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Existence of solutions for a system of singular sum fractional q-differential equations via quantum calculus

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Abstract

In this study, we discuss the existence of positive solutions for the system of m-singular sum fractional q-differential equations

$$D_q^{\alpha_i} x_i + g_i(t, x_1, \dots, x_m, D_q^{\gamma_1} x_1, \dots, D_q^{\gamma_m} x_m) + h_i(t, x_1, \dots, x_m, D_q^{\gamma_1} x_1, \dots, D_q^{\gamma_m} x_m) = 0$$

with boundary conditions $x_i(0) = x_i'(1) = 0$ and $x_i^{(k)}(t) = 0$ whenever $t = 0$, here $2 \le k \le n-1$, where $n = [\alpha_i] + 1$, $\alpha_i \ge 2$, $\gamma_i \in (0,1)$, D_q^{α} is the Caputo fractional q-derivative of order α, here $q \in (0,1)$, function g_i is of Carathéodory type, h_i satisfy the Lipschitz condition and $g_i(t, x_1, \dots, x_{2m})$ is singular at $t = 0$, for $1 \le i \le m$. By means of Krasnoselskii's fixed point theorem, the Arzelà-Ascoli theorem, Lebesgue dominated theorem and some norms, the existence of positive solutions is obtained. Also, we give an example to illustrate the primary effects.

MSC: Primary 34A08; 39A12; secondary 34B16

Keywords: Existence of solutions; Caputo q-derivative; Singularity; Fractional q-differential equations

1 Introduction

Fractional calculus and q-calculus belong to the significant branches in mathematical analysis. In 1910, Jackson introduced the subject of q-difference equations [1]. Later, many researchers studied q-difference equations [2–12]. On the other hand, there appeared recently much work on q-differential equations by using different views and fractional derivatives; young researchers could use the main idea in their work (see, for example, [13–30]).

In 2010, the singular Dirichlet problem $D^{\alpha} x(t) + g(t, x(t), D^{\gamma} x(t)) = 0$ under conditions $x(0) = x(1) = 0$ was investigated by Agarwal *et al.*, where α, γ belong to $(1,2)$, $(0, \alpha - 1)$, respectively, the function g is of Carathéodory type on $[0,1] \times (0,\infty) \times \mathbb{R}$ and D^{α} is the Riemann–Liouville fractional derivative [23]. In 2012, the fractional differential equation ${}^cD^{\alpha} y(t) + w(t, y(t)) = 0$, under boundary conditions $y(0) = y''(0) = 0$ and $y(1) = \lambda \int_0^1 y(s)\,ds$, was investigated, where $t, \alpha, \lambda \in (0,1), (2,3), (0,2)$, respectively, and the function $w : J \times [0,\infty) \to [0,\infty)$ is continuous [24]. Also, in the same year, Ahmad *et al.*, discussed the existence and uniqueness of solutions for the fractional q-difference equations ${}^cD_q^\alpha u(t) = T(t,u(t))$, $\alpha_1 u(0) - \beta_1 D_q u(0) = \gamma_1 u(\eta_1)$, and $\alpha_2 u(1) - \beta_2 D_q u(1) = \gamma_2 u(\eta_2)$, for $t \in I$, where $\alpha \in (1,2]$, $\alpha_i, \beta_i, \gamma_i, \eta_i \in \mathbb{R}$, for $i = 1,2$ and $T \in C([0,1] \times \mathbb{R}, \mathbb{R})$ [6]. In 2013, Zhao *et al.* reviewed the q-integral problem $(D_q^\alpha u)(t) + f(t,u(t)) = 0$, with conditions $u(1)$, $u(0)$ being equal to $\mu I_q^\beta u(\eta)$, 0, respectively, for almost all $t \in (0,1)$, where $q \in (0,1)$, and α, β, η belong to $(1,2]$, $(0,2]$, $(0,1)$, respectively, μ is a positive real number, D_q^α is the q-derivative of Riemann–Liouville and we have the real-valued continuous map u defined on $I \times [0,\infty)$ [10].

In 2014, the singular fractional problem ${}^cD_{0^+}^\alpha x(t) + f(t, x(t), {}^cD_{0^+}^\sigma x(t)) = 0$ with boundary conditions $x(0) = x'(0) = 0$ and $x'(1) = {}^cD_{0^+}^\sigma x(1)$ investigated, where t, α, σ belong to $(0,1)$, $(2,3)$, $(0,1)$, respectively, $f : (0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous with $f(t,x,y)$ may be singular at $t = 0$ and ${}^cD_{0^+}^\alpha$ is the Caputo derivative [31]. In 2017, Aydogan *et al.* and Shabibi *et al.* studied sum-type singular fractional integro-differential equation with k-point boundary conditions together some properties and sum fractional differential system with some conditions, respectively [21, 22]. Also, in the same year, Zhou *et al.*, provided existence criteria for the solutions of p-Laplacian fractional Langevin differential equations with anti-periodic boundary conditions:

$$\begin{cases} D_{0^+}^\beta \phi_p[(D_{0^+}^\alpha + \lambda)x(t)] = f(t, x(t), D_{0^+}^\alpha x(t)), \\ x(0) = -x(1), \qquad D_{0^+}^\alpha x(0) = -D_{0^+}^\alpha x(1), \end{cases}$$

and

$$\begin{cases} {}_qD_{0^+}^\beta \phi_p[(D_{0^+}^\alpha + \lambda)x(t)] = g(t, x(t), {}_qD_{0^+}^\alpha x(t)), \\ x(0) = -x(1), \qquad {}_qD_{0^+}^\alpha x(0) = -{}_qD_{0^+}^\alpha x(1), \end{cases}$$

for $t \in [0,1]$, where $0 < \alpha, \beta \leq 1$, λ is larger than or equal to zero, $1 < \alpha + \beta < 2$, $q \in (0,1)$, and $\phi(p)(s) = |s|^{p-2}s$, with $p \in (1,2]$ [15]. In 2019, Samei *et al.*, investigated existence of solutions for equations and inclusions of multi-term fractional q-integro-differential equations with non-separated and initial boundary conditions [9].

In this article, motivated by main idea of this work and by these achievements, we are working to address the positive solutions for system of singular sum fractional q-differential equations

$$\begin{cases} D_q^{\alpha_1} x_1 + g_1(t, x_1, \dots, x_m, D_q^{\gamma_1} x_1, \dots, D_q^{\gamma_m} x_m) \\ \quad + h_1(t, x_1, \dots, x_m, D_q^{\gamma_1} x_1, \dots, D_q^{\gamma_m} x_m) = 0, \\ D_q^{\alpha_2} x_2 + g_2(t, x_1, \dots, x_m, D_q^{\gamma_1} x_1, \dots, D_q^{\gamma_m} x_m) \\ \quad + h_2(t, x_1, \dots, x_m, D_q^{\gamma_1} x_1, \dots, D_q^{\gamma_m} x_m) = 0, \\ \vdots \\ D_q^{\alpha_m} x_m + g_m(t, x_1, \dots, x_m, D_q^{\gamma_1} x_1, \dots, D_q^{\gamma_m} x_m) \\ \quad + h_m(t, x_1, \dots, x_m, D_q^{\gamma_1} x_1, \dots, D_q^{\gamma_m} x_m) = 0, \end{cases} \tag{1}$$

under some conditions $x_i(0) = 0$, $x_i'(1) = 0$ and $\frac{d^k x_i(t)}{dt^k}|_{t=0} = 0$, for $i \in N_m$ and $k \in N_{n-1} \setminus \{1\}$, where $\alpha_i \geq 2$, $[\alpha_i] = n-1$, $\gamma_i \in J = (0,1)$, D_q^α is the Caputo fractional q-derivative of order α,

Algorithm 1 The proposed method for calculated $(a-b)_q^{(\alpha)}$

Input: a, b, α, n, q

$s \leftarrow 1$
if $n = 0$ **then**
  $p \leftarrow 1$
else
  for $k = 0$ to n **do**
    $s \leftarrow s * (a - b * a^k)/(a - b * q^{\alpha+k})$
  end for
  $p \leftarrow a^\alpha * s$
end if

Output: $(a-b)^{(\alpha)}$

the function g_i is of Carathéodory type, h_i satisfy Lipschitz condition and $g_i(t, x_1, \dots, x_{2m})$ is singular at $t = 0$ for $i \in N_m$, where $N_\kappa = \{1, 2, \dots, \kappa\}$.

The rest of the paper is arranged as follows. In Sect. 2, we recall some preliminary concepts and fundamental results of q-calculus. Section 3 is devoted to the main results, while examples illustrating the obtained results and algorithm for the problems are presented in Sect. 4.

2 Preliminaries

First, we point out some of the materials on the fractional q-calculus and fundamental results of it which are needed in the next sections (for more information, refer to [1, 32, 33]). Then, some well-known theorems as regards the fixed point theorem and definitions are presented.

Assume that $q \in (0,1)$ and $a \in \mathbb{R}$. Define $[a]_q = \frac{1-q^a}{1-q}$ [1]. The power function $(x-y)_q^n$ with $n \in \mathbb{N}_0$ is $(x-y)_q^{(n)} = \prod_{k=0}^{n-1}(x - yq^k)$ and $(x-y)_q^{(0)} = 1$ where x, y are real numbers and $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$ [32]. Also, for real number α and $a \neq 0$, we have $(x-y)_q^{(\alpha)} = x^\alpha \prod_{k=0}^{\infty}(x - yq^k)/(x - yq^{\alpha+k})$. If $y = 0$, then it is clear that $x^{(\alpha)} = x^\alpha$ (Algorithm 1). The q-Gamma function is given by $\Gamma_q(z) = (1-q)^{(z-1)}/(1-q)^{z-1}$, where $z \in \mathbb{R} \setminus \{0, -1, -2, \dots\}$ [1]. Note that $\Gamma_q(z+1) = [z]_q \Gamma_q(z)$. The value of the q-Gamma function, $\Gamma_q(z)$, holds for input values q and z with counting the number of sentences n in summation by simplifying analysis. For this design, we prepare a pseudo-code description of the technique for estimating the q-Gamma function of order n which we show in Algorithm 2. The q-derivative of the function f is defined by $(D_q f)(x) = \frac{f(x) - f(qx)}{(1-q)x}$ and $(D_q f)(0) = \lim_{x \to 0}(D_q f)(x)$, which is shown in Algorithm 3 [2]. Also, the higher order q-derivative of a function f is defined by $(D_q^n f)(x) = D_q(D_q^{n-1} f)(x)$ for all $n \geq 1$, where $(D_q^0 f)(x) = f(x)$ [2]. The operator

$$I_q f(x) = \int_0^x f(s)\, d_q s = x(1-q) \sum_{k=0}^{\infty} q^k f\left(xq^k\right),$$

for $0 \leq x \leq b$, is called the q-integral of a function f, whenever the series is absolutely converges [2]. If a in $[0, b]$, then

$$\int_a^b f(u)\, d_q u = I_q f(b) - I_q f(a) = (1-q) \sum_{k=0}^{\infty} q^k \left[bf\left(bq^k\right) - af\left(aq^k\right)\right],$$

Algorithm 2 The proposed method for calculated $\Gamma_q(x)$

Input: $n, q \in (0,1), x \in \mathbb{R} \setminus \{0,-1,2,\ldots\}$

```
p ← 1
for k = 0 to n do
    p ← p(1 − q^{k+1})(1 − q^{x+k})
end for
Γ_q(x) ← p/(1 − q)^{x−1}
```

Output: $\Gamma_q(x)$

Algorithm 3 The proposed method for calculated $(D_q f)(x)$

Input: $q \in (0,1), f(x), x$

```
syms z
if x = 0 then
    g ← lim((f(z) − f(q ∗ z))/((1 − q)z), z, 0)
else
    g ← (f(x) − f(q ∗ x))/((1 − q)x)
end if
```

