Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Derivation of bounds of several kinds of operators via (s,m)-convexity

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Abstract
The objective of this paper is to derive the bounds of fractional and conformable integral operators for (s,m)-convex functions in a unified form. Further, the upper and lower bounds of these operators are obtained in the form of a Hadamard inequality, and their various fractional versions are presented. Some connections with already known results are obtained.

Keywords: (s,m)-Convex function; Integral operators; Fractional integral operators; Conformable integral operators

1 Introduction

Nobody can deny the importance of convex functions in the field of mathematical analysis, mathematical statistics, and optimization theory. These functions motivate towards the theory of convex analysis, see [17–19].

We start with the definition of convex function.

Definition 1 A function $f:[a,b]\to\mathbb{R}$ is said to be convex if

$$f\big(tx+(1-t)y\big)\le tf(x)+(1-t)f(y) \tag{1.1}$$

holds for all $x,y\in[a,b]$ and $t\in[0,1]$. If inequality (1.1) is reversed, then the function f will be the concave on $[a,b]$.

Convex functions have been generalized theoretically extensively; these generalizations include m-convex function, n-convex function, r-convex function, h-convex function, $(h-m)$-convex function, (α,m)-convex function, s-convex function, and many others. Here we are interested in the generalization of a convex function known as (s,m)-convex function [3].

Definition 2 A function $f:[0,b]\to\mathbb{R}$, $b>0$, is said to be (s,m)-convex, where $(s,m)\in[0,1]^2$ if

$$f\big(tx+m(1-t)y\big)\le t^sf(x)+m(1-t)^sf(y) \tag{1.2}$$

holds for all $x,y\in[0,b]$ and $t\in[0,1]$.

The following remark comprises the functions which can be obtained from the above definition.

Remark 1
(i) If $(s,m) = (1,m)$, then (1.2) produces the definition of m-convex function.
(ii) If $(s,m) = (1,1)$, then (1.2) produces the definition of convex function.
(iii) If $(s,m) = (1,0)$, then (1.2) produces the definition of star-shaped function.

The goal of this paper is to prove generalized integral inequalities for (s,m)-convex functions by the help of generalized integral operator given in Definition 7. This operator has interesting implications in fractional calculus operators. In the following we give definitions associated with Definition 7.

Definition 3 Let $f \in L_1[a,b]$. Then the left-sided and right-sided Riemann–Liouville fractional integral operators of order $\mu \in \mathbb{C}$ ($\mathcal{R}(\mu) > 0$) are defined as follows:

$$^{\mu}I_{a^+}f(x) = \frac{1}{\Gamma(\mu)}\int_a^x (x-t)^{\mu-1}f(t)\,dt, \quad x > a, \tag{1.3}$$

$$^{\mu}I_{b\_}f(x) = \frac{1}{\Gamma(\mu)}\int_x^b (t-x)^{\mu-1}f(t)\,dt, \quad x < b. \tag{1.4}$$

A k-fractional analogue of Riemann–Liouville fractional integral operator is given in [16].

Definition 4 Let $f \in L_1[a,b]$. Then the k-fractional integral operators of f of order $\mu \in \mathbb{C}$, $\mathcal{R}(\mu) > 0$, $k > 0$ are defined as follows:

$$^{\mu}I_{a^+}^{k}f(x) = \frac{1}{k\Gamma_\kappa(\mu)}\int_a^x (x-t)^{\frac{\mu}{k}-1}f(t)\,dt, \quad x > a, \tag{1.5}$$

$$^{\mu}I_{b\_}^{k}f(x) = \frac{1}{k\Gamma_k(\mu)}\int_x^b (t-x)^{\frac{\mu}{k}-1}f(t)\,dt, \quad x < b. \tag{1.6}$$

A more general definition of the Riemann–Liouville fractional integral operators is given in [13].

Definition 5 Let $f : [a,b] \to \mathbb{R}$ be an integrable function. Also, let g be an increasing and positive function on $(a,b]$, having a continuous derivative g' on (a,b). The left-sided and right-sided fractional integrals of a function f with respect to another function g on $[a,b]$ of order $\mu \in \mathbb{C}$ ($\mathcal{R}(\mu) > 0$) are defined as follows:

$$^{\mu}_{g}I_{a^+}f(x) = \frac{1}{\Gamma(\mu)}\int_a^x \bigl(g(x)-g(t)\bigr)^{\mu-1}g'(t)f(t)\,dt, \quad x > a, \tag{1.7}$$

$$^{\mu}_{g}I_{b\_}f(x) = \frac{1}{\Gamma(\mu)}\int_x^b \bigl(g(t)-g(x)\bigr)^{\mu-1}g'(t)f(t)\,dt, \quad x < b, \tag{1.8}$$

where $\Gamma(\cdot)$ is the gamma function.

Definition 6 ([14]) Let $f : [a,b] \to \mathbb{R}$ be an integrable function. Also, let g be an increasing and positive function on $(a,b]$, having a continuous derivative g' on (a,b). The left-sided and right-sided k-fractional integral operators, $k > 0$, of a function f with respect to another function g on $[a,b]$ of order $\mu \in \mathbb{C}$, $\mathcal{R}(\mu) > 0$ are defined as follows:

$$ {}_{g}^{\mu}I_{a^+}^{k} f(x) = \frac{1}{k\Gamma_k(\mu)} \int_a^x \big(g(x) - g(t)\big)^{\frac{\mu}{k}-1} g'(t) f(t)\, dt, \quad x > a, \tag{1.9}$$

$$ {}_{g}^{\mu}I_{b_-}^{k} f(x) = \frac{1}{k\Gamma_k(\mu)} \int_x^b \big(g(t) - g(x)\big)^{\frac{\mu}{k}-1} g'(t) f(t)\, dt, \quad x < b, \tag{1.10}$$

where $\Gamma_k(\cdot)$ is the k-gamma function.

The following generalized integral operator is given in [5].

