Note on the persistence and stability property of a stage-structured prey–predator model with cannibalism and constant attacking rate

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Abstract

We revisit the persistence and stability property of a stage-structured prey–predator model with cannibalism and constant attacking rate. By using the differential inequality theory and Bendixson–Dulac criterion, we show that if the system without cannibalism is permanent, then the system with cannibalism is also permanent. By developing some new analysis technique, we obtain a new set of sufficient conditions which ensure the global asymptotic stability of the nonnegative equilibrium, which means that, under some suitable assumption, prey cannibalism has no influence on the stability property of the predator free equilibrium. Our results essentially improve the corresponding results of Limin Zhang and Chaofeng Zhang.

Keywords: Stage structure; Predator–prey; Cannibalism; Permanence; Global asymptotical stability

1 Introduction

During the last decades, mathematical biology has become one of the important research areas [1–40]. Specially, many scholars investigated the dynamic behaviors of the stage-structured ecosystem, see [1–21, 39, 40], and the references cited therein.

In constructing the stage-structured model, some scholars [1–12] argued that a suitable stage-structured model should incorporate time delay, which reflects the period it takes to immature species to grow up to mature species. For example, Chen et al. [1] and Chen et al. [2] studied the persistence and extinction property of a stage-structured predator–prey system (stage structure for both predator and prey, respectively), they found that due to the influence of stage structure, the extinction of predator species could not directly imply the permanence of the prey species; the extinction of prey species could not lead to the extinction of predator species. Such a phenomenon could not be observed in the predator–prey system without stage structure. They argued that the reason maybe due to the assumption of the system that the predator species has another food resource. Also, Chen et al. [4] showed that due to the influence of the stage structure, the cooperative system could be driven to extinction.

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Some scholars [13–21, 39, 40] also proposed and studied the stage-structured ecosystem without time delay, they assumed that there are proportional numbers of immature species that become mature species at time t. For example, Lei[16] proposed the following stage-structured commensalism system:

$$
\begin{aligned}
\frac{dx_1}{dt} &= \alpha x_2 - \beta x_1 - \delta_1 x_1, \\
\frac{dx_2}{dt} &= \beta x_1 - \delta_2 x_2 - \gamma x_2^2 + dx_2 y, \\
\frac{dy}{dt} &= y(b_2 - a_2 y).
\end{aligned}
\tag{1.1}
$$

He found that if the stage-structured species (i.e., the first species) is extinct, then, depending on the intensity of cooperation, the species may still be extinct or become persistent. If the stage-structured species is permanent, then the final system is always globally asymptotically stable. Thus, he drew the conclusion that increasing the intensity of cooperation between the species is one of the very useful methods to avoid the extinction of the endangered species. Xu, Chaplain, and Davidson [40] proposed the following stage-structured prey–predator model:

$$
\begin{aligned}
\dot{x}_1(t) &= b_1 x_2 - g x_1 - d_1 x_1, \\
\dot{x}_2(t) &= g x_1 - d_2 x_2^2 - \frac{a_1 x_2 y}{my + x_2}, \\
\dot{y}(t) &= \left(\frac{a_2 x_2}{my + x_2} - d_3 \right) y,
\end{aligned}
\tag{1.2}
$$

where $x_1(t)$ represents the density of immature individual preys at time t and $x_2(t)$ denotes the density of mature individual preys at time t, $y(t)$ represents the density of the predator at time t. One could refer to [40] for more details about the construction of the system. Concerned with the persistence property of the system, the authors obtained the following result.

Theorem A *System* (1.2) *is uniformly persistent provided that*
   (H_1)

$$
a_2 > d_3,
\tag{1.3}
$$

   (H_2)

$$
\frac{b_1 g}{g + d_1} > \frac{a_1}{m}.
\tag{1.4}
$$

Cannibalism, which is the act of eating offspring, is observed to occur in some fish [41] and in some spiders [42]. Thus, incorporating cannibalism in the stage-structured models is more realistic and has more practical significance for some cannibals. Based on the work of Xu et al. [40], Zhang and Zhang [39] proposed the following stage-structured

prey–predator model with cannibalism for prey and constant attacking rate for predator:

$$\dot{x}_1(t) = b_1 x_2 - g x_1 - b_2 x_1 x_2 - d_1 x_1,$$

$$\dot{x}_2(t) = g x_1 + s b_2 x_1 x_2 - d_2 x_2^2 - \frac{a_1 x_2 y}{m y + x_2},$$

$$\dot{y}(t) = \left(\frac{a_2 x_2}{m y + x_2} - d_3 \right) y,$$

(1.5)

where all the coefficients have the same meaning as that of system (1.2). b_2 denotes the cannibalism attacking rate of the mature prey population; s is the conversion rate of the immature prey into the mature prey due to cannibalism, according to the biological meaning, $s < 1$. Concerned with the persistence property of system (1.5), the authors obtained the following result.

Theorem B *System* (1.5) *is permanent provided that*
  (H_3)

$$a_2 > d_3,$$

(1.6)

  (H_4)

$$\frac{g}{g + d_1} > \max \left\{ \frac{a_1}{b_1 m}, s \right\}$$

(1.7)

hold.

System (1.5) always admits a predator free equilibrium $E_1(\overline{x}_1, \overline{x}_2, 0)$, where

$$\overline{x}_1 = \frac{b_1 \overline{x}_2}{g + b_2 \overline{x}_2 + d_1}, \qquad \overline{x}_2 = \frac{-\delta + \sqrt{\delta^2 + 4 b_2 d_2 g b_1}}{2 b_2 d_2},$$

(1.8)

here $\delta = (g + d_1) d_2 - s b_2$.

Concerned with the stability property of the nonnegative equilibrium $E_1(\overline{x}_1, \overline{x}_2, 0)$, the authors obtained the following result (Theorem 3.3 in [39]).

Theorem C *Assume that*
  (H_5)

$$a_2 < d_3,$$

(1.9)

  (H_6)

$$\frac{g}{g + d_1} > \max \left\{ \frac{a_1}{b_1 m}, s \right\}$$

(1.10)

hold, then the nonnegative equilibrium $E_1(\overline{x}_1, \overline{x}_2, 0)$ is globally asymptotically stable.

Here, several interesting issues are proposed as follows:

1. Condition (1.7) is independent of coefficient b_2; however, b_2 denotes the cannibalism attacking rate of the mature prey population. Hence, the condition of Theorem B seems curious.

2. Comparing Theorems A and B, we found that to ensure permanence of system (1.5), one needs to require

$$s < \frac{g}{g + d_1} \tag{1.11}$$

holds, what would happen if

$$s \geq \frac{g}{g + d_1} \tag{1.12}$$

holds?

3. Conditions (1.9) and (1.10) are independent of coefficient b_2. Does b_2 really have no influence on the stability property of the nonnegative equilibrium?

4. In Theorem C, the authors only considered the case

$$s < \frac{g}{g + d_1}, \tag{1.13}$$

what would happen if

$$s \geq \frac{g}{g + d_1} \tag{1.14}$$

holds?