Output: $(D_q f)(x)$

whenever the series exists. The operator I_q^n is given by $(I_q^0 f)(x) = f(x)$ and

$$\bigl(I_q^n f\bigr)(x) = \bigl(I_q\bigl(I_q^{n-1} f\bigr)\bigr)(x),$$

for $n \geq 1$ [2]. It has been proved that $(D_q(I_q f))(x) = f(x)$ and $(I_q(D_q f))(x) = f(x) - f(0)$ whenever f is continuous at $x = 0$ [2]. The fractional Riemann–Liouville type q-integral of the function f on $[0,1]$, for $\alpha \geq 0$, is given by $(I_q^0 f)(t) = f(t)$ and

$$\bigl(I_q^\alpha f\bigr)(t) = \frac{1}{\Gamma_q(\alpha)} \int_0^t (t - qs)^{(\alpha-1)} f(s)\, d_q s,$$

for $t \in J$ and $\alpha > 0$ [4, 7]. Also, the fractional Caputo type q-derivative of the function f is given by

$$\begin{aligned} \bigl({}^c D_q^\alpha f\bigr)(t) &= \bigl(I_q^{[\alpha]-\alpha}\bigl(D_q^{[\alpha]} f\bigr)\bigr)(t) \\ &= \frac{1}{\Gamma_q([\alpha]-\alpha)} \int_0^t (t - qs)^{([\alpha]-\alpha-1)} \bigl(D_q^{[\alpha]} f\bigr)(s)\, d_q s, \end{aligned} \tag{2}$$

for $t \in [0,1]$ and $\alpha > 0$ [4, 7]. It has been proved that $(I_q^\beta(I_q^\alpha f))(x) = (I_q^{\alpha+\beta} f)(x)$ and $(D_q^\alpha(I_q^\alpha f))(x) = f(x)$, where $\alpha, \beta \geq 0$ [7]. By using Algorithm 2, we can calculate $(I_q^\alpha f)(x)$ which is shown in Algorithm 4. One can find more details of fractional differential and q-differential equations in [34–37].

Now, we present some necessary notions. Throughout this article, we denote $L^1(0,1)$, $L^1[0,1]$, $C_{\mathbb{R}}(0,1)$, $C_{\mathbb{R}}[0,1]$, $C_{\mathbb{R}}^1[0,1]$ by $\mathcal{L}, \overline{\mathcal{L}}, \mathcal{A}, \overline{\mathcal{A}}, \overline{\mathcal{B}}$, respectively. We say that a map $\theta : \overline{J} \times \mathcal{S} \to \mathbb{R}^n$ is of Carathéodory type whenever the function $t \mapsto \theta(t, r_1, \ldots, r_n)$ is measurable for all $(r_1, \ldots, r_n) \in \mathcal{S}$ and $(r_1, \ldots, r_n) \mapsto \theta(t, r_1, \ldots, r_n)$ is continuous for $t \in \overline{J}$ and for each

Algorithm 4 The proposed method for calculated $(I_q^{\alpha} f)(x)$

Input: $q \in (0,1)$, α, n, $f(x)$, x

```
s ← 0
for i = 0 to n do
    pf ← (1 − q^(i+1))^(α−1)
    s ← s + pf ∗ q^i ∗ f(x ∗ q^i)
end for
g ← (x^α ∗ (1 − q) ∗ s)/(Γ_q(x))
```

Output: $(I_q^{\alpha} f)(x)$

Algorithm 5 The proposed method for calculated $\int_a^b f(r)\,d_q r$

Input: $q \in (0,1)$, α, n, $f(x)$, a, b

```
s ← 0
for i = 0 : n do
    s ← s + q^i ∗ (b ∗ f(b ∗ q^i) − a ∗ f(a ∗ q^i))
end for
g ← (1 − q) ∗ s
```

Output: $\int_a^b f(r)\,d_q r$

compact $C \subseteq \mathcal{S}$ there exists $\psi_C \in \overline{\mathcal{L}}$ such that $|\theta(t, r_1, \ldots, r_n)| \leq \psi_C(t)$ for each $t \in \overline{J}$ and $(r_1, \ldots, e_n) \in C$, here $\mathcal{S} = (0, \infty)^{2m}$. At present, we consider four norms which will be used in the sequel: $\|x\| := \sup\{|x(t)| : t \in \overline{J}\}$, $\|x\|_1 := \int_0^1 |x(t)|\,dt$, $\|(x_1, x_2, \ldots, x_n)\|_* := \max\{\|x_i\| : i \in N_n\}$ and

$$\bigl\|(x_1, x_2, \ldots, x_n)\bigr\|_{**} := \max\bigl\{\|x_1\|, \|x_2\|, \ldots, \|x_n\|, \bigl\|x_1'\bigr\|, \bigl\|x_2'\bigr\|, \ldots, \bigl\|x_n'\bigr\|\bigr\}.$$

The following lemmas can be found in [34, 36–38].

Lemma 1 *If $x \in \overline{\mathcal{A}} \cap \overline{\mathcal{L}}$ with $D_q^{\alpha} x \in \mathcal{A} \cap \mathcal{L}$, then $I_q^{\alpha} D_q^{\alpha} x(t) = x(t) + \sum_{i=1}^{n} c_i t^{\alpha - i}$, where n is the smallest integer greater than or equal to α and c_i is some real number.*

Lemma 2 *Assume that a nonempty subset C of a Banach space $\mathcal{X}$ be a closed, convex. Then, there exists $c \in C$ such that $c = \mathcal{O}_1(c) + \mathcal{O}_2(c)$ whenever the operators $\mathcal{O}_1$ and $\mathcal{O}_2$ are compact and continuous, or a contraction, respectively.*

Lemma 3 *The unique solution for $D_q^{\alpha} x(t) + v(t) = 0$ under conditions $x'(1) = x(0) = x''(0) = \cdots = x^{n-1}(0) = 0$, here $v \in \overline{\mathcal{L}}$, $\alpha \in [2, \infty)$ and $n = [\alpha] + 1$, is $x(t) = \int_0^1 G_{\alpha}(t, qs) v(s)\,d_q s$, where*

$$G_{\alpha}(t, qs) = \frac{t(1 - qs)^{(\alpha - 2)}}{\Gamma_q(\alpha - 1)},$$

whenever $t \leq s$ and

$$G_{\alpha}(t, qs) = \frac{t(1 - qs)^{(\alpha - 2)}}{\Gamma_q(\alpha - 1)} - \frac{(t - qs)^{(\alpha - 1)}}{\Gamma_q(\alpha)},$$

whenever $s \leq t$, for all $t, s \in \overline{J}$.

Proof At first, by applying Lemma 1 and the boundary conditions, we conclude that $x(t) = -I_q^{\alpha} v(t) + c_1 t$ and so $x'(1) = -I_q^{\alpha-1} v(1) + c_1$. Since $x'(1) = 0$, $c_1 = I_q^{\alpha-1} v(1)$. Thus, $x(t) = -I_q^{\alpha} v(t) + t I_q^{\alpha-1} v(1)$. Hence, we obtain

$$x(t) = \int_0^1 G_\alpha(t, qs) v(s)\, ds,$$

where

$$G_\alpha(t, qs) \begin{cases} \frac{t(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}, & t \le s, \\ \frac{t(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} - \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}, & s \le t, \end{cases} \tag{3}$$

for each $t, s \in \overline{J}$. □

Remark 1 Consider a q-Green function as in (3). It can be seen that $G_\alpha(t, qs) > 0$ if $t \le s$ for each $t, s \in J$. Also, $G_\alpha(t, qs) > 0$ whenever $s < t$ if and only if $(t - qs)^{(\alpha-1)} < t(\alpha - 1)(t - qs)^{(\alpha-2)}$ for all $t, s \in J$. In addition

$$\frac{t^\alpha(\alpha - 1)}{\Gamma_q(\alpha + 1)} \le \frac{t}{\Gamma_q(\alpha)} - \frac{t^\alpha}{\Gamma_q(\alpha + 1)} \le \int_0^1 G_\alpha(t, qs)\, d_q s \tag{4}$$

and $\frac{\partial}{\partial t} G_\alpha(t, qs) > 0$ for $t, s \in J$. Moreover, G_α, $\frac{\partial}{\partial t} G_\alpha \in C_{\mathbb{R}}(\overline{J} \times \overline{J})$, $\frac{\partial}{\partial t} G_\alpha(t, qs) \le \frac{1}{\Gamma_q(\alpha-1)}$ and $\int_0^1 \frac{\partial}{\partial t} G_\alpha(t, s) \ge \frac{1-t^{\alpha-1}}{\Gamma(\alpha)}$, for almost all $t, s \in \overline{J}$.

Remark 2 Let $u \in C^1_{\mathbb{R}}(\overline{J})$ and $\gamma \in J$. Since

$$D_q^\gamma u(t) = \frac{1}{\Gamma_q(2 - \gamma)} \int_0^t (t - qs)^{-\gamma} u'(s)\, d_q s,$$

for all $t \in \overline{J}$,

$$\left|D_q^\gamma u\right| \le \frac{\|u'\|}{\Gamma_q(1 - \gamma)} \int_0^t (t - qs)^{-\gamma}\, d_q s = \frac{\|u'\|}{\Gamma_q(2 - \gamma)} t^{1-\gamma} \tag{5}$$

and so $\Gamma_q(2 - \gamma)|D_q^\gamma u| \le \|u'\|$ and $D_q^\gamma u \in C_{\mathbb{R}}(\overline{J})$.

3 Main results

Now, we consider the following assumptions for the problem (1):

(A1) The maps g_i are Carathéodory functions on $\overline{J} \times \mathcal{S}$ and there exist positive constants ℓ_i such that $g_i(t, u_1, \dots, u_{2m}) \ge \ell_i$ for each $t \in \overline{J}$ and all $(u_1, \dots, u_{2m}) \in \mathcal{S}$ where $i \in N_m$.

(A2) The maps h_i are nonnegative and

$$\left|h_i(t, u_1, \dots, u_{2m}) - h_i(t, v_1, \dots, v_{2m})\right| \le \sum_{k=1}^{2m} {}_iM_k |u_k - v_k|, \tag{6}$$

for each t belonging to $\overline{J}$ and for almost all $(u_1, \dots, u_{2m}), (v_1, \dots, v_{2m}) \in \mathcal{S}$, where ${}_iM_j$ in $[0, \infty)$, for $i \in N_m$ and $j \in N_{2m}$, are constants such that

$$\sum_{k=1}^{m} \left({}_iM_k + \frac{{}_iM_{m+k}}{\Gamma_q(2-\gamma_i)} \right) < \Gamma_q(\alpha_i - 1). \tag{7}$$

(A3) There exist some maps $\mu_1, \dots, \mu_m \in \overline{\mathcal{L}}$, some nonincreasing maps $r_1, \dots, r_m \in C_{\mathbb{R}}(\mathcal{S})$ with

$$\int_0^1 r_i \left(L_1 t^{\alpha_1}, \dots, L_m t^{\alpha_m}, \frac{L_1(1-\gamma_1)}{2} t^{1-\gamma_1}, \dots, \frac{L_m(1-\gamma_m)}{2} t^{1-\gamma_m} \right) dt < \infty$$

and some functions $w_1, \dots, w_m \in C_{\mathbb{R}}(\mathcal{S})$ such that w_i is nondecreasing in all components, $\lim_{x\to\infty} \frac{w_i(x,\dots,x)}{x} = 0$ and

$$\begin{aligned} &g_i(t, u_1, \dots, u_{2m}) + h_i(t, u_1, \dots, u_{2m}) \\ &\quad \le r_i(u_1, \dots, u_{2m}) + \mu_i(t) w_i(u_1, \dots, u_{2m}), \end{aligned} \tag{8}$$

for almost all $t \in \overline{J}$ and all $(u_1, \dots, u_{2m}) \in \mathcal{S}$ where $L_i \Gamma_q(\alpha_i + 1) = \ell_i(\alpha_i - 1)$ for all $i \in N_m$.

Now, we prove the following lemma.

