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New approach to solutions of a class of singular fractional q-differential problem via quantum calculus

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Abstract

In the present article, by using the fixed point technique and the Arzelà–Ascoli theorem on cones, we wish to investigate the existence of solutions for a non-linear problems regular and singular fractional q-differential equation

$$({}^cD_q^{\alpha}f)(t) = w(t, f(t), f'(t), ({}^cD_q^{\beta}f)(t)),$$

under the conditions $f(0) = c_1 f(1)$, $f'(0) = c_2({}^cD_q^{\beta}f)(1)$ and $f''(0) = f'''(0) = \cdots = f^{(n-1)}(0) = 0$, where $\alpha \in (n-1, n)$ with $n \geq 3$, $\beta, q \in J = (0, 1)$, $c_1 \in J$, $c_2 \in (0, \Gamma_q(2-\beta))$, the function w is L^{κ}-Carathéodory, $w(t, x_1, x_2, x_3)$ and may be singular and ${}^cD_q^{\alpha}$ the fractional Caputo type q-derivative. Of course, here we applied the definitions of the fractional q-derivative of Riemann–Liouville and Caputo type by presenting some examples with tables and algorithms; we will illustrate our results, too.

MSC: Primary 34A08; 34B16; secondary 39A13

Keywords: Singularity; Caputo q-derivative; Quantum calculus; q-differential

1 Introduction

The fractional calculus and q-calculus deal with the generalization of integration and differentiation of integer order to any order. It is known that fractional calculus is used for a better description of phenomena having both discrete and continuous behaviors, and applying in different sciences and engineering such as mechanics, electricity, biology, control theory, signal and image processing [1–12]. It has an old history and several fractional derivations where defined, such as the Caputo, the Riemann–Liouville and the Caputo and Fabrizio derivations. These derivations appeared recently in much work on integro-differential equations by using different views which young researchers could use for their work [13–27]. The fractional q-calculus has been applied to almost very field of non-linear mathematics analysis [28–38]. This branch of mathematics was introduced by Jackson in 1910 [1, 39]. For earlier work on the topic, we refer to [40, 41], whereas the preliminary concepts on q-fractional calculus can be found in [4]. For some applications of the

q-fractional calculus, see for example [2, 3, 5, 7, 8, 42–44]. Also, there has been a significant increase in knowledge in the field of differential and q-differential equations and inclusions in recent years [45–49].

In 2012, Ahmad *et al.*, studied the existence and uniqueness of solutions for the fractional q-difference equations ${}^cD_q^\alpha u(t) = T(t, u(t))$ with the boundary conditions $\alpha_1 u(0) - \beta_1 D_q u(0) = \gamma_1 u(\eta_1)$ and $\alpha_2 u(1) - \beta_2 D_q u(1) = \gamma_2 u(\eta_2)$, where $\alpha \in (1,2]$, α_i, β_i, γ_i, η_i are real numbers, for $i = 1,2$ and $T \in C(J \times \mathbb{R}, \mathbb{R})$ [34]. In 2013, Baleanu *et al.*, reviewed the nonlinear singular fractional problem $({}^cD^\alpha u)(t) = w(t, u(t), u'(t), ({}^cD^\beta u)(t))$, under the boundary conditions $u(0) = a_1 u(1)$, $u'(0) = a_2({}^cD^\beta u)(1)$ and $u''(0) = u'''(0) = \cdots = u^{(n-1)}(0) = 0$ on cones, where $\alpha \in (n-1, n)$ with an integer number $n \geq 3$, β, a_1, $a_2 \in J = (0,1)$, $(-\infty, 1)$, $(0, \Gamma(2-\beta))$, respectively, and w is a L^κ-Carathéodory function, $\kappa(\alpha - 1) > 1$, with the same conditions, which is was addressed by Agarwal *et al.* [50]. In 2013, Zhao *el al.* [38] reviewed the q-integral problem $(D_q^\alpha u)(t) + f(t, u(t)) = 0$, with the conditions $u(1)$, $u(0)$ equal to $\mu I_q^\beta u(\eta)$, 0, respectively, for almost all $t \in (0,1)$, where $q \in (0,1)$ and α, β, η belong to $(1,2]$, $(0,2]$, $(0,1)$, respectively, μ is positive real number, D_q^α is the q-derivative of Riemann–Liouville and real-values continuous map u defined on $I \times [0,\infty)$. In 2014, Jiang *et al.*, investigated the existence and uniqueness of solution of the problem $D_q^\beta(\phi_p(D_q^\alpha y(x))) + w(x, y(x), D_q^\gamma y(x)) = 0$, under the conditions $y(0) = D_q y(0) = D_q^\alpha y(0) = 0$ and $y(1) = \mu I_q y(\eta)$, by invoking the p-Laplacian operator, where w belongs to $C(E, \mathbb{R})$ with $E = [0,1] \times \mathbb{R}^2$, α and β, q, η, γ belong to in $(2,3)$ and $(0,1)$, respectively, $\mu > 0$ is constant, D_q^α is the fractional q-derivative of the Riemann–Liouville type, D_q and I_q denote the q-derivative and the q-integral, receptively, and ϕ_p is the p-Laplacian operator defined by $\phi_p(s) = |s|^{p-2}s$, with $p > 1$ [51].

Two year later, in 2016, Abdeljawad *et al.* [52] stated and proved a new discrete q-fractional version of the Gronwall inequality: $({}_qC_a^\alpha f)(t) = w(t, f(t))$ and $f(a) = \gamma$ such that $\alpha \in (0,1]$, $a \in \mathbb{T}_q = \{q^n : n \in \mathbb{Z}\}$, t belongs to $\mathbb{T}_a = [0,\infty)_q = \{q^{-i}a : i = 0,1,2,\ldots\}$, ${}_qC_a^\alpha$ means the Caputo fractional difference of order α and $w(t,x)$ fulfills a Lipschitz condition for all t and x. Then, in 2017, Zhou *et al.* [53] provided existence criteria for the solutions of the fractional Langevin differential equation under some conditions:

$$\begin{cases} D_{0^+}^\beta \phi_p[(D_{0^+}^\alpha + \eta)f(t)] = w(t, f(t), D_{0^+}^\alpha f(t)), \\ f(0) = -f(1), \qquad D_{0^+}^\alpha f(0) = -D_{0^+}^\alpha f(1), \end{cases}$$

and

$$\begin{cases} {}_qD_{0^+}^\beta \phi_p[(D_{0^+}^\alpha + \eta)f(t)] = w(t, f(t), {}_qD_{0^+}^\alpha f(t)), \\ f(0) = -f(1), \qquad {}_qD_{0^+}^\alpha f(0) = -{}_qD_{0^+}^\alpha f(1), \end{cases}$$

for each $t \in [0,1]$, where $0 < \alpha, \beta \leq 1$, η is larger than or equal to zero, $1 < \alpha + \beta < 2$, $q \in (0,1)$, and $\phi_p(s) = |s|^{p-2}s$, with $p \in (1,2]$. In 2017, Baleanu *et al.*, presented a new method to investigate some fractional integro-differential equations involving the Caputo–Fabrizio derivation,

$${}^{\mathrm{CF}}D^\alpha u(t) = \frac{(2-\alpha)M(\alpha)}{2(1-\alpha)} \int_0^t \exp\left(\frac{\alpha}{\alpha - 1}(t-s)\right) u'(s)\, ds,$$

where t is used and $M(\alpha)$ is a normalization constant depending on α such that $M(0) = M(1) = 1$; one proved the existence of approximate solutions for these problems [16]. In the same year, they introduced a new operator entitled the infinite coefficient-symmetric Caputo–Fabrizio fractional derivative and applied it to the investigation of the approximate solutions for two infinite coefficient-symmetric Caputo–Fabrizio fractional integro-differential problems [17].

In addition to, Akbari *et al.*, by using the shifted Legendre and Chebyshev polynomials, discussed the existence of solutions for a sum-type fractional integro-differential problem under the Caputo differentiation [19]. Over the past three years, Baleanu and Rezapour *et al.*, by using the Caputo–Fabrizio derivative, achieved innovation, and remarkable and interesting results were found for solutions of fractional differential equations [13–16, 18, 20–25]. In the next year, Rezapour *et al.*, investigated the existence of solutions for the inclusion ${}^cD^\alpha x(t) \in F(x, f(x), {}^cD^\beta f(x), f'(x))$ for each $x \in I$ with the conditions ${}^cD^\beta f(0) - \int_0^{\eta_1} f(r)\,dr = f(0) + f'(0)$ and ${}^cD^\beta f(1) - \int_0^{\eta_2} f(r)\,dr = f(1) + f'(1)$, where the multifunction F maps $[0,1] \times \mathbb{R}^3$ to $2^{\mathbb{R}}$ and is compact valued and ${}^cD^\alpha$ is the Caputo differential operator [54].

