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Existence and uniqueness of solutions for singular fractional differential equation boundary value problem with p-Laplacian

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Abstract

In this paper, we prove the existence and uniqueness of solutions for a singular fractional differential equation boundary value problem with p-Laplacian operator. The main results of this paper are obtained by constructing the monotone iterative sequences of upper and lower solutions and applying the comparison result. Finally, we also provide an illustrative example in support of the existence theorem. Our results generalize some related results in the literature.

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Keywords: Fractional differential equation; Monotone iterative technique; Nonlinear boundary condition; Upper and lower solutions

1 Introduction

The fractional calculus and its varied applications in many fields of science and engineering have gained much attention and developed rapidly in recent decades. Fractional differential equations have been used in the mathematical modeling of process in physics, chemistry, aerodynamics, polymer rheology, fluid flow phenomena, wave propagation, signal theory, electrical circuits, control theory and viscoelastic materials etc. For details, see [1–7] and the references therein.

Many research papers have appeared concerning the existence of solutions for the initial or boundary value problems of fractional differential equations; see [8–19]. We notice that recently a kind of general boundary value conditions, nonlinear boundary value conditions, were investigated in [10, 11]. Moreover, some papers considered recently fractional boundary value problems with p-Laplacian [12–14, 20, 21], and the upper and lower method and the monotone iterative technique are used in [12–14].

By means of the monotone iterative method and lower and upper solutions, Jankowski [11] considered the following fractional differential equations with nonlinear boundary conditions:

$$\begin{cases} (D_T^q u)(t) = f(t, u(t)), & t \in [0, T), T > 0, \\ 0 = g(\overline{u}(0), \overline{u}(T)), \end{cases}$$

where $f \in C([0,T] \times \mathbb{R},\mathbb{R}), g \in C(\mathbb{R} \times \mathbb{R},\mathbb{R})$, and $D^q_T u$ is the right-handed Riemann–Liouville fractional derivative of u with $0 < q < 1, \overline{u}(a) = (T-t)^{1-q}u(t)|_{t=a}, a = 0, T$. One obtained the existence results of a unique solution of the nonlinear fractional differential equations with initial condition at the point T, and got the existence results of a related linear fractional differential problems in terms of the Mittag-Leffler function and the method of successive approximations. On the base of the conclusions, sufficient conditions which guarantee that the problem has extremal solutions were given.

Ding et al. [12] generalized the above problem to the following fractional boundary value problem with p-Laplacian via the upper and lower method and the monotone iterative method:

$$\begin{cases} D^{\beta}_{0^+}(\phi_p(D^{\alpha}_{0^+}u(t))) = f(t,u(t),D^{\alpha}_{0^+}u(t)), & t \in (0,1], \\ t^{\frac{1-\beta}{p-1}} D^{\alpha}_{0^+}u(t)|_{t=0} = 0, & g(\tilde{u}(0),\tilde{u}(1)) = 0, \end{cases}$$

where $0 < \alpha, \beta \leq 1, 1 < \alpha + \beta \leq 2$, $D^{\alpha}_{0^+}$ is the Riemann–Liouville fractional derivative of order α, $\phi_p(t) = |t|^{p-2}t, p > 1$ is the p-Laplacian operator, $f \in C([0,1] \times \mathbb{R} \times \mathbb{R},\mathbb{R}), g \in C(\mathbb{R} \times \mathbb{R},\mathbb{R}), \tilde{u}(0) = t^{1-\alpha}u(t)|_{t=0}$ and $\tilde{u}(1) = t^{1-\alpha}u(t)|_{t=1}$. The existence and uniqueness of extremal solutions are investigated by constructing two well-defined monotone iterative sequences of upper and lower solutions.

Motivated by the above work, in this paper, we investigate the existence and uniqueness of extremal solution for a singular fractional differential equation with p-Laplacian and subjects to more general nonlinear boundary conditions

$$\begin{cases} D^{\beta}_{0^+}(\phi_p(D^{\alpha}_{0^+}u(t))) = f(t,u(t),D^{\alpha}_{0^+}u(t)), & t \in (0,T], \\ g(\tilde{u}(0),\tilde{u}(T)) = 0, \\ h(\overline{D^{\alpha}_{0^+}u}(0),\overline{D^{\alpha}_{0^+}u}(T)) = 0, \end{cases} \tag{1.1}$$

where $0 < \alpha, \beta \leq 1, 1 < \alpha + \beta \leq 2, r = \frac{1-\beta}{p-1}$, $D^{\alpha}_{0^+}$ is the Riemann–Liouville fractional derivative of order α, $\phi_p(t) = |t|^{p-2}t$ $(p > 1)$ is the p-Laplacian operator, $f \in C([0,T] \times \mathbb{R} \times \mathbb{R},\mathbb{R}), g \in C(\mathbb{R} \times \mathbb{R},\mathbb{R}), h \in C(\mathbb{R} \times \mathbb{R},\mathbb{R}), \tilde{u}(c) = t^{1-\alpha}u(t)|_{t=c}$, and $\overline{D^{\alpha}_{0^+}u}(c) = t^r D^{\alpha}_{0^+}u(t)|_{t=c}, c = 0, T$.

In problem (1.1), the boundary value conditions $g(x,y) = 0, h(x,y) = 0$ are a class of general conditions. When $h(x,y) = x$ the problem (1.1) become the problem in [12]. The conditions can cover anti-periodic [22] or other nonlinear boundary conditions. Moreover, the function u and its derivatives may have singularities at both 0 and T. Therefore the problem (1.1) can generalize those problems in [12–14]. Thus our conclusions can be more extensive. We here not only obtain the existence and uniqueness of extremal solutions but also the iterative sequences which converge to the solutions.

For some related results on boundary value problem with p-Laplacian, obtained by means of the monotone iterative method, the monotone type conditions for nonlinear terms f with respect to the functions u or their derivatives are usually required. In this paper, we only consider the functions $f + M\phi_p(D^{\alpha}_{0^+}u(t))$ not f to be enslaved to the monotone type conditions.

The paper is organized as follows. In Sect. 2, we provide some preliminaries, the existence result for linear fractional problems with initial value conditions and a comparison result. In Sect. 3, the existence and uniqueness theorems of extremal solutions are established by constructing two well-defined monotone iterative sequences of upper-lower solutions. Finally, as applications of the theoretical results, an example is given to illustrate the existence result.