Lemma 4 *Suppose that $\mathcal{P}$ is the set of all $(u_1, \dots, u_m)$ belonging to $\overline{\mathcal{B}}^m$ such that $u_i(t)$ and $u_i'(t)$ are larger than or equal to zero for $t \in \overline{J}$ and $i \in \{0\} \cup N_m$. Also, for each natural number n and $i \in N_m$, we define the maps*

$$\begin{aligned} H_i(u_1, \dots, u_m)(t) = \int_0^1 & G_{\alpha_i}(t, qs) \\ & \times h_i\big(s, u_1(s), \dots, u_m(s), D_q^{\gamma_1} u_1(s), \dots, D_q^{\gamma_m} u_m(s)\big)\, d_q s \end{aligned} \tag{9}$$

and

$$H(u_1, \dots, u_m)(t) = \begin{pmatrix} H_1(u_1, \dots, u_m)(t) \\ H_2(u_1, \dots, u_m)(t) \\ \vdots \\ H_m(u_1, \dots, u_m)(t) \end{pmatrix}, \tag{10}$$

for all $(u_1, \dots, u_m) \in \mathcal{P}$. Then the self-map H define on $\mathcal{P}$ is a contraction.

Proof First, by simple review, we can check that $H_i(u_1, \dots, h_m)(t) \ge 0$ and

$$\begin{aligned} H_i'(u_1, \dots, u_m)(t) &= \int_0^1 \frac{\partial}{\partial t} G_{\alpha_i}(t, qs) \\ &\quad \times h_i\big(s, u_1(s), \dots, u_m(s), D_q^{\gamma_1} u_1(s), \dots, D_q^{\gamma_m} u_m(s)\big)\, d_q s \\ &\ge 0 \end{aligned}$$

for all $t \in \bar{J}$, $(u_1, \dots, u_m) \in \mathcal{P}$ and $i \in \{0\} \cup N_m$. On the other hand,

$$\begin{aligned}
&\left\| H_i(u_1, \dots, u_m) - H_i(v_1, \dots, v_m) \right\| \\
&= \sup_{t \in \bar{J}} \left| \int_0^1 G_{\alpha_i}(t, qs) \left[h_i\left(s, u_1(s), \dots, u_m(s), D_q^{\gamma_1} u_1(s), \dots, D_q^{\gamma_m} u_m(s)\right) \right.\right. \\
&\quad \left.\left. - h_i\left(s, v_1(s), \dots, v_m(s), D_q^{\gamma_1} v_1(s), \dots, D_q^{\gamma_m} v_m(s)\right)\right] d_q s \right| \\
&\leq \left| \int_0^1 G_{\alpha_i}(t, qs) \, d_q s \right| \\
&\quad \times \sum_{k=1}^{m} \left({}_iM_k \|u_k - v_k\| + {}_iM_{m+k} \left\| D_q^{\gamma_k} u_k - D_q^{\gamma_k} v_k \right\| \right).
\end{aligned}$$

By using Remark 2, we can conclude that

$$\begin{aligned}
&\left\| H_i(u_1, \dots, u_m) - H_i(v_1, \dots, v_m) \right\| \\
&\leq \frac{1}{\Gamma_q(\alpha_i)} \sum_{k=1}^{m} \left({}_iM_k \|u_k - v_k\| + \frac{{}_iM_{m+k}}{\Gamma_q(2 - \gamma_k)} \left\| u_k' - v_k' \right\| \right) \\
&\leq \frac{\|(u_1, \dots, u_m) - (v_1, \dots, v_m)\|_{**}}{\Gamma_q(\alpha_i)} \sum_{k=1}^{m} \left({}_iM_k + \frac{{}_iM_{m+k}}{\Gamma_q(2 - \gamma_i)} \right) \\
&\leq \frac{\|(u_1, \dots, u_m) - (v_1, \dots, v_m)\|_{**}}{\Gamma_q(\alpha_i - 1)} \sum_{k=1}^{m} \left({}_iM_k + \frac{{}_iM_{m+k}}{\Gamma_q(2 - \gamma_i)} \right)
\end{aligned}$$

for $i \in N_m \cup \{0\}$. Hence,

$$\begin{aligned}
&\left\| H(u_1, \dots, u_m) - H(v_1, \dots, v_m) \right\|_* \\
&= \max_{i \in N_m} \left\| H_i(u_1, \dots, u_m) - H_i(v_1, \dots, v_m) \right\| \\
&\leq \max_{i \in N_m} \left\{ \frac{1}{\Gamma_q(\alpha_i - 1)} \sum_{k=1}^{m} \left({}_iM_k + \frac{{}_iM_{m+k}}{\Gamma_q(2 - \gamma_i)} \right) \right\} \\
&\quad \times \left\| (u_1, \dots, u_m) - (v_1, \dots, v_m) \right\|_{**}.
\end{aligned}$$

In a similar way, we obtain

$$\begin{aligned}
&\left\| H'(u_1, \dots, u_m) - H'(v_1, \dots, v_m) \right\|_* \\
&\leq \max_{i \in N_m} \left\{ \frac{1}{\Gamma_q(\alpha_i - 1)} \sum_{k=1}^{m} \left({}_iM_k + \frac{{}_iM_{m+k}}{\Gamma_q(2 - \gamma_i)} \right) \right\} \\
&\quad \times \left\| (u_1, \dots, u_m) - (v_1, \dots, v_m) \right\|_{**}.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\left\| H(x_1,\dots,u_m) - H(v_1,\dots,v_m) \right\|_{**} \\
&\quad \le \max_{i\in N_m} \left\{ \frac{1}{\Gamma_q(\alpha_i - 1)} \sum_{k=1}^{m} \left({}_iM_k + \frac{{}_iL_{m+k}}{\Gamma_q(2-\gamma_i)} \right) \right\} \\
&\qquad \times \left\| (u_1,\dots,u_m) - (v_1,\dots,v_m) \right\|_{**}.
\end{aligned}$$

By assumption (A2) and inequality (7), we conclude that H is a contraction mapping. □

At present, for $i \in N_m$ and $n \in \mathbb{N}$, we take

$$F_{i,n}(t,u_1,\dots,u_{2m}) = g_i\bigl(t, \chi_1(u_1),\dots,\chi_n(u_{2m})\bigr),$$

where $\chi_n(x) = x$ whenever $x \ge \frac{1}{n}$ and $\chi_n(x) = x$ whenever $x < \frac{1}{n}$. By simple review, we can check that

$$\begin{aligned}
&F_{i,n}(t,u_1,\dots,u_{2m}) + h_i(t,u_1,\dots,u_{2m}) \\
&\quad \le r_i\left(\frac{1}{n},\dots,\frac{1}{n}\right) + \mu_i(t) w_i\left(u_1 + \frac{1}{n},\dots,u_{2m} + \frac{1}{n}\right),
\end{aligned}$$

$F_{i,n}(t,u_1,\dots,u_{2m}) \ge \ell_i$ and

$$\begin{aligned}
&F_{i,n}(t,u_1,\dots,u_{2m}) + h_i(t,u_1,\dots,u_{2m}) \\
&\quad \le r_i(u_1,\dots,u_{2m}) + \mu_i(t) w_i\left(u_1 + \frac{1}{n},\dots,u_{2m} + \frac{1}{n}\right),
\end{aligned}$$

for all $(u_1,\dots,u_n) \in \mathcal{S}$, $i \in N_m$ and each $t \in \overline{J}$.

First, we investigate the system of regular fractional q-differential equations

$$\begin{cases}
D_q^{\alpha_1} x_1 + F_{1,n}(t,x_1,\dots,x_m,D_q^{\gamma_1}x_1,\dots,D_q^{\gamma_m}x_m) = 0, \\
D_q^{\alpha_2} x_2 + F_{2,n}(t,x_1,\dots,x_m,D_q^{\gamma_2}x_1,\dots,D_q^{\gamma_m}x_m) = 0, \\
\vdots \\
D_q^{\alpha_m} x_m + F_{m,n}(t,x_1,\dots,x_m,D_q^{\gamma_1}x_1,\dots,D_q^{\gamma_m}x_m) = 0,
\end{cases} \tag{11}$$

with the same boundary conditions as in (1).

Lemma 5 *Suppose that $\mathcal{P}$ is the set which is defined in Lemma 4 and $i \in \{0\} \cup N_m$. Also, let us, for each natural number n and $i \in N_m$, define the maps*

$$\begin{aligned}
&T_{i,n}(u_1,\dots,u_m)(t) \\
&\quad = \int_0^1 G_{\alpha_i}(t,qs) F_{i,n}\bigl(s,u_1(s),\dots,u_m(s),D_q^{\gamma_1}u_1(s),\dots,D_q^{\gamma_m}u_m(s)\bigr)\, d_qs
\end{aligned} \tag{12}$$

and

$$\Omega_n(u_1,\dots,u_m)(t) = \begin{pmatrix} T_{1,n}(u_1,\dots,u_m)(t) \\ T_{2,n}(u_1,\dots,u_m)(t) \\ \vdots \\ T_{m,n}(u_1,\dots,u_m)(t) \end{pmatrix}, \tag{13}$$

for all $(u_1,\dots,u_m) \in \mathcal{P}$. *Then* Ω_n *is a completely continuous operator on* $\mathcal{P}$ *for each natural number* n.

Proof Assume that $(u_1,\dots,u_m) \in \mathcal{P}$. We choose a positive constant ℓ_i such that

$$F_{i,n}\big(t, u_1(t),\dots,u_m(t), D_q^{\gamma_1}u_1(t),\dots,D_q^{\gamma_m}u_m(t)\big) \ge \ell_i,$$

for almost all $t \in \overline{J}$. Since G_{α_i} and $\frac{\partial}{\partial t}G_{\alpha_i}$ are nonnegative and continuous on $\overline{J}^2$ for each $i \in N_m$, we conclude that $T_{i,n}(u_1,\dots,u_m)(t)$ and $(T_{i,n}(u_1,\dots,u_m))'(t)$ larger than or equal to zero, for all $t \in \overline{J}$ and $i \in N_m$. Indeed, Ω_n maps P into P. Consider a convergent sequence $\{(u_{1,k},\dots,u_{m,k})\} \subseteq \mathcal{P}$ with $\lim_{k\to\infty}(u_{1,k},\dots,u_{m,k}) = (u_1,\dots,u_m)$. In this case, we get $\lim_{k\to\infty} u_{i,k} = u_i$ and $\lim_{k\to\infty} u'_{i,k} = u'_i$ uniformly on $\overline{J}$ $(i \in N_m)$. But

$$\Gamma_q(2-\gamma_i)\big|D_q^{\gamma_i}u_{i,k}(t) - D_q^{\gamma_i}u_i(t)\big| \le \|u'_{i,k} - u'_i\|,$$

for each t in $\overline{J}$ and $i \in N_m$. Hence, $\lim_{k\to\infty} D^{\mu_i}x_{i,k}(t) = D^{\mu_i}x_i(t)$ uniformly on $\overline{J}$. Hence,

$$\begin{aligned} &\lim_{k\to\infty} F_{i,n}\big(t, u_{1,k}(t),\dots,u_{m,k}(t), D_q^{\gamma_1}u_{1,k}(t),\dots,D_q^{\gamma_m}u_{m,k}(t)\big) \\ &\quad = F_{i,n}\big(t, u_1(t),\dots,u_m(t), D_q^{\gamma_1}u_1(t),\dots,D_q^{\gamma_m}u_m(t)\big). \end{aligned}$$