Definition 7 Let $f, g : [a,b] \to \mathbb{R}$, $0 < a < b$, be the functions such that f is positive and $f \in L_1[a,b]$, and g be differentiable and strictly increasing. Also, let $\frac{\phi}{x}$ be an increasing function on $[a,\infty)$. Then, for $x \in [a,b]$, the left and right integral operators are defined as follows:

$$ \big(F_{a^+}^{\phi,g} f\big)(x) = \int_a^x K_g(x,t;\phi) f(t) d\big(g(t)\big), \quad x > a, \tag{1.11}$$

$$ \big(F_{b_-}^{\phi,g} f\big)(x) = \int_x^b K_g(t,x;\phi) f(t) d\big(g(t)\big), \quad x < b, \tag{1.12}$$

where $K_g(x,y;\phi) = \frac{\phi(g(x)-g(y))}{g(x)-g(y)}$.

Integral operators defined in (1.11) and (1.12) produce several fractional and conformable integral operators defined in [1, 2, 8, 9, 11–13, 22, 25].

Remark 2 Integral operators given in (1.11) and (1.12) produce several known fractional and conformable integral operators corresponding to different settings of ϕ and g.

(i) If we consider $\phi(t) = \frac{t^{\frac{\mu}{k}}}{k\Gamma_k(\mu)}$, then (1.11) and (1.12) integral operators coincide with (1.9) and (1.10) fractional integral operators.

(ii) If we consider $\phi(t) = \frac{t^{\mu}}{\Gamma(\mu)}$, $\mu > 0$, then (1.11) and (1.12) integral operators coincide with (1.7) and (1.8) fractional integral operators.

(iii) If we consider $\phi(t) = \frac{t^{\frac{\mu}{k}}}{k\Gamma_k(\mu)}$ and g as an identity function, then (1.11) and (1.12) integral operators coincide with (1.5) and (1.6) fractional integral operators.

(iv) If we consider $\phi(t) = \frac{t^{\mu}}{\Gamma(\mu)}$, $\mu > 0$, and g the identity function, then (1.11) and (1.12) integral operators coincide with (1.3) and (1.4) fractional integral operators.

(v) If we consider $\phi(t) = \frac{t^{\mu}}{\Gamma(\mu)}$ and $g(x) = \frac{x^{\rho}}{\rho}$, $\rho > 0$, then (1.11) and (1.12) produce Katugampola fractional integral operators defined by Chen et al. in [1].

(vi) If we consider $\phi(t) = \frac{t^{\mu}}{\Gamma(\mu)}$ and $g(x) = \frac{x^{\tau+s}}{\tau+s}$, $s > 0$, then (1.11) and (1.12) produce generalized conformable integral operators defined by Khan et al. in [11].

(vii) If we consider $\phi(t) = \frac{t^{\frac{\mu}{k}}}{k\Gamma_k(\mu)}$ and $g(x) = \frac{(x-a)^s}{s}$, $s > 0$, in (1.11) and $\phi(t) = \frac{t^{\frac{\mu}{k}}}{k\Gamma_k(\mu)}$ and $g(x) = -\frac{(b-x)^s}{s}$, $s > 0$, in (1.12) respectively, then conformable (k,s)-fractional integrals are achieved as defined by Habib et al. in [8].

(viii) If we consider $\phi(t) = \frac{t^{\frac{\mu}{k}}}{k\Gamma_k(\mu)}$ and $g(x) = \frac{x^{1+s}}{1+s}$, then (1.11) and (1.12) produce conformable fractional integrals defined by Sarikaya et al. in [22].

(ix) If we consider $\phi(t) = \frac{t^\mu}{\Gamma(\mu)}$ and $g(x) = \frac{(x-a)^s}{s}$, $s > 0$, in (1.11) and $\phi(t) = \frac{t^\mu}{\Gamma(\mu)}$ and $g(x) = -\frac{(b-x)^s}{s}$, $s > 0$, in (1.12) respectively, then conformable fractional integrals are achieved as defined by Jarad et al. in [9].

(x) If we consider $\phi(t) = t^{\frac{\lambda}{k}}\mathcal{F}^{\sigma}_{\rho,\lambda}(w(t)^\rho)$, then (1.11) and (1.12) produce generalized k-fractional integral operators defined by Tunc et al. in [25].

(xi) If we consider $\phi(t) = \frac{\exp(-At)}{\mu}$, $A = \frac{1-\mu}{\mu}$, $\mu > 0$, then the following generalized fractional integral operators with exponential kernel are obtained [2]:

$$ {}^{\mu}_{g}E_{a^+}f(x) = \frac{1}{\mu}\int_a^x \exp\biggl(-\frac{1-\mu}{\mu}\bigl(g(x)-g(t)\bigr)\biggr)f(t)\,dt, \quad x > a, \tag{1.13}$$

$$ {}^{\mu}_{g}E_{b^-}f(x) = \frac{1}{\mu}\int_x^b \exp\biggl(-\frac{1-\mu}{\mu}\bigl(g(x)-g(t)\bigr)\biggr)f(t)\,dt, \quad x < b. \tag{1.14}$$

(xii) If we consider $\phi(t) = \frac{t^\mu}{\Gamma(\mu)}$ and $g(t) = \ln t$, then Hadamard fractional integral operators will be obtained [12, 13].

(xiii) If we consider $\phi(t) = \frac{t^\mu}{\Gamma(\mu)}$ and $g(t) = -t^{-1}$, then Harmonic fractional integral operators defined in [13] will be obtained and given as follows:

$$ {}^{\mu}R_{a^+}f(x) = \frac{t^\mu}{\Gamma(\mu)}\int_a^x (x-t)^{\mu-1}\frac{f(t)}{t^{\mu+1}}\,dt, \quad x > a, \tag{1.15}$$

$$ {}^{\mu}R_{b^-}f(x) = \frac{t^\mu}{\Gamma(\mu)}\int_a^x (t-x)^{\mu-1}\frac{f(t)}{t^{\mu+1}}\,dt, \quad x < b. \tag{1.16}$$

