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The controllability of fractional differential system with state and control delay

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Abstract

In this research work, we investigate the controllability of linear fractional differential control systems with state and control delay. By using an explicit solution formula, a rank criterion for controllability is established. For the controllability criteria, we establish necessary and sufficient conditions of a fractional differential systems with state and control delay. In the end, a numerical example is constructed to support the results.

MSC: 34k05; 26A33

Keywords: Controllability; Delayed Mittag-Leffler type matrix; State delay; Control delay

1 Introduction

The fractional differential equation is a mathematical model which is useful for the explanation of hereditary characteristics and memory of different processes and materials. A variety of research work is based on the basic study of fractional differential equations [1–6] as in further work various researchers considered control problems; for example, see [7–9].

The controllability shows a major presence in the advancement of modern mathematical control theory and engineering which has a close connection with structural decomposition, quadratic optimal and so on; see [10–17]. Controllability is a qualitative property of fractional delay dynamical system, so one needs to find its representation of a solution. He and Wei [18, 19] gave a representation of a solution and discussed the controllability and then for a fractional control delay system obtained necessary and sufficient conditions, Nirmala [11] give a representation of a solution by using Laplace transform and Mittag-Leffler function and established controllability criteria for fractional delay dynamical system. Moreover, Khusainov et al. [20] obtained the representation of a solution of a Cauchy problem for a linear differential equation with pure delay by using the delayed Mittag-Leffler function, Shukla et al. [21–24] discussed the complete and approximate controllability of semilinear stochastic systems with delays in the state and control function with non-Lipschitz coefficients, the Schauder fixed point theorem, sequence methods and by the theory of the strongly continuous z-order cosine family, and the fixed point theorem, respectively. In a most recent work [25] the authors discussed the relative controllability problem and an explicit representation of solutions is given with the use of delayed Mittag-Leffler function, Li and Wang [26] discussed the controllability criteria of a fractional differential system with state delay by using an explicit solution formula. By following this study we consider a fractional differential system with state and control delay and discussed its controllability by giving its necessary and sufficient conditions. Li and Wang [27] considered pure delay for linear fractional differential equations and gave a representation of a solution by using a delayed Mittag-Leffler type matrix:

$$\begin{cases} {}^cD^{\alpha}_{0^+}x(t) = Ax(t-h), \quad x(t) \in \mathcal{R}^n, t \in J := [0,t_1], h > 0, \\ x(t) = \varphi(t), \quad -h \le t \le 0, \varphi \in \mathcal{C}^1_h := \mathcal{C}^1([-h,0],\mathcal{R}^n), \end{cases} \tag{1}$$

where ${}^cD^{\alpha}_{0^+}x(t)$ stands for the αth order Caputo fractional derivative of $x(t)$ where zero is a lower limit, t_1 is the integral multiple of h, $A \in \mathcal{R}^{n\times n}$, $h > 0$ is a time delay, $n \in \mathcal{N}$ stands for a constant matrix. $\mathcal{E}^{A.\alpha}_h$ is a new notation (delayed Mittag-Leffler type matrix) being reported in Definition 2.3 [28], any solution $x \in C([-h,t_1],\mathcal{R}^n)$ of (1) can be established by Li:

$$x(t) = \mathcal{E}^{At^{\alpha}}_h \varphi(-h) + \int_{-h}^{0} \mathcal{E}^{A(t-h-\tau)^{\alpha}}_h \varphi'(\tau)\,d\tau. \tag{2}$$

Motivated by the previous study, in this research work we deal with the fractional differential systems with state and control delay by using of an explicit formula governed by

$$\begin{cases} {}^cD^{\alpha}_{0^+}x(t) = Ax(t-h) + Bu(t) + Cu(t-h), \quad x(t) \in J := [0,t_1], h > 0, t_1 \ge 0, \\ x(t) = \varphi(t), \quad -h \le t \le 0, \\ u(t) = \psi(t), \quad -h \le t \le 0, \end{cases} \tag{3}$$

where $x : [-h,t_1] \to \mathcal{R}^n$ is a continuous differentiable on $[0,t_1]$ with $t_1 > (n-1)h$, $0 < \alpha \le 1$, $A \in \mathcal{R}^{n\times n}$, $B, C \in \mathcal{R}^{n\times m}$ are any matrices, $h > 0$ shows the time delay, $x(t) \in \mathcal{R}^n$ denotes the state vector, $u(t) \in \mathcal{R}^m$ shows the control vector, $\varphi(t)$ shows the initial state function and $\psi(t)$ shows the initial control function $\varphi \in \mathcal{C}^1_h := \mathcal{C}^1([-h,0],\mathcal{R}^n)$. The lay-out of this article as follows, Sect. 2 includes some useful definitions, preliminary results, and lemmas about delayed Mittag-Leffler type matrix to establish the controllability of fractional differential systems with state and control delay. In Sect. 3 we obtain necessary and sufficient conditions for controllability criteria for the above fractional differential delay system (3). Section 4 presents an example to explain the applicability of the theoretical results.