Since $F_{i,n} \in C(\overline{J} \times \mathbb{R}^{2m})$, the sequence $\{(u_{1,k},\dots,u_{m,k})\} \subseteq \overline{\mathcal{B}}^m$ is bounded, there exists a map $\mu_i \in \overline{\mathcal{L}}$ such that

$$\ell_i \le F_{i,n}\big(t, u_{1,k}(t),\dots,u_{m,k}(t), D_q^{\gamma_1}u_{1,k}(t),\dots,D_q^{\gamma_m}u_{m,k}(t)\big) \le \mu_i(t), \tag{14}$$

for almost all $t \in \overline{J}$, $i \in N_m$ and $k \in \mathbb{N}$. By using the dominated convergence theorem of Lebesgue, we conclude that

$$\begin{aligned} &\big|T_{i,n}(u_{1,k},\dots,u_{m,k})(t) - T_{i,n}(u_1,\dots,u_m)(t)\big| \\ &\quad \le \frac{1}{\Gamma_q(\alpha_i)} \int_0^1 \big|F_{i,n}\big(s, u_{1,k}(s),\dots,u_{m,k}(s), D_q^{\gamma_1}u_{1,k}(s),\dots,D_q^{\gamma_m}u_{m,k}(s)\big) \\ &\qquad - F_{i,n}\big(s, u_1(s),\dots,u_m(s), D_q^{\gamma_1}u_1(s),\dots,D_q^{\gamma_m}u_m(s)\big)\big|\,ds \end{aligned}$$

and

$$\begin{aligned} &\big|\big(T_{i,n}(u_{1,k},\dots,u_{m,k})\big)'(t) - \big(T_{i,n}(u_1,\dots,u_m)\big)'(t)\big| \\ &\quad \le \frac{1}{\Gamma_q(\alpha_i - 1)} \end{aligned}$$

$$\times \int_0^1 \big| F_{i,n}\big(s, u_{1,k}(s), \dots, u_{m,k}(s), D_q^{\gamma_1} u_{1,k}(s), \dots, D_q^{\gamma_m} u_{m,k}(s)\big) - F_{i,n}\big(s, u_1(s), \dots, u_m(s), D_q^{\gamma_1} u_1(s), \dots, D_q^{\gamma_m} u_m(s)\big) \big| \, ds.$$

Hence,

$$\lim_{k\to\infty} \big| \big(T_{i,n}(u_{1,k}, \dots, u_{m,k})\big)^{j}(t) - \big(T_{i,n}(u_1, \dots, u_m)\big)^{j}(t) \big| = 0,$$

uniformly on $\overline{J}$ for $j = 0, 1$. Thus,

$$\big\| \Omega_n(u_{1,k}, \dots, u_{m,k})(t) - \Omega_n(u_1, \dots, u_m)(t) \big\|_{**} \to 0$$

and so Ω_n is continuous. Let $\{(u_{1,k}, \dots, u_{m,k})\} \subseteq \mathcal{P}$ be a bounded sequence. We choose a positive number M such that $\|u_{i,k}\|$ and $\|u'_{i,k}\|$ are smaller than or equal to M for all $i \in N_m$ and $k \geq 1$. Since $\|D_q^{\gamma_i} u_{i,k}\| \Gamma_q(2 - \gamma_i) \leq 1$ for each $i \in N_m$, there exists a map $\mu_i \in \overline{\mathcal{L}}$ such that inequality (14) holds for almost all $t \in \overline{J}$, $i \in N_m$ and $k \geq 1$. On the other hand,

$$\begin{aligned} 0 &\leq T_{i,n}(u_{1,k}, \dots, u_{m,k})(t) \\ &= \int_0^1 G_{\alpha_i}(t, qs) \\ &\quad \times F_{i,n}\big(s, u_{1,k}(s), \dots, u_{m,k}(s), D_q^{\gamma_1} u_{1,k}(s), \dots, D_q^{\gamma_m} u_{m,k}(s)\big) \, ds \\ &\leq \frac{1}{\Gamma_q(\alpha_i)} \int_0^1 \mu_i(s) \, d_q s = \frac{\|\mu_i\|_1}{\Gamma_q(\alpha_i)} \end{aligned}$$

and

$$\begin{aligned} 0 &\leq \big(T_{i,n}(u_{1,k}, \dots, u_{m,k})\big)'(t) \\ &= \int_0^1 \frac{\partial}{\partial t} G_{\alpha_i}(t, qs) \\ &\quad \times F_{i,n}\big(s, u_{1,k}(s), \dots, u_{m,k}(s), D_q^{\gamma_1} u_{1,k}(s), \dots, D_q^{\gamma_m} x_{m,k}(s)\big) \, d_q s \\ &\leq \frac{1}{\Gamma_q(\alpha_i - 1)} \int_0^1 \mu_i(s) \, d_q s = \frac{\|\mu_i\|_1}{\Gamma_q(\alpha_i - 1)} \end{aligned}$$

for all $i \in N_m$. Hence,

$$\big\| \Omega_n(u_{1,k}, \dots, u_{m,k})(t) \big\|_{**} \leq \max_{i \in N_m} \frac{\|\mu_i\|_1}{\Gamma_q(\alpha_i - 1)}.$$

Indeed, $\{\Omega_n(u_{1,k}, \dots, u_{m,k})\}$ is bounded in $\overline{\mathcal{B}}^m$. Assume that $t_1, t_2 \in \overline{J}$ such that $t_1 \leq t_2$ and $i \in N_m$. Then, we have

$$\begin{aligned} &\big| \big(T_{i,n}(u_{1,k}, \dots, u_{m,k})\big)'(t_2) - \big(T_{i,n}(u_{1,k}, \dots, u_{m,k})\big)'(t_1) \big| \\ &\quad \leq \frac{t_2 - t_1}{\Gamma_q(\alpha_i - 1)} \int_0^1 (1 - qs)^{(\alpha_i - 2)} \end{aligned}$$

$$
\begin{aligned}
&\times F_{i,n}\big(s, u_{1,k}(s), \ldots, u_{m,k}(s), D_q^{\gamma_1} u_{1,k}(s), \ldots, D_q^{\gamma_m} x_{m,k}(s)\big)\, d_q s \\
&+ \frac{1}{\Gamma_q(\alpha_i)} \bigg| \int_0^{t_2} (t_2 - qs)^{(\alpha_i - 1)} \\
&\times F_{i,n}\big(s, u_{1,k}(s), \ldots, u_{m,k}(s), D_q^{\gamma_1} u_{1,k}(s), \ldots, D_q^{\gamma_m} u_{m,k}(s)\big)\, d_q s \\
&- \int_0^{t_1} (t_1 - qs)^{(\alpha_i - 1)} \\
&\times F_{i,n}\big(s, u_{1,k}(s), \ldots, u_{m,k}(s), D_q^{\gamma_1} u_{1,k}(s), \ldots, D_q^{\gamma_m} u_{m,k}(s)\big)\, d_q s \bigg| \\
\leq{}& \frac{\|F_{i,n}\|_1}{\Gamma_q(\alpha_i - 1)} (t_2 - t_1) + \frac{1}{\Gamma_q(\alpha_i)} \bigg[\int_0^{t_1} \big((t_2 - qs)^{(\alpha_i - 1)} - (t_1 - s)^{\alpha_i - 1} \big) \\
&\times F_{i,n}\big(s, u_{1,k}(s), \ldots, u_{m,k}(s), D_q^{\gamma_1} u_{1,k}(s), \ldots, D_q^{\gamma_m} u_{m,k}(s)\big)\, d_q s \\
&+ \int_{t_1}^{t_2} (t_2 - qs)^{(\alpha_i - 2)} \\
&\times F_{i,n}\big(s, u_{1,k}(s), \ldots, u_{m,k}(s), D_q^{\gamma_1} u_{1,k}(s), \ldots, D_q^{\gamma_m} u_{m,k}(s)\big)\, d_q s \bigg] \\
\leq{}& \frac{\|\mu_i\|_1}{\Gamma_q(\alpha_i - 1)} (t_2 - t_1) \\
&+ \frac{1}{\Gamma_q(\alpha_i)} \bigg[\int_0^{t_1} \big((t_2 - qs)^{(\alpha_i - 1)} - (t_1 - qs)^{(\alpha_i - 1)} \big) \mu_i(s)\, d_q s + (t_2 - t_1)^{\alpha_i - 1} \|\mu_i\|_1 \bigg].
\end{aligned}
$$

Since the function $|t - qs|^{(\alpha_i - 1)}$ is uniformly continuous on $\overline{J}^2$, there exists $\delta > 0$ such that $(t_2 - qs)^{(\alpha_i - 1)} - (t_1 - qs)^{(\alpha_i - 1)} < \varepsilon$ for all $t_1, t_2 \in \overline{J}$ with $t_1 \leq t_2$, $t_2 - t_1 < \delta$ and $s \in [0, t_1]$, where $\varepsilon > 0$ be given. Take $t_2 - t_1 < \min\{\delta, \varepsilon\}$, then we have

$$
\big| \big(T_{i,n}(u_{1,k}, \ldots, u_{m,k})\big)'(t_2) - \big(T_{i,n}(u_{1,k}, \ldots, u_{m,k})\big)'(t_1) \big| < \frac{3\varepsilon \|\mu_i\|_1}{\Gamma_q(\alpha_i)}.
$$

Thus,

$$
\big\| \Omega_n'(u_{1,k}, \ldots, u_{m,k})(t_2) - \Omega_n'(u_{1,k}, \ldots, u_{m,k})(t_1) \big\|_* < \max_{i \in N_m} \frac{3\varepsilon \|\mu_i\|_1}{\Gamma_q(\alpha_i)}.
$$

This implies that $\{\Omega_n'(u_{1,k}, \ldots, u_{m,k})\}$ is equi-continuous on $\overline{J}$. At present, by using the Arzelà-Ascoli theorem, $\{\Omega_n(u_{1,k}, \ldots, u_{m,k})\}$ is relatively compact. Therefore Ω_n is completely continuous. □

Now, we are ready to provide our main results about the problem (1).

Theorem 6 *The problem* (11) *under boundary conditions in* (1) *has a solution* $(u_{1,n}, \ldots, u_{m,n})$ *belongs to* $\mathcal{P}$ *such that* $u_{i,n}(t)\Gamma(\alpha_i + 1) \geq \ell_i t^{\alpha_i}(\alpha_i - 1)$, *for all* $t \in \overline{J}$ *and* $i \in N_m$, *whenever assumptions* (A1) *and* (A2) *hold.*

Proof The mapping $H : P \to P$ is a contraction and the operator $\Omega_n : P \to P$ is completely continuous, by employing Lemma 4 and Lemma 5, respectively. Now, by applying

Lemma 2, there exists $(u_{1,n}, \ldots, u_{m,n}) \in \mathcal{P}$ such that $(u_{1,n}, \ldots, u_{m,n}) = \Omega_n(u_{1,n}, \ldots, u_{m,n}) + H(u_{1,n}, \ldots, u_{m,n})$. Therefore, $u_{i,n} = T_{i,n}(u_{1,n}, \ldots, u_{m,n}) + H_i(u_{1,n}, \ldots, u_{m,n})$ for all $i \in N_m$. Hence,

$$\begin{aligned} u_{i,n}(t) &= \int_0^1 G_{\alpha_i}(t, qs) F_{i,n}\big(s, u_1(s), \ldots, u_m(s), D^{\mu_1} u_1(s), \ldots, D_q^{\gamma_m} u_m(s)\big)\, d_q s \\ &\quad + \int_0^1 G_{\alpha_i}(t, qs) h_i\big(s, u_1(s), \ldots, u_m(s), D_q^{\gamma_1} u_1(s), \ldots, D_q^{\gamma_m} u_m(s)\big)\, d_q s \end{aligned}$$

for all $i \in N_m$. By applying the hypothesis, we obtain $u_{i,n}(t)\Gamma(\alpha_i + 1) \geq \ell_i t^{\alpha_i}(\alpha_i - 1)$ for all $t \in \overline{J}$ and $i \in N_m$. By simple review, we can see that the element $(u_{1,n}, \ldots, u_{m,n}) \in \mathcal{P}$ is a solution of the problem (11) under the boundary conditions in (1). □

Lemma 7 *Let the element* $(u_{1,n}, \ldots, u_{m,n})$ *be a solution for the problem* (11) *under the boundary conditions in* (1). *Then* $\{(u_{1,n}, \ldots, u_{m,n})\}_{n \geq 1}$ *is relatively compact in* $\mathcal{P}$ *whenever assumptions* (A1), (A2) *and* (A3) *hold.*