2 Preliminaries

Let $J = [0, T]$ be a closed interval on the real axis $\mathbb{R}$. It is well known that $C[0, T]$ is a Banach space of continuous functions from $[0, T]$ into $\mathbb{R}$ with the norm $\|u\|_C = \max_{t\in[0,T]} |u(t)|$. Denote $C_\lambda[0, T]$ by

$$C_\lambda[0,T] = \{u \in C(0,T) : t^\lambda u \in C[0,T]\},$$

where $\lambda \in [0, 1)$. Then $C_\lambda[0, T]$ is also a Banach space with the norm $\|u\|_{C_\lambda} = \|t^\lambda u\|_C$. It is clear that $C[0, T] := C_0[0, T] \subset C_\lambda[0, T] \subset C_\delta[0, T]$ for $0 \le \lambda \le \delta < 1$ and $C_\lambda[0, T] \subset L[0, T]$ ($L[0, T]$ is a space of Lebesgue integrable real functions defined on $[0, T]$). Define $C_r^\alpha[0, T]$ by

$$C_r^\alpha[0,T] = \{u(t) \in C_{1-\alpha}[0,T] : (D_{0^+}^\alpha u)(t) \in C_r[0,T]\},$$

where $0 < \alpha, \beta \le 1, r = \frac{1-\beta}{p-1}, p > 1$ and $p + \beta > 2$. It is a Banach space with the norm $\|u\|_{C_r^\alpha} = \|u\|_{C_{1-\alpha}} + \|D_{0^+}^\alpha u\|_{C_r}$ (see Lemma 2.2 in [14]).

We introduce some useful definitions and fundamental facts of fractional calculus theory; for more details, see [1, 2].

Definition 2.1 ([1]) The Riemann–Liouville fractional integral $I_{0^+}^\alpha$ is given by

$$I_{0^+}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\,ds$$

and the fractional derivative $D_{0^+}^\alpha$ is defined by

$$D_{0^+}^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} f(s)\,ds = \left(\frac{d}{dt}\right)^n \left(I_{0^+}^{n-\alpha} f\right)(t),$$

where $n - 1 < \alpha \le n, n \in \mathbb{N}$, provided the integrals exist.

Lemma 2.1 ([1]) *Assume that we have the function $u \in C(0, T] \cap L(0, T]$ with a fractional derivative of order α $(0 < \alpha \le 1)$ that belongs to $C(0, T] \cap L(0, T]$. Then*

$$I_{0^+}^\alpha D_{0^+}^\alpha u(t) = u(t) + ct^{\alpha-1} \quad \text{for some } c \in \mathbb{R};$$
$$D_{0^+}^\alpha I_{0^+}^\alpha u(t) = u(t).$$

Lemma 2.2 ([12, Lemma 2.1]) *Assume that $0 < \beta \le 1, M \in \mathbb{R}, \kappa \in \mathbb{R}, u(t) \in C_{1-\beta}[0, T]$ and $h(t) \in C_{1-\beta}[0, T]$. Then the linear fractional initial value problem*

$$\begin{cases} D_{0^+}^\beta u(t) + Mu(t) = h(t), & t \in (0, T], \\ t^{1-\beta} u(t)|_{t=0} = \kappa, \end{cases}$$

has the following solution of integral representation:

$$u(t) = \Gamma(\beta)\kappa t^{\beta-1}E_{\beta,\beta}\bigl(-Mt^{\beta}\bigr) + \int_0^t (t-s)^{\beta-1}E_{\beta,\beta}\bigl(-M(t-s)^{\beta}\bigr)h(s)\,ds,$$

where $E_{\beta,\beta}(x) = \sum_{k=0}^{\infty}\frac{x^k}{\Gamma(k\beta+\beta)}$ *is the Mittag-Leffler function. It is continuous and nonnegative* [1, 8].

Lemma 2.3 *Suppose that* $0 < \alpha, \beta \leq 1$, M *is a constant*, $k_0, h_0 \in \mathbb{R}$, $u(t) \in C_r^{\alpha}[0,T]$ *and* $\eta(t) \in C_{1-\beta}[0,T]$. *Then the following linear fractional initial value problem*:

$$\begin{cases} D_{0^+}^{\beta}(\phi_p(D_{0^+}^{\alpha}u(t))) + M\phi_p(D_{0^+}^{\alpha}u(t)) = \eta(t), & t \in (0,T], \\ \tilde{u}(0) = k_0, \qquad \overline{D_{0^+}^{\alpha}u}(0) = h_0, \end{cases} \tag{2.1}$$

has a unique solution with the integral form

$$\begin{aligned} u(t) = k_0 t^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\phi_q\biggl[\Gamma(\beta)\phi_p(h_0)s^{\beta-1}E_{\beta,\beta}\bigl(-Ms^{\beta}\bigr) \\ + \int_0^s (s-\tau)^{\beta-1}E_{\beta,\beta}\bigl(-M(s-\tau)^{\beta}\bigr)\eta(\tau)\,d\tau\biggr]ds, \end{aligned} \tag{2.2}$$

where ϕ_q *is the inverse function of* ϕ_p.

Proof Let $v(t) = \phi_p(D_{0^+}^{\alpha}u(t))$, then we have $\phi_p(t^r D_{0^+}^{\alpha}u(t)) = t^{1-\beta}v(t), 0 < t \leq T$. Thus the problem (2.1) is converted to the following fractional initial value problem:

$$\begin{cases} D_{0^+}^{\beta}v(t) + Mv(t) = \eta(t), & t \in (0,T], \\ t^{1-\beta}v(t)|_{t=0} = \phi_p(h_0). \end{cases}$$

From Lemma 2.2, we find

$$v(t) = \Gamma(\beta)\phi_p(h_0)t^{\beta-1}E_{\beta,\beta}\bigl(-Mt^{\beta}\bigr) + \int_0^t (t-s)^{\beta-1}E_{\beta,\beta}\bigl(-M(t-s)^{\beta}\bigr)\eta(s)\,ds \tag{2.3}$$

and $v(t) \in C_{1-\beta}[0,T]$, thus

$$D_{0^+}^{\alpha}u(t) = \phi_q\biggl[\Gamma(\beta)\phi_p(h_0)t^{\beta-1}E_{\beta,\beta}\bigl(-Mt^{\beta}\bigr) + \int_0^t (t-s)^{\beta-1}E_{\beta,\beta}\bigl(-M(t-s)^{\beta}\bigr)\eta(s)\,ds\biggr]. \tag{2.4}$$