To find out the answer to the above issues, it seems better for us to consider some numeric examples, which may give some hints.
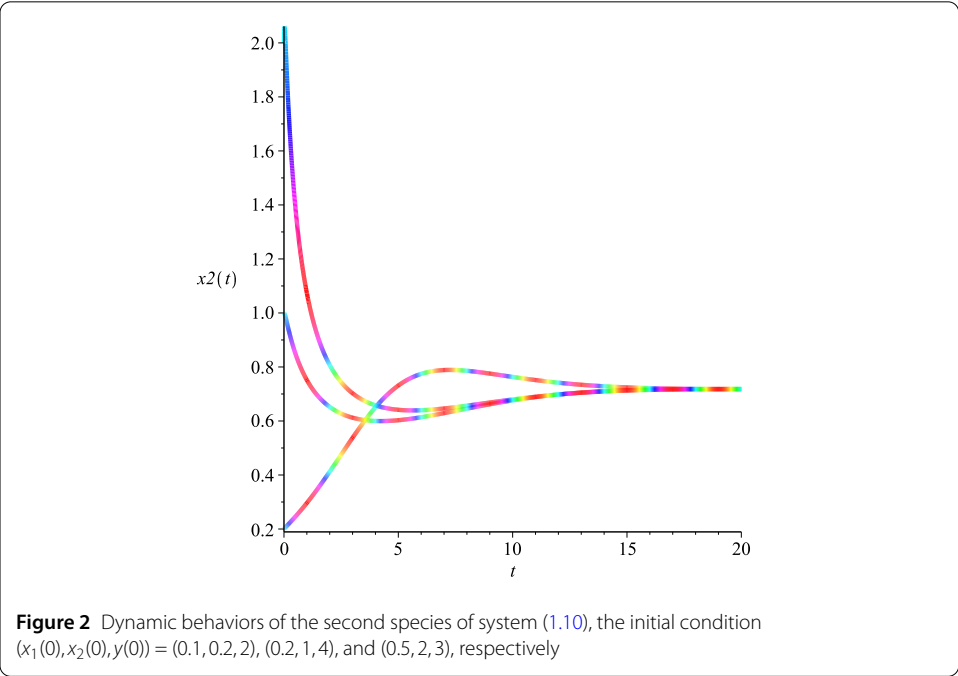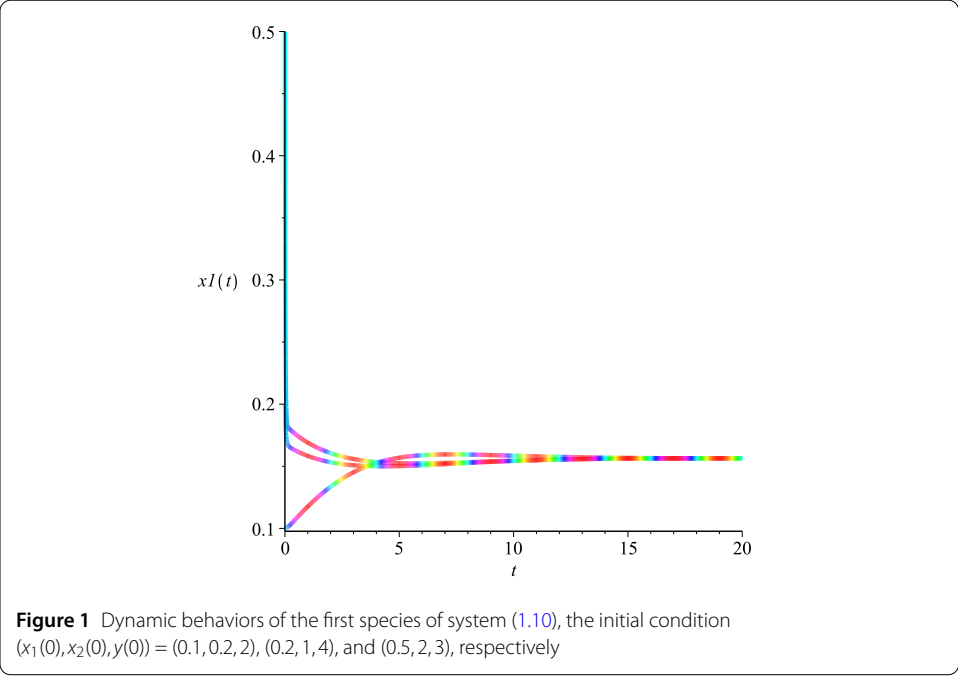
Example 1.1 Consider the following system:

$$
\begin{aligned}
\dot{x}_1(t) &= 4x_2 - 3x_1 - 10x_1x_2 - x_1, \\
\dot{x}_2(t) &= 3x_1 + \frac{1}{2}10x_1x_2 - x_2^2 - \frac{2x_2y}{y + x_2}, \\
\dot{y}(t) &= \left(\frac{x_2}{y + x_2} - \frac{1}{4}\right)y.
\end{aligned}
\tag{1.15}
$$

Here, we choose $b_1 = 4$, $m = 1$, $s = \frac{1}{2}$, $d_1 = 1$, $g = 3$, $a_2 = 1$, $d_3 = \frac{1}{4}$, $d_2 = 1$, $b_2 = 10$. Hence,

$$
\begin{aligned}
a_2 &= 1 > \frac{1}{4} = d_3, \\
\frac{g}{g + d_1} &= \frac{3}{3 + 1} > \frac{1}{2} = \max\left\{\frac{a_1}{b_1m}, s\right\}.
\end{aligned}
\tag{1.16}
$$

That is, the conditions of Theorem B hold. Also, here we choose $b_2 = 10$, which is very large when compared to the other coefficients. Numeric simulations (Figs. 1–3) show that in this case the system is permanent. Hence, we conjecture that the cannibalism attacking rate of the mature prey population has no influence on the persistence property of the system if the system without cannibalism is permanent.

Figure 1 Dynamic behaviors of the first species of system (1.10), the initial condition $(x_1(0), x_2(0), y(0)) = (0.1, 0.2, 2)$, $(0.2, 1, 4)$, and $(0.5, 2, 3)$, respectively

Figure 2 Dynamic behaviors of the second species of system (1.10), the initial condition $(x_1(0), x_2(0), y(0)) = (0.1, 0.2, 2)$, $(0.2, 1, 4)$, and $(0.5, 2, 3)$, respectively

Example 1.2　Consider the following system:

$$\dot{x}_1(t) = 4x_2 - x_1 - x_1 x_2 - x_1,$$

$$\dot{x}_2(t) = x_1 + \frac{3}{4}x_1 x_2 - x_2^2 - \frac{2x_2 y}{2y + x_2},$$

$$\dot{y}(t) = \left(\frac{x_2}{2y + x_2} - \frac{1}{4}\right)y.$$

(1.17)