2 Preliminaries and essential lemmas

This part includes some basic definitions and results used throughout this paper and some lemmas for the main results. We recall some well-known definitions. For more details, see [3, 5].

Definition 2.1 ([29]) We consider a function $f : [0,\infty) \to \mathcal{R}$ where its Caputo fractional derivative of order $(0 < \alpha < 1)$ is defined as

$$\left({}^cD^{\alpha}_{0^+}x\right)(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{x'(\theta)}{(t-\theta)^{\alpha}}\,d\theta, \quad t > 0.$$

Here the Gamma function is denoted by $\Gamma(\cdot)$.

Definition 2.2 ([29]) We consider a function $f:[0,\infty)\to\mathcal{R}$ where its fractional integral of order $\alpha>0$ is defined as

$$\bigl(I_{0^+}^{\alpha}f\bigr)(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-\theta)^{\alpha-1}f(\theta)\,d\theta.$$

Here $\Gamma(\cdot)$ denotes the Gamma function.

Definition 2.3 ([26]) A matrix $\mathcal{E}_h^{A.^{\alpha}}:\mathcal{R}\to\mathcal{R}^{n\times n}$ known as a delayed Mittag-Leffler type matrix is defined as

$$\mathcal{E}_h^{At^{\alpha}}=\begin{cases}\Theta, & -\infty<t<-h,\\ I, & -h\le t\le 0,\\ I+A\frac{(t)^{\alpha}}{\Gamma(\alpha+1)}+A^2\frac{(t-h)^{2\alpha}}{\Gamma(2\alpha+1)}+\cdots+A^k\frac{(t-(k-1)h)^{k\alpha}}{\Gamma(k\alpha+1)}, & (k-1)h\le t\le kh, k\in\mathcal{N},\end{cases}\tag{4}$$

where zero and identity matrices are shown by Θ and I, respectively.

Definition 2.4 The system (3) is said to be controllable on $J=[0,t_1]$ if one can reach any state from any allowed initial state $x(t)=\varphi(t)$ and initial control $u(t)=\psi(t)$.

Lemma 2.5 ([26]) *Let $f:J\to\mathcal{R}^n$ be a continuous vector value function. A solution $x\in C([-h,t_1],\mathcal{R}^n)$ of the following system*:

$$\begin{cases}{}^cD_{0^+}^{\alpha}x(t)=Ax(t-h)+f(t), & x(t)\in\mathcal{R}^n, t\in J:=[0,t_1], h>0,\\ x(t)=\varphi(t), & -h\le t\le 0, \varphi\in\mathcal{C}_h^1,\end{cases}\tag{5}$$

can be written in the form of an integral equation by using the method in [26];

$$x(t)=\mathcal{E}_h^{At^{\alpha}}\varphi(-h)+\int_{-h}^0\mathcal{E}_h^{A(t-h-\tau)^{\alpha}}\varphi'(\tau)\,d\tau+\int_0^t\mathcal{E}_h^{A(t-h-\tau)^{\alpha}}f(\tau)\,d\tau.$$

By Lemma 2.8 in [26], *a solution $x\in C([-h,t_1],\mathcal{R}^n)$ of system* (3) *can be composed in the form*

$$\begin{aligned}x(t)={}&\mathcal{E}_h^{At^{\alpha}}\varphi(-h)+\int_{-h}^0\mathcal{E}_h^{A(t-h-\tau)^{\alpha}}\varphi'(\tau)\,d\tau\\&+\int_0^t\mathcal{E}_h^{A(t-h-\tau)^{\alpha}}Bu(\tau)\,d\tau+\int_0^t\mathcal{E}_h^{A(t-h-\tau)^{\alpha}}Cu(\tau-h)\,d\tau.\end{aligned}\tag{6}$$

Lemma 2.6 ([18]) *From Lemma 2.5 for system* (3), *a general solution can be composed as*

$$\begin{aligned}x(t)={}&\mathcal{E}_h^{At^{\alpha}}\varphi(-h)+\int_{-h}^0\mathcal{E}_h^{A(t-h-\tau)^{\alpha}}\varphi'(\tau)\,d\tau+\int_0^{t-h}\mathcal{E}_h^{A(t-h-\tau)^{\alpha}}Bu(\tau)\,d\tau\\&+\int_{t-h}^t\mathcal{E}_h^{A(t-h-\tau)^{\alpha}}Bu(\tau)\,d\tau+\int_0^{t-h}\mathcal{E}_h^{A(t-2h-\tau)^{\alpha}}Cu(\tau)\,d\tau\\&+\int_{-h}^0\mathcal{E}_h^{A(t-2h-\tau)^{\alpha}}C\psi(\tau)\,d\tau.\end{aligned}\tag{7}$$

Definition 2.7 We call the set in [18] $R(\varphi,\psi) = \{\nu \mid \text{there exists } t_1 > 0, u(t) \in C^{l-1}$, such that the solution of the system (3) $x(t,\varphi,\psi)$ satisfies $x(t_1,\varphi,\psi) = \nu\}$ the reachable set of (3) with $x(t) = \varphi(t)$ and $u(t) = \psi(t)$ at $-h \le t \le 0$.