In 2019, Samei *et al.*, discussed the fractional hybrid q-differential inclusions ${}^cD_q^\alpha(x/F(t, x, I_q^{\alpha_1}x, \dots, I_q^{\alpha_n}x)) \in T(t, x, I_q^{\beta_1}x, \dots, I_q^{\beta_k}x)$, with the boundary conditions $x(0) = x_0$ and $x(1) = x_1$, where $1 < \alpha \leq 2$, $q \in (0,1)$, $x_0, x_1 \in \mathbb{R}$, $\alpha_i > 0$, for $i = 1, 2, \dots, n$, $\beta_j > 0$, for $j = 1, 2, \dots, k$, $n, k \in \mathbb{N}$, ${}^cD_q^\alpha$ denotes a Caputo type q-derivative of order α, I_q^β denotes the Riemann–Liouville type q-integral of order β, $F : J \times \mathbb{R}^n \to (0, \infty)$ is continuous and T mapping $J \times \mathbb{R}^k$ to $P(\mathbb{R})$ is a multifunction [32]. Also, they discussed the existence of solutions for the fractional q-derivative inclusions ${}^cD_q^\alpha x(t) \in F(t, x(t), x'(t), {}^cD_q^\beta x(t))$, $x(0) + x'(0) + {}^cD_q^\beta x(0) = \int_0^{\eta_1} x(s)\,ds$, and $x(1) + x'(1) + {}^cD_q^\beta x(1) = \int_0^{\eta_2} x(s)\,ds$, for any t in I and $q, \eta_1, \eta_2, \beta \in (0,1)$, where F maps $I \times \mathbb{R}^3$ into $2^{\mathbb{R}}$ is a compact valued multifunction and ${}^cD_q^\alpha$ is the fractional Caputo type q-derivative operator of order $\alpha \in (1,2]$, and $\Gamma_q(2-\beta)(\eta^2\nu - \nu^2\eta - \eta^2 + \nu^2 + 4\eta - 2\nu - 2) + 2(1-\eta) \neq 0$, such that $\alpha - \beta > 1$ [49]. In 2019, Samei *et al.* [32, 36], investigated the fractional hybrid q-difference inclusion, and also equations and inclusions of multi-term fractional q-integro-differential equations with non-separated and initial boundary conditions.

In this article, motivated by the main idea of the literature, we are going to investigate the problems of the fractional q-differential equation

$$\begin{cases} ({}^cD_q^\alpha f)(t) = w(t, f(t), f'(t), ({}^cD_q^\beta f)(t)), \\ f(0) = c_1 f(1), \\ f'(0) = c_2 ({}^cD_q^\beta f)(1), \\ f''(0) = f'''(0) = \cdots = f^{(n-1)}(0) = 0, \end{cases} \tag{1}$$

where $\alpha \in (n-1, n)$ with $n \geq 3$, $\beta, q \in J = (0,1)$, $c_1 \in J$, $c_2 \in (0, B)$ with $B = \Gamma_q(2-\beta)$, the function w is L^κ-Carathéodory being positive real valued and $\kappa(\alpha - 1) > 1$, $w(t, x_1, x_2, x_3)$ may be singular at the value 0 of its space variables x_1, x_2, x_3; ${}^cD_q^\alpha$ is the fractional Caputo type q-derivative.

This manuscript is organized as follows: In Sect. 2, we recall some preliminary concepts and fundamental results of q-calculus. Section 3 is devoted to the main results, while examples illustrating the obtained results and algorithm for the problems are presented in Sect. 4.

2 Preliminaries

First of all, we summarize the basic definitions and properties of q-calculus and q-fractional integrals and derivatives. One can find more information about them in [1–6, 8].

Suppose that $q \in (0,1)$ and $a \in \mathbb{R}$. Define $[a]_q = \frac{1-q^a}{1-q}$ [1]. The power function $(x-y)_q^n$ with $n \in \mathbb{N}_0$ is $(x-y)_q^{(n)} = \prod_{k=0}^{n-1}(x-yq^k)$ and $(x-y)_q^{(0)} = 1$ where $x, y \in \mathbb{R}$ and $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$ [1–3]. Also, for $\alpha \in \mathbb{R}$ and $a \neq 0$, we have $(x-y)_q^{(\alpha)} = x^\alpha \prod_{k=0}^{\infty}(x-yq^k)/(x-yq^{\alpha+k})$. If $y=0$, then it is clear that $x^{(\alpha)} = x^\alpha$ (Algorithm 1). The q-Gamma function is given by $\Gamma_q(z) = (1-q)^{(z-1)}/(1-q)^{z-1}$, where $z \in \mathbb{R}\setminus\{0,-1,-2,\dots\}$ [1, 2, 55, 56]. Note that $\Gamma_q(z+1) = [z]_q\Gamma_q(z)$. We show in Algorithm 2, a pseudo-code for estimating the q-Gamma function. The q-derivative of the function f, is defined by $(D_q f)(x) = \frac{f(x)-f(qx)}{(1-q)x}$ and $(D_q f)(0) = \lim_{x\to 0}(D_q f)(x)$ [2, 6, 57]. One can find in Algorithm 3 a pseudo-code for calculating the q-derivative of the function f. The higher-order q-derivative of a function f is defined by $(D_q^n f)(x) = D_q(D_q^{n-1}f)(x)$ for all $n \geq 1$, where $(D_q^0 f)(x) = f(x)$ [57]. The q-integral of a function f defined on $[0,b]$ is defined by

$$I_q f(x) = \int_0^x f(s)\, d_q s = x(1-q)\sum_{k=0}^{\infty} q^k f\left(xq^k\right),$$

for $x \in [0,b]$, provided that the series absolutely converges, which is shown in Algorithm 4 [57, 58]. If a is in $[0,b]$, then

$$\int_a^b f(u)\, d_q u = I_q f(b) - I_q f(a) = (1-q)\sum_{k=0}^{\infty} q^k\left[bf\left(bq^k\right) - af\left(aq^k\right)\right],$$

whenever the series exists. The operator I_q^n is given by $(I_q^0 h)(x) = h(x)$ and

$$\left(I_q^n h\right)(x) = \left(I_q\left(I_q^{n-1}h\right)\right)(x),$$

for $n \geq 1$ and $g \in C([0,b])$ which is shown in Algorithm 5 [57]. It has been proved that $(D_q(I_q f))(x) = f(x)$ and $(I_q(D_q f))(x) = f(x) - f(0)$ whenever f is continuous at $x = 0$ [2, 57, 58]. The fractional Riemann–Liouville type q-integral of the function f on J, of $\alpha \geq 0$ is given by $(I_q^0 f)(t) = f(t)$ and

$$\left(I_q^\alpha f\right)(t) = \frac{1}{\Gamma_q(\alpha)}\int_0^t (t-qs)^{(\alpha-1)} f(s)\, d_q s,$$

for $t \in J$ and $\alpha > 0$ [35, 55, 59]. Also, the fractional Caputo type q-derivative of the function f is given by

$$\begin{aligned}\left({}^c D_q^\alpha f\right)(t) &= \left(I_q^{[\alpha]-\alpha}\left(D_q^{[\alpha]} f\right)\right)(t) \\ &= \frac{1}{\Gamma_q([\alpha]-\alpha)}\int_0^t (t-qs)^{([\alpha]-\alpha-1)}\left(D_q^{[\alpha]} f\right)(s)\, d_q s,\end{aligned} \tag{2}$$

for $t \in J$ and $\alpha > 0$ [35, 59]. It has been proved that $(I_q^\beta(I_q^\alpha f))(x) = (I_q^{\alpha+\beta} f)(x)$ and $(D_q^\alpha(I_q^\alpha f))(x) = f(x)$, where α and β in $[0,\infty)$ [2, 35, 55, 59].

Let $\bar{J} = [0,1]$ and A be a subset of $\mathbb{R}^3$. We denote the space of functions whose κth powers of modulus are integrable on $\bar{J}$, endowed with norm $\|u\|_\kappa = (\int_0^1 |u(t)|^\kappa\, dt)^{1/\kappa}$ and the set of absolutely continuous functions on $\bar{J}$, by $L^\kappa(\bar{J})$ and $AC(\bar{J})$, respectively, where $\kappa \in [1,\infty)$.

Definition 1 We say that f is multi-singular when it is singular at more than one point t. Also, a real-valued and non-continuous function f on the interval $I = [a,b]$ is said to be singular whenever f is non-constant on I, and there exists a set S of measure 0 such that for x outside of S the derivative $f'(x)$ exists and is zero, that is, the derivative of f vanishes almost everywhere.

Definition 2 A function w is called L^κ-Carathéodory on $\overline{J} \times A$ whenever the real-valued function $w(\cdot, x_1, x_2, x_3)$ on $\overline{J}$ is measurable for all (x_1, x_2, x_3) belonging to A, the real-valued function $w(t, \cdot, \cdot, \cdot)$ defined on A is continuous for each t and belongs to $(0,1]$ and for each compact set $C \subset A$, there exists $\varphi_C \in L^\kappa(\overline{J})$, such that $|w(t, x_1, x_2, x_3)| \leq \varphi_C(t)$, for t belonging to $\overline{J}$ and $(x_1, x_2, x_3) \in C$.

Definition 3 A real value function f define on $\overline{J}$ is called a positive solution for problem (1), whenever $f(t)$ is more than to zero, ${}^cD_q^\alpha f$ is a function in $L^\kappa(\overline{J})$ and f satisfies the boundary conditions for all $t \in \overline{J}$.

Throughout the paper, we suppose that the function w in (1) has the following conditions:

(H1) The map w is an L^κ-Carathéodory on $\overline{J} \times A$, where $\kappa(\alpha - 1) > 1$ and it fulfills the estimate

$$w(t, x_1, x_2, x_3) \leq g_1(x_1) + g_2(|x_2|) + g_3(|x_3|) + \gamma(t)\theta(x_1, |x_2|, |x_3|),$$

for $t \in \overline{J}$ and (x_1, x_2, x_3) belonging to A, where positive valued functions g_1, g_2, g_3 in $C(\mathbb{R}^{>0})$ are decreasing, γ and θ in $L^\kappa(\overline{J})$ and $C(E)$ where $E = [0,\infty) \times [0,\infty) \times [0,\infty)$, respectively, are positive, w is increasing in all its arguments and $\lim_{y\to\infty} \frac{w(y,y,y)}{y} = 0$ and $\Gamma_q(\alpha)(I_q^\alpha g_i^\kappa)(1) < \infty$ for $i = 1, 2, 3$.

(H2) For each $t \in \overline{J}$ and (x_1, x_2, x_3) belongs to A, there exists $m > 0$ such that $m \leq w(t, x_1, x_2, x_3)$.