For $v(t) \in C(0,T] \cap L(0,T]$, we have $D_{0^+}^{\alpha}u(t) \in C_r[0,T] \subset C(0,T] \cap L(0,T]$. Lemma 2.1 yields

$$\begin{aligned} u(t) = ct^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\phi_q\biggl[\Gamma(\beta)\phi_p(h_0)s^{\beta-1}E_{\beta,\beta}\bigl(-Ms^{\beta}\bigr) \\ + \int_0^s (s-\tau)^{\beta-1}E_{\beta,\beta}\bigl(-M(s-\tau)^{\beta}\bigr)\eta(\tau)\,d\tau\biggr]ds. \end{aligned}$$

By virtue of $\tilde{u}(0) = k_0$, we get $c = k_0$ and

$$u(t) = k_0 t^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \phi_q \biggl[\Gamma(\beta)\phi_p(h_0) s^{\beta-1} E_{\beta,\beta}\bigl(-Ms^\beta\bigr) + \int_0^s (s-\tau)^{\beta-1} E_{\beta,\beta}\bigl(-M(s-\tau)^\beta\bigr)\eta(\tau)\,d\tau \biggr]\,ds. \tag{2.5}$$

Conversely, it is obvious that $u(t) \in C_{1-\alpha}[0,T]$ and $\tilde{u}(0) = k_0$. Noting that $D^\alpha_{0^+} t^{\alpha-1} = 0$, $D^\alpha_{0^+} I^\alpha u = u$, $\forall u \in C(0,T] \cap L(0,T]$ and differentiating (2.5) with order α, we arrive at (2.4). Since $\eta(t) \in C_{1-\beta}[0,T]$, it is clear that $\phi_p(D^\alpha_{0^+} u(t)) \in C_{1-\beta}[0,T]$, and $D^\alpha_{0^+} u(t) \in C_r[0,T]$. Using ϕ_p to (2.4) and then multiply by $t^{1-\beta}$, we get

$$t^{1-\beta}\phi_p\bigl(D^\alpha_{0^+} u(t)\bigr) = \Gamma(\beta)\phi_p(h_0) E_{\beta,\beta}\bigl(-Mt^\beta\bigr) + t^{1-\beta}\int_0^t (t-s)^{\beta-1} E_{\beta,\beta}\bigl(-M(t-s)^\beta\bigr)\eta(s)\,ds,$$

and $t^r D^\alpha_{0^+} u(t)|_{t=0} = h_0$. Differentiating the above equation with order β, from Lemma 2.2, we find

$$D^\beta_{0^+}\bigl(\phi_p\bigl(D^\alpha_{0^+} u(t)\bigr)\bigr) + M\phi_p\bigl(D^\alpha_{0^+} u(t)\bigr) = \eta(t).$$

This completes the proof. □

Lemma 2.4 (Comparison result) *If $u(t) \in C^\alpha_r[0,T]$ and satisfies*

$$\begin{cases} D^\beta_{0^+}(\phi_p(D^\alpha_{0^+} u(t))) + M\phi_p(D^\alpha_{0^+} u(t)) \geq 0, & t \in (0,T], \\ \tilde{u}(0) \geq 0, \quad \overline{D^\alpha_{0^+} u}(0) \geq 0, \end{cases}$$

where M is a constant, then $D^\alpha_{0^+} u(t) \geq 0$ and $u(t) \geq 0$ for $t \in (0,T]$.

Proof Let $w(t) = \phi_p(D^\alpha_{0^+} u(t))$, then $w(t) \in C_{1-\beta}[0,T]$ and satisfies

$$\begin{cases} D^\beta_{0^+} w(t) + Mw(t) \geq 0, & t \in (0,T], \\ t^{1-\beta} w(t)|_{t=0} \geq 0, \end{cases}$$

hence $w(t) \geq 0$ for $t \in (0,T]$, by Lemma 2.2. Since $\phi_p(x)$ is nondecreasing, $u(t)$ satisfies

$$\begin{cases} D^\alpha_{0^+} u(t) \geq 0, & t \in (0,T], \\ \widetilde{u}(0) \geq 0. \end{cases}$$

Therefore we get $u(t) \geq 0$, $t \in (0,T]$ from Lemma 2.1. This lemma is complete. □

3 Main results and an example

We introduce the definition of a pair of lower and upper solutions for using the monotone iterative method.

Definition 3.1 A function $u(t) \in C_r^\alpha[0,T]$ is called a lower solution of problem (1.1) if it satisfies

$$\begin{cases} D_{0^+}^{\beta}(\phi_p(D_{0^+}^{\alpha}u(t))) \le f(t,u(t),D_{0^+}^{\alpha}u(t)), & t \in (0,T], \\ g(\tilde{u}(0),\tilde{u}(T)) \ge 0, & h(\overline{D_{0^+}^{\alpha}u}(0),\overline{D_{0^+}^{\alpha}u}(T)) \ge 0. \end{cases} \tag{3.1}$$

A function $v(t) \in C_r^\alpha[0,T]$ is called an upper solution of problem (1.1) if it satisfies

$$\begin{cases} D_{0^+}^{\beta}(\phi_p(D_{0^+}^{\alpha}v(t))) \ge f(t,v(t),D_{0^+}^{\alpha}v(t)), & t \in (0,T], \\ g(\tilde{v}(0),\tilde{v}(T)) \le 0, & h(\overline{D_{0^+}^{\alpha}v}(0),\overline{D_{0^+}^{\alpha}v}(T)) \le 0. \end{cases} \tag{3.2}$$

We need the following assumptions for our main results.

(H_1) Assume that $u_0, v_0 \in C_r^\alpha[0,T]$ are lower and upper solutions of the problem (1.1), respectively, and $u_0(t) \le v_0(t), t \in (0,T]$.

(H_2) There exists a constant M such that

$$f\bigl(t,u(t),D_{0^+}^{\alpha}u(t)\bigr) - f\bigl(t,v(t),D_{0^+}^{\alpha}v(t)\bigr) \le M\bigl[\phi_p\bigl(D_{0^+}^{\alpha}v(t)\bigr) - \phi_p\bigl(D_{0^+}^{\alpha}u(t)\bigr)\bigr]$$

for $u_0(t) \le u(t) \le v(t) \le v_0(t), D_{0^+}^{\alpha}u_0(t) \le D_{0^+}^{\alpha}u(t) \le D_{0^+}^{\alpha}v(t) \le D_{0^+}^{\alpha}v_0(t), t \in (0,T]$.