Lemma 2.8 ([18]) *For the beta function*

$$\mathcal{B}(p,q) = \int_0^1 s^{p-1}(1-s)^{q-1}\,ds \quad \bigl(Re(p) > 0, Re(q) > 0\bigr),$$

we have

$$\mathcal{B}(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

Lemma 2.9 ([28]) *Let* $(k-1)h \le t \le kh$, $k \in \mathcal{N}$, *we have*

$$\int_{(k-1)h}^{t} (t-s)^{-\alpha}\bigl(s-(k-1)h\bigr)^{k\alpha-1}\,ds = \bigl(t-(k-1)h\bigr)^{(k-1)\alpha}\mathcal{B}[1-\alpha, k\alpha],$$

where $\mathcal{B}$ *is the beta function*; *see Lemma* 2.8.

Lemma 2.10 *For a delayed Mittag-Leffler type matrix* $\mathcal{E}_h^{A.^{\alpha}} : \mathcal{R} \to \mathcal{R}^{n\times n}$, *one has*

$$^cD_{0^+}^{\alpha}\bigl(\mathcal{E}_h^{At^{\alpha}}\bigr) = A\mathcal{E}_h^{A(t-h)^{\alpha}}, \tag{8}$$

i.e., $\mathcal{E}_h^{At^{\alpha}}$ *is a solution of* $({}^cD_{0^+}^{\alpha}x)(t) = Ax(t-h)$ *that satisfies the initial conditions* $\mathcal{E}_h^{At^{\alpha}} = I$, $-h \le t \le 0$.

Proof For arbitrary $t \in (-\infty,-h]$, $\mathcal{E}_h^{At^{\alpha}} = \mathcal{E}_h^{A(t-h)^{\alpha}} = \Theta$. Obviously, (8) holds. Next for $t \in (-h,0]$, $\mathcal{E}_h^{At^{\alpha}} = I$ and $\mathcal{E}_h^{A(t-h)^{\alpha}} = \Theta$. which shows ${}^cD_{0^+}^{\alpha}I = \Theta = A\Theta$. Thus, (8) holds.

For arbitrary $t \in ((k-1)h, Kh]$, $k \in \mathcal{N}$, we follow mathematical induction to establish our result.

(1) For $k = 1$, $0 \le t \le h$, we have

$$x(t) = \mathcal{E}_h^{At^{\alpha}} = I + \frac{A(t)^{\alpha}}{\Gamma(\alpha+1)}, \qquad x'(t) = \frac{\alpha A(t)^{\alpha-1}}{\Gamma(\alpha+1)}. \tag{9}$$

Next by using the Caputo fractional differentiation expression of $\mathcal{E}_h^{A.^{\alpha}}$ via (9) and Lemma 2.9, we obtain

$$^cD_{0^+}^{\alpha}\bigl(\mathcal{E}_h^{As^{\alpha}}\bigr)(t) = \frac{\alpha A}{\Gamma(\alpha+1)\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}(s)^{\alpha-1}\,ds = A. \tag{10}$$

(2) For $k = 2$, $h \le t \le 2h$, we have

$$\begin{aligned} x(t) &= \mathcal{E}_h^{At^{\alpha}} = I + \frac{A(t)^{\alpha}}{\Gamma(\alpha+1)} + \frac{A^2(t-h)^{2\alpha}}{\Gamma(2\alpha+1)}, \\ x'(t) &= \frac{\alpha A(t)^{\alpha-1}}{\Gamma(\alpha+1)} + \frac{2\alpha A^2(t-h)^{2\alpha-1}}{\Gamma(2\alpha+1)}. \end{aligned} \tag{11}$$

Next by using the Caputo fractional differentiation expression of $\mathcal{E}_h^{A.^\alpha}$ via (11), (10) and Lemma 2.9, we obtain

$$\begin{aligned} {}^cD_{0^+}^\alpha\big(\mathcal{E}_h^{As^\alpha}\big)(t) &= A + \frac{2\alpha A^2}{\Gamma(2\alpha+1)\Gamma(1-\alpha)}\int_h^t (t-s)^{-\alpha}(s-h)^{2\alpha-1}\,ds \\ &= A + \frac{A^2(t-h)^\alpha}{\Gamma(\alpha+1)}. \end{aligned}$$