Proof As we found in Theorem 6,

$$\begin{aligned} u_{i,n}(t) &= \int_0^1 G_{\alpha_i}(t, qs) \\ &\quad \times F_{i,n}\big(s, u_{1,n}(s), \ldots, u_{m,n}(s), D_q^{\gamma_1} u_{1,n}(s), \ldots, D_q^{\gamma_m} u_{m,n}(s)\big)\, d_q s \\ &\quad + \int_0^1 G_{\alpha_i}(t, qs) \\ &\quad \times h_i\big(s, u_{1,n}(s), \ldots, u_{m,n}(s), D_q^{\gamma_1} u_{1,n}(s), \ldots, D_q^{\gamma_m} u_{m,n}(s)\big)\, d_q s \end{aligned}$$

for all $n \in \mathbb{N}$, $t \in \overline{J}$ and $i \in N_m$. Hence,

$$\frac{m_i(1 - t^{\alpha i - 1})}{\Gamma_q(\alpha_i)} \leq \ell_i \int_0^1 \frac{\partial}{\partial t} G_{\alpha_i}(t, qs)\, d_q s \leq u'_{i,n}(t),$$

for $t \in \overline{J}$ and so,

$$\begin{aligned} D_q^{\gamma_i} u_{i,n}(t) &= \frac{1}{\Gamma_q(1 - \gamma_i)} \int_0^t (t - qs)^{(-\gamma_i)} u'_{i,n}(s)\, d_q s \\ &\geq \frac{\ell_i}{\Gamma_q(\alpha_i)\Gamma_q(1 - \gamma_i)} \int_0^t (t - qs)^{(-\gamma_i)} (1 - qs)^{(\alpha_i - 1)}\, d_q s \\ &> \frac{\ell_i}{\Gamma_q(\alpha_i)\Gamma_q(1 - \gamma_i)} \int_0^t (t - qs)^{(-\gamma_i)} (1 - qs)\, d_q s. \end{aligned}$$

Thus,

$$\begin{aligned} D_q^{\gamma_i} u_{i,n}(t) &> \frac{\ell_i t^{1 - \gamma_i}}{\Gamma_q(\alpha_i)\Gamma_q(2 - \gamma_i)} - \frac{\ell_i t^{2 - \gamma_i}}{\Gamma_q(\alpha_i)\Gamma_q(3 - \gamma_i)} \\ &= \frac{\ell_i t^{1 - \gamma_i}}{\Gamma_q(\alpha_i)} \left(\frac{\Gamma_q(3 - \gamma_i) - t\Gamma_q(2 - \gamma_i)}{\Gamma_q(2 - \gamma_i)\Gamma_q(3 - \gamma_i)} \right) \end{aligned}$$

$$= \frac{\ell_i t^{1-\gamma_i}}{\Gamma_q(\alpha_i)}\left(\frac{2-\gamma_i-t}{\Gamma_q(3-\gamma_i)}\right) \geq \frac{\ell_i t^{1-\gamma_i}(1-\gamma_i)}{\Gamma_q(\alpha_i)\Gamma_q(3-\gamma_i)}$$

for $t \in \overline{J}$. Since $\Gamma_q(3-\gamma_i) \leq 2$, we have $2\Gamma_q(\alpha_i)D_q^{\gamma_i}u_{i,n}(t) \geq \ell_i t^{1-\gamma_i}(1-\gamma_i)$. Now, put

$$L_i = \ell_i \min\left\{\frac{1}{\Gamma_q(\alpha_i)}, \frac{\alpha_i - 1}{\Gamma_q(\alpha_i+1)}\right\}.$$

Therefore, $u_{i,n}(t) \geq L_i t^{\alpha_i}$ and $2D_q^{\gamma_i}u_{i,n}(t) \geq L_i(1-\gamma_i)t^{1-\gamma_i}$ for all $n \geq 1$, $t \in \overline{J}$ and $i \in N_m$. Indeed,

$$r_i\big(u_{1,n}(t),\dots,u_{m,n}(t),D_q^{\gamma_1}u_{1,n}(t),\dots,D_q^{\gamma_m}u_{m,n}(t)\big) \leq r_i\left(L_1 t^{\alpha_1},\dots,L_m t^{\alpha_m},\frac{L_1(1-\gamma_1)}{2}t^{1-\gamma_1},\dots,\frac{L_m(1-\gamma_m)}{2}t^{1-\gamma_m}\right)$$

for $n \in \mathbb{N}$, $t \in \overline{J}$ and $i \in N_m$. Hence, we conclude that

$$\begin{aligned} 0 &\leq u'_{i,n}(t) \\ &= \int_0^1 \frac{\partial}{\partial t}G_{\alpha_i}(t,qs) \\ &\quad \times F_{i,n}\big(s,u_{1,n}(s),\dots,u_{m,n}(s),D_q^{\gamma_1}u_{1,n}(s),\dots,D_q^{\gamma_m}u_{m,n}(s)\big)\,d_qs \\ &\quad + \int_0^1 \frac{\partial}{\partial t}G_{\alpha_i}(t,qs) \\ &\quad \times h_i\big(s,u_{1,n}(s),\dots,u_{m,n}(s),D_q^{\gamma_1}u_{1,n}(s),\dots,D_q^{\gamma_m}u_{m,n}(s)\big)\,d_qs \\ &\leq \frac{1}{\Gamma_q(\alpha_i-1)}\int_0^1 r_i\left(K_1 s^{\alpha_1},\dots,L_m s^{\alpha_m},\frac{L_1(1-\gamma_1)}{2}s^{1-\gamma_1},\dots,\frac{L_m(1-\gamma_m)}{2}s^{1-\gamma_m}\right)d_qs \\ &\quad + \frac{1}{\Gamma_q(\alpha_i-1)}\int_0^1 \gamma_i(s) \\ &\quad \times w_i\big(u_{1,n}(s),\dots,u_{m,n}(s),D_q^{\gamma_1}u_{1,n}(s),\dots,D_q^{\gamma_m}u_{m,n}(s)\big)\,d_qs \end{aligned}$$

for all $t \in \overline{J}$, $n \geq 1$ and $i \in N_m$. Also, for all $i \in N_m$, we obtain

$$\Lambda_i = \int_0^1 r_i\left(L_1 s^{\alpha_1},\dots,L_m s^{\alpha_m},\frac{L_1(1-\gamma_1)}{2}s^{1-\gamma_1},\dots,\frac{L_m(1-\gamma_m)}{2}s^{1-\gamma_m}\right)d_qs < \infty.$$

Assume that $\lambda_n = \|(u_{1,n},\dots,u_{m,n})\|_{**}$. Then, for all i and n, we have $\|u_{i,n}\|$ and $\|u'_{i,n}\|$ smaller than or equal to λ_n. Thus, $\Gamma_q(2-\gamma_i)|D_q^{\gamma_i}u_{i,n}(t)| \leq \lambda_n$ for each $n \in \mathbb{N}$, $t \in \overline{J}$ and $i \in N_m$. Hence

$$\begin{aligned} 0 &\leq u'_{i,n}(t) \\ &\leq \frac{1}{\Gamma_q(\alpha_i-1)}\Bigg(\Lambda_i + w_i\bigg(1+\lambda_n,\dots,1+\lambda_n, \\ &\quad 1+\frac{\lambda_n}{\Gamma_q(2-\gamma_1)},\dots,1+\frac{\lambda_n}{\Gamma_q(2-\gamma_m)}\bigg)\Bigg)\|\mu_i\|_1 \end{aligned}$$

and $0 \le u_{i,n}(t) = \int_0^t u'_{i,n}(s)\,ds$ for $n \in \mathbb{N}$, $t \in \overline{J}$ and $i \in N_m$. By a similar method, we get

$$\begin{aligned} 0 &\le u_{i,n}(t) \\ &\le \frac{1}{\Gamma_q(\alpha_i - 1)}\Bigg(\Lambda_i + w_i\Bigg(1 + \lambda_n, \dots, 1 + \lambda_n, \\ &\quad 1 + \frac{\lambda_n}{\Gamma_q(2-\gamma_1)}, \dots, 1 + \frac{\lambda_n}{\Gamma_q(2-\gamma_m)}\Bigg)\Bigg)\|\mu_i\|_1, \\ \lambda_n &\le \frac{1}{\Gamma_q(\alpha_i - 1)}\Bigg(\Lambda_i + w_i\Bigg(1 + \lambda_n, \dots, 1 + \lambda_n, \\ &\quad 1 + \frac{\lambda_n}{\Gamma_q(2-\gamma_1)}, \dots, 1 + \frac{\lambda_n}{\Gamma_q(2-\gamma_m)}\Bigg)\Bigg)\|\mu_i\|_1, \end{aligned}$$

for all i. Since $\lim_{x\to\infty} \frac{w_i(x,\dots,x)}{x} = 0$ for all $i \in N_m$, there exists $M_i > 0$ such that

$$\begin{aligned} &\frac{1}{\Gamma_q(\alpha_i - 1)}\Bigg(\Lambda_i + w_i\Bigg(1 + \gamma_i, \dots, 1 + \nu_i, \\ &\quad 1 + \frac{\nu_i}{\Gamma_q(2-\gamma_1)}, \dots, 1 + \frac{\nu_i}{\Gamma_q(2-\gamma_m)}\Bigg)\Bigg)\|\mu_i\|_1 < \nu_i, \end{aligned}$$

for all $\nu_i > M_i$. We take $M = \max\{M_1, \dots, M_m\}$. Then we have

$$\begin{aligned} &\frac{1}{\Gamma_q(\alpha_i - 1)}\Bigg(\Lambda_i + w_i\Bigg(1 + \nu, \dots, 1 + \nu, \\ &\quad 1 + \frac{\nu}{\Gamma_q(2-\gamma_1)}, \dots, 1 + \frac{\nu}{\Gamma_q(2-\gamma_m)}\Bigg)\Bigg)\|\mu_i\|_1 < \nu, \end{aligned}$$

for all $\nu > M$. Thus,

$$\lambda_n = \big\|(u_{1,n}, \dots, u_{m,n})\big\|_{**} = \max_{i\in N_m}\big\{\|u_{i,n}\|, \big\|u'_{i,n}\big\|\big\} < M,$$

which implies $\{\|(u_{1,n}, \dots, u_{m,n})\|_{**}\}$ is bounded in $\overline{\mathcal{B}}_m$. Now, take

$$C_i = w_i\Bigg(1 + M, \dots, 1 + M, 1 + \frac{M}{\Gamma_q(2-\gamma_1)}, \dots, 1 + \frac{M}{\Gamma_q(2-\gamma_m)}\Bigg)$$

and

$$U_i(t) = r_i\Bigg(L_1 t^{\alpha_1}, \dots, L_m t^{\alpha_m}, \frac{L_1(1-\gamma_1)}{2}t^{1-\gamma_1} \dots, \frac{L_m(1-\gamma_m)}{2}t^{1-\gamma_m}\Bigg),$$

for all i and each $t \in \overline{J}$. Then, we have $\Lambda_i = \int_0^1 U_i(t)\,dt$ and

$$\begin{aligned} &F_{i,n}\big(t, u_{1,n}(t), \dots, u_{m,n}(t), D_q^{\gamma_1}u_{1,n}(t), \dots, D_q^{\gamma_m}u_{m,n}(t)\big) \\ &\qquad + h_i\big(t, u_{1,n}(t), \dots, u_{m,n}(t), D_q^{\gamma_{1,n}}u_1(t), \dots, D_q^{\gamma_m}u_{m,n}(t)\big) \\ &\quad \le U_i(t) + C_i\mu_i(t). \end{aligned}$$