(H_3) There exist constants $\lambda_1 > 0, \lambda_2 \ge 0$ such that

$$g(x_1,y_1) - g(x_2,y_2) \le \lambda_1(x_2 - x_1) - \lambda_2(y_2 - y_1)$$

for $\tilde{u}_0(0) \le x_1 \le x_2 \le \tilde{v}_0(0)$ and $\tilde{u}_0(T) \le y_1 \le y_2 \le \tilde{v}_0(T)$.

(H_4) There exist constants $\mu_1 > 0, \mu_2 \ge 0$ such that

$$h(x_1,y_1) - h(x_2,y_2) \le \mu_1(x_2 - x_1) - \mu_2(y_2 - y_1)$$

for $\overline{D_{0^+}^{\alpha}u_0}(0) \le x_1 \le x_2 \le \overline{D_{0^+}^{\alpha}v_0}(0)$ and $\overline{D_{0^+}^{\alpha}u_0}(T) \le y_1 \le y_2 \le \overline{D_{0^+}^{\alpha}v_0}(T)$.

Theorem 3.1 *Assume that $f \in C([0,T] \times \mathbb{R} \times \mathbb{R}, \mathbb{R}), g \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R}), h \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ and (H_1)–(H_4) hold. Then there exist sequences $\{u_n(t)\}, \{v_n(t)\} \subset C_r^\alpha[0,T]$ such that $\lim_{n\to\infty} u_n = x, \lim_{n\to\infty} v_n = y$ on $(0,T]$ and x, y are minimal and maximal solutions on the interval $[u_0, v_0]$ of the problem (1.1), respectively, where*

$$[u_0,v_0] = \bigl\{u \in C_r^\alpha[0,T] : u_0(t) \le u(t) \le v_0(t), t \in (0,T], \tilde{u}_0(0) \le \tilde{u}(0) \le \tilde{v}_0(0)\bigr\}.$$

That is, for any solution $u \in [u_0, v_0]$,

$$u_0 \le u_1 \le \cdots \le u_n \le \cdots \le x \le u \le y \le \cdots \le v_n \le \cdots \le v_1 \le v_0$$

and

$$\begin{aligned} D_{0^+}^{\alpha}u_0 &\le D_{0^+}^{\alpha}u_1 \le \cdots \le D_{0^+}^{\alpha}u_n \le \cdots \le D_{0^+}^{\alpha}x \le D_{0^+}^{\alpha}u \le D_{0^+}^{\alpha}y \le \cdots \le D_{0^+}^{\alpha}v_n \le \cdots \\ &\le D_{0^+}^{\alpha}v_1 \le D_{0^+}^{\alpha}v_0. \end{aligned}$$

Proof Let $F(u(t)) := f(t, u(t), D^{\alpha}_{0^+}u(t))$. For $n = 1, 2, \dots$, we define

$$\begin{cases} D^{\beta}_{0^+}(\phi_p(D^{\alpha}_{0^+}u_n(t))) + M\phi_p(D^{\alpha}_{0^+}u_n(t)) = F(u_{n-1}(t)) + M\phi_p(D^{\alpha}_{0^+}u_{n-1}(t)), \\ \quad t \in (0, T], \\ \tilde{u}_n(0) = \tilde{u}_{n-1}(0) + \frac{1}{\lambda_1}g(\tilde{u}_{n-1}(0), \tilde{u}_{n-1}(T)), \\ \overline{D^{\alpha}_{0^+}u_n}(0) = \overline{D^{\alpha}_{0^+}u_{n-1}}(0) + \frac{1}{\mu_1}h(\overline{D^{\alpha}_{0^+}u_{n-1}}(0), \overline{D^{\alpha}_{0^+}u_{n-1}}(T)), \end{cases} \tag{3.3}$$

and

$$\begin{cases} D^{\beta}_{0^+}(\phi_p(D^{\alpha}_{0^+}v_n(t))) + M\phi_p(D^{\alpha}_{0^+}v_n(t)) = F(v_{n-1}(t)) + M\phi_p(D^{\alpha}_{0^+}v_{n-1}(t)), \\ \quad t \in (0, T], \\ \tilde{v}_n(0) = \tilde{v}_{n-1}(0) + \frac{1}{\lambda_1}g(\tilde{v}_{n-1}(0), \tilde{v}_{n-1}(T)), \\ \overline{D^{\alpha}_{0^+}v_n}(0) = \overline{D^{\alpha}_{0^+}v_{n-1}}(0) + \frac{1}{\mu_1}h(\overline{D^{\alpha}_{0^+}v_{n-1}}(0), \overline{D^{\alpha}_{0^+}v_{n-1}}(T)). \end{cases} \tag{3.4}$$

From $u_0, v_0 \in C^{\alpha}_r[0, T]$, we have $D^{\alpha}_{0^+}u_0(t), D^{\alpha}_{0^+}v_0(t) \in C_r[0, T]$ and $F(u_0(t)) + \phi_p(D^{\alpha}_{0^+}u_0(t))$, $F(v_0(t)) + \phi_p(D^{\alpha}_{0^+}v_0(t)) \in C_{1-\beta}[0, T]$. In view of Lemma 2.3, the functions u_1, v_1 are well defined in the space $C^{\alpha}_r[0, T]$. By induction, we can infer that u_n, v_n are well defined in the space $C^{\alpha}_r[0, T]$.

Firstly, we prove that $u_0(t) \le u_1(t) \le v_1(t) \le v_0(t)$ and $D^{\alpha}_{0^+}u_0(t) \le D^{\alpha}_{0^+}u_1(t) \le D^{\alpha}_{0^+}v_1(t) \le D^{\alpha}_{0^+}v_0(t)$ for $t \in (0, T]$.