(xiv) If we consider $\phi(t) = t^\mu \ln t$, then left- and right-sided logarithmic fractional integrals defined in [2] will be obtained and given as follows:

$$ {}^{\mu}_{g}\mathcal{L}_{a^+}f(x) = \int_a^x \bigl(g(x)-g(t)\bigr)^{\mu-1}\ln\bigl(g(x)-g(t)\bigr)g'(t)\,dt, \quad x > a, \tag{1.17}$$

$$ {}^{\mu}_{g}\mathcal{L}_{b^-}f(x) = \int_a^x \bigl(g(t)-g(x)\bigr)^{\mu-1}\ln\bigl(g(x)-g(t)\bigr)g'(t)\,dt, \quad x < b. \tag{1.18}$$

In recent decades fractional and conformable integral operators have been used by many researchers to obtain corresponding operator versions of well-known inequalities. For some recent work, we refer the reader to [1, 2, 7, 8, 10, 20, 21, 24–26]. In the upcoming section we derive the bounds of sum of the left- and right-sided integral operators defined in (1.11) and (1.12) for (s,m)-convex functions. These bounds lead to producing results for several kinds of well-known operators for convex function, m-convex function, s-convex function, and star-shaped function. Further, in Sect. 3, bounds are presented in the form of a Hadamard inequality, from which several fractional Hadamard inequalities are deduced.

2 Bounds of integral operators and their consequences

Theorem 1 *Let $f : [a,b] \to \mathbb{R}$ be a positive (s,m)-convex function with $m \in (0,1]$, and let $g : [a,b] \to \mathbb{R}$ be a differentiable and strictly increasing function. Also, let $\frac{\phi}{x}$ be an increasing function on $[a,b]$. Then, for $x \in [a,b]$, the following inequality for integral operators* (1.11) *and* (1.12) *holds*:

$$\begin{aligned}
&\left(F_{a^+}^{\phi,g}f\right)(x)+\left(F_{b^-}^{\phi,g}f\right)(x)\\
&\quad\le\frac{K_g(x,a;\phi)}{(x-a)^s}\left((x-a)^s\left(mf\left(\frac{x}{m}\right)g(x)-f(a)g(a)\right)\right.\\
&\qquad\left.-\Gamma(s+1)\left(mf\left(\frac{x}{m}\right)^sI_{x^-}g(a)-f(a)^sI_{a^+}g(x)\right)\right)\\
&\qquad+\frac{K_g(b,x;\phi)}{(b-x)^s}\left((b-x)^s\left(f(b)g(b)-mf\left(\frac{x}{m}\right)g(x)\right)\right.\\
&\qquad\left.-\Gamma(s+1)\left(f(b)^sI_{b^-}g(x)-mf\left(\frac{x}{m}\right)^sI_{x^+}g(b)\right)\right).
\end{aligned}\tag{2.1}$$

Proof For the kernel of integral operator (1.11), we have

$$K_g(x,t;\phi)g'(t)\le K_g(x,a;\phi)g'(t),\quad x\in(a,b]\text{ and }t\in[a,x).\tag{2.2}$$

An (s,m)-convex function satisfies the following inequality:

$$f(t)\le\left(\frac{x-t}{x-a}\right)^sf(a)+m\left(\frac{t-a}{x-a}\right)^sf\left(\frac{x}{m}\right),\quad m\in(0,1].\tag{2.3}$$

Inequalities (2.2) and (2.3) lead to the following integral inequality:

$$\begin{aligned}
&\int_a^xK_g(x,t;\phi)g'(t)f(t)\,dt\\
&\quad\le K_g(x,a;\phi)\left(f(a)\int_a^x\left(\frac{x-t}{x-a}\right)^sg'(t)\,dt+mf\left(\frac{x}{m}\right)\int_a^x\left(\frac{t-a}{x-a}\right)^sg'(t)\,dt\right),
\end{aligned}\tag{2.4}$$

while (2.4) gives

$$\begin{aligned}
\left(F_{a^+}^{\phi,g}f\right)(x)&\le\frac{K_g(x,a;\phi)}{(x-a)^s}\left((x-a)^s\left(mf\left(\frac{x}{m}\right)g(x)-f(a)g(a)\right)\right.\\
&\quad\left.-\Gamma(s+1)\left(mf\left(\frac{x}{m}\right)^sI_{x^-}g(a)-f(a)^sI_{a^+}g(x)\right)\right).
\end{aligned}\tag{2.5}$$

Again, for the kernel of integral operator (1.12), we have

$$K_g(t,x;\phi)g'(t)\le K_g(b,x;\phi)g'(t),\quad t\in(x,b]\text{ and }x\in[a,b).\tag{2.6}$$

An (s,m)-convex function satisfies the following inequality:

$$f(t)\le\left(\frac{t-x}{b-x}\right)^sf(b)+m\left(\frac{b-t}{b-x}\right)^sf\left(\frac{x}{m}\right),\quad m\in(0,1].\tag{2.7}$$

Inequalities (2.6) and (2.7) lead to the following integral inequality:

$$\begin{aligned}
&\int_x^bK_g(t,x;\phi)g'(t)f(t)\,dt\\
&\quad\le K_g(b,x;\phi)\left(f(b)\int_x^b\left(\frac{t-x}{b-x}\right)^sg'(t)\,dt+mf\left(\frac{x}{m}\right)\int_x^b\left(\frac{b-t}{b-x}\right)^sg'(t)\,dt\right),
\end{aligned}\tag{2.8}$$

while (2.8) further gives

$$\big(F_{b^-}^{\phi,g}f\big)(x) \le \frac{K_g(b,x;\phi)}{(b-x)^s}\bigg((b-x)^s\bigg(f(b)g(b)-mf\bigg(\frac{x}{m}\bigg)g(x)\bigg) \\ -\Gamma(s+1)\bigg(f(b)^sI_{b^-}g(x)-mf\bigg(\frac{x}{m}\bigg)^sI_{x^+}g(b)\bigg)\bigg). \tag{2.9}$$

By adding (2.5) and (2.9), (2.1) can be obtained. □

The following remark connects the above theorem with already known results.