Figure 3 Dynamic behaviors of the third species of system (1.10), the initial condition $(x_1(0), x_2(0), y(0)) = (0.1, 0.2, 2)$, $(0.2, 1, 4)$, and $(0.5, 2, 3)$, respectively
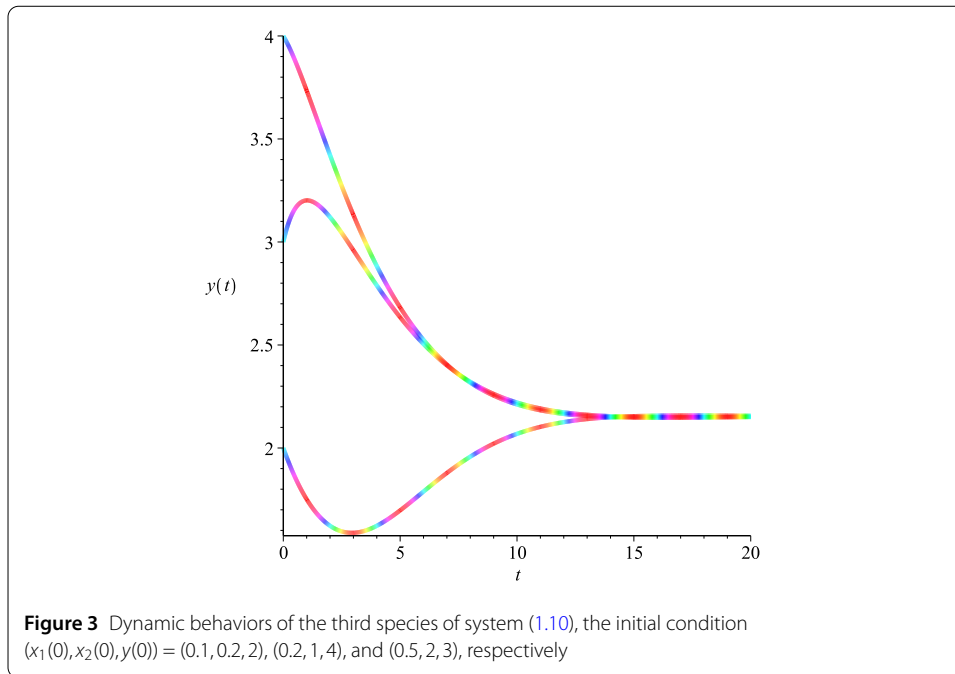
Here, we choose $b_1 = 4$, $m = 2$, $s = \frac{3}{4}$, $d_1 = 1$, $g = 1$, $a_2 = 1$, $d_3 = \frac{1}{4}$, $d_2 = 1$, $b_2 = 1$, $a_1 = 2$, hence

$$a_2 = 1 > \frac{1}{4} = d_3, \tag{1.18}$$

$$\frac{g}{g + d_1} = \frac{1}{1 + 1} > \frac{1}{4} = \frac{2}{4 \times 2} = \frac{a_1}{b_1 m}. \tag{1.19}$$
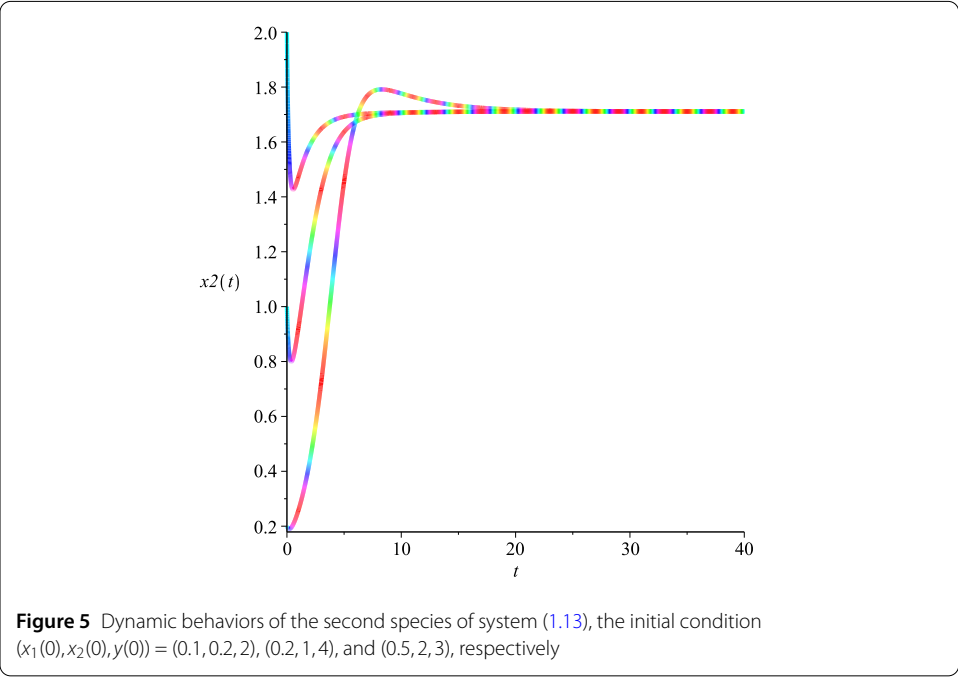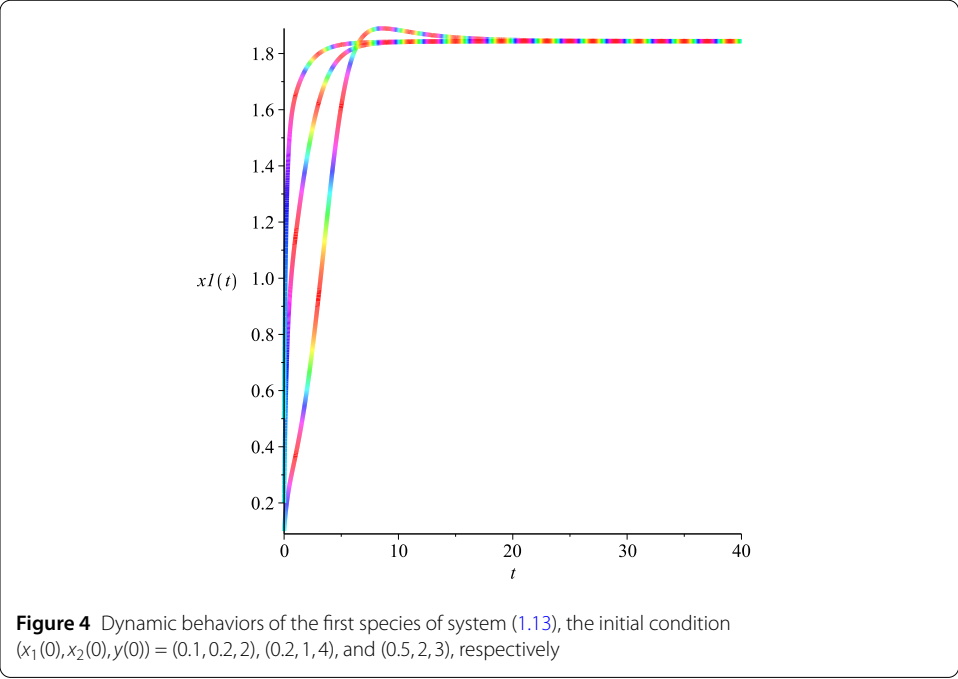
However,

$$\frac{g}{g + d_1} = \frac{1}{1 + 1} = \frac{1}{2} < \frac{3}{4} = s. \tag{1.20}$$

That is, the conditions of Theorem A hold, while the second inequality in Theorem B does not hold. Numeric simulations (Figs. 4–6) show that in this case the system is permanent.

