. **355**(7), 2973–2989 (2003)
9. de Figueiredo, D.G., Felmer, P.L.: On superquadratic elliptic systems. Trans. Am. Math. Soc. **343**, 119–123 (1994)
10. Bartsch, T., de Figueiredo, D.G.: Infinitely many solutions of nonlinear elliptic systems. Progress in Nonlinear Differential Equations and Their Applications **35** (1999)
11. Wang, J., Xu, J., Zhang, F.: Existence and multiplicity of solutions for asymptotically Hamiltonian elliptic systems in $\mathbb{R}^N$. J. Math. Anal. Appl. **367**, 193–203 (2010)
12. Clark, D.C.: A variant of the Lusternik–Schnirelman theory. Indiana Univ. Math. J. **22**, 65–74 (1972)
13. Costa, D.G.: An Invitation to Variational Methods in Differential Equations. Birkhäuser, Boston (2007)
14. de Morais Filho, D.C., Souto, M.A.S.: Systems of p-Laplacian equations involving homogeneous nonlinearities with critical Sobolev exponent degrees. Commun. Partial Differ. Equ. **24**, 1537–1553 (1999)
15. Bianchi, G., Chabrowski, J., Szulkin, A.: On symmetric solutions of an elliptic equation with nonlinearity involving critical Sobolev exponent. Nonlinear Anal. **25**, 41–59 (1995)