Since we imagine that problem (1) is singular, that is, $w(t, x_1, x_2, x_3)$ may be singular at the value zero of its space variables x_1, x_2 and x_3, we use regularization and sequential techniques for the existence of positive solutions of the problem. For this purpose, for each natural number n, define the function w_n on $\overline{J} \times A$ by

$$w_n(t, x_1, x_2, x_3) = w\big(t, \xi_n^+(x_1), \xi_n^+(x_2), \xi_n^+(x_3)\big),$$

where $\xi_n^+(u) = u$, whenever $f \geq \frac{1}{n}$ and $\xi_n^+(u) = \frac{1}{n}$, whenever $u < \frac{1}{n}$.

Remark 1 Since w is L^κ-Carathéodory, obviously w_n is an L^κ-Carathéodory function on $\overline{J} \times A$ and by assumption (H1), for each n, we get

$$\begin{aligned} w_n(t, x_1, x_2, x_3) \leq{}& g_1\left(\frac{1}{n}\right) + g_2\left(\frac{1}{n}\right) + g_3\left(\frac{1}{n}\right) \\ &+ \gamma(t)\theta\big(1 + x_1, 1 + |x_2|, 1 + |x_3|\big) \end{aligned}$$

and $w_n(t, x_1, x_2, x_3) \leq g_1(x_1) + g_2(|x_2|) + g_3(|x_3|) + \gamma(t)\theta(1 + x_1, 1 + |x_2|, 1 + |x_3|)$. Also, the condition (H2) entails that there exists a natural number m such that $m \leq w_n(t, x_1, x_2, x_3)$.

Lemma 4 ([60]) *Suppose that τ belongs to $L^{\kappa}(\bar{J})$ and $t_1, t_2 \in \bar{J}$. Then*

$$\bigl|\Gamma_q(\alpha-1)\bigl(I_q^{\alpha-1}\tau\bigr)(t)\bigr| \le \left(\frac{t^d}{d}\right)^{1/p}\|\tau\|_{\kappa},$$

for almost all t belongs to $\bar{J}$ and

$$\begin{aligned}&\left|\int_0^{t_2}(t_2-qs)^{(\alpha-2)}\tau(s)\,d_qs-\int_0^{t_1}(t_1-qs)^{(\alpha-2)}\tau(s)\,d_qs\right|\\&\quad\le\left(\frac{t_1^d+(t_2-t_1)^d-t_2^d}{d}\right)^{1/p}\|\tau\|_{\kappa}+\left(\frac{(t_2-t_1)^d}{d}\right)^{1/p}\|\tau\|_{\kappa},\end{aligned}$$

whenever $t_1 \le t_2$, here $d-1=(\alpha-2)p$ with $p=\frac{\kappa-1}{\kappa}$.

3 Main results

At present, we discuss the existence of solutions of problem (1). Foremost, we prove the key result.

Lemma 5 *Suppose that v belongs to $C(\bar{J})$. Then the boundary value problem*

$$\begin{cases}({}^cD_q^{\alpha}f)(t)=v(t),\\ f(0)=c_1f(1),\\ f'(0)=c_2({}^cD_q^{\beta}f)(1),\\ f''(0)=f'''(0)=\cdots=f^{(n-1)}(0)=0,\end{cases}\tag{3}$$

for each $t\in J$, where $c_1\in(n-1,n)$ with $n\ge 3$ and $c_2\in(0,B)$ with $B=\Gamma_q(2-\beta)$, is equivalent to the fractional integral equation $f(t)=\int_0^1 G_q(t,s)v(s)\,d_qs$, for all $s,t\in\bar{J}$, where

$$G_q(t,s)=\begin{cases}\frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}+\frac{c_1(1-qs)^{(\alpha-1)}}{(1-c_1)\Gamma_q(\alpha)}+\frac{c_2B(c_1+t-c_1t)(1-qs)^{(\alpha-\beta-1)}}{(1-a)\Gamma_q(\alpha-\beta)(B-c_2)}, & s\le t,\\ \frac{c_1(1-qs)^{(\alpha-1)}}{(1-c_1)\Gamma_q(\alpha)}+\frac{c_2B(c_1+t-c_1t)(1-qs)^{(\alpha-\beta-1)}}{(1-c_1)\Gamma_q(\alpha-\beta)(B-c_2)}, & t\le s.\end{cases}\tag{4}$$

Proof From $({}^cD_q^{\alpha})f(t)=v(t)$, for all t belonging to $(0,1)$ and the boundary conditions $f''(0)=f'''(0)=f^{(n-1)}(0)=0$, we obtain

$$\begin{aligned}f(t)&=\bigl(I_q^{\alpha}v\bigr)(t)+f(0)+f'(0)t+\frac{f''(0)}{2!}t^2+\cdots+\frac{f^{(n-1)}(0)}{(n-1)!}t^{n-1}\\&=\bigl(I_q^{\alpha}v\bigr)(t)+f(0)+f'(0)t.\end{aligned}$$

So, we obtain

$$\bigl({}^cD_q^{\beta}f\bigr)(t)=\bigl(I_q^{\alpha-\beta}v\bigr)(t)+\bigl({}^cD_q^{\beta}\bigr)\bigl(f(0)+f'(0)t\bigr)=\bigl(I_q^{\alpha-\beta}v\bigr)(t)+\frac{1}{B}f'(0)t^{1-\beta}.$$

Therefore, $f(1)=(I_q^{\alpha}v)(1)+f(0)+f'(0)$, and $({}^cD_q^{\beta}f)(1)=(I_q^{\alpha-\beta}v)(1)+\frac{1}{B}f'(0)$. By using the conditions of problem (3), we get $f(0)=c_1((I_q^{\alpha}v)(1)+f(0)+f'(0))$ and $f'(0)=c_2((I_q^{\alpha-\beta}v)(1)+$

$\frac{1}{B}f'(0))$. Hence,

$$f(0) = \frac{c_1}{(1-c_1)}\big(I_q^{\alpha}v\big)(1) + \frac{c_1c_2B}{(1-c_1)(B-c_2)}\big(I_q^{\alpha-\beta}v\big)(1)$$

and $f'(0) = \frac{c_2B}{B-c_2}(I_q^{\alpha-\beta}v)(1)$. We simply observe that

$$\begin{aligned} f(t) &= \big(I_q^{\alpha}v\big)(t)\,d_qs + f(0) + f'(0)t \\ &= \int_0^t \biggl[\frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} + \frac{c_1(1-qs)^{(\alpha-1)}}{(1-c_1)\Gamma_q(\alpha)} \\ &\quad + \frac{c_2B(c_1+t-c_1t)(1-qs)^{(\alpha-\beta-1)}}{(1-c_1)\Gamma_q(\alpha-\beta)(B-c_2)}\biggr]v(s)\,d_qs \\ &\quad + \int_t^1 \biggl[\frac{c_1(1-qs)^{(\alpha-1)}}{(1-c_1)\Gamma_q(\alpha)} \\ &\quad + \frac{c_2B(c_1+t-c_1t)(1-qs)^{(\alpha-\beta-1)}}{(1-c_1)\Gamma_q(\alpha-\beta)(B-c_2)}\biggr]v(s)\,d_qs \\ &= \int_0^1 G_q(t,s)v(s)\,d_qs. \end{aligned}$$

This completes our proof. □

For unification, we put $p = \frac{\kappa-1}{\kappa}$ with $\kappa \geq 1$, $d = (\alpha-2)p+1$,

$$\begin{aligned} \Lambda_1 &= \frac{1}{(1-c_1)\Gamma_q(\alpha)} + \frac{\Gamma_q(\alpha-\beta)(B-c_2) + c_2\Gamma_q(2-\beta)}{(1-c_1)\Gamma_q(\alpha-\beta)(\Gamma_q(2-\beta)-c_2)} \\ &= \frac{\Gamma_q(\alpha-\beta)(B-c_2) + c_2B\Gamma_q(\alpha)}{(1-c_1)\Gamma_q(\alpha)\Gamma_q(\alpha-\beta)(B-c_2)} \end{aligned} \tag{5}$$

and

$$\Lambda_2 = \frac{c_1c_2B}{(1-c_1)\Gamma_q(\alpha-\beta)(B-c_2)}. \tag{6}$$

Lemma 6 *The q-Green function $G_q(t,s)$ in Lemma 5, which belongs to $C(\overline{J}\times\overline{J})$, for all $(t,s)\in\overline{J}\times\overline{J}$, satisfies the conditions*:

(i) $G_q(t,s) \leq \Lambda_1(1-qs)^{(\alpha-\beta-1)} \leq 1$,

(ii) $G_q(t,s) \geq \Lambda_2(1-qs)^{(\alpha-\beta-1)}$.