(3) Let $k = M$, $(M-1)h \le t \le Mh$ and $M \in \mathcal{N}$; the following relation holds:

$$\begin{aligned} {}^cD_{0^+}^\alpha\big(\mathcal{E}_h^{As^\alpha}\big)(t) &= A + \frac{A^2(t-h)^\alpha}{\Gamma(\alpha+1)} + \frac{A^3(t-2h)^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots \\ &\quad + \frac{A^M(t-(M-1)h)^{(M-1)\alpha}}{\Gamma((M-1)\alpha+1)}. \end{aligned}$$

Next let $k = M+1, Mh \le t \le (M+1)h$; by elementary computation, we get

$$\begin{aligned} x'(t) &= \frac{\alpha A(t)^{\alpha-1}}{\Gamma(\alpha+1)} + \frac{2\alpha A^2(t-h)^{2\alpha-1}}{\Gamma(2\alpha+1)} + \cdots \\ &\quad + \frac{(M+1)\alpha A^{(M+1)}(t-Mh)^{(M+1)\alpha-1}}{\Gamma((M+1)\alpha+1)}. \end{aligned} \tag{12}$$

Now taking the Caputo fractional differentiation expression of $\mathcal{E}_h^{A.^\alpha}$ via (12) and Lemma 2.9, we obtain

$$\begin{aligned} &{}^cD_{0^+}^\alpha\big(\mathcal{E}_h^{As^\alpha}\big)(t) \\ &\quad = \frac{\alpha A}{\Gamma(\alpha+1)\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}s^{\alpha-1}\,ds \\ &\qquad + \frac{2\alpha A^2}{\Gamma(2\alpha+1)\Gamma(1-\alpha)}\int_h^t (t-s)^{-\alpha}(s-h)^{2\alpha-1}\,ds + \cdots \\ &\qquad + \frac{(M+1)\alpha A^{(M+1)}}{\Gamma(1-\alpha)\Gamma((M+1)\alpha+1)}\int_{Mh}^t (t-s)^{-\alpha}(s-Mh)^{(M+1)\alpha-1}\,ds \\ &\quad = A + \frac{A^2(t-h)^\alpha}{\Gamma(\alpha+1)} + \frac{A^3(t-2h)^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots + \frac{A^{(M+1)}(t-Mh)^{M\alpha}}{\Gamma(M\alpha+1)}. \end{aligned}$$

This shows that Eq. (8) is satisfied for any $(k-1)h \le t \le kh$ and $k \in \mathcal{N}$. The proof is completed. From Lemma 2.10, we have

$$ {}^cD_{0^+}^\alpha\big(\mathcal{E}_h^{A(t-h-\tau)^\alpha}\big) = A\mathcal{E}_h^{A(t-2h-\tau)^\alpha}. \tag{13}$$

□

3 Main results

In this part for the controllability of system (3) necessary and sufficient conditions are given. Firstly we prove a lemma, then by using this lemma the main results are constructed.

Remark 3.1 Let

$$\langle A|B,C\rangle = \alpha + A\alpha + A^2\alpha + \cdots + A^{n-1}\alpha + \beta + B\beta + B^2\beta + \cdots + B^{n-1}\beta,$$

where α = Image B, β = Image C and n stands for order of A. Then the space $\langle A|B,C\rangle$ is spanned by the columns of the matrix

$$[B,AB,A^2B,\dots,A^{n-1}B,C,AC,A^2C,A^3C,\dots,A^{n-1}C].$$

Lemma 3.2 *For any $z \in \mathcal{R}^n$, define $W(t):\mathcal{R}^n \to \mathcal{R}^n$ by*

$$\begin{aligned} W(t) = {} & \int_0^{t-h} \bigl[\bigl(\mathcal{E}_h^{A(t-h-\tau)^\alpha}B+\mathcal{E}_h^{A(t-2h-\tau)^\alpha}C\bigr)\bigl(\mathcal{E}_h^{A(t-h-\tau)^\alpha}B+\mathcal{E}_h^{A(t-2h-\tau)^\alpha}C\bigr)^T\bigr]z\,d\tau \\ & + \int_{t-h}^{t} \bigl[\bigl(\mathcal{E}_h^{A(t-h-\tau)^\alpha}\bigr)BB^T\bigl(\mathcal{E}_h^{A(t-h-\tau)^\alpha}\bigr)^T\bigr]z\,d\tau. \end{aligned} \tag{14}$$