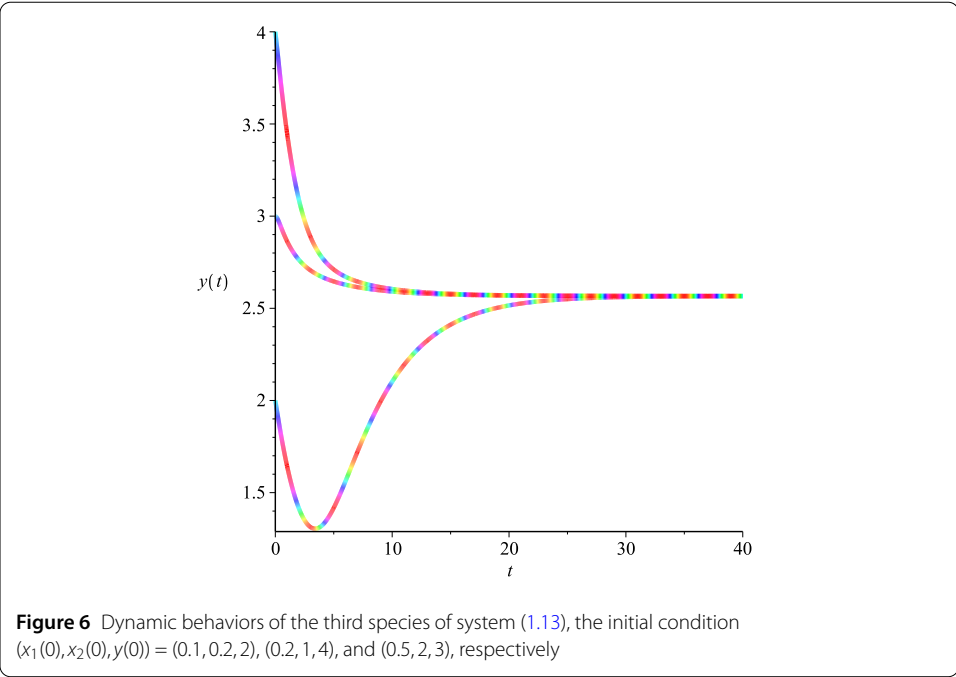Example 1.2 shows that maybe parameter s is not an essential coefficient to ensure the permanence of system (1.5), while Example 1.1 shows that maybe parameter b_2 is not an essential coefficient to ensure the permanence of system (1.5). This leads us to making the following conjecture.

Conjecture A Maybe in system (1.5) cannibalism has no influence on the persistence property of the system.

Figure 4 Dynamic behaviors of the first species of system (1.13), the initial condition $(x_1(0), x_2(0), y(0)) = (0.1, 0.2, 2)$, $(0.2, 1, 4)$, and $(0.5, 2, 3)$, respectively

Figure 5 Dynamic behaviors of the second species of system (1.13), the initial condition $(x_1(0), x_2(0), y(0)) = (0.1, 0.2, 2)$, $(0.2, 1, 4)$, and $(0.5, 2, 3)$, respectively

Example 1.3 Consider the following system:

$$
\begin{aligned}
\dot{x}_1(t) &= 4x_2 - x_1 - x_1 x_2 - x_1, \\
\dot{x}_2(t) &= x_1 + \frac{3}{4} x_1 x_2 - x_2^2 - \frac{2x_2 y}{2y + x_2}, \\
\dot{y}(t) &= \left(\frac{1}{2} \frac{x_2}{2y + x_2} - 1 \right) y.
\end{aligned}
\tag{1.21}
$$

Figure 6 Dynamic behaviors of the third species of system (1.13), the initial condition $(x_1(0), x_2(0), y(0)) = (0.1, 0.2, 2)$, $(0.2, 1, 4)$, and $(0.5, 2, 3)$, respectively

Here, we choose $b_1 = 4$, $m = 2$, $s = \frac{3}{4}$, $d_1 = 1$, $g = 1$, $a_2 = \frac{1}{2}$, $d_3 = \frac{1}{4}$, $d_2 = 1$, $b_2 = 1$, $a_1 = 2$, $d_3 = 1$, hence

$$a_2 = \frac{1}{2} < 1 = d_3, \tag{1.22}$$

$$\frac{g}{g + d_1} = \frac{1}{1 + 1} = \frac{1}{2} > \frac{1}{4} = \frac{2}{4 \times 2} = \frac{a_1}{b_1 m}. \tag{1.23}$$

However,

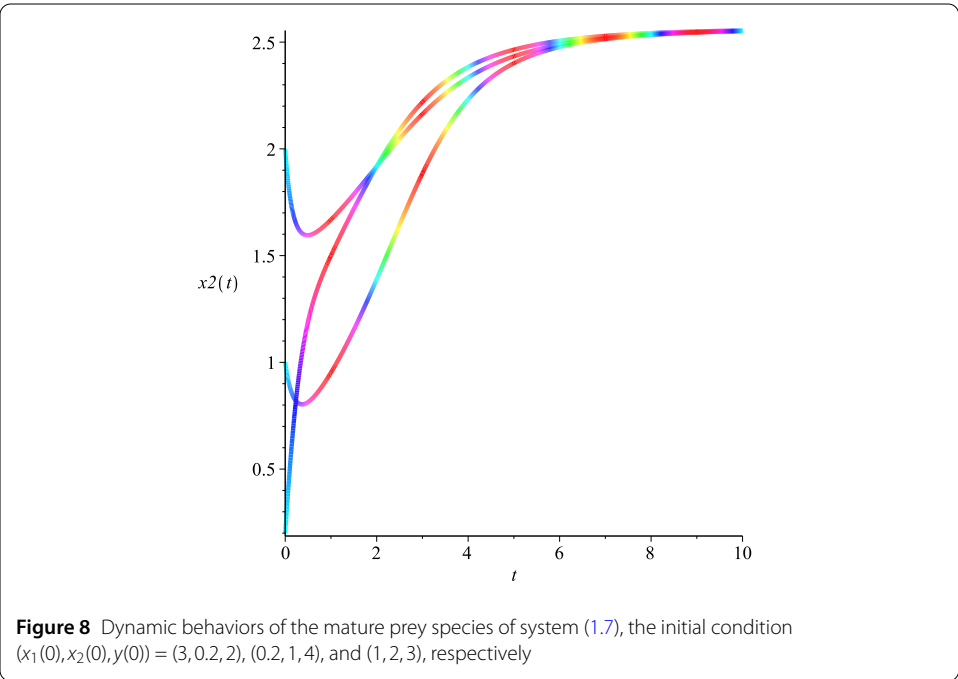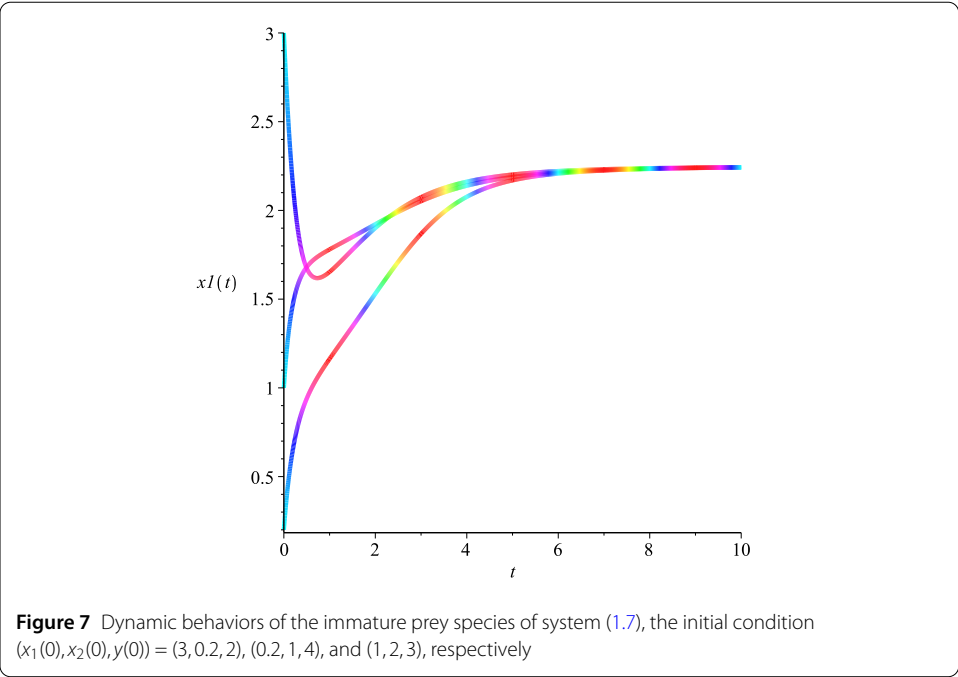$$\frac{g}{g + d_1} = \frac{1}{1 + 1} = \frac{1}{2} < \frac{3}{4} = s. \tag{1.24}$$