Proof One can easy to check that $G_q(t,s) > 0$ on $\overline{J}\times\overline{J}$. Then from (5) and (6), we have

$$\begin{aligned} &\frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} + \frac{c_1(1-qs)^{(\alpha-1)}}{(1-c_1)\Gamma_q(\alpha)} + \frac{c_2B(c_1+t-c_1t)(1-qs)^{(\alpha-\beta-1)}}{(1-c_1)\Gamma_q(\alpha-\beta)(B-c_2)} \\ &\quad \leq \frac{(1-qs)^{(\alpha-\beta-1)}(\Gamma_q(\alpha-\beta)(B)-c_2) + c_2B\Gamma_q(\alpha)}{(1-c_1)\Gamma_q(\alpha)\Gamma_q(\alpha-\beta)(B-c_2)} \\ &\quad = \Lambda_1(1-qs)^{(\alpha-\beta-1)} \end{aligned}$$

and

$$\begin{aligned}
&\frac{c_1(1-qs)^{(\alpha-1)}}{(1-c_1)\Gamma_q(\alpha)} + \frac{c_2B(c_1+t-c_1t)(1-qs)^{(\alpha-\beta-1)}}{(1-c_1)\Gamma_q(\alpha-\beta)(B-c_2)} \\
&\quad \le \frac{(1-qs)^{(\alpha-\beta-1)}(c_1(1-qs)^{(-\beta)}}{(1-c_1)\Gamma_q(\alpha)} \\
&\qquad + \frac{c_2B(c_1+t-c_1t)}{(1-c_1)\Gamma_q(\alpha-\beta)(B-c_2)} \\
&\quad \le \Lambda_1(1-qs)^{(\alpha-\beta-1)},
\end{aligned}$$

whenever $s \le t$ and $t \le s$, respectively, for t and s in $\overline{J}$. Therefore, $G_q(t,s) \le \Lambda_1(1-qs)^{(\alpha-\beta-1)}$, for all (t,s) belonging to $\overline{J} \times \overline{J}$. Finally, it is observed that

$$\begin{aligned}
&(1-c_1)(B-c_2)\Gamma_q(\alpha-\beta)\Gamma_q(\alpha)G_q(t,s) \\
&\quad \ge c_2B\Gamma_q(\alpha)(c_1+t-c_1t)(1-qs)^{(\alpha-\beta-1)} \\
&\quad \ge c_1c_2B\Gamma_q(\alpha)(1-qs)^{(\alpha-\beta-1)}.
\end{aligned}$$

Therefore, $G_q(t,s) \ge \Lambda_2(1-qs)^{(\alpha-\beta-1)}$ for all (t,s) belonging to $\overline{J} \times \overline{J}$. □

Consider the Banach space $X = C^1(\overline{J})$ endowed with the norm $\|u\|_* = \max\{\|u\|, \|u'\|\}$ and the cone P on X, containing all the functions u belonging to X such that $u(t) \ge 0$ and $u'(t) \ge 0$ for all t. Now, we define an operator Θ_n on P by

$$(\Theta_n u)(t) = \int_0^1 G_q(t,s)T_n\big(s,f(s),f'(s),\big({}^cD_q^\beta f\big)(s)\big)\,d_qs.$$

At present, we show that the operator Θ_n is completely continuous [61].

Lemma 7 *Θ_n is a completely continuous operator, whenever the Θ_n satisfy conditions* $(H1)$ *and* $(H2)$ *for all natural number sn.*

Proof Consider an element $u \in P$. Then $u \in C(\overline{J})$. Also, u and u' are larger than or equal to zero. Therefore by the definition of ${}^cD_q^\beta$, we get $({}^cD_q^\beta u)(t) \in C(\overline{J})$ and $({}^cD_q^\beta u)(t) \ge 0$. Now, define $\tau(t) = w_n(t,f(t),f'(t),({}^cD_q^\beta f)(t))$. Then $\tau \in L^\kappa(\overline{J})$ and $\tau(t)$ higher than or equal to m for almost all $t \in \overline{J}$. It follows from $G_q(t,s) \ge 0$ for all (t,s) belonging to $\overline{J} \times \overline{J}$, from the equality

$$\begin{aligned}
(\Theta_n u)(t) &= \frac{a\Gamma_q(\alpha-\beta)(B-c_2)(1-qs)^{(\alpha-1)}}{(1-c_1)\Gamma_q(\alpha-\beta)(\Gamma_q(2-\beta)-c_2)}\big(I_q^\alpha\tau\big)(1) \\
&\quad + \frac{c_2B\Gamma_q(\alpha)(c_1+t-c_1t)(1-qs)^{(\alpha-\beta-1)}}{(1-c_1)\Gamma_q(\alpha)(B-c_2)}\big(I_q^{\alpha-\beta}\tau\big)(1) + \big(I_q^\alpha\tau\big)(t).
\end{aligned}$$

From the properties of I_q^α that $\Theta_n u \in C(\overline{J})$ and $(\Theta_n u)(t) \ge 0$ for all $t \in \overline{J}$ we have $(\Theta_n u)'(t) = (I_q^{\alpha-1}\tau)(t)$. Hence, $(\Theta_n u)' \in C(\overline{J})$ and $(\Theta_n u)'$ higher than or equal to zero, on $\overline{J}$. We test that the operator Θ_n is continuous. Suppose that the sequence $u_m \subset P$ is convergent and

$\lim_{m\to\infty} u_m = u$. Thus, $\lim_{m\to\infty} u_m^{(i)}(t) = u^{(i)}(t)$ uniformly on $\overline{J}$ for $i = 0, 1$. Since

$$\left({}^cD_q^\beta u\right)(t) = \frac{d}{dt}\left(I_q^{1-\beta}\right)\left(u(t) - u(0)\right) = \left(I_q^{1-\beta} u'\right)(t), \tag{7}$$

we get

$$\left|\left({}^cD_q^\beta u_m\right)(t) - \left({}^cD_q^\beta u\right)(t)\right| \le \frac{\|u'_m - u'\|}{\Gamma_q(1-\beta)} \int_0^t (t-qs)^{(-\beta)}\, d_qs \le \frac{\|u_m - u\|_*}{\Gamma_q(\beta)}$$

and $\lim_{m\to\infty}({}^cD_q^\beta u_m)(t) = ({}^cD_q^\beta u)(t)$ uniformly on $\overline{J}$. In addition, by using (7), we have $|({}^cD_q^\beta u_m)(t)| \le \frac{u'_m}{\Gamma_q(\beta)}$ and so

$$\left\|\left({}^cD_q^\beta u_m\right)\right\| \le \frac{\|u'_m\|}{\Gamma_q(\beta)}. \tag{8}$$

Put $\tau_m(t) = w_n(t, u_m(t), u'_m(t), ({}^cD_q^\beta u_m)(t))$ and $\tau(t) = w_n(t, u(t), u'(t), ({}^cD_q^\beta u)(t))$. Then $\lim_{m\to\infty} \tau_m(t) = \tau(t)$ and there exists $\mu \in L^\kappa(\overline{J})$ such that $0 \le \tau_m(t) \le \mu(t)$, for each t in $\overline{J}$ and natural number m. Since w_n is a L^κ-Carathéodory function, $\{u_m\}$, $\{({}^cD_q^\beta u_m)(t)\}$ are bounded in $C^1(\overline{J})$, $C(\overline{J})$, respectively. So, $\lim_{m\to\infty}(\Theta_n u_m)(t) = (\Theta_n u)(t)$ uniformly on $\overline{J}$. Since $\{\tau_m\}$ is L^κ-convergent on $\overline{J}$, we conclude that $\lim_{m\to\infty}(\Theta_n u_m)'(t) = \lim_{m\to\infty}(I_q^{\alpha-1}\tau_m)(t) = (\Theta_n u)'(t)$, uniformly on $\overline{J}$. Hence, the operator Θ_n is a continuous. We choose a positive constant r such that both $\|u_m\|$ and $\|u'_m\|$ are less than or equal to r for each natural number m, thus, we have $\Gamma_q(\beta)\|({}^cD_q^\beta u_m)(t)\| \le r$ and

$$\begin{aligned}
\left|\int_0^t (t-qs)^{(\alpha-2)} \tau_m(s)\, d_qs\right| &\le \left(\int_0^t (t-qs)^{((\alpha-2)p)}\, d_qs\right)^{\frac{1}{p}} \left(\int_0^t \left|\tau_m(s)\right|^\kappa d_qs\right)^{\frac{1}{\kappa}} \\
&\le \left(\frac{t^d}{d}\right)^{\frac{1}{p}} \|\tau_m\|_\kappa,
\end{aligned} \tag{9}$$

for all m. On the other hand, the relations

$$0 \le (\Theta_n u_m)(t) = \int_0^1 G_q(t,s)\tau_m(s)\, d_qs \le \int_0^1 G_q(t,s)\mu(s)\, d_qs \le \frac{\|\mu\|_1}{\Gamma_q(\alpha)}$$

and

$$0 \le (\Theta_n u_m)'(t) = \left(I_q^{\alpha-1}\tau_m\right)(t) \le \left(I_q^{\alpha-1}\mu\right)(t) \le \frac{1}{\Gamma_q(\alpha-1)}\left[\frac{1}{(\alpha-2)p+1}\right]^{\frac{1}{p}} \|\mu\|_\kappa,$$

hold for each t and m and so $\{\Theta_n u_m\}$ is bounded in $C(\overline{J})$. Moreover, it follows from Lemma 4 that

$$\begin{aligned}
\left|(\Theta_n u_m)'(t_2) - (\Theta_n u_m)'(t_1)\right| &= \left|\left(I_q^{\alpha-1}\right)\left(\tau_m(t_2) - \tau_m(t_1)\right)\right| \\
&\le \frac{\|\tau_m\|_\kappa}{\Gamma_q(\alpha-1)}\left[\left(\frac{t_1^d + (t_2-t_1)^d - t_2^d}{d}\right)^{\frac{1}{p}} + \left(\frac{(t_2-t_1)^d}{d}\right)^{\frac{1}{p}}\right] \\
&\le \frac{\|\mu\|_\kappa}{\Gamma_q(\alpha-1)}\left[\left(\frac{t_1^d + (t_2-t_1)^d - t_2^d}{d}\right)^{\frac{1}{p}} + \left(\frac{(t_2-t_1)^d}{d}\right)^{\frac{1}{p}}\right],
\end{aligned}$$

for each t_1 and t_2 belonging to $\overline{J}$ such that $t_1 \leq t_2$ is fulfilled. As a result, $\{(\Theta_n u_m)'\}$ is equicontinuous on $\overline{J}$. Consequently, based on the Arzelà–Ascoli theorem, $\{\Theta_n u_m\}$ is relatively compact in $C^1(\overline{J})$. Also, since Θ_n is continuous, we conclude that the operator Θ_n is completely continuous. □

Lemma 8 ([61, 62]) *Let X be a Banach space, $P \subset X$ a cone and $\mathcal{O}_1$ and $\mathcal{O}_2$ bounded open balls in X centered at the origin with $\overline{\mathcal{O}}_1 \subset \mathcal{O}_2$. A completely continuous operator w mapping $P \cap (\overline{\mathcal{O}}_2 \backslash \mathcal{O}_1)$ into P has a fixed point whenever $\|w(u)\| \geq \|u\|$ and $\|w(u)\| \leq \|u\|$ for $u \in P \cap \partial\mathcal{O}_1$ and $u \in P \cap \partial\mathcal{O}_2$, respectively.*

Theorem 9 *Let w satisfy conditions* (H1) *and* (H2). *Then problem* (1) *has a solution f_n in P such that*

$$f_n \geq \frac{m\Lambda_2}{\alpha-\beta}, \qquad f_n'(t) \geq \frac{mt^{\alpha-1}}{\Gamma_q(\alpha)}, \quad \text{and} \quad \left({}^cD_q^\beta f_n\right)(t) \geq \frac{mt^{\alpha-\beta}}{\Gamma_q(\alpha-\beta+1)}, \tag{10}$$

for all t belonging to $\overline{J}$ and the natural number n.