Then

$$\operatorname{Im} W(t) = \langle A|B,C\rangle. \tag{15}$$

Proof Showing $\operatorname{Im} W(t) = \langle A|B,C\rangle$ is equivalent to

$$\operatorname{Ker} W(t) = \bigcap_{i=0}^{n-1} \operatorname{Ker} B^T\bigl(A^T\bigr)^i \bigcap_{j=0}^{n-1} \operatorname{Ker} C^T\bigl(A^T\bigr)^j. \tag{16}$$

If $x \in \ker W(t)$ and $x \neq 0$ then

$$\begin{aligned} 0 &= x^T W(t) x \\ &= \int_0^{t-h} \bigl\| \bigl(\mathcal{E}_h^{A(t-h-\tau)^\alpha}B+\mathcal{E}_h^{A(t-2h-\tau)^\alpha}C\bigr)^T x\bigr\|^2\,d\tau \\ &\quad + \int_{t-h}^{t} \bigl\|B^T\bigl(\mathcal{E}_h^{A(t-h-\tau)^\alpha}\bigr)^T x\bigr\|^2\,d\tau, \end{aligned}$$

that is

$$\begin{cases} 0 = (\mathcal{E}_h^{A(t-h-\tau)^\alpha}B+\mathcal{E}_h^{A(t-2h-\tau)^\alpha}C)^T x, & 0 \le \tau \le t-h, \\ 0 = B^T(\mathcal{E}_h^{A(t-h-\tau)^\alpha})^T x, & t-h \le \tau < t. \end{cases} \tag{17}$$

For the second equation of (17) by taking its Caputo derivative from Lemma 2.10 we have

$$\begin{aligned} 0 &= B^T\bigl({}^cD_{0^+}^{\alpha}\mathcal{E}_h^{A(t-h-\tau)^\alpha}\bigr)^T x \\ &= B^T\bigl(\mathcal{E}_h^{A(t-2h-\tau)^\alpha}\bigr)^T A^T x. \end{aligned} \tag{18}$$

Let $\tau = t-h$; we have

$$0 = B^T A^T x.$$

For the second equation of (17) by performing repeatedly Caputo's differentiation, we get

$$0 = B^T A^T x, \quad \text{for } k = 0,1,2,3,\dots,n-1. \tag{19}$$

Using the Cayley–Hamiltonian theorem [18]

$$\mathcal{E}_h^{Au^\alpha} = \sum_{k=0}^{n-1} \frac{A^k(u-(k-1)h)^{(k+1)\alpha-1}}{\Gamma(k\alpha+\beta)}, \tag{20}$$

where $u = t - h - \tau$. Then when $0 \le \tau \le t - h$

$$0 = B^T\left(\mathcal{E}_h^{A(t-h-\tau)^\alpha}\right)^T A^T x = \sum_{k=0}^{n-1} \gamma_k(t-h-\tau) B^T\left(A^T\right)^k x = 0.$$

By taking it into the first equation of (17)

$$0 = C^T\left(\mathcal{E}_h^{A(t-2h-\tau)^\alpha}\right)^T x, \quad 0 \le \tau \le t-h.$$

By taking its Caputo derivative and letting $\tau = t - 2h$, we get

$$0 = C^T\left(\mathcal{E}_h^{A(t-3h-\tau)^\alpha}\right)^T A^T x.$$

By performing repeatedly Caputo's differentiation, we get

$$0 = C^T A^T x, \quad \text{for } k = 0, 1, 2, 3, \dots, n-1. \tag{21}$$

Using (19) and (21) we get

$$x \in \bigcap_{i=0}^{n-1} \ker B^T\left(A^T\right)^i \bigcap_{j=0}^{n-1} \ker C^T\left(A^T\right)^j.$$

That is,

$$\ker W(t) \subset \bigcap_{i=0}^{n-1} \ker B^T\left(A^T\right)^i \bigcap_{j=0}^{n-1} \ker C^T\left(A^T\right)^j. \tag{22}$$

Conversely, suppose

$$x \in \bigcap_{i=0}^{n-1} \ker B^T\left(A^T\right)^i \bigcap_{j=0}^{n-1} \operatorname{Ker} C^T\left(A^T\right)^j,$$

then (19) and (21) hold.