If $t_1, t_2 \in \overline{J}$ such that $t_1 \leq t_2$, then we obtain

$$\begin{aligned}
\left|u'_{i,n}(t_2) - u'_{i,n}(t_1)\right| &= \left|\int_0^1 \left(\frac{\partial}{\partial t} G_{\alpha_i}(t_2, qs) - \frac{\partial}{\partial t} G_{\alpha_i}(t_2, qs)\right)\right. \\
&\quad \times \left[F_{i,n}\left(s, u_{1,n}(s), \ldots, u_{m,n}(s), D_q^{\gamma_1} u_{1,n}(s), \ldots, D_q^{\gamma_m} u_{m,n}(s)\right)\right. \\
&\quad \left.\left. + h_i\left(s, u_{1,n}(s), \ldots, u_{m,n}(s), D_q^{\gamma_1} u_{1,n}(s), \ldots, D_q^{\gamma_m} u_{m,n}(s)\right)\right] d_q s\right| \\
&\leq \frac{1}{\Gamma_q(\alpha_i - 1)} \left[(t_2 - t_1) \int_0^1 U_i(s) + C_i \mu_i(s)\, d_q s\right. \\
&\quad + \int_0^{t_1} \left((t_2 - qs)^{(\alpha_i - 2)} - (t_1 - qs)^{(\alpha_i - 2)}\right) \\
&\quad \times \left(U_i(s) + C_i \mu_i(s)\right) d_q s \\
&\quad \left. + \int_{t_1}^{t_2} (t_2 - qs)^{(\alpha_i - 2)} \left(U_i(s) + C_i \mu_i(s)\right) d_q s\right] \\
&\leq \frac{1}{\Gamma_q(\alpha_i - 1)} \left[(t_2 - t_1)\left(\Lambda_i + C_i \|\mu_i\|_1\right)\right. \\
&\quad + \int_0^{t_1} \left((t_2 - qs)^{(\alpha_i - 2)} - (t_1 - qs)^{(\alpha_i - 2)}\right) \\
&\quad \times \left(U_i(s) + C_i \mu_i(s)\right) d_q s \\
&\quad \left. + (t_2 - t_1)^{\alpha_i - 2}\left(\Lambda_i + C_i \|\mu_i\|_1\right)\right].
\end{aligned}$$

Let $\varepsilon_i > 0$ be given. Choose $\delta(\varepsilon_i) > 0$ such that

$$(t_2 - qs)^{(\alpha_i - 2)} - (t_1 - qs)^{(\alpha_i - 2)} < \varepsilon_i,$$

for all $0 \leq t_1 < t_2 \leq 1$ with $t_2 - t_1 < \delta(\varepsilon_i)$ and $s \in (0, t]$. Take

$$\delta < \min\left\{\delta(\varepsilon_1), \ldots, \delta(\varepsilon_m), \sqrt[\alpha_1 - 2]{\varepsilon_1}, \ldots, \sqrt[\alpha_m - 2]{\varepsilon_m}\right\},$$

then $\Gamma_q(\alpha_i - 1)|u'_{i,n}(t_2) - u'_{i,n}(t_1)| \leq 3\varepsilon_i(\Lambda_i + C_i\|\mu_i\|_1)$, for all $i \in N_m$. Hence, $\{(u_{1,n}, \ldots, u_{m,n})'\}$ is equi-continuous. Indeed, $\{(u_{1,n}, \ldots, u_{m,n})\}_{n \geq 1} \subseteq \overline{\mathcal{B}}^m$ is relatively compact. □

Theorem 8 *The system* (1) *has a solution* $(u_1, \ldots, u_m) \in \mathcal{P}$ *such that* $2D_q^{\gamma_i} u_i(t) \geq L_i(1 - \gamma_i)t^{1 - \gamma_i}$ *and* $u_i(t) \geq L_i t^{\alpha_i}$ *for all* $t \in \overline{J}$ *and* $i \in N_m$ *whenever the assumptions* (A1), (A2) *and* (A3) *hold.*

Proof As we found in Theorem 6, for each natural number n the system (11) under the boundary conditions in (1) has a solution $(u_{1,n}, \ldots, u_{m,n})$ in $\mathcal{P}$. By applying Lemma 7, we have $\{(u_{1,n}, \ldots, u_{m,n})\}_{n \geq 1}$ is relatively compact in $\overline{\mathcal{B}}^m$. Also, by employing the Arzelá–Ascoli theorem, $(u_1, \ldots, u_m)$ exists such that $\lim_{n \to \infty}(u_{1,n}, \ldots, u_{m,n}) = (u_1, \ldots, u_m)$. It is obvious that $(u_1, \ldots, u_m)$ satisfies the boundary conditions of the problem (1), $D_q^{\gamma_i} u_{i,n} \to D_q^{\gamma_i} u_i$

and

$$\begin{aligned}
&\lim_{n\to\infty} F_{i,n}\big(t,u_{1,n}(t),\dots,u_{m,n}(t),D_q^{\gamma_1}u_{1,n}(t),\dots,D_q^{\gamma_m}u_{m,n}(t)\big)\\
&\qquad + h_i\big(t,u_{1,n}(t),\dots,u_{m,n}(t),D_q^{\gamma_1}u_{1,n}(t),\dots,D_q^{\gamma_m}u_{m,n}(t)\big)\\
&\quad = g_i\big(t,u_1(t),\dots,u_m(t),D_q^{\mu_1}u_1(t),\dots,D_q^{\gamma_m}u_m(t)\big)\\
&\qquad + h_i\big(t,u_1(t),\dots,u_m(t),D_q^{\mu_1}u_1(t),\dots,D_q^{\gamma_m}u_m(t)\big)
\end{aligned}$$

for each t belonging to $\overline{J}$ and $i \in N_m$. Thus $(u_1,\dots,u_m) \in \mathcal{P}$. At present, suppose that $K = \sup_{n\ge 1}\|(u_{1,n},\dots,u_{m,n})\|_{**}$. Then we have $\|D_q^{\gamma_i}u_{i,n}\| \le \frac{K}{\Gamma_q(2-\gamma_i)}$ for all n and $i \in N_m$. Hence,

$$\begin{aligned}
0 &\le G_{\alpha_i}(t,qs)\big[F_{i,n}\big(s,u_{1,n}(s),\dots,u_{m,n}(s),D_q^{\gamma_1}u_{1,n}(s),\dots,D_q^{\gamma_m}u_{m,n}(s)\big)\\
&\quad + h_i\big(s,u_{1,n}(s),\dots,u_{m,n}(s),D_q^{\gamma_1}u_{1,n}(s),\dots,D_q^{\gamma_m}u_{m,n}(s)\big)\big]\\
&\le \frac{1}{\Gamma_q(\alpha_i-1)}\bigg[U_i(s)\\
&\quad + w_i\bigg(1+K,\dots,1+K,1+\frac{K}{\Gamma_q(2-\gamma_i)},\dots,1+\frac{K}{\Gamma_q(2-\gamma_i)}\bigg)\mu_i(s)\bigg]
\end{aligned}$$

for almost all $(t,qs) \in \overline{J}^2$, $n \ge 1$ and $i \in N_m$. At present, the dominated theorem of Lebesgue implies that

$$\begin{aligned}
u_i(t) &= \int_0^1 G_{\alpha_i}(t,qs)g_i\big(s,u_1(s),\dots,u_m(s),D_q^{\gamma_1}u_1(s),\dots,D_q^{\gamma_m}u_m(s)\big)\,d_qs\\
&\quad + \int_0^1 G_{\alpha_i}(t,qs)g_i\big(s,u_1(s),\dots,u_m(s),D_q^{\gamma_1}x_1(s),\dots,D_q^{\gamma_m}u_m(s)\big)\,d_qs
\end{aligned}$$

for all $i \in N_m$ and $t \in \overline{J}$. This completes the proof. □

4 Example and numerical check technique for the problems

In this part, we give complete computational techniques for illustrating of the problem (1), in Theorems 8, such that it covers all the problems, and present numerical examples which solve the problems perfectly. Foremost, we present a simplified analysis that can be executed to calculate the value of q-Gamma function, $\Gamma_q(x)$, for input values q and x by counting the number of sentences n in summation. To this aim, we consider a pseudo-code description of the method for the calculated q-Gamma function of order n in Algorithm 2 (for more details, see the following link: https://en.wikipedia.org/wiki/Q-gamma_function). Table 1 shows that when q is constant, the q-Gamma function is an increasing function. Also, for smaller values of x, an approximate result is obtained with smaller values of n. It is shown by underlined rows. Table 2 shows that the q-Gamma function for values q close to 1 is obtained with higher values of n in comparison with other columns. They have been underlined in line 8 of the first column, line 17 of the second column and line 29 of third columns of Table 2. Also, Table 3 is the same as Table 2, but x values increase in 3. Similarly, the q-Gamma function for values q near to one is obtained with more values of n in comparison with other columns. Furthermore, we provide Algorithm 3, which calculates $(D_q^{\alpha}f)(x)$.

Table 1 Some numerical results for calculation of $\Gamma_q(x)$ with $q = \frac{1}{3}$ that is constant, $x = 4.5, 8.4, 12.7$ and $n = 1, 2, \ldots, 15$ of Algorithm 2

n	$x = 4.5$	$x = 8.4$	$x = 12.7$	n	$x = 4.5$	$x = 8.4$	$x = 12.7$
1	2.472950	11.909360	68.080769	9	2.340263	11.257158	64.351366
2	2.383247	11.468397	65.559266	10	2.340250	11.257095	64.351003
3	2.354446	11.326853	64.749894	11	2.340245	11.257074	64.350881
4	2.344963	11.280255	64.483434	12	2.340244	11.257066	64.350841
5	2.341815	11.264786	64.394980	13	2.340243	11.257064	64.350828
6	2.340767	11.259636	64.365536	14	2.340243	11.257063	64.350823
7	2.340418	11.257921	64.355725	15	2.340243	11.257063	64.350822
8	2.340301	11.257349	64.352456				

Table 2 Some numerical results for calculation of $\Gamma_q(x)$ with $q = \frac{1}{3}, \frac{1}{2}, \frac{2}{3}$, $x = 5$ and $n = 1, 2, \ldots, 35$ of Algorithm 2

n	$q = \frac{1}{3}$	$q = \frac{1}{2}$	$q = \frac{2}{3}$	n	$q = \frac{1}{3}$	$q = \frac{1}{2}$	$q = \frac{2}{3}$
1	3.016535	6.291859	18.937427	18	2.853224	4.921884	8.476643
2	2.906140	5.548726	14.154784	19	2.853224	4.921879	8.474597
3	2.870699	5.222330	11.819974	20	2.853224	4.921877	8.473234
4	2.859031	5.069033	10.537540	21	2.853224	4.921876	8.472325
5	2.855157	4.994707	9.782069	22	2.853224	4.921876	8.471719
6	2.853868	4.958107	9.317265	23	2.853224	4.921875	8.471315
7	2.853438	4.939945	9.023265	24	2.853224	4.921875	8.471046
8	2.853295	4.930899	8.833940	25	2.853224	4.921875	8.470866
9	2.853247	4.926384	8.710584	26	2.853224	4.921875	8.470747
10	2.853232	4.924129	8.629588	27	2.853224	4.921875	8.470667
11	2.853226	4.923002	8.576133	28	2.853224	4.921875	8.470614
12	2.853224	4.922438	8.540736	29	2.853224	4.921875	8.470578
13	2.853224	4.922157	8.517243	30	2.853224	4.921875	8.470555
14	2.853224	4.922016	8.501627	31	2.853224	4.921875	8.470539
15	2.853224	4.921945	8.491237	32	2.853224	4.921875	8.470529
16	2.853224	4.921910	8.484320	33	2.853224	4.921875	8.470522
17	2.853224	4.921893	8.479713	34	2.853224	4.921875	8.470517

Table 3 Some numerical results for calculation of $\Gamma_q(x)$ with $x = 8.4$, $q = \frac{1}{3}, \frac{1}{2}, \frac{2}{3}$ and $n = 1, 2, \ldots, 40$ of Algorithm 2

n	$q = \frac{1}{3}$	$q = \frac{1}{2}$	$q = \frac{2}{3}$	n	$q = \frac{1}{3}$	$q = \frac{1}{2}$	$q = \frac{2}{3}$
1	11.909360	63.618604	664.767669	21	11.257063	49.065390	260.033372
2	11.468397	55.707508	474.800503	22	11.257063	49.065384	260.011354
3	11.326853	52.245122	384.795341	23	11.257063	49.065381	259.996678
4	11.280255	50.621828	336.326796	24	11.257063	49.065380	259.986893
5	11.264786	49.835472	308.146441	25	11.257063	49.065379	259.980371
6	11.259636	49.448420	290.958806	26	11.257063	49.065379	259.976023
7	11.257921	49.256401	280.150029	27	11.257063	49.065379	259.973124
8	11.257349	49.160766	273.216364	28	11.257063	49.065378	259.971192
9	11.257158	49.113041	268.710272	29	11.257063	49.065378	259.969903
10	11.257095	49.089202	265.756606	30	11.257063	49.065378	259.969044
11	11.257074	49.077288	263.809514	31	11.257063	49.065378	259.968472
12	11.257066	49.071333	262.521127	32	11.257063	49.065378	259.968090
13	11.257064	49.068355	261.666471	33	11.257063	49.065378	259.967836
14	11.257063	49.066867	261.098587	34	11.257063	49.065378	259.967666
15	11.257063	49.066123	260.720833	35	11.257063	49.065378	259.967553
16	11.257063	49.065751	260.469369	36	11.257063	49.065378	259.967478
17	11.257063	49.065564	260.301890	37	11.257063	49.065378	259.967427
18	11.257063	49.065471	260.190310	38	11.257063	49.065378	259.967394
19	11.257063	49.065425	260.115957	39	11.257063	49.065378	259.967371
20	11.257063	49.065402	260.066402	40	11.257063	49.065378	259.967357

Here, we provide an example to illustrate the results of Theorem 8.