Remark 3
1. For $\phi(t)=\frac{t^\mu}{\Gamma(\mu)}$, $\mu>0$, and $(s,m)=(1,1)$ in (2.1), [6, Theorem 1] can be achieved.
2. For $\phi(t)=\frac{t^\mu}{\Gamma(\mu)}$, $\mu>0$, $g(x)=x$, and $(s,m)=(1,1)$ in (2.1), [4, Theorem 1] can be achieved.
3. For $(s,m)=(1,1)$ in (2.1), [15, Theorem 1] can be achieved.

The following results indicate upper bounds of several known fractional and conformable integral operators.

Proposition 1 *Let* $\phi(t)=\frac{t^\mu}{\Gamma(\mu)}$, $\mu>0$. *Then* (1.11) *and* (1.12) *produce the fractional integral operators* (1.7) *and* (1.8) *as follows*:

$$\big(F_{a^+}^{\frac{t^\mu}{\Gamma(\mu)},g}f\big)(x):={}_g^{\mu}I_{a^+}f(x), \qquad \big(F_{b^-}^{\frac{t^\mu}{\Gamma(\mu)},g}f\big)(x):={}_g^{\mu}I_{b^-}f(x). \tag{2.10}$$

Further, they satisfy the following bound for $\mu\ge1$:

$$\begin{aligned}&\big({}_g^{\mu}I_{a^+}f\big)(x)+\big({}_g^{\mu}I_{b^-}f\big)(x)\\ &\quad\le\frac{(g(x)-g(a))^{\mu-1}}{(x-a)^s\Gamma(\mu)}\bigg((x-a)^s\bigg(mf\bigg(\frac{x}{m}\bigg)g(x)-f(a)g(a)\bigg)\\ &\qquad-\Gamma(s+1)\bigg(mf\bigg(\frac{x}{m}\bigg)^sI_{x^-}g(a)-f(a)^sI_{a^+}g(x)\bigg)\bigg)\\ &\qquad+\frac{(g(b)-g(x))^{\mu-1}}{(b-x)^s\Gamma(\mu)}\bigg((b-x)^s\bigg(f(b)g(b)-mf\bigg(\frac{x}{m}\bigg)g(x)\bigg)\\ &\qquad-\Gamma(s+1)\bigg(f(b)^sI_{b^-}g(x)-mf\bigg(\frac{x}{m}\bigg)^sI_{x^+}g(b)\bigg)\bigg).\end{aligned}$$

Proposition 2 *Let* $g(x)=I(x)=x$. *Then* (1.11) *and* (1.12) *produce integral operators defined in* [23] *as follows*:

$$\big(F_{a^+}^{\phi,I}f\big)(x):=({}_{a^+}I_\phi f)(x)=\int_a^x\frac{\phi(x-t)}{(x-t)}f(t)\,dt, \tag{2.11}$$

$$\big(F_{b^-}^{\phi,I}f\big)(x):=({}_{b^-}I_\phi f)(x)=\int_x^b\frac{\phi(t-x)}{(t-x)}f(t)\,dt. \tag{2.12}$$

Further, they satisfy the following bound:

$$
\begin{aligned}
&({}_{a^+}I_\phi f)(x) + ({}_{b^-}I_\phi f)(x) \\
&\quad\le \frac{\phi(x-a)}{(x-a)^{s+1}}\left((x-a)^s\left(mf\left(\frac{x}{m}\right)g(x) - f(a)g(a)\right)\right. \\
&\qquad \left. - \Gamma(s+1)\left(mf\left(\frac{x}{m}\right){}^sI_{x^-}g(a) - f(a){}^sI_{a^+}g(x)\right)\right) \\
&\qquad + \frac{\phi(b-x)}{(b-x)^{s+1}}\left((b-x)^s\left(f(b)g(b) - mf\left(\frac{x}{m}\right)g(x)\right)\right. \\
&\qquad \left. - \Gamma(s+1)\left(f(b){}^sI_{b^-}g(x) - mf\left(\frac{x}{m}\right){}^sI_{x^+}g(b)\right)\right).
\end{aligned}
$$

Corollary 1 *If we take* $\phi(t) = \frac{t^{\frac{\mu}{k}}}{k\Gamma_k(\mu)}$, *then* (1.11) *and* (1.12) *produce the fractional integral operators* (1.9) *and* (1.10) *as follows*:

$$
\left(F_{a^+}^{\frac{t^{\frac{\mu}{k}}}{k\Gamma_k(\mu)},g} f\right)(x) := {}^{\mu}_{g}I^{k}_{a^+}f(x), \qquad \left(F_{b^-}^{\frac{t^{\frac{\mu}{k}}}{k\Gamma_k(\mu)},g} f\right)(x) := {}^{\mu}_{g}I^{k}_{b^-}f(x). \tag{2.13}
$$

Moreover, from (2.1) *the following bound holds for* $\mu \ge k$:

$$
\begin{aligned}
&\left({}^{\mu}_{g}I^{k}_{a^+}f\right)(x) + \left({}^{\mu}_{g}I^{k}_{b^-}f\right)(x) \\
&\quad\le \frac{(g(x)-g(a))^{\frac{\mu}{k}-1}}{(x-a)^s(k\Gamma_k(\mu))}\left((x-a)^s\left(mf\left(\frac{x}{m}\right)g(x) - f(a)g(a)\right)\right. \\
&\qquad \left. - \Gamma(s+1)\left(mf\left(\frac{x}{m}\right){}^sI_{x^-}g(a) - f(a){}^sI_{a^+}g(x)\right)\right) \\
&\qquad + \frac{(g(b)-g(x))^{\frac{\mu}{k}-1}}{(b-x)^s(k\Gamma_k(\mu))}\left((b-x)^s\left(f(b)g(b) - mf\left(\frac{x}{m}\right)g(x)\right)\right. \\
&\qquad \left. - \Gamma(s+1)\left(f(b){}^sI_{b^-}g(x) - mf\left(\frac{x}{m}\right){}^sI_{x^+}g(b)\right)\right).
\end{aligned}
$$