Proof By using Lemma 7, one can conclude that the operator $\Theta_n : P \to P$ is completely continuous. A function f is a solution of problem (1), whenever f solves the operator equation $f = \Theta_n f$. Finally, we demonstrate w_n in P is a fixed point of Θ_n with desired continuousness. For this purpose, it is observed that

$$\begin{aligned}(\Theta_n u)(t) &= \int_0^1 G_q(t,s) w_n\big(s, u(s), u'(s), \left({}^cD_q^\beta u\right)(s)\big)\, d_qs \\ &\geq m \int_0^1 G_q(t,s)\, d_qs \geq m \int_0^1 (1-t)^\alpha (1-qs)^{(\alpha-\beta-1)}\, d_qs \\ &= \frac{m\Lambda_2}{\alpha-\beta}\end{aligned} \tag{11}$$

and so $\|\Theta_n u\|_* \geq \|\Theta_n u\| \geq \frac{m\Lambda_2}{\alpha-\beta}$. Put

$$\mathcal{O}_1 = \left\{u \in X : \|u\|_* < \frac{m\Lambda_2}{\alpha-\beta}\right\}.$$

Then $\|\Theta_n u\|_* \geq \|u\|_*$ for all u belonging to $P \cap \partial\mathcal{O}_1$. Let $v_n = g_1(\frac{1}{n}) + g_2(\frac{1}{n}) + g_3(\frac{1}{n})$. Inequality (7) implies that

$$\begin{aligned}\left|(\Theta_n u)(t)\right| &\leq \left|\int_0^1 G_q(t,s) w_n\big(s, f(s), f'(s), \left({}^cD_q^\beta f\right)(s)\big)\, d_qs\right| \\ &\leq \int_0^1 \left|G_q(t,s)\right|\left[v_n + \gamma(s)\theta\big(1 + |u(s)|, 1 + |u'(s)|, 1 + \left|\left({}^cD_q^\beta u\right)(s)\right|\big)\right] d_qs \\ &\leq \Lambda_1\big(v_n + w\big(1 + \|u\|, 1 + \|u'\|, 1 + \left\|\left({}^cD^\beta u\right)\right\|\big)\|\gamma\|_1\big)\end{aligned}$$

and

$$\begin{aligned}\left|(\Theta_n u)'(t)\right| &= \left|\left(I_q^{\alpha-1} w_n\right)\big(t, u(t), u'(t), \left({}^cD_q^\beta u\right)(t)\big)\, d_qs\right| \\ &\leq \left(I_q^{\alpha-1}\big(v_n + \gamma(t)\theta\big(1 + |u(t)|, 1 + |u'(t)|, 1 + \left|\left({}^cD_q^\beta u\right)(t)\right|\big)\big)\right)\end{aligned}$$

$$+\frac{\nu_n t^{\alpha-1}}{(\alpha-1)\Gamma_q(\alpha-1)}$$
$$+ w\left(1+\|u\|, 1+\|u'\|, 1+\left\|\left({}^cD_q^{\beta}u\right)\right\|\right)\left(I_q^{\alpha-1}\gamma\right)(t),$$

for each $u \in P$ and all $t \in \overline{J}$, because w is increasing in all its arguments. Since $\|u\|$ and $\|u'\|$ are less than or equal to $\|u\|_*$, $\|({}^cD_q^{\beta}u)\| \le \frac{\|u'\|}{\Gamma_q(\beta)} \le \frac{\|u\|_*}{\Gamma_q(\beta)}$ and by inequality (9), $\int_0^t (t-qs)^{(\alpha-2)}\gamma(s)\,d_qs \le (\frac{1}{d})^{1/p}\|\gamma\|_\kappa$, we have

$$\left\|\Theta_n(x)\right\| \le \Lambda_1\left[\nu_n + w\left(1+\|u\|_*, 1+\|u\|_*, 1+\frac{\|u\|_*}{\Gamma_q(\beta)}\right)\|\gamma\|_1\right]$$

and

$$\left\|(\Theta_n u)'\right\| \le \frac{1}{\Gamma_q(\alpha-1)}\left[\frac{\nu_n}{\alpha-1} + w\left(1+\|u\|_*, 1+\|u\|_*, 1+\frac{\|u\|_*}{\Gamma_q(\beta)}\right)\left(\frac{1}{d}\right)^{1/p}\|\gamma\|_\kappa\right].$$

Therefore,

$$\|\Theta_n u\|_* \le M\left[\frac{\nu_n}{\alpha-1} + Nw\left(1+\|u\|_*, 1+\|u\|_*, 1+\frac{\|u\|_*}{\Gamma_q(\beta)}\right)\right],$$

where N and M are $\max\{\|\gamma\|_1, (\frac{1}{d})^{1/p}\|\gamma\|_\kappa\}$ and $\max\{\Lambda_1, \frac{1}{\Gamma_q(\alpha-1)}\}$, respectively. Since

$$\lim_{\nu\to\infty}\frac{w(1+\nu, 1+\nu, 1+\nu)}{\nu}$$

is equal to zero, by condition (H1), there exists a positive constant L such that

$$M\left[\frac{\nu_n}{\alpha-1} + Nw\left(1+\nu, 1+\nu, \frac{\nu}{\Gamma(\beta)}\right)\right] < \nu,$$

for each ν higher than or equal to L. Hence, $\|\Theta_n u\|_* < \|u\|_*$ for all u in P with $\|u\|_* \ge L$. Let $\mathcal{O}_2 = \{u \in X : \|u\|_* < L\}$, then $\|\theta_n u\|_* < \|u\|_*$ for $u \in P \cap \partial\Omega_2$. Now applying the last result, with X and $w = \Theta_n$, we conclude that Θ_n has a fixed point f_n in $P \cap (\overline{\mathcal{O}}_2 \setminus \mathcal{O}_1)$. Consequently, f_n is a solution of Problem (1). The first inequality follows from (11), $f_n = (\Theta_n f_n)(t) \ge \frac{m\Lambda_2}{\alpha-\beta}$, the second one follows from the relation

$$(\Theta_n u)'(t) = \left(I_q^{\alpha-1}w_n\right)\left(t, u(t), u'(t), \left({}^cD_q^{\beta}u\right)(t)\right) \ge \left(I_q^{\alpha-1}m\right) = \frac{mt^{\alpha-1}}{\Gamma_q(\alpha)},$$

for $t \in \overline{J}$ and u belongs to P. Finally, using the second inequality and $(I_q^{1-\beta}u)(t) = \frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha-\beta+1)}t^{\alpha-\beta}$, where $u(t) = t^{\alpha-1}$, we obtain

$$\left({}^cD_q^{\beta}f_n\right)(t) = \left(I_q^{1-\beta}f_n'\right)(t) \ge \frac{m}{\Gamma_q(\alpha)}\left(I_q^{1-\beta}h\right)(t) = \frac{mt^{\alpha-\beta}}{\Gamma_q(\alpha-\beta+1)},$$

for each t. This completes our proof. □

Theorem 10 *The problem* (1) *has a solution* f *such that* $(\alpha-\beta)f(t) \ge m\Lambda_2$, $\Gamma_q(\alpha)f'(t) \ge mt^{\alpha-1}$ *and* $\Gamma_q(\alpha-\beta+1)({}^cD_q^{\beta}f)(t) \ge mt^{\alpha-\beta}$, *for all* $t \in \overline{J}$, *whenever conditions* (H1) *and* (H2) *hold.*

Proof By using Theorem 9, for each n, problem (1) has a solution $f_n \in P$ which satisfies inequality (10). Hence

$$g_1\big(f_n(t)\big) \le g_1\left(\frac{m\Lambda_2}{\alpha-\beta}\right), \qquad g_2\big(\big|f_n'(t)\big|\big) \le g_2\left(\frac{mt^{\alpha-1}}{\Gamma_q(\alpha)}\right)$$

and

$$g_3\big(\big|\big({}^cD_q^\beta f_n\big)(t)\big|\big) \le g_3\left(\frac{mt^{\alpha-\beta}}{\Gamma_q(\alpha-\beta+1)}\right),$$

for each $t \in \overline{J}$ and all natural number n. In addition, it follows from (8) that $\|({}^cD_q^\beta f_n)\| \le \frac{\|f_n'\|}{\Gamma_q(\beta)}$. We put

$$F(t) = g_1\left(\frac{m\Lambda_2}{\alpha-\beta}\right) + g_2\left(\frac{mt^{\alpha-1}}{\Gamma_q(\alpha)}\right) + g_3\left(\frac{mt^{\alpha-\beta}}{\Gamma_q(\alpha-\beta+1)}\right). \tag{12}$$