That is, the first inequality in Theorem C holds, while the second inequality in Theorem C does not hold. Numeric simulations (Figs. 7–9) show that in this case the nonnegative equilibrium $E_1(2.246211251, 2.246211251, 0)$ is still globally asymptotically stable.

Example 1.3 shows that maybe parameter s is not an essential coefficient to ensure the stability of the nonnegative equilibrium of system (1.1). This leads us to proposing the following conjecture.

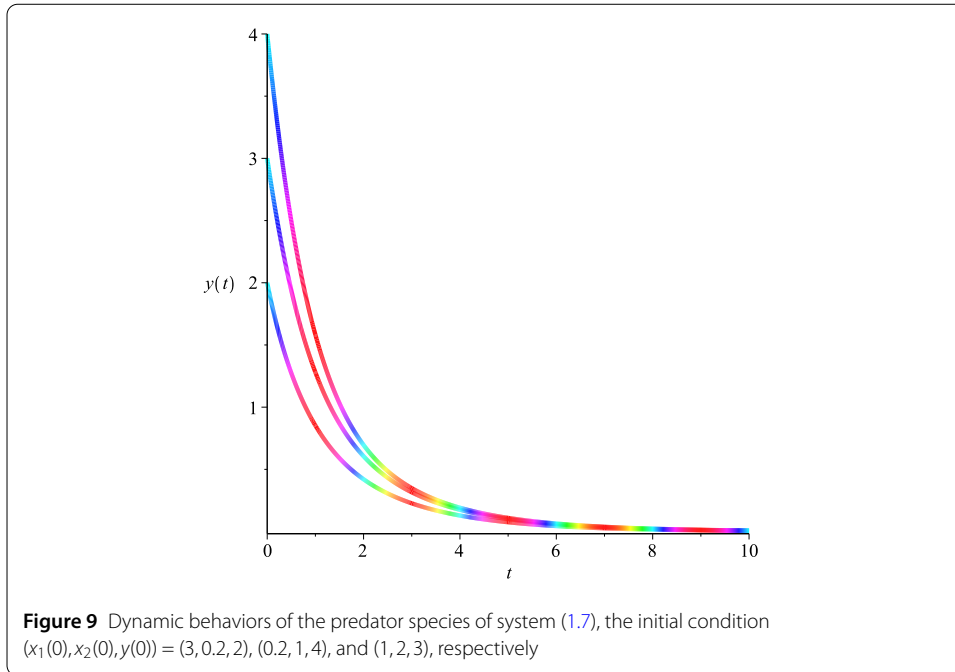**Conjecture B** In Theorem C, inequality (1.13) is not an essential one, maybe it is redundant and could be dropped.

The aim of this paper is to give an affirmative answer to the above two conjectures. Indeed, we will establish the following results.

Theorem 1.1 *Assume that* (H_1) *and* (H_2) *hold, then system* (1.5) *is permanent.*

Figure 7 Dynamic behaviors of the immature prey species of system (1.7), the initial condition $(x_1(0), x_2(0), y(0)) = (3, 0.2, 2)$, $(0.2, 1, 4)$, and $(1, 2, 3)$, respectively

Figure 8 Dynamic behaviors of the mature prey species of system (1.7), the initial condition $(x_1(0), x_2(0), y(0)) = (3, 0.2, 2)$, $(0.2, 1, 4)$, and $(1, 2, 3)$, respectively

Remark 1.1 Compared with Theorem A, B, and 1.1, Theorem 1.1 shows that if the system without cannibalism is permanent, then cannibalism has no influence on the persistence property of the system.

Theorem 1.2 *Assume that* (H_5) *and* (H_2) *hold, then the nonnegative equilibrium* $E_1(\overline{x}_1, \overline{x}_2, 0)$ *of system* (1.5) *is globally asymptotically stable.*

Figure 9 Dynamic behaviors of the predator species of system (1.7), the initial condition $(x_1(0), x_2(0), y(0)) = (3, 0.2, 2)$, $(0.2, 1, 4)$, and $(1, 2, 3)$, respectively

Remark 1.2 Compared with Theorem C and Theorem 1.2, our result shows that under assumptions (H_1) and (H_2') cannibalism has no influence on the stability property of the boundary equilibrium of system (1.5).

We prove Theorems 1.1 and 1.2 in the next section and Sect. 3, respectively. We end this paper with a brief discussion.

2 Proof of Theorem 1.1

Before we begin the proof of Theorem 1.1, we need to establish some useful lemmas.

We first introduce lemmas from [39], which give the upper bound of the solutions of system (1.5).