Corollary 2 *If we take* $\phi(t) = \frac{t^\mu}{\Gamma(\mu)}$, $\mu > 0$, *and* $g(x) = \frac{x^\rho}{\rho}$, $\rho > 0$, *then* (1.11) *and* (1.12) *produce the fractional integral operators defined in* [1] *as follows*:

$$
\left(F_{a^+}^{\frac{t^\mu}{\Gamma(\mu)},g} f\right)(x) = \left({}^{\rho}I^{\mu}_{a^+}f\right)(x) = \frac{\rho^{1-\mu}}{\Gamma(\mu)}\int_a^x \left(x^\rho - t^\rho\right)^{\mu-1} t^{\rho-1} f(t)\,dt, \tag{2.14}
$$

$$
\left(F_{b^-}^{\frac{t^\mu}{\Gamma(\mu)},g} f\right)(x) = \left({}^{\rho}I^{\mu}_{b^-}f\right)(x) = \frac{\rho^{1-\mu}}{\Gamma(\mu)}\int_x^b \left(t^\rho - x^\rho\right)^{\mu-1} t^{\rho-1} f(t)\,dt. \tag{2.15}
$$

Moreover, from (2.1) *they satisfy the following bound*:

$$
\begin{aligned}
&\left({}^{\rho}I^{\mu}_{a^+}f\right)(x) + \left({}^{\rho}I^{\mu}_{b^-}f\right)(x) \\
&\le \frac{(x^\rho - a^\rho)^{\mu-1}}{(x-a)^s(\Gamma(\mu))(\rho^{\mu-1})}\left((x-a)^s\left(mf\left(\frac{x}{m}\right)g(x) - f(a)g(a)\right)\right. \\
&\left. - \Gamma(s+1)\left(mf\left(\frac{x}{m}\right){}^sI_{x^-}g(a) - f(a){}^sI_{a^+}g(x)\right)\right)
\end{aligned}
$$

$$+\frac{(b^{\rho}-x^{\rho})^{\mu-1}}{(b-x)^{s}(\Gamma(\mu))(\rho^{\mu-1})}\left((b-x)^{s}\left(f(b)g(b)-mf\left(\frac{x}{m}\right)g(x)\right)\right.$$
$$\left.-\Gamma(s+1)\left(f(b)^{s}I_{b^-}g(x)-mf\left(\frac{x}{m}\right)^{s}I_{x^+}g(b)\right)\right).$$

Corollary 3 *If we take* $\phi(t)=\frac{t^{\mu}}{\Gamma(\mu)}$, $\mu>0$, *and* $g(x)=\frac{x^{n+1}}{n+1}$, $n>0$, *then* (1.11) *and* (1.12) *produce the fractional integral operators defined as follows*:

$$\left(F_{a^+}^{\frac{t^{\mu}}{\Gamma(\mu)},g}f\right)(x)=\left({}^{n}I_{a^+}^{\mu}f\right)(x)=\frac{(n+1)^{1-\mu}}{\Gamma(\mu)}\int_{a}^{x}\left(x^{n+1}-t^{n+1}\right)^{\mu-1}t^{n}f(t)\,dt,\tag{2.16}$$
$$\left(F_{b^-}^{\frac{t^{\mu}}{\Gamma(\mu)},g}f\right)(x)=\left({}^{n}I_{b^-}^{\mu}f\right)(x)=\frac{(n+1)^{1-\mu}}{\Gamma(\mu)}\int_{x}^{b}\left(t^{n+1}-x^{n+1}\right)^{\mu-1}t^{n}f(t)\,dt.\tag{2.17}$$

Moreover, from (2.1) *they satisfy the following bound*:

$$\begin{aligned}&\left({}^{n}I_{a^+}^{\mu}f\right)(x)+\left({}^{n}I_{b^-}^{\mu}f\right)(x)\\&\quad\le\frac{(x^{n+1}-a^{n+1})^{\mu-1}}{(x-a)^{s}(\Gamma(\mu))(n+1)^{\mu-1}}\left((x-a)^{s}\left(mf\left(\frac{x}{m}\right)g(x)-f(a)g(a)\right)\right.\\&\quad\left.-\Gamma(s+1)\left(mf\left(\frac{x}{m}\right)^{s}I_{x^-}g(a)-f(a)^{s}I_{a^+}g(x)\right)\right)\\&\quad+\frac{(b^{n+1}-x^{n+1})^{\mu-1}}{(b-x)^{s}(\Gamma(\mu))(n+1)^{\mu-1}}\left((b-x)^{s}\left(f(b)g(b)-mf\left(\frac{x}{m}\right)g(x)\right)\right.\\&\quad\left.-\Gamma(s+1)\left(f(b)^{s}I_{b^-}g(x)-mf\left(\frac{x}{m}\right)^{s}I_{x^+}g(b)\right)\right).\end{aligned}$$

Remark 4 The bounds of Riemann–Liouville fractional and k-fractional integrals can be computed by setting $\phi(t)=\frac{t^{\mu}}{\Gamma(\mu)}$, $g(t)=t$ and $\phi(t)=\frac{t^{\frac{\mu}{k}}}{k\Gamma_k(\mu)}$, $g(t)=t$ respectively in (2.1), we leave it for the reader.

For the function f which is differentiable and $|f'|$ is (s,m)-convex, the following result holds.