For $t - h \le \tau < t$, from (17 and 20),

$$B^T\left(\mathcal{E}_h^{A(t-h-\tau)^\alpha}\right)^T A^T x = \sum_{k=0}^{n-1} \gamma_k(t-h-\tau) B^T\left(A^T\right)^k x = 0,$$

for $0 \leq \tau \leq t-h$,

$$\begin{aligned}\left(\mathcal{E}_h^{A(t-h-\tau)^\alpha}B + \mathcal{E}_h^{A(t-2h-\tau)^\alpha}C\right)^T x &= \sum_{k=0}^{n-1}\gamma_k(t-h-\tau)B^T\left(A^T\right)^k x \\ &\quad + \sum_{k=0}^{n-1}\gamma_k(t-2h-\tau)C^T\left(A^T\right)^k x \\ &= 0.\end{aligned}$$

Therefore, $x \in \ker W(t)$, that is,

$$\operatorname{Ker} W(t) \supset \bigcap_{i=0}^{n-1}\operatorname{Ker} B^T\left(A^T\right)^i \bigcap_{j=0}^{n-1}\operatorname{Ker} C^T\left(A^T\right)^j. \tag{23}$$

From (22) and (23), it is proven that (16) holds, completing the proof of the lemma. □

Theorem 3.3 ([18]) *For system* (3) *the fractional differential control system with state and control delay is controllable iff*

$$\operatorname{rank}\left[B, AB, A^2B, \dots, A^{n-1}B, C, AC, A^2C, A^3C, \dots, A^{n-1}C\right] = n.$$

That is, in Theorem 3.3 the conditions are equivalent to $\langle A|B, C\rangle = \mathcal{R}^n$.

By using Lemmas 2.8, 2.10, 3.2 we will prove Theorem 3.3.

Proof of Theorem 3.3 Firstly we show that $R(0,0) = \langle A|B, C\rangle$.

Actually, let $x \in R(0,0)$, from Lemma 2.6 and Eq. (20), we get

$$\begin{aligned} x &= \int_0^{t_1-h}\left(\mathcal{E}_h^{A(t_1-h-\tau)^\alpha}B + \mathcal{E}_h^{A(t_1-2h-\tau)^\alpha}C\right)u(\tau)\,d\tau \\ &\quad + \int_{t_1-h}^{t_1}\mathcal{E}_h^{A(t_1-h-\tau)^\alpha}Bu(\tau)\,d\tau, \\ x &= \int_0^{t_1}\mathcal{E}_h^{A(t_1-h-\tau)^\alpha}Bu(\tau)\,d\tau + \int_0^{t_1-h}\mathcal{E}_h^{A(t_1-2h-\tau)^\alpha}Cu(\tau)\,d\tau \\ &= \sum_{i=0}^{n-1}\int_0^{t_1}\gamma_i(t_1-h-s)A^iBu(s)\,ds + \sum_{j=0}^{n-1}\int_0^{t_1-h}\gamma_j(t_1-2h-s)A^jCu(s)\,ds, \end{aligned}$$

which implies $x \in \langle A|B, C\rangle$.

Thus,

$$\langle A|B, C\rangle \supset R(0,0). \tag{24}$$

On the other hand, we show $\langle A|B,C\rangle \subset R(0,0)$. Let $\hat{x} \in \langle A|B,C\rangle$, let $x(t)$ be a solution of system (3) at $t > 0$ from Lemma 2.6 we get

$$\begin{aligned} x(t) &= \int_0^{t-h} \big(\mathcal{E}_h^{A(t-h-\tau)^\alpha} B + \mathcal{E}_h^{A(t-2h-\tau)^\alpha} C\big) u(\tau)\, d\tau \\ &\quad + \int_{t-h}^{t} \mathcal{E}_h^{A(t-h-\tau)^\alpha} Bu(\tau)\, d\tau. \end{aligned}$$

For $x \in \langle A|B,C\rangle$ from Lemma 3.2 there exists $z \in \mathcal{R}^n$, s.t.

$$\hat{x} = W(t)z.$$

Let

$$u(s) = \begin{cases} (\mathcal{E}_h^{A(t-h-\tau)^\alpha} B + \mathcal{E}_h^{A(t-2h-\tau)^\alpha} C)^T z, & 0 \leq s \leq t-h, \\ B^T \mathcal{E}_h^{A(t-h-\tau)^\alpha} Z, & t-h \leq s < t, \\ 0, & -h \leq s \leq 0. \end{cases}$$

Then

$$\begin{aligned} &\int_0^{t} \mathcal{E}_h^{A(t-h-s)^\alpha} Bu(s)\, ds + \int_0^{t} \mathcal{E}_h^{A(t-h-s)^\alpha} Cu(s-h)\, ds \\ &\quad = \int_0^{t-h} \big[\big(\mathcal{E}_h^{A(t-h-s)^\alpha} B + \mathcal{E}_h^{A(t-2h-s)^\alpha} C\big) + \big(\mathcal{E}_h^{A(t-h-s)^\alpha} B + \mathcal{E}_h^{A(t-2h-s)^\alpha} C\big)\big]^T z\, ds \\ &\qquad + \int_{t-h}^{t} \big(\mathcal{E}_h^{A(t-h-s)^\alpha}\big) BB^T \big(\mathcal{E}_h^{A(t-h-s)^\alpha}\big) z\, ds \\ &\quad = W(t)z = \hat{x}. \end{aligned}$$