Lemma 2.1 *Any solutions of system* (1.5) *with positive initial conditions are ultimately bounded.*

Lemma 2.2 *Consider the following system*:

$$\frac{du_1}{dt} = b_1 u_2 - g u_1 - b_2 u_1 u_2 - d_1 u_1 \overset{\text{def}}{=} f(u_1, u_2),$$

$$\frac{du_2}{dt} = g u_1 + s b_2 u_1 u_2 - d_2 u_2^2 - \frac{a_1}{m} u_2 \overset{\text{def}}{=} g(u_1, u_2).$$

$$(2.1)$$

Under the assumption of Theorem 1.1, *system* (2.1) *admits a boundary equilibrium* $E_0(0, 0)$, *which is a saddle point, and a positive equilibrium* $E^*(u_1^*, u_2^*)$, *which is locally asymptotically stable and globally asymptotically stable.*

Proof The equilibria of system (2.1) satisfy the equations

$$b_1 u_2 - g u_1 - b_2 u_1 u_2 - d_1 u_1 = 0,$$
$$g u_1 + s b_2 u_1 u_2 - d_2 u_2^2 - \frac{a_1}{m} u_2 = 0. \tag{2.2}$$

Obviously, under the assumption of Theorem 1.1, the system admits the boundary equilibrium $E_0(0,0)$ and the unique positive equilibrium $E^*(u_1^*, u_2^*)$, where

$$u_1^* = \frac{b_1 u_2^*}{b_2 u_2 + d_1 + g}, \qquad u_2^* = \frac{-\delta_2 + \sqrt{\delta_2^2 - 4\delta_1 \delta_3}}{2\delta_2}, \tag{2.3}$$

where

$$\delta_1 = b_2 d_2 m, \qquad \delta_2 = m d_2 (d_1 + g) + b_2 (a_1 - b_1 m s), \qquad \delta_3 = a_1 (d_1 + g) - b_1 g m < 0.$$

The variation matrix of the continuous-time system (2.1) at an equilibrium solution (u_1, u_2) is

$$J(u_1, u_2) = \begin{pmatrix} f_{u_1}(u_1, u_2) & f_{u_2}(u_1, u_2) \\ g_{u_1}(u_1, u_2) & g_{u_2}(u_1, u_2) \end{pmatrix}$$
$$= \begin{pmatrix} -b_2 u_2 - d_1 - g & -b_2 u_1 + b_1 \\ b_2 u_2 s + g & s b_2 u_1 - 2 d_2 u_2 - \frac{a_1}{m} \end{pmatrix}. \tag{2.4}$$

Thus, at $E_0(0,0)$

$$J(0,0) = \begin{pmatrix} -(d_1 + g) & b_1 \\ g & -\frac{a_1}{m} \end{pmatrix}. \tag{2.5}$$

Consequently, the characteristic equation is

$$\lambda^2 + \left(d_1 + g + \frac{a_1}{m} \right) \lambda + (d_1 + g) \frac{a_1}{m} - b_1 g = 0. \tag{2.6}$$

Note that from condition (H_2) one has

$$(d_1 + g) \frac{a_1}{m} - b_1 g < 0,$$

hence, Eq. (2.6) has one positive solution, that is, $J(0,0)$ has one positive eigenvalue, hence $E_0(0,0)$ is unstable.

At $E^*(u_1^*, u_2^*)$

$$J(u_1^*, u_2^*) = \begin{pmatrix} -b_2 u_2^* - d_1 - g & -b_2 u_1^* + b_1 \\ b_2 u_2^* s + g & s b_2 u_1^* - 2 d_2 u_2^* - \frac{a_1}{m} \end{pmatrix}.$$

Note that

$$\operatorname{tr}\big(J(u_1^*, u_2^*)\big) = -\frac{b_1 u_2^*}{u_1^*} - \frac{g u_1^*}{u_2^*} - d_2 u_2^* < 0, \tag{2.7}$$

and

$$
\begin{aligned}
\det\big(J\big(u_1^*, u_2^*\big)\big) &= b_1 \frac{u_2^*}{u_1^*}\left(\frac{gu_1^*}{u_2^*} + d_2 u_2^*\right) - \big(-b_2 u_1^* + b_1\big)\big(b_2 u_2^* s + g\big) \\
&= b_1 \frac{u_2^*}{u_1^*}\left(\frac{gu_1^*}{u_2^*} + d_2 u_2^*\right) - (d_1 + g)\frac{u_1^*}{u_2^*}\frac{d_2 (u_2^*)^2 + \frac{a_1}{m} u_2^*}{u_1^*} \\
&= b_1 g + \frac{b_1 d_2 (u_2^*)^2}{u_1^*} - (d_1 + g)d_2 u_2^* - (d_1 + g)\frac{a_1}{m}.
\end{aligned}
$$

Since

$$
b_1 g - (d_1 + g)\frac{a_1}{m} > 0,
$$

$$
\begin{aligned}
&\frac{b_1 d_2 (u_2^*)^2}{u_1^*} - (d_1 + g)d_2 u_2^* \\
&= u_2^*\left(\frac{b_1 d_2 u_2^*}{u_1^*} - (d_1 + g)d_2\right) \\
&= u_2^*\big(\big(g + b_2 u_2^* + d_1\big)d_2 - (d_1 + g)d_2\big) \\
&= b_2 \big(u_2^*\big)^2 u_1^* > 0,
\end{aligned}
$$

then

$$
\det\big(J\big(u_1^*, u_2^*\big)\big) > 0. \tag{2.8}
$$

It follows from (2.7) and (2.8) that both eigenvalues of $J(u_1^*, u_2^*)$ have negative real parts; consequently, this steady-state solution is locally asymptotically stable.

In what follows, we will take the idea and method of Wu and Chen[38] to investigate the global asymptotic stability property of the positive equilibrium. We need to determine the existence or non-existence of the limit cycle in the first quadrant.

For $E^*(u_1^*, u_2^*)$, which is the unique stable equilibrium in the first quadrant, let AB be the line segment of $L_1 : u_1 = p$ and BC be the line segment of $L_2 : u_2 = q$, where $A(p, 0)$, $B(p, q)$, $C(0, q)$, and p, q are positive constants which satisfy $p > u_1^*$, and

$$
\max\left\{\frac{sb_2}{d_2}, \frac{spm}{a_1}\right\} < q < \frac{gp + d_1 p}{b_1}.
$$

By simple calculation, we have

$$
\dot{u}_1|_{AB} = b_1 u_2 - gp - b_2 p u_2 - d_1 p|_{0 \le u_2 \le q} < 0,
$$

$$
\dot{u}_2|_{BC} = gu_1 + sb_2 u_1 q - d_2 q^2 - \frac{a_1}{m}q|_{0 \le u_1 \le p} < 0,
$$

thus AB, BC are the transversals of system (2.1). It is not hard to check that OA, OC are the transversals of system (1.1), and any trajectory enters region $OABCO$ from its exterior to interior.

Let $G = \frac{1}{u_1 u_2}$, it follows from (2.1) that

$$\frac{\partial Gf}{\partial u_1} + \frac{\partial Gg}{\partial u_2} = -\frac{d_2 u_1 u_2^2 + b_1 u_2^2 + g u_1^2}{u_1^2 u_2^2} < 0.$$

By Poincare–Bendixson theorem, there are no limit cycles in the first quadrant, thus $E_*(x_1^*, x_2^*)$ is globally asymptotically stable if it exists. This ends the proof of Lemma 2.2.   □

Now we are in a position to prove Theorem 1.1.