Theorem 2 *Let* $f:I\to\mathbb{R}$ *be a differentiable function if* $|f'|$ *is* (s,m)*-convex with* $m\in(0,1]$, *and let* $g:I\to\mathbb{R}$ *be a differentiable and strictly increasing function. Also, let* $\frac{\phi}{x}$ *be an increasing function on* I, *then for* $a,b\in I$, $a<b$ *the following inequalities for integral operators hold*:

$$\begin{aligned}\left|F_{a^+}^{\phi,g}(f*g)(x)\right|&\le\frac{K_g(x,a;\phi)}{(x-a)^{s}}\left((x-a)^{s}\left(m\left|f'\left(\frac{x}{m}\right)\right|g(x)-\left|f'(a)\right|g(a)\right)\right.\\&\quad\left.-\Gamma(s+1)\left(m\left|f'\left(\frac{x}{m}\right)\right|^{s}I_{x^-}g(a)-\left|f'(a)\right|^{s}I_{a^+}g(x)\right)\right),\end{aligned}\tag{2.18}$$
$$\begin{aligned}\left|F_{b^-}^{\phi,g}(f*g)(x)\right|&\le\frac{K_g(b,x;\phi)}{(b-x)^{s}}\left((b-x)^{s}\left(\left|f'(b)\right|g(b)-m\left|f'\left(\frac{x}{m}\right)\right|g(x)\right)\right.\\&\quad\left.-\Gamma(s+1)\left(\left|f'(b)\right|^{s}I_{b^-}g(x)-m\left|f'\left(\frac{x}{m}\right)\right|^{s}I_{x^+}g(b)\right)\right),\end{aligned}\tag{2.19}$$

where

$$F^{\phi,g}_{a^+}(f*g)(x)=\int_a^x K_g(x,t;\phi)g'(t)f'(t)\,dt,\qquad F^{\phi,g}_{b^-}(f*g)(x)=\int_x^b K_g(t,x;\phi)g'(t)f'(t)\,dt.$$

Proof An (s,m)-convex function $|f'|$ satisfies the following inequality:

$$\bigl|f'(t)\bigr|\le\left(\frac{x-t}{x-a}\right)^s\bigl|f'(a)\bigr|+m\left(\frac{t-a}{x-a}\right)^s\left|f'\left(\frac{x}{m}\right)\right|,\quad m\in(0,1],\tag{2.20}$$

from which we can write

$$f'(t)\le\left(\frac{x-t}{x-a}\right)^s\bigl|f'(a)\bigr|+m\left(\frac{t-a}{x-a}\right)^s\left|f'\left(\frac{x}{m}\right)\right|.\tag{2.21}$$

Inequalities (2.2) and (2.21) lead to the following integral inequality:

$$\begin{aligned}&\int_a^x K_g(x,t;\phi)g'(t)f'(t)\,dt\\&\quad\le K_g(x,a;\phi)\left(\bigl|f'(a)\bigr|\int_a^x\left(\frac{x-t}{x-a}\right)^s g'(t)\,dt\right.\\&\qquad\left.+m\left|f'\left(\frac{x}{m}\right)\right|\int_a^x\left(\frac{t-a}{x-a}\right)^s g'(t)\,dt\right),\end{aligned}\tag{2.22}$$

while (2.22) further gives

$$\begin{aligned}F^{\phi,g}_{a^+}(f*g)(x)\le{}&\frac{K_g(x,a;\phi)}{(x-a)^s}\left((x-a)^s\left(m\left|f'\left(\frac{x}{m}\right)\right|g(x)-\bigl|f'(a)\bigr|g(a)\right)\right.\\&\left.-\Gamma(s+1)\left(m\left|f'\left(\frac{x}{m}\right)\right|{}^sI_{x^-}g(a)-\bigl|f'(a)\bigr|{}^sI_{a^+}g(x)\right)\right).\end{aligned}\tag{2.23}$$

From (2.20) we can write

$$f'(t)\ge-\left(\left(\frac{x-t}{x-a}\right)^s\bigl|f'(a)\bigr|+m\left(\frac{t-a}{x-a}\right)^s\left|f'\left(\frac{x}{m}\right)\right|\right).\tag{2.24}$$

Adopting the same method as we did for (2.21), the following integral inequality holds:

$$\begin{aligned}F^{\phi,g}_{a^+}(f*g)(x)\ge{}&-\frac{K_g(x,a;\phi)}{(x-a)^s}\left((x-a)^s\left(m\left|f'\left(\frac{x}{m}\right)\right|g(x)-\bigl|f'(a)\bigr|g(a)\right)\right.\\&\left.-\Gamma(s+1)\left(m\left|f'\left(\frac{x}{m}\right)\right|{}^sI_{x^-}g(a)-\bigl|f'(a)\bigr|{}^sI_{a^+}g(x)\right)\right).\end{aligned}\tag{2.25}$$

From (2.23) and (2.25), (2.18) can be obtained.

An (s,m)-convex function $|f'|$ satisfies the following inequality:

$$\bigl|f'(t)\bigr|\le\left(\frac{t-x}{b-x}\right)^s\bigl|f'(b)\bigr|+m\left(\frac{b-t}{b-x}\right)^s\left|f'\left(\frac{x}{m}\right)\right|,\quad m\in(0,1],\tag{2.26}$$

from which we can write

$$f'(t) \le \left(\frac{t-x}{b-x}\right)^s \left|f'(b)\right| + m\left(\frac{b-t}{b-x}\right)^s \left|f'\left(\frac{x}{m}\right)\right|. \tag{2.27}$$

Inequalities (2.6) and (2.27) lead to the following integral inequality:

$$\begin{aligned}
&\int_x^b K_g(t,x;\phi)g'(t)f'(t)\,dt \\
&\quad\le K_g(b,x;\phi)\left(\left|f'(b)\right|\int_x^b \left(\frac{x-t}{b-x}\right)^s g'(t)\,dt\right. \\
&\qquad\left. + m\left|f'\left(\frac{x}{m}\right)\right|\int_x^b \left(\frac{b-t}{b-x}\right)^s g'(t)\,dt\right),
\end{aligned} \tag{2.28}$$

while (2.28) further gives

$$\begin{aligned}
F_{b^-}^{\phi,g}(f*g)(x) \le{}& \frac{K_g(b,x;\phi)}{(b-x)^s}\left((b-x)^s\left(\left|f'(b)\right|g(b) - m\left|f'\left(\frac{x}{m}\right)\right|g(x)\right)\right. \\
&\left. - \Gamma(s+1)\left(\left|f'(b)\right|{}^sI_{b^-}g(x) - m\left|f'\left(\frac{x}{m}\right)\right|{}^sI_{x^+}g(b)\right)\right).
\end{aligned} \tag{2.29}$$

From (2.26) we can write

$$f'(t) \ge -\left(\left(\frac{t-x}{b-x}\right)^s \left|f'(b)\right| + m\left(\frac{b-t}{b-x}\right)^s \left|f'\left(\frac{x}{m}\right)\right|\right). \tag{2.30}$$