Let $\delta(t) := \phi_p(D^{\alpha}_{0^+}u_1(t)) - \phi_p(D^{\alpha}_{0^+}u_0(t))$. The definition of u_1 and the assumption that u_0 is a lower solution imply

$$D^{\beta}_{0^+}\delta(t) + M\delta(t) = F\big(u_0(t)\big) - D^{\beta}_{0^+}(\phi_p\big(D^{\alpha}_{0^+}u_0(t)\big) \ge 0,$$

and $\tilde{u}_1(0) - \tilde{u}_0(0) = \frac{1}{\lambda_1}g(\tilde{u}_0(0), \tilde{u}_0(T)) \ge 0$, $t^rD^{\alpha}_{0^+}u_1(0) - t^rD^{\alpha}_{0^+}u_0(0) = \frac{1}{\mu_1}h(t^rD^{\alpha}_{0^+}u_0(0), t^rD^{\alpha}_{0^+}u_0(T)) \ge 0$, thus we have $D^{\alpha}_{0^+}u_0(t) \le D^{\alpha}_{0^+}u_1(t)$ and $u_1(t) \ge u_0(t), t \in (0, T]$ by Lemma 2.4.

Using a similar method, we can show that $v_1(t) \le v_0(t)$ and $D^{\alpha}_{0^+}v_1(t) \le D^{\alpha}_{0^+}v_0(t)$ for all $t \in (0, T]$. Now, we put $\xi(t) = \phi_p(D^{\alpha}_{0^+}v_1(t)) - \phi_p(D^{\alpha}_{0^+}u_1(t))$. From (3.3), (3.4) and (H_2), we have

$$D^{\beta}_{0^+}\xi(t) + M\xi(t) = F\big(v_0(t)\big) - F\big(u_0(t)\big) + M\big[\phi_p\big(D^{\alpha}_{0^+}v_0(t)\big) - \phi_p\big(D^{\alpha}_{0^+}u_0(t)\big)\big] \ge 0. \tag{3.5}$$

We find, by (H_3) and (H_1),

$$\begin{aligned} \tilde{v}_1(0) - \tilde{u}_1(0) &= \tilde{v}_0(0) + \frac{1}{\lambda_1}g\big(\tilde{v}_0(0), \tilde{v}_0(T)\big) - \left[u_0(0) + \frac{1}{\lambda_1}g\big(\tilde{u}_0(0), \tilde{u}_0(T)\big)\right] \\ &= \frac{1}{\lambda_1}\big[\lambda\big(\tilde{v}_0(0) - \tilde{u}_0(0)\big) + g\big(\tilde{v}_0(0), \tilde{v}_0(T)\big) - g\big(\tilde{u}_0(0), \tilde{u}_0(T)\big)\big] \\ &\ge \frac{1}{\lambda_1}\big[\lambda_1\big(\tilde{v}_0(0) - \tilde{u}_0(0)\big) - \lambda_1\big(\tilde{v}_0(0) - \tilde{u}_0(0)\big) + \lambda_2(\tilde{v}_0(T) - \tilde{u}_0(T)\big] \\ &= \frac{\lambda_2}{\lambda_1}\big(\tilde{v}_0(T) - \tilde{u}_0(T)\big) \ge 0. \end{aligned} \tag{3.6}$$

Similarly,

$$\overline{D^{\alpha}_{0^+}v_1}(0) - \overline{D^{\alpha}_{0^+}u_1}(0) \geq \frac{\mu_2}{\mu_1}\bigl(\overline{D^{\alpha}_{0^+}v_0}(T) - \overline{D^{\alpha}_{0^+}u_0}(T)\bigr) \geq 0. \tag{3.7}$$

It follows from (3.5)–(3.7) and Lemma 2.4 that $D^{\alpha}_{0^+}v_1(t) \geq D^{\alpha}_{0^+}u_1(t)$ and $v_1(t) \geq u_1(t), t \in (0, T]$.

Next, we show that u_1, v_1 are lower and upper solutions of problem (1.1), respectively. From (3.3) and conditions (H_2)–(H_4), we have

$$\begin{aligned} D^{\beta}_{0^+}\bigl(\phi_p\bigl(D^{\alpha}_{0^+}u_1(t)\bigr)\bigr) &= F\bigl(u_0(t)\bigr) - F\bigl(u_1(t)\bigr) + F\bigl(u_1(t)\bigr) \\ &\quad - M\bigl[\phi_p\bigl(D^{\alpha}_{0^+}u_1(t)\bigr) - \phi_p\bigl(D^{\alpha}_{0^+}u_0(t)\bigr)\bigr] \\ &\leq M\bigl[\phi_p\bigl(D^{\alpha}_{0^+}u_1(t)\bigr) - \phi_p\bigl(D^{\alpha}_{0^+}u_0(t)\bigr)\bigr] \\ &\quad - M\bigl[\phi_p\bigl(D^{\alpha}_{0^+}u_1(t)\bigr) - \phi_p\bigl(D^{\alpha}_{0^+}u_0(t)\bigr)\bigr] + F\bigl(u_1(t)\bigr) \\ &= F\bigl(u_1(t)\bigr) \end{aligned}$$

and

$$\begin{aligned} 0 &= g\bigl(\tilde{u}_0(0), \tilde{u}_0(T)\bigr) - g\bigl(\tilde{u}_1(0), \tilde{u}_1(T)\bigr) + g\bigl(\tilde{u}_1(0), \tilde{u}_1(T)\bigr) - \lambda_1\bigl[\tilde{u}_1(0) - \tilde{u}_0(0)\bigr] \\ &\leq g\bigl(\tilde{u}_1(0), \tilde{u}_1(T)\bigr) - \lambda_2\bigl(\tilde{u}_1(T) - \tilde{u}_0(T)\bigr), \\ 0 &= h\bigl(\overline{D^{\alpha}_{0^+}u_0}(0), \overline{D^{\alpha}_{0^+}u_0}(T)\bigr) - h\bigl(\overline{D^{\alpha}_{0^+}u_1}(0), \overline{D^{\alpha}_{0^+}u_1}(T)\bigr) \\ &\quad + h\bigl(\overline{D^{\alpha}_{0^+}u_1}(0), \overline{D^{\alpha}_{0^+}u_1}(T)\bigr) - \mu_1\bigl[\overline{D^{\alpha}_{0^+}u_1}(0) - \overline{D^{\alpha}_{0^+}u_0}(0)\bigr] \\ &\leq h\bigl(\overline{D^{\alpha}_{0^+}u_1}(0), \overline{D^{\alpha}_{0^+}u_1}(T)\bigr) - \mu_2\bigl(\overline{D^{\alpha}_{0^+}u_1}(T) - \overline{D^{\alpha}_{0^+}u_0}(T)\bigr). \end{aligned}$$

Since $\tilde{u}_1(T) \geq \tilde{u}_0(T), t^r D^{\alpha}_{0^+}u_1(T) \geq t^r D^{\alpha}_{0^+}u_0(T)$, the above inequality implies

$$g\bigl(\tilde{u}_1(0), \tilde{u}_1(T)\bigr) \geq 0, \qquad h\bigl(\overline{D^{\alpha}_{0^+}u_1}(0), \overline{D^{\alpha}_{0^+}u_1}(T)\bigr) \geq 0.$$

This proves that u_1 is a lower solution of the problem (1.1). Similarly, we can prove that v_1 is an upper solution of (1.1).