Proof of Theorem 1.1 Let $(x_1(t), x_2(t), y(t))$ be any positive solution of system (1.5), its initially condition is $(x_1(0), x_2(0), y(0))$. From system (1.5) it immediately follows that

$$\dot{x}_1(t) = b_1 x_2 - g x_1 - b_2 x_1 x_2 - d_1 x_1,$$
$$\dot{x}_2(t) \geq g x_1 + s b_2 x_1 x_2 - d_2 x_2^2. \tag{2.9}$$

Now let us consider the following auxiliary system:

$$\frac{du_1}{dt} = b_1 u_2 - g u_1 - b_2 u_1 u_2 - d_1 u_1,$$
$$\frac{du_2}{dt} = g u_1 + s b_2 u_1 u_2 - d_2 u_2^2 - \frac{a_1}{m} u_2. \tag{2.10}$$

Let $(u_1(t), u_2(t))$ be the positive solution of system (2.10) that satisfies the initial condition $(u_1(0), u_2(0)) = (x_1(0), x_2(0))$. Then, by the standard comparison argument, it follows that

$$x_1(t) \geq u_1(t), \qquad x_2(t) \geq u_2(t). \tag{2.11}$$

It follows from Lemma 2.2 that under the assumption of Theorem 1.1 system (2.10) admits a unique positive equilibrium $E^*(u_1^*, u_2^*)$ which is globally asymptotically stable. That is,

$$\lim_{t \to +\infty} u_1(t) = u_1^*, \qquad \lim_{t \to +\infty} u_2(t) = u_2^*. \tag{2.12}$$

Therefore, for $\varepsilon > 0$ small enough, without loss of generality, we may assume that

$$\varepsilon < \frac{1}{2} \min\{u_1^*, u_2^*\}.$$

There exists $T > 0$ such that, for all $t \geq T$,

$$u_1(t) \geq u_1^* - \frac{\varepsilon}{2}, \qquad u_2(t) \geq u_2^* - \frac{\varepsilon}{2}. \tag{2.13}$$

Combining with (2.11) and (2.13) leads to

$$x_1(t) \geq u_1^* - \frac{\varepsilon}{2}, \qquad x_2(t) \geq u_2^* - \frac{\varepsilon}{2}. \tag{2.14}$$

Hence,

$$\lim_{t \to +\infty} x_1(t) \geq \frac{1}{2} u_1^*, \qquad \lim_{t \to +\infty} x_2(t) \geq \frac{1}{2} u_2^*. \tag{2.15}$$

It follows from (2.14) that, for T large enough, $x_2(t) \geq \frac{1}{2}u_2^*$. This, together with the third equation of system (1.5), leads to

$$\dot{y} \geq \left(\frac{a_2 \frac{u_2^*}{2}}{my + \frac{u_2^*}{2}} - d_3 \right) y, \tag{2.16}$$

from (2.16) one could easily obtain that

$$\liminf_{t \to +\infty} y(t) \geq \frac{u_2^*(a_2 - d_3)}{2md_3}. \tag{2.17}$$

Lemma 2.1, (2.15), and (2.17) show that system (1.5) is permanent under assumptions (H_1) and (H_2), this ends the proof of Theorem 1.1. □

3 Proof of Theorem 1.2

We first establish several lemmas which will be used in the proof of Theorem 1.2.

Lemma 3.1 *Consider the following system*:

$$\begin{aligned}
\frac{du_1}{dt} &= b_1 u_2 - g u_1 - b_2 u_1 u_2 - d_1 u_1, \\
\frac{du_2}{dt} &= g u_1 + s b_2 u_1 u_2 - d_2 u_2^2 - \gamma u_2.
\end{aligned} \tag{3.1}$$

Assume that

$$\frac{g}{g + d_1} > \frac{\gamma}{b_1}$$

holds, then system (3.2) *admits a boundary equilibrium* $E_0(0,0)$, *which is a saddle point, and a positive equilibrium* $E^*(u_1^*, u_2^*)$, *which is locally asymptotically stable and globally asymptotically stable.*

Proof The proof of this lemma is similar to the proof of Lemma 2.2, with some minor revision, and we omit the details here. □

Lemma 3.2 *Consider the following system*:

$$\begin{aligned}
\frac{du_1}{dt} &= b_1 u_2 - g u_1 - b_2 u_1 u_2 - d_1 u_1, \\
\frac{du_2}{dt} &= g u_1 + s b_2 u_1 u_2 - d_2 u_2^2.
\end{aligned} \tag{3.2}$$

System (3.2) *admits a globally asymptotically stable positive equilibrium* $E_*(\bar{x}_1, \bar{x}_2)$, *where* $\bar{x}_1, \bar{x}_2$ *is defined by* (1.2).

Now we are in a position to prove Theorem 1.2.

Proof of Theorem 1.2 Let $(x_1(t), x_2(t), y(t))$ be a positive solution of system (1.5) with initial conditions $x_1(0) > 0$, $x_2(0) > 0$, $y(0) > 0$.

It follows from the third equation of system (1.1) that

$$\dot{y} \leq y(a_2 - d_3), \tag{3.3}$$

hence

$$y(t) \leq y(0) \exp\{(a_2 - d_3)t\}.$$

Thus, under the assumption $a_2 < d_3$, we have

$$\lim_{t \to +\infty} y(t) = 0. \tag{3.4}$$

From the first two equations of system (1.5), we have

$$\dot{x}_1(t) = b_1 x_2 - g x_1 - b_2 x_1 x_2 - d_1 x_1,$$
$$\dot{x}_2(t) \leq g x_1 + s b_2 x_1 x_2 - d_2 x_2^2. \tag{3.5}$$

Now let us consider the following auxiliary equation:

$$\dot{u}_1(t) = b_1 u_2 - g u_1 - b_2 u_1 x_2 - d_1 u_1,$$
$$\dot{u}_2(t) = g u_1 + s b_2 u_1 u_2 - d_2 u_2^2. \tag{3.6}$$

By Lemma 3.2, system (3.6) has a unique globally attractive positive equilibrium $E_*(\overline{x}_1, \overline{x}_2)$. Let $(u_1(t), u_2(t))$ be the solution of system (3.6) with $(u_1(0), u_2(0)) = (x_1(0), x_2(0))$, by comparison theorem, we have

$$x_i(t) \leq u_i(t). \tag{3.7}$$

Moreover, from the global asymptotic stability of $E_2(\overline{x}_1, \overline{x}_2)$, for any small enough given ε $(0 < \varepsilon < 1)$, there exists $T_1 > 0$ such that

$$\left| u_i(t) - \overline{x}_i \right| < \varepsilon \quad \text{for all } t \geq T_1. \tag{3.8}$$

Equation (3.8) combined with (3.7) leads to

$$x_i(t) \leq \overline{x}_i + \varepsilon \quad \text{for all } t \geq T_1. \tag{3.9}$$

Hence,

$$\limsup_{t \to +\infty} x_i(t) \leq \overline{x}_i + \varepsilon. \tag{3.10}$$

Since ε is arbitrary small enough positive constant, setting $\varepsilon \to 0$ in (3.10) leads to

$$\limsup_{t \to +\infty} x_i(t) \leq \overline{x}_i. \tag{3.11}$$

From the first two equations of system (1.5), we also have

$$\dot{x}_1(t) = b_1 x_2 - g x_1 - b_2 x_1 x_2 - d_1 x_1,$$
$$\dot{x}_2(t) \geq g x_1 + s b_2 x_1 x_2 - d_2 x_2^2 - \frac{a_1 x_2}{m}.$$

Now let us consider the following auxiliary equation:

$$\dot{v}_1(t) = b_1 v_2 - g v_1 - b_2 v_1 v_2 - d_1 v_1,$$
$$\dot{v}_2(t) = g v_1 + s b_2 v_1 v_2 - d_2 v_2^2 - \frac{a_1 v_2}{m}. \tag{3.12}$$