Therefore, we conclude that

$$\begin{aligned} m &\le w_n\big(t, f_n(t), f_n'(t), \big({}^cD_q^\beta f_n\big)(t)\big) \\ &\le F(t) + \gamma(t)\theta\big(1 + \|f_n\|, 1 + \big\|f_n'\big\|, 1 + \big\|\big({}^cD_q^\beta f_n\big)\big\|\big) \\ &\le F(t) + \gamma(t)\theta\left(1 + \|f_n\|_*, 1 + \big\|f_n'\big\|_*, 1 + \frac{\|f_n\|_*}{\Gamma_q(\beta)}\right). \end{aligned}$$

Since we have a positive value $G_q(t,s) \le \Lambda_1$, we get

$$\begin{aligned} 0 \le f_n(t) &= \int_0^1 G_q(t,s) w_n\big(s, f_n(s), f_n'(s), \big({}^cD_q^\beta f_n\big)(s)\big)\, d_qs \\ &\le \Lambda_1\left[\int_0^1 F(qs)\, d_qs + w\left(1 + \|f_n\|_*, 1 + \|f_n\|_*, 1 + \frac{\|f_n\|_*}{\Gamma_q(\beta)}\right)\|\gamma\|_1\right] \end{aligned}$$

and

$$0 \le f_n'(t) \le \big(I_q^{\alpha-1}F\big)(t) + w\left(1 + \|f_n\|_*, 1 + \|f_n\|_*, 1 + \frac{\|f_n\|_*}{\Gamma_q(\beta)}\right)\big(I_q^{\alpha-1}\gamma\big)(t).$$

At present, we show that $\int_0^t (t-qs)^{(\alpha-2)} F(s)\, d_qs$ is bounded on $[0,1]$. By using the Hölder inequality, we get

$$\begin{aligned} &\int_0^1 (t-qs)^{(\alpha-2)} g_1\left(\frac{m\Lambda_2}{\alpha-\beta}\right) d_qs \\ &\quad = g_1\left(\frac{m\Lambda_2}{\alpha-\beta}\right)\int_0^1 (t-qs)^{(\alpha-2)}\, d_qs = \frac{1}{\alpha-1} g_1\left(\frac{m(1-t)^\alpha}{\alpha-\beta}\right) =: \lambda_1, \\ &\int_0^t (t-qs)^{(\alpha-2)} g_2\left(\frac{ms^{\alpha-1}}{\Gamma_q(\alpha)}\right) d_qs \\ &\quad = \left(\frac{1}{d}\right)^{1/p}\left(\frac{\Gamma_q(\alpha)}{m}\right)^{\frac{1}{(\alpha-1)\kappa}}\left[\int_0^{(\frac{m}{\Gamma_q(\alpha)})^{\frac{1}{\alpha-1}}} g_2^\kappa\big(s^{\alpha-1}\big)\, d_qs\right]^{1/\kappa} =: \lambda_2, \end{aligned}$$

and analogously

$$\begin{aligned}
&\int_0^t (t-qs)^{(\alpha-2)} g_3\left(\frac{ms^{\alpha-\beta}}{\Gamma_q(\alpha-\beta+1)}\right) d_qs \\
&= \left(\frac{1}{d}\right)^{1/p} \left(\frac{\Gamma_q(\alpha-\beta+1)}{m}\right)^{\frac{1}{(\alpha-\beta)\kappa}} \left[\int_0^{(\frac{m}{\Gamma_q(\alpha-\beta+1)})^{\frac{1}{\alpha-\beta}}} g_3^\kappa\left(s^{\alpha-\beta}\right) d_qs\right]^{1/\kappa} \\
&=: \lambda_3.
\end{aligned}$$

Note that (H1) guarantees $\lambda_j < \infty$ for $j = 1, 2$ and 3. Hence, for all $t \in \overline{J}$, we obtain

$$\int_0^t (t-qs)^{(\alpha-2)} F(s)\, d_qs \le \lambda,$$

where $\lambda = \lambda_1 + \lambda_2 + \lambda_3$. Also, we have

$$\begin{aligned}
\int_0^1 F(qs)\, d_qs \le{}& \frac{1}{\alpha-1} g_1\left(\frac{m\Lambda_2}{\alpha-\beta}\right) + \left(\frac{\Gamma_q(\alpha)}{m}\right)^{\frac{1}{\alpha-1}} \int_0^{(\frac{m}{\Gamma_q(\alpha)})^{\frac{1}{\alpha-1}}} g_2\left(s^{\alpha-1}\right) ds \\
&+ \left(\frac{\Gamma_q(\alpha-\beta+1)}{m}\right)^{\frac{1}{\alpha-\beta}} \int_0^{(\frac{m}{\Gamma_q(\alpha-\beta+1)})^{\frac{1}{\alpha-\beta}}} g_3\left(s^{\alpha-\beta}\right) d_qs \\
<{}& \infty.
\end{aligned}$$

Now, we conclude from the estimates

$$\|f_n\| = \Lambda_1\left[\int_0^1 F(qs)\, d_qs + w\left(1 + \|f_n\|_*, 1 + \|f_n\|_*, 1 + \frac{\|f_n\|_*}{\Gamma_q(\beta)}\right)\|\gamma\|_1\right]$$

and

$$\|f_n'\| \le \frac{1}{\Gamma_q(\alpha-1)}\left[\lambda + w\left(1 + \|f_n\|_*, 1 + \|f_n\|_*, 1 + \frac{\|f_n\|_*}{\Gamma_q(\beta)}\right)\left(\frac{1}{d}\right)^{1/p}\|\gamma\|_\kappa\right]$$

to the inequality

$$\|f_n\|_* \le M\left[\eta_1 + \eta_2 w\left(1 + \|f_n\|_*, 1 + \|f_n\|_*, 1 + \frac{\|f_n\|_*}{\Gamma_q(\beta)}\right)\right], \tag{13}$$

holding, for $n \ge 1$, where $M = \max\{\Lambda_1, \frac{1}{\Gamma_q(\alpha-1)}\}$, $\eta_1 = \max\{\lambda, \int_0^1 F(qs)\, d_qs\}$ and

$$\eta_2 = \max\left\{\|\gamma\|_1, \left(\frac{1}{d}\right)^{1/p}\|\gamma\|_\kappa\right\}.$$

Now, by condition (H1), there exists a positive constant L such that

$$M\left[\eta_1 + \eta_2 w\left(1 + \nu, 1 + \nu, 1 + \frac{\nu}{\Gamma_q(\beta)}\right)\right] < \nu,$$

for each ν higher than or equal to L. Now, inequality (13) gives $\|f_n\|_* < L$, for all n. Therefore

$$w_n\big(t,f_n(t),f_n'(t),\big({}^cD_q^\beta f_n\big)(t)\big) \le R(t),$$

where $R(t) = F(t) + \gamma(t)\theta(1+L, 1+L, 1+\frac{L}{\Gamma_q(\beta)})$. Note that, from condition (H1), R in $L^\kappa(\bar{J})$. Let

$$\tau_n(t) = w_n\big(t,f_n(t),f_n'(t),\big({}^cD_q^\beta f_n\big)(t)\big)$$

and $t_1, t_2 \in [0,\delta]$ such that $t_1 \le t_2$. Then

$$\begin{aligned}
\big|f_n'(t_2) - f_n'(t_1)\big| &= \big(I_q^{\alpha-1}\big)\big|\tau_n(t_2) - \tau_n(t_1)\big| \\
&\le \frac{1}{\Gamma_q(\alpha-1)}\bigg[\int_0^{t_1}\big((t_1-qs)^{(\alpha-2)} - (t_2-qs)^{(\alpha-2)}\big)\tau_n(s)\,d_qs \\
&\quad + \int_{t_1}^{t_2}(t_2-qs)^{(\alpha-2)}\tau_n(s)\,d_qs\bigg] \\
&\le \frac{1}{\Gamma_q(\alpha-1)}\bigg[\int_0^{t_1}\big((t_1-qs)^{(\alpha-2)} - (t_2-qs)^{(\alpha-2)}\big)R(s)\,d_qs \\
&\quad + \int_{t_1}^{t_2}(t_2-qs)^{(\alpha-2)}R(s)\,d_qs\bigg]
\end{aligned}$$

and so, by applying Lemma 4, we get

$$\big|f_n'(t_2) - f_n'(t_1)\big| \le \frac{\|R\|_\kappa}{\Gamma_q(\alpha-1)}\bigg[\bigg(\frac{t_1^d + (t_2-t_1)^d - t_2^d}{d}\bigg)^{1/p} + \bigg(\frac{(t_2-t_1)^d}{d}\bigg)^{1/p}\bigg].$$

As a consequence, $\{f_n'\}$ is equicontinuous on $\bar{J}$. Since $\{f_n\}$ is bounded in $C(\bar{J})$, without less of generality, we may assume that $\{f_n\}$ is convergent in $C(\bar{J})$ by the Arzelà–Ascoli theorem. Let $\lim_{n\to\infty} f_n = f$, then passing to the limit as $n \to \infty$, we obtain $({}^cD_q^\beta f_n)(t) = (I_q^{\alpha-1}f_n')(t)$ and using Eq. (7), we have

$$\lim_{n\to\infty}\big({}^cD_q^\beta f_n\big)(t) = \frac{1}{\Gamma_q(\alpha-1)}\int_0^t (t-qs)^{(-\beta)}f'(s)\,d_qs,$$

uniformly on $\bar{J}$. The last relation yields $\lim_{n\to\infty}({}^cD_q^\beta f_n)(t) = ({}^cD_q^\beta f)(t)$ in $C(\bar{J})$. Hence,

$$\lim_{n\to\infty} w_n\big(t,f_n(t),f_n'(t),\big({}^cD_q^\beta f_n\big)(t)\big) = w\big(t,f(t),f'(t),\big({}^cD_q^\beta f\big)(t)\big).$$