Using mathematical induction, we know that

$$\begin{aligned} &u_0(t) \leq u_1(t) \leq \cdots \leq u_n(t) \leq u_{n+1}(t) \leq v_{n+1}(t) \leq v_n(t) \leq \cdots \leq v_1(t) \leq v_0(t), \\ &D^{\alpha}_{0^+}u_0 \leq D^{\alpha}_{0^+}u_1 \leq \cdots \leq D^{\alpha}_{0^+}u_n \leq D^{\alpha}_{0^+}u_{n+1} \\ &\qquad \leq D^{\alpha}_{0^+}v_{n+1} \leq D^{\alpha}_{0^+}v_n \leq \cdots \leq D^{\alpha}_{0^+}v_1 \leq D^{\alpha}_{0^+}v_0, \end{aligned} \tag{3.8}$$

for $t \in (0, T]$ and $n = 1, 2, 3, \ldots$.

The sequences $\{t^{1-\alpha}u_n\}$ and $\{t^r D^{\alpha}_{0^+}u_n\}$ are uniformly bounded and equi-continuous [14]. Similarly, we can prove that the sequences $\{t^{1-\alpha}v_n\}$ and $\{t^r D^{\alpha}_{0^+}v_n\}$ are uniformly bounded and equi-continuous. The Arzela–Ascoli theorem guarantees that $\{t^{1-\alpha}u_n\}$ and $\{t^{1-\alpha}v_n\}$ converge to $t^{1-\alpha}x(t)$ and $t^{1-\alpha}y(t)$ uniformly on $[0, T]$, respectively; $\{t^r D^{\alpha}_{0^+}u_n\}$ and $\{t^r D^{\alpha}_{0^+}v_n\}$ converge to $\{t^r D^{\alpha}_{0^+}x(t)\}$ and $\{t^r D^{\alpha}_{0^+}y(t)\}$ uniformly on $[0, T]$, respectively. Therefore $\|u_n - x\|_{C^{\alpha}_r} \to 0, \|v_n - y\|_{C^{\alpha}_r} \to 0$ $(n \to \infty)$.

By the integral representation (2.2) for the linear fractional problem, the solution $u_n(t)$ of problem (3.3) can be expressed as

$$u_n(t) = t^{\alpha-1}k_{n-1} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\phi_q\biggl[\Gamma(\beta)\phi_p(h_{n-1})s^{\beta-1}E_{\beta,\beta}\bigl(-Ms^\beta\bigr) + \int_0^s (s-\tau)^{\beta-1}E_{\beta,\beta}\bigl(-M(s-\tau)^\beta\bigr)\eta_{n-1}(\tau)\biggr]\,d\tau, \quad t\in(0,T],$$

where $k_{n-1} = \widetilde{u}_{n-1}(0) + \frac{1}{\lambda}g(\widetilde{u}_{n-1}(0),\widetilde{u}_{n-1}(T))$, $\eta_{n-1}(s) = F(u_{n-1}(s)) + M\phi_p(D_{0^+}^\alpha u_{n-1}(s))$ and $h_{n-1} = \overline{D_{0^+}^\alpha u_{n-1}}(0) + \frac{1}{\mu_1}h(\overline{D_{0^+}^\alpha u_{n-1}}(0),\overline{D_{0^+}^\alpha u_{n-1}}(T))$.

By the assumption of f and applying the dominated convergence theorem, $x(t)$ satisfies the following integral equation:

$$x(t) = t^{\alpha-1}\widetilde{x}(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\phi_q\biggl[\Gamma(\beta)\phi_p(h_0)s^{\beta-1}E_{\beta,\beta}\bigl(-Ms^\beta\bigr) + \int_0^s (s-\tau)^{\beta-1}E_{\beta,\beta}\bigl(-M(s-\tau)^\beta\bigr)\eta(\tau)\biggr]\,d\tau, \quad t\in(0,T],$$

where $h_0 = \overline{D_{0^+}^\alpha x}(0)$, $\eta(s) = F(x(s)) + M\phi_p(D_{0^+}^\alpha x(s))$. By Lemma 2.3, we know $x(t)$ is a solution of problem (1.1). In the same way as above, we can prove that $y(t)$ is also a solution of problem (1.1), and satisfies $u_0 \le x \le y \le v_0$ on $(0,T]$.

To prove that $x(t), y(t)$ are extremal solutions of (1.1), let $u \in [u_0, v_0]$ be any solution of the problem (1.1). We suppose that $u_n \le u \le v_n, t \in (0,T]$ for some n. Let $\zeta(t) = \phi_p(D_{0^+}^\alpha u(t)) - \phi_p(D_{0^+}^\alpha u_{n+1}(t))$, $\varrho(t) = \phi_p(D_{0^+}^\alpha v_{n+1}(t)) - \phi_p(D_{0^+}^\alpha u(t))$. Then, by condition (H_2), we see that

$$D_{0^+}^\beta \zeta(t) + M\zeta(t) = F\bigl(u(t)\bigr) - F\bigl(u_n(t)\bigr) + M\bigl[\phi_p\bigl(D_{0^+}^\alpha u\bigr) - \phi_p\bigl(D_{0^+}^\alpha u_n\bigr)\bigr] \ge 0$$

and

$$D_{0^+}^\beta \varrho(t) + M\varrho(t) = F\bigl(v_n(t)\bigr) - F\bigl(u(t)\bigr) + M\bigl[\phi_p\bigl(D_{0^+}^\alpha v_n\bigr) - \phi_p\bigl(D_{0^+}^\alpha v\bigr)\bigr] \ge 0.$$