Example 1 Consider the system as (1) with $m = 2$:

$$\begin{cases} D_q^{\frac{5}{2}} u_1 + \frac{1}{\sqrt[3]{t^2}}(2 + c_1 u_1 + c_2 u_2 + c_3 D_q^{\frac{1}{3}} u_1 + c_4 D_q^{\frac{1}{2}} u_2) \\ \quad + (0.1 e^{\frac{1}{1+u_1}} + 0.2 e^{\frac{1}{1+u_2}} + 0.1 e^{\frac{1}{1+D_q^{\frac{1}{3}} u_1}} + 0.2 e^{\frac{1}{1+D_q^{\frac{1}{2}} u_2}}) = 0, \\ D_q^{\frac{7}{3}} u_2 + \frac{1}{\sqrt{t}}(1 + d_1 u_1 + d_2 u_2 + d_3 D_q^{\frac{1}{3}} u_1 + d_4 D_q^{\frac{1}{2}} u_2) \\ \quad + (0.2 e^{\frac{1}{1+u_1}} + 0.2 e^{\frac{1}{1+u_2}} + 0.3 e^{\frac{1}{1+D_q^{\frac{1}{3}} u_1}} + 0.1 e^{\frac{1}{1+D_q^{\frac{1}{2}} u_2}}) = 0, \end{cases} \tag{15}$$

under boundary conditions $u_1(0) = u_2(0) = 0$, $u_1'(1) = u_2'(1) = 0$ and $u_1''(0) = u_2''(0) = 0$, where c_i, d_i are positive constants, for $i = 1, 2, 3, 4$. Note that $\mathcal{S} = (0, \infty)^4$. We take the functions

$$\begin{aligned} g_1(t, u_1, u_2, u_3, u_4) &= \frac{1}{\sqrt[3]{t^2}}(2 + c_1 u_1 + c_2 u_2 + c_3 u_3 + c_4 u_4), \\ g_2(t, u_1, u_2, u_3, u_4) &= \frac{1}{\sqrt{t}}(1 + d_1 u_1 + d_2 u_2 + d_3 u_3 + d_4 u_4), \\ h_1(t, u_1, u_2, u_3, u_4) &= r_1(u_1, u_2, u_3, u_4) \\ &= 0.1 e^{\frac{1}{1+u_1}} + 0.2 e^{\frac{1}{1+u_2}} + 0.1 e^{\frac{1}{1+u_3}} + 0.2 e^{\frac{1}{1+u_4}}, \\ h_2(t, u_1, u_2, u_3, u_4) &= r_2(u_1, u_2, u_3, u_4) \\ &= 0.2 e^{\frac{1}{1+u_1}} + 0.2 e^{\frac{1}{1+u_2}} + 0.3 e^{\frac{1}{1+u_3}} + 0.1 e^{\frac{1}{1+u_4}}, \\ w_1(u_1, u_2, u_3, u_4) &= 2 + c_1 u_1 + c_2 u_2 + c_3 u_3 + c_4 u_4, \\ w_2(u_1, u_2, u_3, u_4) &= 1 + d_1 u_1 + d_2 u_2 + d_3 u_3 + d_4 u_4, \end{aligned}$$

$\lambda_1(t) = \frac{1}{\sqrt[3]{t^2}}$ and $\lambda_2(t) = \frac{1}{\sqrt{t}}$. Put $m = 2$, $\alpha_1 = \frac{5}{2}$, $\alpha_2 = \frac{7}{3}$, $\gamma_1 = \frac{1}{2}$, $\gamma_2 = \frac{1}{3}$,

$${}_iM_j = \begin{bmatrix} 0.1 & 0.2 & 0.1 & 0.2 \\ 0.2 & 0.2 & 0.3 & 0.1 \end{bmatrix},$$

$\ell_1 = 2$ and $\ell_2 = 1$. By simple review, we can see that g_1 and g_2 are Carathéodory functions, $g_1(t, u_1, u_2, u_3, u_4) \geq 2$, $g_2(t, u_1, u_2, u_3, u_4) \geq 1$ for all $(u_1, u_2, u_3, u_4) \in \mathcal{S}$ and each $t \in \overline{J}$, h_1 and h_2 are nonnegative and h_1, h_2 satisfy inequality (6):

$$\begin{aligned} \left| h_1(t, u_1, u_2, u_3, u_4) - h_1(t, v_1, v_2, v_3, v_4) \right| &\leq \sum_{i=1}^{4} {}_1M_i |u_i - v_i|, \\ \left| h_2(t, u_1, u_2, u_3, u_4) - h_2(t, v_1, v_2, v_3, v_4) \right| &\leq \sum_{i=1}^{4} {}_2M_i |u_i - v_i|, \end{aligned}$$

Table 4 Some numerical results of η_1 and $\Gamma_q(\alpha_1 - 1)$ from inequality (16) in Example 1 for $q \in \{\frac{1}{8}, \frac{1}{2}, \frac{8}{9}\}$. One can check that $\frac{\eta_1}{\Gamma_q(\alpha_1-1)}$ by approximation is smaller than 1

n	$q=\frac{1}{8}$			$q=\frac{1}{2}$			$q=\frac{8}{9}$		
	η_1	$\Gamma_q(\alpha_1-1)$	$\frac{\eta_1}{\Gamma_q(\alpha_1-1)}$	η_1	$\Gamma_q(\alpha_1-1)$	$\frac{\eta_1}{\Gamma_q(\alpha_1-1)}$	η_1	$\Gamma_q(\alpha_1-1)$	$\frac{\eta_1}{\Gamma_q(\alpha_1-1)}$
1	0.6075	1.0323	0.5885	0.5909	1.0035	0.5888	0.4428	0.5904	0.7500
2	0.6079	1.0336	0.5882	0.6062	1.0455	0.5799	0.4683	0.6701	0.6988
3	0.6080	1.0338	0.5881	0.6138	1.0659	0.5759	0.4894	0.7338	0.6670
4	0.6080	1.0338	0.5881	0.6175	1.0759	0.5739	0.5073	0.7861	0.6453
5	0.6080	1.0338	0.5881	0.6194	1.0809	0.5730	0.5225	0.8299	0.6296
6	0.6080	1.0338	0.5881	0.6203	1.0834	0.5726	0.5356	0.8670	0.6178
7	0.6080	1.0338	0.5881	0.6208	1.0847	0.5723	0.5470	0.8986	0.6087
8	0.6080	1.0338	0.5881	0.6210	1.0853	0.5722	0.5569	0.9259	0.6014
9	0.6080	1.0338	0.5881	0.6211	1.0856	0.5722	0.5655	0.9494	0.5956
10	0.6080	1.0338	0.5881	0.6212	1.0858	0.5721	0.5730	0.9699	0.5908
11	0.6080	1.0338	0.5881	0.6212	1.0858	0.5721	0.5797	0.9877	0.5869
12	0.6080	1.0338	0.5881	0.6212	1.0859	0.5721	0.5855	1.0033	0.5836
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
48	0.6080	1.0338	0.5881	0.6213	1.0859	0.5721	0.6296	1.1187	0.5628
49	0.6080	1.0338	0.5881	0.6213	1.0859	0.5721	0.6296	1.1188	0.5628
50	0.6080	1.0338	0.5881	0.6213	1.0859	0.5721	0.6297	1.1190	0.5627
51	0.6080	1.0338	0.5881	0.6213	1.0859	0.5721	0.6298	1.1191	0.5627
52	0.6080	1.0338	0.5881	0.6213	1.0859	0.5721	0.6298	1.1193	0.5627
53	0.6080	1.0338	0.5881	0.6213	1.0859	0.5721	0.6299	1.1194	0.5627
54	0.6080	1.0338	0.5881	0.6213	1.0859	0.5721	0.6299	1.1195	0.5627

for $(u_1, u_2, u_3, u_4), (v_1, v_2, v_3, v_4) \in (0, \infty)^4$ and $t \in \overline{J}$. Also, by putting values in the problem in (7), we have

$$\begin{aligned}
\eta_1 &= \sum_{k=1}^{2} {}_1M_k + \frac{{}_1M_{2+k}}{\Gamma_q(2-\gamma_1)} = 0.1 + 0.2 + \frac{0.1}{\Gamma_q(\frac{5}{3})} + \frac{0.2}{\Gamma_q(\frac{5}{3})} \\
&< \Gamma_q\left(\frac{3}{2}\right) = \Gamma_q(\alpha_1 - 1), \qquad (16)\\
\eta_2 &= \sum_{k=1}^{2} {}_2M_k + \frac{{}_2M_{2+k}}{\Gamma_q(2-\gamma_2)} = 0.2 + 0.2 + \frac{0.3}{\Gamma_q(\frac{3}{2})} + \frac{0.1}{\Gamma_q(\frac{3}{2})} \\
&< \Gamma_q\left(\frac{4}{3}\right) = \Gamma_q(\alpha_2 - 1). \qquad (17)
\end{aligned}$$

Tables 4 and 5 show the values of inequalities (16) and (17), respectively. On the other hand, the maps r_1 and r_2 are nonincreasing with respect to all components. If

$$\begin{aligned}
L_1 &= \ell_1 \frac{\alpha_1 - 1}{\Gamma_q(\alpha_1 + 1)} = 2 \times \frac{\frac{3}{2}}{\Gamma_q(\frac{7}{2})} = \frac{3}{\Gamma_q(\frac{7}{2})}, \\
L_2 &= \ell_2 \frac{\alpha_2 - 1}{\Gamma_q(\alpha_2 + 1)} = 1 \times \frac{\frac{4}{3}}{\Gamma_q(\frac{10}{3})} = \frac{4}{3\Gamma_q(\frac{10}{3})},
\end{aligned}$$

then

$$\int_0^1 r_1\left(L_1 t^{\alpha_1}, L_2 t^{\alpha_2}, \frac{L_1(1-\gamma_1)}{2} t^{1-\gamma_1}, \frac{L_2(1-\gamma_2)}{2} t^{1-\gamma_2}\right) dt < \infty,$$