Adopting the same method as we did for (2.27), the following inequality holds:

$$\begin{aligned}
F_{b^-}^{\phi,g}(f*g)(x) \ge{}& -\frac{K_g(b,x;\phi)}{(b-x)^s}\left((b-x)^s\left(\left|f'(b)\right|g(b) - m\left|f'\left(\frac{x}{m}\right)\right|g(x)\right)\right. \\
&\left. - \Gamma(s+1)\left(\left|f'(b)\right|{}^sI_{b^-}g(x) - m\left|f'\left(\frac{x}{m}\right)\right|{}^sI_{x^+}g(b)\right)\right).
\end{aligned} \tag{2.31}$$

From (2.29) and (2.31), (2.19) can be obtained. □

3 Hadamard type inequalities for (s,m)-convex function

In order to prove our next result, we need the following lemma.

Lemma 1 *Let $f:[0,\infty]\to\mathbb{R}$ be an (s,m)-convex function with $m\in(0,1]$. If $0\le a<b$ and $f(x)=f(\frac{a+b-x}{m})$, then the following inequality holds:*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{2^s}(1+m)f(x), \quad x\in[a,b]. \tag{3.1}$$

Proof Since f is (s,m)-convex, the following inequality is valid:

$$\begin{aligned}
&f\left(\frac{a+b}{2}\right) \le \frac{1}{2^s}f\left(\frac{x-a}{b-a}b + \frac{b-x}{b-a}\right) + m\left(1-\frac{1}{2}\right)^s f\left(\frac{\frac{x-a}{b-a}a + \frac{b-x}{b-a}b}{m}\right), \\
&f\left(\frac{a+b}{2}\right) \le \frac{1}{2^s}\left(f(x) + mf\left(\frac{a+b-x}{m}\right)\right).
\end{aligned}$$

By using $f(x)=f(\frac{a+b-x}{m})$ in the above inequality, we get (3.1). □

By applying Lemma 1, we prove the following Hadamard type inequality.

Theorem 3 *Let $f:[a,b]\to\mathbb{R}$ be a positive (s,m)-convex function with $m\in(0,1]$, $f(x)=f(\frac{a+b-x}{m})$ and $g:[a,b]\to\mathbb{R}$ be a differentiable and strictly increasing function. Also, let $\frac{\phi}{x}$ be an increasing function on $[a,b]$. Then, for $(\alpha,m)\in[0,1]^2$, the following inequality holds*:

$$\begin{aligned}
&\frac{2^s f(\frac{a+b}{2})}{(m+1)}\big(\big(F_{b^-}^{\phi,g}1\big)(a)+\big(F_{a^+}^{\phi,g}1\big)(b)\big)\\
&\quad\le \big(F_{b^-}^{\phi,g}f\big)(a)+\big(F_{a^+}^{\phi,g}f\big)(b)\\
&\quad\le 2K_g(b,a;\phi)\left(f(b)g(b)-mf\left(\frac{a}{m}\right)g(a)\right.\\
&\quad\quad\left.-\frac{\Gamma(s+1)}{(b-a)^s}\left(f(b)^sI_{b^-}g(a)-mf\left(\frac{a}{m}\right)^sI_{a^+}g(b)\right)\right).
\end{aligned}\tag{3.2}$$

Proof For the kernel of integral operator (1.11), we have

$$K_g(x,a;\phi)g'(x)\le K_g(b,a;\phi)g'(x),\quad x\in(a,b].\tag{3.3}$$

An (s,m)-convex function satisfies the following inequality:

$$f(x)\le\left(\frac{x-a}{b-a}\right)^s f(b)+m\left(\frac{b-x}{b-a}\right)^s f\left(\frac{a}{m}\right),\quad m\in(0,1].\tag{3.4}$$

Inequalities (3.3) and (3.4) lead to the following integral inequality:

$$\begin{aligned}
&\int_a^b K_g(x,a;\phi)g'(x)f(x)\,dx\\
&\quad\le K_g(b,a;\phi)\left(f(b)\int_a^b\left(\frac{x-a}{b-a}\right)^s g'(x)\,dx+mf\left(\frac{a}{m}\right)\int_a^b\left(\frac{b-x}{b-a}\right)^s g'(x)\,dx\right),
\end{aligned}\tag{3.5}$$

while (3.5) further gives

$$\begin{aligned}
\big(F_{b^-}^{\phi,g}f\big)(a)\le{}&\frac{K_g(b,a;\phi)}{(b-a)^s}\left(\left(f(b)g(b)-mf\left(\frac{a}{m}\right)g(a)\right)(b-a)^s\right.\\
&\left.-\Gamma(s+1)\left(f(b)^sI_{b^-}g(a)-mf\left(\frac{a}{m}\right)^sI_{a^+}g(b)\right)\right).
\end{aligned}\tag{3.6}$$

On the other hand, for the kernel of integral operator (1.12), we have

$$K_g(b,x;\phi)g'(x)\le K_g(b,a;\phi)g'(x).\tag{3.7}$$

Inequalities (3.4) and (3.7) lead to the following integral inequality:

$$\begin{aligned}
&\int_a^b K_g(b,x;\phi)g'(x)f(x)\,dx\\
&\quad\le K_g(b,a;\phi)\left(f(b)\int_a^b\left(\frac{x-a}{b-a}\right)^s g'(x)\,dx+mf\left(\frac{a}{m}\right)\int_a^b\left(\frac{b-x}{b-a}\right)^s g'(x)\,dx\right),
\end{aligned}$$

while the above inequality gives

$$\begin{aligned}\left(F_{a^+}^{\phi,g}f\right)(b) \leq \frac{K_g(b,a;\phi)}{(b-a)^s}\Bigg(&\left(f(b)g(b)-mf\left(\frac{a}{m}\right)g(a)\right)(b-a)^s \\ &-\Gamma(s+1)\left(f(b)^s I_{b^-}g(a)-mf\left(\frac{a}{m}\right)^s I_{a^+}g(b)\right)\Bigg).\end{aligned} \tag{3.8}$$