In addition, by condition (H_3), we have

$$\begin{aligned}\tilde{u}(0) - \tilde{u}_{n+1}(0) &= \tilde{u}(0) + \frac{1}{\lambda_1}g\bigl(\tilde{u}(0),\tilde{u}(T)\bigr) - \biggl[\tilde{u}_n(0) + \frac{1}{\lambda_1}g\bigl(\tilde{u}_n(0),\tilde{u}_n(T)\bigr)\biggr]\\ &= \frac{1}{\lambda_1}\bigl[\lambda_1\tilde{u}(0) + g\bigl(\tilde{u}(0),\tilde{u}(T)\bigr) - \bigl(\lambda_1\tilde{u}_n(0) + g\bigl(\tilde{u}_n(0),\tilde{u}_n(T)\bigr)\bigr)\bigr]\\ &\ge \frac{\lambda_2}{\lambda_1}\bigl(\tilde{u}(T) - \tilde{u}_n(T)\bigr) \ge 0\end{aligned}$$

and

$$\begin{aligned}\tilde{v}_{n+1}(0) - \tilde{u}(0) &= \tilde{v}_n(0) + \frac{1}{\lambda_1}g\bigl(\tilde{v}_n(0),\tilde{v}_n(T)\bigr) - \biggl[\tilde{u}(0) + \frac{1}{\lambda-1}g\bigl(\tilde{u}(0),\tilde{u}(T)\bigr)\biggr]\\ &= \frac{1}{\lambda_1}\bigl[\lambda_1\tilde{v}_n(0) + g\bigl(\tilde{u}(0),\tilde{u}(T)\bigr) - (\lambda_1\tilde{u}(0) + g\bigl(\tilde{u}_n(0),\tilde{u}_n(T)\bigr)\bigr]\end{aligned}$$

$$\geq \frac{\lambda_2}{\lambda_1}\big(\tilde{v}_n(T) - \tilde{u}(T)\big) \geq 0.$$

By condition (H_4), we have

$$\begin{aligned}
\overline{D_{0^+}^{\alpha} u}(0) - \overline{D_{0^+}^{\alpha} u_{n+1}}(0) &= \overline{D_{0^+}^{\alpha} u}(0) + \frac{1}{\mu_1} h\big(\overline{D_{0^+}^{\alpha} u}(0), \overline{D_{0^+}^{\alpha} u}(T)\big) \\
&\quad - \left[\overline{D_{0^+}^{\alpha} u_n}(0) + \frac{1}{\mu_1} h\big(\overline{D_{0^+}^{\alpha} u_n}(0), \overline{D_{0^+}^{\alpha} u_n}(T)\big)\right] \\
&= \frac{1}{\mu_1}\big[\mu_1 \overline{D_{0^+}^{\alpha} u}(0) + h\big(\overline{D_{0^+}^{\alpha} u}(0), \overline{D_{0^+}^{\alpha} u}(T)\big) \\
&\quad - \big(\mu_1 \overline{D_{0^+}^{\alpha} u_n}(0) + h\big(\overline{D_{0^+}^{\alpha} u_n}(0), \overline{D_{0^+}^{\alpha} u_n}(T)\big)\big)\big] \\
&\geq \frac{\mu_2}{\mu_1}\big(\overline{D_{0^+}^{\alpha} u}(T) - \overline{D_{0^+}^{\alpha} u_n}(T)\big) \geq 0
\end{aligned}$$

and

$$\overline{D_{0^+}^{\alpha} v_{n+1}}(0) - \overline{D_{0^+}^{\alpha} u}(0) \geq \frac{\mu_2}{\mu_1}\big(\overline{D_{0^+}^{\alpha} v_{n+1}}(T) - \overline{D_{0^+}^{\alpha} u}(T)\big) \geq 0.$$

Therefore, $D_{0^+}^{\alpha} u_{n+1}(t) \leq D_{0^+}^{\alpha} u(t) \leq D_{0^+}^{\alpha} v_{n+1}(t)$ and $u_{n+1}(t) \leq u(t) \leq v_{n+1}(t), t \in (0, T]$, furthermore, by induction $x(t) \leq u(t) \leq y(t), D_{0^+}^{\alpha} x \leq D_{0^+}^{\alpha} u \leq D_{0^+}^{\alpha} y$ on $(0, T]$ by taking $n \to \infty$. The proof is complete. □

Theorem 3.2 *The assumptions in Theorem 3.1 hold and there exists a constant N such that*

$$f\big(t, u(t), D_{0^+}^{\alpha} u(t)\big) - f\big(t, v(t), D_{0^+}^{\alpha} v(t)\big) \geq -N\big[\phi_p\big(D_{0^+}^{\alpha} u(t)\big) - \phi_p\big(D_{0^+}^{\alpha} v(t)\big)\big] \tag{3.9}$$

for $u_0(t) \leq u(t) \leq v(t) \leq v_0(t), D_{0^+}^{\alpha} u_0(t) \leq D_{0^+}^{\alpha} u(t) \leq D_{0^+}^{\alpha} v(t) \leq D_{0^+}^{\alpha} v_0(t), t \in (0, T]$, and $\widetilde{u}_0(0) = \widetilde{v}_0(0)$, $\overline{D_{0^+}^{\alpha} u_0}(0) = \overline{D_{0^+}^{\alpha} v_0}(0)$. Then problem (1.1) has a unique solution in the order interval $[u_0, v_0]$.

Proof From Theorem 3.1, we know $x(t)$ and $y(t)$ are extremal solutions and $x(t) \leq y(t), t \in (0, T]$. It is sufficient to prove $x(t) \geq y(t), t \in (0, T]$.

In fact, by (3.8) and $\overline{D_{0^+}^{\alpha} u_0}(0) = \overline{D_{0^+}^{\alpha} v_0}(0)$, we know $\overline{D_{0^+}^{\alpha} x}(0) = \overline{D_{0^+}^{\alpha} y}(0)$. Let $w(t) = \phi_p(D_{0^+}^{\alpha} x(t)) - \phi_p(D_{0^+}^{\alpha} y(t)), t \in (0, T]$, we have, from (3.9),

$$\begin{cases} D_{0^+}^{\beta} w(t) = F(x(t)) - F(y(t)) \geq -N[\phi_p(D_{0^+}^{\alpha} x(t)) - \phi_p(D_{0^+}^{\alpha} y(t))] = -Nw(t), \\ t^{1-\beta} w(t)|_{t=0} = 0. \end{cases}$$

Then $w(t) \geq 0, t \in (0, T]$, i.e. $D_{0^+}^{\alpha} x(t) \geq D_{0^+}^{\alpha} y(t), t \in (0, T]$. And also by (3.8) and $\widetilde{u}_0(0) = \widetilde{v}_0(0)$, we have $\widetilde{x}(0) = \widetilde{y}(0)$, Lemma 2.4 implies $x(t) \geq y(t), t \in (0, T]$. Thus, we obtain $x(t) = y(t)$. The problem (1.1) has a unique solution. The proof is complete. □

Finally, we present an example to illustrate Theorem 3.1.