That is

$$R(0,0) \supset \langle A|B,C\rangle. \tag{25}$$

Using (24) and (25) we get

$$R(0,0) = \langle A|B,C\rangle.$$

Immediately we show the necessity of Theorem 3.3. Assuming that, for any $x \in \mathcal{R}^n$, system (3) is controllable, by Definition 2.4, via the initial state $\varphi = 0$ and the initial control $\psi = 0$, there occurs a control $u(s)$ such that

$$= \int_0^{t-h} \big(\mathcal{E}_h^{A(t-h-s)^\alpha} B + \mathcal{E}_h^{A(t-2h-s)^\alpha} C\big) u(s)\, ds + \int_{t-h}^{t} \big(\mathcal{E}_h^{A(t-h-s)^\alpha}\big) Bu(s)\, ds.$$

Using Eq. (20) we get $x \in \langle A|B,C\rangle$. That is, $\mathcal{R}^n \subset \langle A|B,C\rangle$. Thus $\mathcal{R}^n = \langle A|B,C\rangle$, and the conditions of Theorem 3.3 are satisfied. At last, we show the sufficiency. Suppose the conditions of Theorem 3.3 are satisfied, then $\mathcal{R}^n = \langle A|B,C\rangle$. For any $\overline{x} \in \mathcal{R}^n$ and any initial state φ and initial control ψ, let

$$\begin{aligned} k &= \overline{x} - \mathcal{E}_h^{At^\alpha}\varphi(-h) - \int_{-h}^{0}\mathcal{E}_h^{A(t-h-s)^\alpha}\varphi'(s)\,ds \\ &\quad - \int_0^{t-h}\bigl(\mathcal{E}_h^{A(t-h-s)^\alpha}B + \mathcal{E}_h^{A(t-2h-s)^\alpha}C\bigr)\psi(0)\,ds \\ &\quad - \int_{t-h}^{t}\mathcal{E}_h^{A(t-h-s)^\alpha}B\psi(0)\,ds - \int_{-h}^{0}\mathcal{E}_h^{A(t-2h-s)^\alpha}C\psi(s)\,ds. \end{aligned}$$

For $k \in \mathcal{R}^n = \langle A|B,C\rangle$, that is, $k \in R(0,0)$, there exists a control $u^*(s)$ such that

$$\begin{aligned} k &= \int_0^{t-h}\bigl(\mathcal{E}_h^{A(t-h-s)^\alpha}B + \mathcal{E}_h^{A(t-2h-s)^\alpha}C\bigr)u^*(s)\,ds \\ &\quad + \int_{t-h}^{t}\mathcal{E}_h^{A(t-h-s)^\alpha}Bu^*(s)\,ds + \int_{-h}^{0}\mathcal{E}_h^{A(t-2h-s)^\alpha}C\psi(s)\,ds. \end{aligned}$$

Let $u(s) = u^*(s) + \psi(0)$ then we have

$$\begin{aligned} \overline{x} &= \mathcal{E}_h^{At^\alpha}\varphi(-h) + \int_{-h}^{0}\mathcal{E}_h^{A(t-h-s)^\alpha}\varphi'(s)\,ds \\ &\quad + \int_0^{t-h}\bigl(\mathcal{E}_h^{A(t-h-s)^\alpha}B + \mathcal{E}_h^{A(t-2h-s)^\alpha}C\bigr)u(s)\,ds \\ &\quad + \int_{t-h}^{t}\mathcal{E}_h^{A(t-h-s)^\alpha}Bu(s)\,ds + \int_{-h}^{0}\mathcal{E}_h^{A(t-2h-s)^\alpha}C\psi(s)\,ds. \end{aligned}$$

So the fractional control system (3) with state and control delay is controllable. Sufficiency is proved. This completes the result of Theorem 3.3. □

4 Example

Now, we will apply the conditions which we obtained in the previous section for a fractional differential system with state and control delay;

$${}^cD_{0^+}^{\alpha}x(t) = Ax(t-h) + Bu(t) + Cu(t-h),$$

$\alpha = 0.5$, $h = 1$, where

$$A = \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}, \qquad B = \begin{pmatrix} 5 \\ 0 \end{pmatrix}, \qquad C = \begin{pmatrix} 0 \\ 3 \end{pmatrix},$$

$${}^cD_{0^+}^{0.5}x(t) = \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}x(t-1) + \begin{pmatrix} 5 \\ 0 \end{pmatrix}u(t) + \begin{pmatrix} 0 \\ 3 \end{pmatrix}u(t-1),$$

where $x \in \mathcal{R}^n$ by simple calculations shows that

$$(BABCAC) = \begin{pmatrix} 2 & 15 & 0 & 0 \\ 0 & 0 & 3 & 12 \end{pmatrix}$$

and $\operatorname{rank}(BABCAC) = 2$.