From (3.6) and (3.8), the following inequality can be obtained:

$$\begin{aligned}\left(F_{a^+}^{\phi,g}f\right)(b)+\left(F_{b^-}^{\phi,g}f\right)(a) \leq 2\frac{K_g(b,a;\phi)}{(b-a)^s}\Bigg(&\left(f(b)g(b)-mf\left(\frac{a}{m}\right)g(a)\right)(b-a)^s \\ &-\Gamma(s+1)\left(f(b)^s I_{b^-}g(a)-mf\left(\frac{a}{m}\right)^s I_{a^+}g(b)\right)\Bigg).\end{aligned} \tag{3.9}$$

Now, using Lemma 1 and multiplying (3.1) with $K_g(x,a;\phi)g'(x)$, then integrating over $[a,b]$, we have

$$\int_a^b K_g(x,a;\phi)f\left(\frac{a+b}{2}\right)g'(x)\,dx \leq \frac{1}{2^s}(1+m)\int_a^b K_g(x,a;\phi)g'(x)f(x)\,dx, \tag{3.10}$$

from which we get

$$f\left(\frac{a+b}{2}\right)\left(F_{b^-}^{\phi,g}1\right)(a) \leq \frac{1}{2^s}(1+m)\left(F_{b^-}^{\phi,g}f\right)(a). \tag{3.11}$$

Again using Lemma 1 and multiplying (3.1) with $K_g(b,x;\phi)g'(x)$, then integrating over $[a,b]$, we have

$$\int_a^b K_g(b,x;\phi)f\left(\frac{a+b}{2}\right)g'(x)\,dx \leq \frac{1}{2^s}(1+m)\int_a^b K_g(b,x;\phi)g'(x)f(x)\,dx,$$

from which we get

$$f\left(\frac{a+b}{2}\right)\left(F_{a^+}^{\phi,g}1\right)(b) \leq \frac{1}{2^s}(1+m)\left(F_{a^+}^{\phi,g}f\right)(b). \tag{3.12}$$

From (3.11) and (3.12), the following inequality can be achieved:

$$f\left(\frac{a+b}{2}\right)\left(\left(F_{b^-}^{\phi,g}1\right)(a)+\left(F_{a^+}^{\phi,g}1\right)(b)\right) \leq \frac{1}{2^s}(1+m)\left(\left(F_{b^-}^{\phi,g}f\right)(a)+\left(F_{a^+}^{\phi,g}f\right)(b)\right). \tag{3.13}$$

From (3.9) and (3.13), (3.2) can be obtained. □

Remark 5 For $(s,m)=(1,1)$, in (3.2), [15, Theorem 3] can be obtained.

Corollary 4 *If we put* $\phi(t)=\frac{t^{\frac{\mu}{k}}}{k\Gamma_k(\mu)}$, *then inequality* (3.2) *produces the following Hadamard inequality*:

$$\begin{aligned}&\frac{2^s f(\frac{a+b}{2})}{(m+1)}\left({}_g^{\mu}I_{b_-}^k(1)(a)+{}_g^{\mu}I_{a^+}^k(1)(b)\right) \\ &\quad\leq {}_g^{\mu}I_{b\_}^k f(a)+{}_g^{\mu}I_{a^+}^k f(b)\end{aligned}$$

$$\begin{aligned}&\le \frac{2(g(b)-g(a))^{\frac{\mu}{k}-1}}{k\Gamma_k(\mu)}\left(f(b)g(b)-mf\left(\frac{a}{m}\right)g(a)\right.\\&\quad\left.-\frac{\Gamma(s+1)}{(b-a)^s}\left(f(b)^sI_{b^-}g(a)-mf\left(\frac{a}{m}\right)^sI_{a^+}g(b)\right)\right).\end{aligned}\tag{3.14}$$

Corollary 5 *If we put $\phi(t)=\frac{t^\mu}{\Gamma(\mu)}$, then inequality* (3.2) *produces the following Hadamard inequality*:

$$\begin{aligned}&\frac{2^sf(\frac{a+b}{2})}{(m+1)}\left({}_g^\mu I_{b_-}(1)(a)+{}_g^\mu I_{a^+}(1)(b)\right)\\&\quad\le {}_g^\mu I_{b\_}f(a)+{}_g^\mu I_{a^+}f(b)\\&\quad\le \frac{2(g(b)-g(a))^{\mu-1}}{\Gamma(\mu)}\left(f(b)g(b)-mf\left(\frac{a}{m}\right)g(a)\right.\\&\quad\quad\left.-\frac{\Gamma(s+1)}{(b-a)^s}\left(f(b)^sI_{b^-}g(a)-mf\left(\frac{a}{m}\right)^sI_{a^+}g(b)\right)\right).\end{aligned}\tag{3.15}$$

Remark 6 The Hadamard inequality for Riemann–Liouville fractional and k-fractional integrals can be computed by setting $\phi(t)=\frac{t^\mu}{\Gamma(\mu)}$, $g(t)=t$ and $\phi(t)=\frac{t^{\frac{\mu}{k}}}{k\Gamma_k(\mu)}$, $g(t)=t$ respectively in (3.2), we leave it for the reader.

4 Concluding remarks

This work produces some generalized integral operator inequalities via (s,m)-convex function. From these inequalities the bounds of all integral operators defined in Remark 2 can be established for convex function, m-convex function, s-convex function, and star-shaped function. The reader can produce a plenty of Hadamard type inequalities for fractional and conformable integral operators deduced in Remark 2 by applying Theorem 3.

Acknowledgements
We thank the editor and referees for their careful reading and valuable suggestions to make the article reader friendly.

Funding
This work was supported by the Dong-A University Research Fund.

Availability of data and materials
There is no additional data required for the finding of results of this paper.

Competing interests
It is declared that the authors have no competing interests.

Authors' contributions
All authors have equal contribution in this article. All authors read and approved the final manuscript.

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Received: 10 October 2019 Accepted: 16 December 2019 Published online: 03 January 2020

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