Example 3.1 Consider the following fractional periodic boundary value problem:

$$\begin{cases} D_{0^+}^{\beta}(\phi_p(D_{0^+}^{\alpha}u(t))) = t^{1/2}(1-t) - 2[D_{0^+}^{\alpha}u(t)]^2 + u(t), \quad t \in (0,1], \\ \widetilde{u}(0)(\frac{\Gamma(5/6)}{2\Gamma(4/3)} - \widetilde{u}(1)) = 0, \\ (\frac{1}{2} + \overline{D_{0^+}^{\alpha}u}(0))(1 - \overline{D_{0^+}^{\alpha}u}(1)) = 0, \end{cases} \tag{3.10}$$

where $\alpha = 1/2, \beta = 2/3, p = 3, T = 1, f(t,u,D_{0^+}^{\alpha}u) = t^{1/2}(1-t) - 2[D_{0^+}^{\alpha}u(t)]^2 + u(t), g(x,y) = x(\frac{\Gamma(5/6)}{2\Gamma(4/3)} - y)$, and $h(x,y) = (\frac{1}{2} + x)(1-y)$.

Set

$$u_0(t) \equiv 0, \qquad v_0(t) = \frac{\Gamma(5/6)}{\Gamma(4/3)}t^{1/3}, \quad t \in [0,1].$$

It is easily verified that $D_{0^+}^{1/2}u_0(t) \equiv 0, D_{0^+}^{1/2}v_0(t) = t^{-1/6}$ for $t \in (0,1]$ and

$$t^{1/6}D_{0^+}^{1/2}u_0(t)|_{t=0} = t^{1/6}D_{0^+}^{1/2}u_0(t)|_{t=1} = 0, \qquad t^{1/6}D_{0^+}^{1/2}v_0(t)|_{t=0} = t^{1/6}D_{0^+}^{1/2}v_0(t)|_{t=1} = 1.$$

Therefore,

$$\begin{aligned} &D_{0^+}^{2/3}\big(\phi_3\big(D_{0^+}^{1/2}u_0(t)\big)\big) \equiv 0 \le f\big(t,u_0,D_{0^+}^{1/2}u_0\big) = t^{1/2}(1-t), \\ &g\big(\widetilde{u}_0(0),\widetilde{u}_0(1)\big) = 0, \qquad h\big(t^{1/6}D_{0^+}^{1/2}u_0(t)|_{t=0}, t^{1/6}D_{0^+}^{1/2}u_0(t)|_{t=1}\big) = \frac{1}{2}. \end{aligned}$$

These show that u_0 is a lower solution of (3.10). We have

$$\begin{aligned} D_{0^+}^{2/3}\big(\phi_3\big(D_{0^+}^{1/2}v_0(t)\big)\big) &= D_{0^+}^{2/3}\big(t^{-1/3}\big) = 0 \ge f\big(t,v_0,D_{0^+}^{1/2}v_0\big) \\ &= t^{1/2}(1-t) - 2t^{-1/3} + \frac{\Gamma(5/6)}{\Gamma(4/3)}t^{1/3}, \\ g\big(\widetilde{v}_0(0),\widetilde{v}_0(1)\big) = 0, h\big(&t^{1/6}D_{0^+}^{1/2}v_0(t)|_{t=0}, t^{1/6}D_{0^+}^{1/2}v_0(t)|_{t=1}\big) = 0. \end{aligned}$$

These show that v_0 is an upper solution of (3.10), and $u_0(t) \le v_0(t)$ on $[0,1]$.

For $u_0 \le u \le v \le v_0$, we have $\phi_3(D_{0^+}^{1/2}v) - \phi_3(D_{0^+}^{1/2}u) = (D_{0^+}^{1/2}v)^2 - (D_{0^+}^{1/2}u)^2$ and

$$f\big(t,u,D_{0^+}^{1/2}u\big) + 2\phi_3\big(D_{0^+}^{1/2}u\big) - \big[f\big(t,v,D_{0^+}^{1/2}v\big) + 2\phi_3\big(D_{0^+}^{1/2}v\big)\big] = u - v \le 0.$$

Thus, $f(t,u,D_{0^+}^{1/2}u) - f(t,v,D_{0^+}^{1/2}v) \le 2[\phi_3(D_{0^+}^{1/2}v) - \phi_3(D_{0^+}^{1/2}u)]$.

In addition, $\frac{\partial g(x,y)}{\partial x} = \frac{\Gamma(5/6)}{2\Gamma(4/3)} - y \ge -\frac{\Gamma(5/6)}{2\Gamma(4/3)}, \frac{\partial g(x,y)}{\partial y} = -x$ for $\widetilde{u}_0(0) \le x \le \widetilde{v}_0(0), y \in [\widetilde{u}_0(1), \widetilde{v}_0(1)] = [0, \frac{\Gamma(5/6)}{\Gamma(4/3)}]$. Therefore, $g(u_1,v_1) - g(u_2,v_2) \le \frac{\Gamma(5/6)}{2\Gamma(4/3)}(u_2 - u_1)$ for $\widetilde{u}_0(0) \le u_1 \le u_2 \le \widetilde{v}_0(0), \widetilde{u}_0(1) \le v_1 \le v_2 \le \widetilde{v}_0(1)$. In the same way, $h(u_1,v_1) - h(u_2,v_2) \le \frac{1}{2}(u_2 - u_1)$, for $t^{1/6}D_{0^+}^{1/2}u_0(t)|_{t=0} \le u_1 \le u_2 \le t^{1/6}D_{0^+}^{1/2}v_0(t)|_{t=0}, t^{1/6}D_{0^+}^{1/2}u_0(t)|_{t=1} \le v_1 \le v_2 \le t^{1/6}D_{0^+}^{1/2}v_0(t)|_{t=1}$.

Hence, conditions (H_1)–(H_4) are satisfied. There exist two monotone iterative sequences $\{u_k\}$ and $\{v_k\}$, which converge uniformly to the minimal and maximal solutions of problem (3.10) in $[u_0, v_0]$ by Theorem 3.1.

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