Since $R \in L^\kappa(\bar{J})$, by taking $n \to \infty$ in the equality

$$f_n(t) = \int_0^1 G_q(t,s)w_n\big(s,f_n(s),f_n'(s),\big({}^cD_q^\beta f_n\big)(s)\big)\,d_qs.$$

By using the dominated convergence theorem for $L^\kappa(\bar{J})$, we get

$$f(t) = \int_0^1 G_q(t,s)w\big(s,f(s),f'(s),\big({}^cD_q^\beta f\big)(s)\big)\,d_qs.$$

Consequently, f is a solution of problem (1), satisfying the boundary conditions. This completes our proof. □

4 Algorithms and examples

In this section, we give some algorithms to illustrate problem (1), in Theorems 10 and present numerical examples. Foremost, we present a simplified analysis that can be executed to calculate the value of q-Gamma function, $\Gamma_q(x)$, for input q, x and different values of n. To this aim, we consider a pseudo-code description of the method for calculating the q-Gamma function of order n in Algorithm 2 (for details, see the link https://en.wikipedia.org/wiki/Q-gamma_function). Now we give some examples to illustrate our results. Table 1 shows that when q is constant, the q-Gamma function is an increasing function. Also, for smaller values of x, an approximate result is obtained with smaller values of n. It is shown by underlined rows. Table 2 shows that the q-Gamma function for values q close to 1 is obtained with higher values of n in comparison with other columns. They have been underlined in line 8 of the first column, line 17 of the second column and line 29 of third columns of Table 2. Also, Table 3 is the same as Table 2, but x values increased in 3. Similarly, the q-Gamma function for values of q close to 1 is obtained with higher values of n in comparison with other columns.

Here, we provide an example to illustrate our main result.

Algorithm 1 The proposed method for calculating $(a-b)_q^{(\alpha)}$

Input: a, b, α, n, q

$s \leftarrow 1$
if $n = 0$ **then**
    $p \leftarrow 1$
else
    for $k = 0$ to n **do**
        $s \leftarrow s * (a - b * a^k)/(a - b * q^{\alpha+k})$
    end for
    $p \leftarrow a^\alpha * s$
end if

Output: $(a-b)^{(\alpha)}$

Algorithm 2 The proposed method for calculating $\Gamma_q(x)$

Input: $n, q \in (0,1), x \in \mathbb{R} \backslash \{0, -1, 2, \dots\}$

$p \leftarrow 1$
for $k = 0$ to n **do**
    $p \leftarrow p(1 - q^{k+1})(1 - q^{x+k})$
end for
$\Gamma_q(x) \leftarrow p/(1-q)^{x-1}$

Output: $\Gamma_q(x)$

Table 1 Some numerical results for calculation of $\Gamma_q(x)$ with $q = \frac{1}{3}$ that is constant, $x = 4.5, 8.4, 12.7$ and $n = 1, 2, \ldots, 15$ of Algorithm 2

n	$x = 4.5$	$x = 8.4$	$x = 12.7$	n	$x = 4.5$	$x = 8.4$	$x = 12.7$
1	2.472950	11.909360	68.080769	9	2.340263	11.257158	64.351366
2	2.383247	11.468397	65.559266	10	2.340250	11.257095	64.351003
3	2.354446	11.326853	64.749894	11	2.340245	11.257074	64.350881
4	2.344963	11.280255	64.483434	12	2.340244	11.257066	64.350841
5	2.341815	11.264786	64.394980	13	2.340243	11.257064	64.350828
6	2.340767	11.259636	64.365536	14	2.340243	11.257063	64.350823
7	2.340418	11.257921	64.355725	15	2.340243	11.257063	64.350822
8	2.340301	11.257349	64.352456				

Table 2 Some numerical results for calculation of $\Gamma_q(x)$ with $q = \frac{1}{3}, \frac{1}{2}, \frac{2}{3}$, $x = 5$ and $n = 1, 2, \ldots, 35$ of Algorithm 2

n	$q = \frac{1}{3}$	$q = \frac{1}{2}$	$q = \frac{2}{3}$	n	$q = \frac{1}{3}$	$q = \frac{1}{2}$	$q = \frac{2}{3}$
1	3.016535	6.291859	18.937427	18	2.853224	4.921884	8.476643
2	2.906140	5.548726	14.154784	19	2.853224	4.921879	8.474597
3	2.870699	5.222330	11.819974	20	2.853224	4.921877	8.473234
4	2.859031	5.069033	10.537540	21	2.853224	4.921876	8.472325
5	2.855157	4.994707	9.782069	22	2.853224	4.921876	8.471719
6	2.853868	4.958107	9.317265	23	2.853224	4.921875	8.471315
7	2.853438	4.939945	9.023265	24	2.853224	4.921875	8.471046
8	2.853295	4.930899	8.833940	25	2.853224	4.921875	8.470866
9	2.853247	4.926384	8.710584	26	2.853224	4.921875	8.470747
10	2.853232	4.924129	8.629588	27	2.853224	4.921875	8.470667
11	2.853226	4.923002	8.576133	28	2.853224	4.921875	8.470614
12	2.853224	4.922438	8.540736	29	2.853224	4.921875	8.470578
13	2.853224	4.922157	8.517243	30	2.853224	4.921875	8.470555
14	2.853224	4.922016	8.501627	31	2.853224	4.921875	8.470539
15	2.853224	4.921945	8.491237	32	2.853224	4.921875	8.470529
16	2.853224	4.921910	8.484320	33	2.853224	4.921875	8.470522
17	2.853224	4.921893	8.479713	34	2.853224	4.921875	8.470517

Table 3 Some numerical results for calculation of $\Gamma_q(x)$ with $x = 8.4$, $q = \frac{1}{3}, \frac{1}{2}, \frac{2}{3}$ and $n = 1, 2, \ldots, 40$ of Algorithm 2

n	$q = \frac{1}{3}$	$q = \frac{1}{2}$	$q = \frac{2}{3}$	n	$q = \frac{1}{3}$	$q = \frac{1}{2}$	$q = \frac{2}{3}$
1	11.909360	63.618604	664.767669	21	11.257063	49.065390	260.033372
2	11.468397	55.707508	474.800503	22	11.257063	49.065384	260.011354
3	11.326853	52.245122	384.795341	23	11.257063	49.065381	259.996678
4	11.280255	50.621828	336.326796	24	11.257063	49.065380	259.986893
5	11.264786	49.835472	308.146441	25	11.257063	49.065379	259.980371
6	11.259636	49.448420	290.958806	26	11.257063	49.065379	259.976023
7	11.257921	49.256401	280.150029	27	11.257063	49.065379	259.973124
8	11.257349	49.160766	273.216364	28	11.257063	49.065378	259.971192
9	11.257158	49.113041	268.710272	29	11.257063	49.065378	259.969903
10	11.257095	49.089202	265.756606	30	11.257063	49.065378	259.969044
11	11.257074	49.077288	263.809514	31	11.257063	49.065378	259.968472
12	11.257066	49.071333	262.521127	32	11.257063	49.065378	259.968090
13	11.257064	49.068355	261.666471	33	11.257063	49.065378	259.967836
14	11.257063	49.066867	261.098587	34	11.257063	49.065378	259.967666
15	11.257063	49.066123	260.720833	35	11.257063	49.065378	259.967553
16	11.257063	49.065751	260.469369	36	11.257063	49.065378	259.967478
17	11.257063	49.065564	260.301890	37	11.257063	49.065378	259.967427
18	11.257063	49.065471	260.190310	38	11.257063	49.065378	259.967394
19	11.257063	49.065425	260.115957	39	11.257063	49.065378	259.967371
20	11.257063	49.065402	260.066402	40	11.257063	49.065378	259.967357

Example 1 Let $\bar{J} = [0,1]$, τ_1 and τ_2 belongs to $L^\kappa(\bar{J})$ and $\tau_1(t)$ higher than or equal to positive real number m for all $t \in \bar{J}$. Also, let

$$w(t,x_1,x_2,x_3) = \tau_1(t) + \frac{1}{x_1^{2/5} - r} + \frac{1}{x_2^{1/4}} + \frac{1}{x_3^{1/4}} + \left|\tau_2(t)\right|\left(x_1^{2/5} + x_2^{1/4} + x_3^{1/4}\right),$$

on $\bar{J} \times A$ with $A = [0,\infty) \times [0,\infty) \times [0,\infty)$, $g_1(u) = \frac{1}{u^{2/5}-r}$ whenever $u^{2/5} \geq r$ and $g_1(u) = 0$ whenever $u^{2/5} < r$, $g_2(u) = \frac{1}{u^{1/4}}$, $g_3(u) = \frac{1}{u^{1/4}}$,

$$w(x_1,x_2,x_3) = x_1^{2/5} + x_2^{1/4} + x_3^{1/4} + 1$$

and $\gamma(t) = \tau_1(t) + |\tau_2(t)|$, where $r = (af(1))^{2/5}$. Since w satisfies conditions (H1) and (H2), Theorem 10 guarantees that problem (1) has a positive solution.