Table 5 Some numerical results of η_2 and $\Gamma_q(\alpha_2 - 1)$ from (17) in Example 1 for $q \in \{\frac{1}{8}, \frac{1}{2}, \frac{8}{9}\}$. One can heck that $\frac{\eta_2}{\Gamma_q(\alpha_2-1)}$ by approximation is smaller than 1

n	$q = \frac{1}{8}$			$q = \frac{1}{2}$			$q = \frac{8}{9}$		
	η_2	$\Gamma_q(\alpha_2 - 1)$	$\frac{\eta_2}{\Gamma_q(\alpha_2-1)}$	η_2	$\Gamma_q(\alpha_2 - 1)$	$\frac{\eta_2}{\Gamma_q(\alpha_2-1)}$	η_2	$\Gamma_q(\alpha_2 - 1)$	$\frac{\eta_2}{\Gamma_q(\alpha_2-1)}$
1	0.5097	1.0329	0.4934	0.5010	1.0233	0.4896	0.3771	0.7200	0.5238
2	0.5101	1.0339	0.4933	0.5136	1.0534	0.4876	0.4010	0.7854	0.5106
3	0.5101	1.0341	<u>0.4933</u>	0.5198	1.0679	0.4867	0.4201	0.8356	0.5028
4	0.5101	1.0341	0.4933	0.5228	1.075	0.4863	0.4358	0.8757	0.4977
5	0.5101	1.0341	0.4933	0.5243	1.0786	0.4861	0.4490	0.9085	0.4942
6	0.5101	1.0341	0.4933	0.5250	1.0803	0.4860	0.4601	0.9359	0.4916
7	0.5101	1.0341	0.4933	0.5254	1.0812	0.486	0.4696	0.9589	0.4897
8	0.5101	1.0341	0.4933	0.5256	1.0816	0.4859	0.4778	0.9784	0.4883
9	0.5101	1.0341	0.4933	0.5257	1.0818	0.4859	0.4848	0.9952	0.4872
10	0.5101	1.0341	0.4933	0.5257	1.0819	<u>0.4859</u>	0.4910	1.0096	0.4863
11	0.5101	1.0341	0.4933	0.5258	1.0820	0.4859	0.4963	1.0221	0.4856
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
34	0.5101	1.0341	0.4933	0.5258	1.0821	0.4859	0.5336	1.1071	0.4820
35	0.5101	1.0341	0.4933	0.5258	1.0821	0.4859	0.5339	1.1077	0.4819
36	0.5101	1.0341	0.4933	0.5258	1.0821	0.4859	0.5341	1.1083	<u>0.4819</u>
37	0.5101	1.0341	0.4933	0.5258	1.0821	0.4859	0.5343	1.1088	0.4819
38	0.5101	1.0341	0.4933	0.5258	1.0821	0.4859	0.5345	1.1092	0.4819
39	0.5101	1.0341	0.4933	0.5258	1.0821	0.4859	0.5347	1.1096	0.4819
40	0.5101	1.0341	0.4933	0.5258	1.0821	0.4859	0.5348	1.1099	0.4819

$$\int_0^1 r_2\left(L_1 t^{\alpha_1}, L_2 t^{\alpha_2}, \frac{L_1(1-\gamma_1)}{2} t^{1-\gamma_1}, \frac{L_2(1-\gamma_2)}{2} t^{1-\gamma_2}\right) dt < \infty.$$

Also, the functions w_1 and w_2 are nondecreasing with respect to all components and

$$\lim_{x\to\infty} \frac{w_1(x,\dots,x)}{x} = \lim_{x\to\infty} \frac{2 + c_1x + c_2x + c_3x + c_4x}{x} = 0,$$
$$\lim_{x\to\infty} \frac{w_2(x,\dots,x)}{x} = \lim_{x\to\infty} \frac{1 + d_1x + d_2x + d_3x + d_4x}{x} = 0.$$

Therefore, Theorem 8 implies that the problem (15) has a positive solution.

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References

1. Jackson, F.H.: q-Difference equations. Am. J. Math. **32**, 305–314 (1910). https://doi.org/10.2307/2370183
2. Adams, C.R.: The general theory of a class of linear partial q-difference equations. Trans. Am. Math. Soc. **26**, 283–312 (1924)
3. Adams, C.R.: Note on the integro-q-difference equations. Trans. Am. Math. Soc. **31**(4), 861–867 (1929)
4. Ahmad, B., Etemad, S., Ettefagh, M., Rezapour, S.: On the existence of solutions for fractional q-difference inclusions with q-antiperiodic boundary conditions. Bull. Math. Soc. Sci. Math. Roum. **59(107)**(2), 119–134 (2016)
5. Ahmad, B., Nieto, J.J., Alsaedi, A., Al-Hutami, H.: Existence of solutions for nonlinear fractional q-difference integral equations with two fractional orders and nonlocal four-point boundary conditions. J. Franklin Inst. **351**, 2890–2909 (2014). https://doi.org/10.1016/j.jfranklin.2014.01.020
6. Ahmad, B., Ntouyas, S.K., Purnaras, I.K.: Existence results for nonlocal boundary value problems of nonlinear fractional q-difference equations. Adv. Differ. Equ. **2012**, Article ID 140 (2012). https://doi.org/10.1186/1687-1847-2012-140
7. Ferreira, R.A.C.: Nontrivial solutions for fractional q-difference boundary value problems. Electron. J. Qual. Theory Differ. Equ. **70**, Article ID 70 (2010)
8. Kalvandi, V., Samei, M.E.: New stability results for a sum-type fractional q-integro-differential equation. J. Adv. Math. Stud. **12**(2), 201–209 (2019)
9. Samei, M.E., Ranjbar, G.K., Hedayati, V.: Existence of solutions for equations and inclusions of multi-term fractional q-integro-differential with non-separated and initial boundary conditions. J. Inequal. Appl. **2019**, Article ID 273 (2019). https://doi.org/10.1186/s13660-019-2224-2
10. Zhao, Y., Chen, H., Zhang, Q.: Existence results for fractional q-difference equations with nonlocal q-integral boundary conditions. Adv. Differ. Equ. **2013**, Article ID 48 (2013). https://doi.org/10.1186/1687-1847-2013-48
11. Samei, M.E., Hedayati, V., Rezapour, S.: Existence results for a fraction hybrid differential inclusion with Caputo–Hadamard type fractional derivative. Adv. Differ. Equ. **2019**, Article ID 163 (2019). https://doi.org/10.1186/s13662-019-2090-8
12. Hedayati, V., Samei, M.E.: Positive solutions of fractional differential equation with two pieces in chain interval and simultaneous Dirichlet boundary conditions. Bound. Value Probl. **2019**, Article ID 141 (2019). https://doi.org/10.1186/s13661-019-1251-8
13. Atici, F., Eloe, P.W.: Fractional q-calculus on a time scale. J. Nonlinear Math. Phys. **14**(3), 341–352 (2007). https://doi.org/10.2991/jnmp.2007.14.3.4
14. Baleanu, D., Agarwal, R.P., Mohammadi, H., Rezapour, S.: Some existence results for a nonlinear fractional differential equation on partially ordered Banach spaces. Bound. Value Probl. **2013**, Article ID 112 (2013). https://doi.org/10.1186/1687-2770-2013-112
15. Zhou, H., Alzabut, J., Yang, L.: On fractional Langevin differential equations with anti-periodic boundary conditions. Eur. Phys. J. Spec. Top. **226**, 3577–3590 (2017). https://doi.org/10.1140/epjst/e2018-00082-0
16. Samei, M.E., Khalilzadeh Ranjbar, G.: Some theorems of existence of solutions for fractional hybrid q-difference inclusion. J. Adv. Math. Stud. **12**(1), 63–76 (2019)
17. Ntouyas, S.K., Samei, M.E.: Existence and uniqueness of solutions for multi-term fractional q-integro-differential equations via quantum calculus. Adv. Differ. Equ. **2019**, Article ID 475 (2019). https://doi.org/10.1186/s13662-019-2414-8
18. Samei, M.E., Khalilzadeh Ranjbar, G., Hedayati, V.: Existence of solutions for a class of Caputo fractional q-difference inclusion on multifunctions by computational results. Kragujev. J. Math. **45**(4), 543–570 (2021)
19. Liang, S., Zhang, J.: Existence of three positive solutions of m-point boundary value problems for some nonlinear fractional differential equations on an infinite interval. Comput. Math. Appl. **61**, 3343–3354 (2011). https://doi.org/10.1016/j.camwa.2011.04.018
20. Abdeljawad, T., Alzabut, J.: The q-fractional analogue for Gronwall-type inequality. J. Funct. Spaces Appl. **2013**, Article ID 7 (2013). https://doi.org/10.1155/2013/543839
21. Aydogan, M., Nazemi, S.Z., Rezapour, S.: Positive solutions for a sum-type singular fractional integro-differential equation with m-point boundary conditions. UPB Sci. Bull., Ser. A **79**(1), 109–118 (2017)
22. Shabibi, M., Postolache, M., Rezapour, S.: Positive solutions for a singular sum fractional differential system. Int. J. Anal. Appl. **13**(1), 108–118 (2017)
23. Agarwal, R.P., O'Regan, D., Staněk, S.: Positive solutions for Dirichlet problems of singular nonlinear fractional differential equations. J. Math. Anal. Appl. **371**(1), 57–68 (2010). https://doi.org/10.1016/j.jmaa.2010.04.034
24. Cabada, A., Wang, G.: Positive solution of nonlinear fractional differential equations with integral boundary value conditions. J. Math. Anal. Appl. **389**(1), 403–411 (2012). https://doi.org/10.1016/j.jmaa.2011.11.065
25. Jafari, H., Haghbin, A., Johnston, S.J., Baleanu, D.: A new algorithm for solving dynamic equations on a time scale. J. Comput. Appl. Math. **312**, 167–173 (2017). https://doi.org/10.1016/j.cam.2016.02.047
26. Al-Omari, S.K.Q., Baleanu, D., Purohit, S.D.: Some results for Laplace-type integral operator in quantum calculus. Adv. Differ. Equ. **2018**, Article ID 124 (2018). https://doi.org/10.1186/s13662-018-1567-1
27. Agarwal, R.P., Baleanu, D., Hedayati, V., Rezapour, S.: Two fractional derivative inclusion problems via integral boundary condition. Appl. Math. Comput. **257**, 205–212 (2015). https://doi.org/10.1016/j.amc.2014.10.082
28. Aydogan, S.M., Baleanu, D., Mousalou, A., Rezapour, S.: On high order fractional integro-differential equations including the Caputo–Fabrizio derivative. Bound. Value Probl. **2018**, Article ID 90 (2018). https://doi.org/10.1186/s13661-018-1008-9
29. Baleanu, D., Ghafarnezhad, K., Rezapour, S., Shabibi, M.: On the existence of solutions of a three steps crisis integro-differential equation. Adv. Differ. Equ. **2018**, Article ID 135 (2018). https://doi.org/10.1186/s13662-018-1583-1
30. Baleanu, D., Mohammadi, H., Rezapour, S.: The existence of solutions for a nonlinear mixed problem of singular fractional differential equations. Adv. Differ. Equ. **2013**, Article ID 359 (2013). https://doi.org/10.1186/1687-1847-2013-359
31. Li, R.: Existence of solutions for nonlinear fractional equation with fractional derivative condition. Adv. Differ. Equ. **2014**, Article ID 292 (2014). https://doi.org/10.1186/1687-1847-2014-292
32. Rajković, P.M., Marinković, S.D., Stanković, M.S.: Fractional integrals and derivatives in q-calculus. Appl. Anal. Discrete Math. **1**, 311–323 (2007)

33. Annaby, M.H., Mansour, Z.S.: *q*-Fractional Calculus and Equations. Springer, Berlin (2012). https://doi.org/10.1007/978-3-642-30898-7
34. Podlubny, I.: Fractional Differential Equations. Academic Press, San Diego (1999)
35. Hilfer, R.: Applications of Fractional Calculus in Physics. World Scientific, Singapore (2000). https://doi.org/10.1142/3779
36. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. North-Holland Mathematics Studies. Elsevier, Amsterdam (2006)
37. Kac, V., Cheung, P.: Quantum Calculus. Universitext. Springer, New York (2002). https://doi.org/10.1007/978-1-4613-0071-7-1
38. Smart, D.R.: Fixed Point Theorems. Cambridge University Press, New York (1980)