From (1.4) and Lemma 3.1, system (3.12) has a unique globally attractive positive equilibrium $E^*(v_1^*, v_2^*)$. Let $(v_1(t), v_2(t))$ be the solution of system (3.12) with $(v_1(0), v_2(0)) = (x_1(0), x_2(0))$, by comparison theorem, we have

$$x_i(t) \geq v_i(t). \tag{3.13}$$

Moreover, from the global asymptotic stability of $E^*(v_1^*, v_2^*)$, for any small enough given ε $(0 < \varepsilon < \frac{1}{2} v_i^*)$, there exists $T_2 > T_1$ such that

$$\left| v_i(t) - v_i^* \right| < \varepsilon \quad \text{for all } t \geq T_2. \tag{3.14}$$

Equation (3.14) combined with (3.13) leads to

$$x_i(t) \geq v_i^* - \varepsilon > \frac{v_i^*}{2} \quad \text{for all } t \geq T_2. \tag{3.15}$$

For any small enough positive constant ε_1 $(0 < \varepsilon_1 < \frac{a_1}{m})$, from (3.4) and (3.15), there exists $T_3 > T_2$ such that, for all $t \geq T_3$,

$$\frac{a_1 y(t)}{m y(t) + x_2(t)} < \varepsilon_1 \tag{3.16}$$

holds. From (3.16) and the first and second equations of system (1.5), we have

$$\dot{x}_1(t) = b_1 x_2 - g x_1 - b_2 x_1 x_2 - d_1 x_1,$$
$$\dot{x}_2(t) \geq g x_1 + s b_2 x_1 x_2 - d_2 x_2^2 - \varepsilon_1 x_2. \tag{3.17}$$

Now let us consider the following auxiliary equation:

$$\dot{w}_1(t) = b_1 w_2 - g w_1 - b_2 w_1 w_2 - d_1 w_1,$$
$$\dot{w}_2(t) = g w_1 + s b_2 w_1 w_2 - d_2 w_2^2 - \varepsilon_1 w_2. \tag{3.18}$$

Condition (1.4) implies that

$$\frac{g}{g + d_1} > \frac{\varepsilon_1}{b_1}. \tag{3.19}$$

Hence, by Lemma 3.1, system (3.18) has a unique globally attractive positive equilibrium $E_{\varepsilon_1}(w_1^{\varepsilon_1}, w_2^{\varepsilon_1})$, where

$$w_1^{*\varepsilon_1} = \frac{b_1 w_2^{\varepsilon_1}}{b_2 w_2^{\varepsilon_1} + d_1 + g}, \qquad w_2^{*\varepsilon_1} = \frac{-\delta_1 + \sqrt{\delta_1^2 - 4 b_2 d_2(-b_1 g + \varepsilon_1(d_1 + g))}}{2 b_2 d_2}, \qquad (3.20)$$

where $\delta_1 = -b_1 b_2 s + b_2 \varepsilon_1 + d_2(d_1 + g)$. Obviously,

$$\delta_1 \to \delta \quad \text{as } \varepsilon_1 \to 0, \qquad (3.21)$$

where δ is defined by (1.8), and

$$-4 b_2 d_2 \big(-b_1 g + \varepsilon_1(d_1 + g)\big) \to 4 b_2 d_2 b_1 g \quad \text{as } \varepsilon_1 \to 0. \qquad (3.22)$$

Hence, it follows from (3.20)–(3.22) that

$$w_1^{*\varepsilon_1} \to \overline{x}_1, w_2^{*\varepsilon_1} \to \overline{x}_2 \quad \text{as } \varepsilon_1 \to 0, \qquad (3.23)$$

where $\overline{x}_1, \overline{x}_2$ are defined by (1.8).

Let $(w_1(t), w_2(t))$ be the solution of system (3.18) with $(w_1(0), w_2(0)) = (x_1(0), x_2(0))$. By the comparison theorem, we have

$$x_i(t) \geq w_i(t). \qquad (3.24)$$

Moreover, from the global asymptotic stability of $E_{\varepsilon_1}(w_1^{*\varepsilon_1}, w_2^{*\varepsilon_1})$, for above ε_1, there exists $T_3 > T_2$ such that

$$\big|w_i(t) - w_i^{*\varepsilon_1}\big| < \varepsilon_1 \quad \text{for all } t \geq T_3. \qquad (3.25)$$

Equation (3.24) combined with (3.25) leads to

$$x_i(t) \geq w_i^{*\varepsilon_1} - \varepsilon_1 \quad \text{for all } t \geq T_3. \qquad (3.26)$$

Hence,

$$\liminf_{t \to +\infty} x_i(t) \geq w_i^{*\varepsilon_1} - \varepsilon_1. \qquad (3.27)$$

Since ε_1 is arbitrary small enough positive constant, setting $\varepsilon_1 \to 0$ in (3.27), it follows from (3.23) that

$$\liminf_{t \to +\infty} x_i(t) \geq \overline{x}_i. \qquad (3.28)$$

Equation (3.11) together with (3.27)leads to

$$\overline{x}_i \leq \liminf_{t \to +\infty} x_i(t) \leq \limsup_{t \to +\infty} x_i(t) \leq \overline{x}_i.$$

That is,

$$\lim_{t \to +\infty} x_i(t) = \overline{x}_i, \quad i = 1, 2. \tag{3.29}$$

Equation (3.4) and (3.29) show that the nonnegative equilibrium $E_1(\overline{x}_1, \overline{x}_2, 0)$ of system (1.5) is globally asymptotically stable. This ends the proof of Theorem 1.2. □

4 Conclusion

Based on the work of Xu et al. [40], Zhang and Zhang [39] tried to incorporate the cannibalism to system (1.1), and they proposed system (1.5). By constructing some suitable Lyapunov function, they obtained the conditions (Theorem B) which ensure the permanence of system (1.5). The condition is independent of b_2, which leads us to finding out the reason behind this phenomenon. By numeric simulations (Example 1.1 and 1.2), we found that maybe both coefficients b_2 and s have no influence on the persistence property of the system. Indeed, we propose a conjecture (Conjecture A). Obviously, one could not prove this conjecture by directly applying the method of [39, 40]. By developing the analysis technique of Wu et al. [13], we finally give a strict proof of Theorem 1.1. Obviously, Theorem 1.1 essentially improves the main result of [39] since our condition is cannibalism independent, which means that if the original system is permanent, then cannibalism has no influence on the persistence property of the system.

Concerned with the stability property of the predator free equilibrium, Zhang and Zhang[39] obtained Theorem C, which is cannibalism dependent. However, numeric simulations (Example 1.3) show that their result still has room to improve. We first propose a conjecture (Conjecture B), then, by developing the analysis technique of differential inequality theory, we finally establish some new result about the stability property of the predator free equilibrium, some unnecessary restriction of Theorem C is dropped. Obviously, Theorem 1.2 essentially improves the main result of [39] since our condition is cannibalism independent, which means that if inequalities $(H_5)$and (H_2) hold, cannibalism has no influence on the persistence property of the system.

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