Example 2 In this example, we choose a problem similar to (1),

$$\begin{cases} {}^cD_q^{9/4}f(t) = t + 1 + \frac{1}{(f(t))^{2/5}-\lambda} + \frac{1}{(f'(t))^{1/4}} + \frac{1}{[({}^cD_q^{1/4}f)(t)]^{1/4}} \\ \qquad + 2(f(t)^{2/5} + f'(t)^{1/4} + [({}^cD_q^{1/4}f)(t)]^{1/4} + 1), \\ f(0) = \frac{1}{4}f(1), \\ f'(0) = \frac{1}{3}({}^cD_q^{1/4}f)(1), \\ f''(0) = f'''(0) = \cdots = f^{(n-1)}(0) = 0, \end{cases}$$

where $\lambda = (\frac{1}{4}f(1))^{1/3}$. here $\alpha = \frac{9}{4} \in (2,3)$, with $n = 3$, $\beta = \frac{1}{4} \in (0,1)$, $c_1 = \frac{1}{4} \in (0,1)$, $c_2 = \frac{1}{3} \in (0, \Gamma_q(\frac{7}{4}))$ for all $q \in (0,1)$ and $\kappa(\frac{9}{4} - 1) = \frac{4}{5} > 1$. Then

$$\begin{aligned} w\left(t,f(t),f'(t),\left({}^cD_q^{1/4}f\right)(t)\right) &= t + 1 + \frac{1}{f(t)^{1/3} - \lambda} + \frac{1}{f'(t)^{1/4}} \\ &\quad + \frac{1}{[({}^cD_q^{1/4}f)(t)]^{1/4}} \\ &\quad + 2\left(f(t)^{1/3} + r'(t)^{1/4} + \left[{}^cD_q^{1/4}f(t)\right]^{1/4} + 1\right), \end{aligned}$$

and w may be singular at $t = 0$ and satisfies conditions (H1) and (H2), for $g_1(h) = \frac{1}{h^{1/3}-\lambda}$ whenever $h^{1/3} - k \geq 0$ and $g_1(h) = 0$ whenever $h^{1/3} - \lambda < 0$, $g_2(h) = \frac{1}{h^{1/4}}$, $g_3(h) = \frac{1}{h^{1/4}}$,

$$w(x_1,x_2,x_3) = x_1^{1/3} + x_2^{1/4} + x_3^{1/4} + 1$$

and $\tau_1(t) = t + 1 > 1 = m$, $\tau_2(t) = 2$ and $\gamma(t) = \tau_1(t) + |\tau_2(t)|$, Theorem 10 guarantees that problem (1) has a positive solution. Now, we investigate the computational complexity of Example 2 of Algorithm 6 and 7. Note that n in Algorithms 6 and 7 is used for calculating $\Gamma_q(x)$. Tables 4, 5 and 6 show the values of Λ_1 and Λ_2 for $q = \frac{1}{3}, \frac{1}{2}$ and $\frac{3}{4}$, respectively, an approximate result is obtained with less than four decimal places indicated by underlining.

Algorithm 3 The proposed method for calculating $(D_q f)(x)$

Input: $q \in (0,1), f(x), x$

syms z
if $x = 0$ **then**
  $g \leftarrow \lim((f(z) - f(q * z))/((1-q)z), z, 0)$
else
  $g \leftarrow (f(x) - f(q * x))/((1-q)x)$
end if

Output: $(D_q f)(x)$

Algorithm 4 The proposed method for calculating $(I_q^\alpha f)(x)$

Input: $q \in (0,1), \alpha, n, f(x), x$

$s \leftarrow 0$
for $i = 0$ to n **do**
  $pf \leftarrow (1 - q^{i+1})^{\alpha-1}$
  $s \leftarrow s + pf * q^i * f(x * q^i)$
end for
$g \leftarrow (x^\alpha * (1-q) * s)/(\Gamma_q(x))$

Output: $(I_q^\alpha f)(x)$

Algorithm 5 The proposed method for calculating $\int_a^b f(r)\, d_q r$

Input: $q \in (0,1), \alpha, n, f(x), a, b$

$s \leftarrow 0$
for $i = 0 : n$ **do**
  $s \leftarrow s + q^i * (b * f(b * q^i) - a * f(a * q^i))$
end for
$g \leftarrow (1-q) * s$

Output: $\int_a^b f(r)\, d_q r$

Algorithm 6 The proposed method for calculating Λ_1

Input: $n, q \in (0,1), c_1, c_2, \alpha, \beta$

for $k = 0$ to n **do**
  $s_1 \leftarrow \Gamma_q(\alpha - \beta) * (B - c_2) + c_2 * \Gamma_q(\alpha) * \Gamma_q(2 - \beta)$
  $s_2 \leftarrow (1 - c_1) * \Gamma_q(\alpha) * \Gamma_q(\alpha - \beta) * (\Gamma_q(2 - \beta) - c_2)$
  $s \leftarrow s_1/s_2$
end for

Output: $\Lambda_1 = s$

Algorithm 7 The proposed method for calculating Λ_2

Input: $n, q \in (0,1), c_1, c_2, \alpha, \beta$

for $k = 0$ to n **do**
  $s_1 \leftarrow c_1 * c_2 * \Gamma_q(2 - \beta)$
  $s_2 \leftarrow (1 - c_1) * \Gamma_q(\alpha - \beta) * (\Gamma_q(2 - \beta) - c_2)$
  $s \leftarrow s_1/s_2$
end for

Output: $\Lambda_2 = s$

Table 4 Some numerical results for calculation of Λ_1 and Λ_2 with $q = \frac{1}{3}$ and $n = 1, 2, \ldots, 12$ of Example 2

n	$\Gamma_q(2-\beta)$	$\Gamma_q(\alpha-\beta)$	$\Gamma_q(\alpha)$	Λ_1	Λ_2
1	0.988977	1.038462	1.105539	−1.110973	−0.871523
2	0.968078	1.0125	1.074674	−1.146096	−0.90379
3	0.961333	1.004132	1.064736	−1.15789	−0.914692
4	0.959108	1.001374	1.061461	−1.16183	−0.918342
5	0.958369	1.000457	1.060373	−1.163145	−0.919561
6	0.958123	1.000152	1.060011	−1.163583	−0.919967
7	0.958041	1.000051	1.059891	−1.16373	−0.920103
8	0.958014	1.000017	1.05985	−1.163778	−0.920148
9	0.958005	1.000006	1.059837	−1.163794	−0.920163
10	0.958002	1.000002	1.059832	−1.1638	−0.920168
11	0.958001	1.000001	1.059831	−1.163802	−0.92017
12	0.958	1	1.05983	−1.163802	−0.92017

Table 5 Some numerical results for calculation of Λ_1 and Λ_2 with $q = \frac{1}{2}$ and $n = 1, 2, \ldots, 19$ of Example 2

n	$\Gamma_q(2-\beta)$	$\Gamma_q(\alpha-\beta)$	$\Gamma_q(\alpha)$	Λ_1	Λ_2
1	1.05421	1.142857	1.261962	−0.97516	−0.76776
2	0.996499	1.066667	1.165469	−1.062079	−0.845235
3	0.970276	1.032258	1.122114	−1.106468	−0.885437
4	0.957751	1.015873	1.101521	−1.128899	−0.905919
5	0.951628	1.007874	1.09148	−1.140174	−0.916256
6	0.9486	1.003922	1.086522	−1.145827	−0.921449
7	0.947094	1.001957	1.084058	−1.148657	−0.924052
8	0.946343	1.000978	1.08283	−1.150073	−0.925354
9	0.945968	1.000489	1.082217	−1.150782	−0.926006
10	0.945781	1.000244	1.081911	−1.151136	−0.926332
11	0.945687	1.000122	1.081758	−1.151313	−0.926495
12	0.945641	1.000061	1.081681	−1.151401	−0.926577
13	0.945617	1.000031	1.081643	−1.151446	−0.926618
14	0.945606	1.000015	1.081624	−1.151468	−0.926638
15	0.9456	1.000008	1.081614	−1.151479	−0.926648
16	0.945597	1.000004	1.081609	−1.151485	−0.926653
17	0.945595	1.000002	1.081607	−1.151487	−0.926656
18	0.945595	1.000001	1.081606	−1.151489	−0.926657
19	0.945594	1.000000	1.081605	−1.151489	−0.926658

Table 6 Some numerical results for calculation of Λ_1 and Λ_2 with $q = \frac{3}{4}$ and $n = 1, 2, \ldots, 30$ of Example 2

n	$\Gamma_q(2-\beta)$	$\Gamma_q(\alpha-\beta)$	$\Gamma_q(\alpha)$	Λ_1	Λ_2
2	1.253179	1.462857	1.751525	–0.705095	–0.558789
3	1.149887	1.31114	1.536689	–0.807011	–0.644426
4	1.084407	1.216513	1.40468	–0.886016	–0.712105
5	1.040678	1.154047	1.318456	–0.946732	–0.764915
6	1.010469	1.111251	1.259837	–0.993102	–0.805725
7	0.989113	1.08118	1.218879	–1.028354	–0.83703
8	0.973772	1.059674	1.189708	–1.055062	–0.860912
9	0.962624	1.044098	1.168644	–1.075245	–0.879054
10	0.954455	1.032713	1.153282	–1.090469	–0.892792
11	0.948434	1.024335	1.141999	–1.101936	–0.903171
12	0.943976	1.018141	1.133666	–1.110563	–0.910997
13	0.940664	1.013544	1.127488	–1.117049	–0.916891
14	0.938198	1.010124	1.122894	–1.121923	–0.921325
15	0.936358	1.007574	1.119471	–1.125583	–0.924658
16	0.934984	1.00567	1.116915	–1.12833	–0.927162
17	0.933956	1.004246	1.115006	–1.130393	–0.929043
18	0.933187	1.003181	1.113578	–1.13194	–0.930455
19	0.932611	1.002384	1.112509	–1.133102	–0.931514
20	0.93218	1.001787	1.111708	–1.133973	–0.932309
21	0.931857	1.00134	1.111109	–1.134627	–0.932906
22	0.931615	1.001004	1.110659	–1.135117	–0.933354
23	0.931433	1.000753	1.110322	–1.135485	–0.933689
24	0.931297	1.000565	1.11007	–1.13576	–0.933941
25	0.931195	1.000423	1.10988	–1.135967	–0.93413
26	0.931118	1.000318	1.109738	–1.136122	–0.934272
27	0.931061	1.000238	1.109632	–1.136239	–0.934378
28	0.931018	1.000179	1.109552	–1.136326	–0.934458
29	0.930986	1.000134	1.109492	–1.136392	–0.934518
30	0.930961	1.0001	1.109447	–1.136441	–0.934562

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