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References
1. Miller, K.S., Ross, B.: An Introduction to the Fractional Calculus and Fractional Differential Equations (1993)
2. Diethelm, K.: The Analysis of Fractional Differential Equations: An Application-Oriented Exposition Using Differential Operators of Caputo Type. Springer, Berlin (2010)
3. Hale, J.K., Lunel, S.M.V.: Introduction to functional differential equations. In: Introduction to Functional Differential Equations, pp. 1–10. Springer, Berlin (1993)
4. Jiang, W.: The Degeneration Differential Systems with Delay. Anhui University Press, Hefei (1998)
5. Podlubny, I.: Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications. Academic Press, San Diego (1998)
6. Kolmanovskii, V., Myshkis, A.: Applied Theory of Functional Differential Equations. Springer, Berlin (2012)
7. Dai, L.: Singular Control Systems. Springer, Berlin (1989)
8. Wei, J., Wenzhong, S.: Controllability of singular systems with control delay. Automatica **37**, 1873–1877 (2001)
9. Diblík, J., Fečkan, M., Pospíšil, M.: On the new control functions for linear discrete delay systems. SIAM J. Control Optim. **52**, 1745–1760 (2014)
10. Dauer, J.P., Gahl, R.D.: Controllability of nonlinear delay systems. J. Optim. Theory Appl. **21**, 59–70 (1977)
11. Nirmala, R.J., Balachandran, K., Rodríguez-Germa, L., Trujillo, J.J.: Controllability of nonlinear fractional delay dynamical systems. Rep. Math. Phys. **77**, 87–104 (2016)
12. Zhou, X.F., Wei, J., Hu, L.G.: Controllability of a fractional linear time-invariant neutral dynamical system. Appl. Math. Lett. **26**, 418–424 (2013)
13. Diblík, J., Khusainov, D.Y., Ružicková, M.: Controllability of linear discrete systems with constant coefficients and pure delay. SIAM J. Control Optim. **47**, 1140–1149 (2008)
14. Liang, C., Wang, J., O'Regan, D.: Controllability of nonlinear delay oscillating systems. Electron. J. Qual. Theory Differ. Equ. **2017**, Article ID 47 (2017)
15. Pospíšil, M.: Relative controllability of neutral differential equations with a delay. SIAM J. Control Optim. **55**, 835–855 (2017)
16. Khusainov, D.Y., Shuklin, G.V.: Relative controllability in systems with pure delay. Int. Appl. Mech. **41**, 210–221 (2005)
17. Wei, J.: On the interval controllability of fractional systems with control delay. J. Math. Res. **9**, 87–96 (2017)
18. Wei, J.: The controllability of fractional control systems with control delay. Comput. Math. Appl. **64**, 3153–3159 (2012)
19. He, B.B., Zhou, H.C., Kou, C.H.: The controllability of fractional damped dynamical systems with control delay. Commun. Nonlinear Sci. Numer. Simul. **32**, 190–198 (2016)
20. Khusainov, D.Y., Diblík, J., Růžičková, M., Lukáčová, J.: Representation of a solution of the Cauchy problem for an oscillating system with pure delay. Nonlinear Oscil. **11**, 276–285 (2008)
21. Shukla, A., Sukavanam, N., Pandey, D.N.: Complete controllability of semilinear stochastic systems with delay in both state and control. Math. Rep. **18**, 247–259 (2016)
22. Shukla, A., Sukavanam, N., Pandey, D.N.: Controllability of semilinear stochastic control system with finite delay. IMA J. Math. Control Inf. **35**, 427–449 (2016)
23. Shukla, A., Sukavanam, N., Pandey, D.N.: Approximate controllability of semilinear system with state delay using sequence method. J. Franklin Inst. **352**, 5380–5392 (2015)
24. Shukla, A., Sukavanam, N., Pandey, D.N.: Approximate controllability of fractional semilinear stochastic system of order $\alpha \in (1,2]$. J. Dyn. Control Syst. **23**, 679–691 (2017)
25. Diblík, J., Khusainov, D.Ya., Lukáčová, J.: Control of oscillating systems with a single delay. Adv. Differ. Equ. **2010**, Article ID 108218 (2010)
26. Li, M., Wang, J.: Relative controllability in fractional differential equations with pure delay. Math. Methods Appl. Sci. **41**(18), 8906–8914 (2018)
27. Li, M., Wang, J.: Representation of solution of a Riemann–Liouville fractional differential equation with pure delay. Appl. Math. Lett. **85**, 118–124 (2018)
28. Li, M., Wang, J.: Finite time stability of fractional delay differential equations. Appl. Math. Lett. **64**, 170–176 (2017)
29. Kilbas, A.A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. Elsevier, Amsterdam (2006)