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# On the solution of two-dimensional fractional Black–Scholes equation for European put option

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**Abstract**

The purpose of this paper was to investigate the dynamics of the option pricing in the market through the two-dimensional time fractional-order Black–Scholes equation for a European put option. The Liouville–Caputo derivative was used to improve the ordinary Black–Scholes equation. The analytic solution is a powerful tool for describing the behavior of the option price in the European style market. In this study, analytic solution is carried out by the Laplace homotopy perturbation method. Moreover, the obtained solution showed that the Laplace homotopy perturbation method was an efficient method for finding an analytic solution of two-dimensional fractional-order differential equation.

**Keywords:** Analytical solutions; Fractional Black–Scholes equation; European put option; Laplace homotopy perturbation method

## 1 Introduction

Fractional calculus is widely used to model many real-life problems in various fields such as physics, engineering, biology, earth science, chemistry, finance, and so on [1–8]. The fractional calculus was introduced more than 300 years ago [9]. In the beginning, it was only in a theoretical sense. Recently, fractional calculus was wildly studied in many ways [5, 9–14]. Moreover, the fractional calculus has shown to be more accurate for describing real complicated phenomena than the ordinary calculus [15, 16]. There are many version of fractional calculus such as Riemann–Liouville [5], Hadamard [17], Atangana–Baleanu [18], Liouville–Caputo [19], Riesz [14], etc.

Actually, one cannot find exact solutions to most fractional-order differential equations, so the approximate solutions are investigated to solve linear and nonlinear fractional-order differential equations. There are many researches dealing with fractional order differential equations in many different fields [20–23].

The main feature of fractional order differential equations is memory. The variables in financial problems have long memories. Therefore, the fractional order differential equations are fitted to describe the financial problems. Many researches have been done to investigate the fractional-order differential equations in financial problems [24–26].

The Black–Scholes equation was proposed by Fisher Black and Myron Scholes in 1973 [27] to express the behavior of the option price in the European style market. Several papers have investigated how the Black–Scholes equation describes the behavior of the market [28–32]. Moreover, the Black–Scholes equation has been extended to express the behavior in another market, such as the American-style market and Asian-style market [33–36].

In general, the two-dimensional Black–Scholes equations for a European put option can be written as follows:

$$\frac{\partial P}{\partial \tau} + \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2}\sigma_i\sigma_j\rho_{ij}x_ix_j\frac{\partial^2 P}{\partial x_i\,\partial x_j} + \sum_{i=1}^{2}(r-q_i)x_i\frac{\partial P}{\partial x_i} - rP = 0,$$
$$x_1, x_2 \in [0,\infty), \qquad \tau \in [0,T]$$

with the terminal condition

$$P(x_1,x_2,T) = \max\left(E - \sum_{i=1}^{2}\tilde{\beta}_i x_i, 0\right),$$

where $E = \max\{E_1, E_2\}$, and the boundary conditions:

$$P(x_1,x_2,\tau) = 0 \quad \text{as } (x_1,x_2) \to (0,0),$$
$$P(x_1,x_2,\tau) = \sum_{i=1}^{2}\tilde{\beta}_i x_i - Ee^{-r(T-\tau)} \quad \text{as } x_1 \to \infty \text{ or } x_2 \to \infty,$$

where $P$ denotes the value of a put option of the underlying stock prices $\{x_1, x_2\}$ at time $\tau$, $q_i$ denotes the dividend yield on the $i$th underlying stock, $\rho_{ij}$ denotes the correlation between the $i$th and $j$th underlying stock prices, $T$ denotes the expiration date, $r$ denotes the risk-free interest rate, $\sigma_i$ denotes the volatility of the $i$th underlying stock, $E_i$ denotes the strike price of the $i$th underlying stock, and $\tilde{\beta}_i$ denotes a coefficient so that prices of all the risky assets are at the same level.

There are many researchers who investigated analytical and approximate solutions of the Black–Scholes equation. Various effective methods have been used to solve the Black–Scholes equation, for example, the finite difference method [37], finite element method [15, 38], homotopy perturbation method [39, 40], the Mellin transform method [41], Adomian decomposition method [42], the variational iteration method [43, 44], radial basis function partition of unity method (RBF-PUM) [45, 46], and adaptive moving mesh method [47].

In this paper, we apply the Laplace homotopy perturbation method (LHPM) to get the analytical solution of the fractional-order Black–Scholes equation. This method is a combination of the Laplace transform and homotopy perturbation method [48]. It provides an analytical solution in the form of a convergent series. From the mathematical point of view, the analytical solution is a useful tool to describe the behavior of the solution, particularly the financial behavior of the solution to fractional-order differential equations.

The paper is structured as follows: the basic knowledge about fractional calculus is introduced in Sect. 2, and the two-dimensional time-fractional Black–Scholes equation is presented in Sect. 3. In Sect. 4, the analytical solution of the model is obtained by LHPM, followed by the conclusions in Sect. 5.

## 2 Mathematical background

### 2.1 Fractional calculus

In this section, we present the definitions of fractional calculus which are used in this paper.

**Definition 1** The Riemann–Liouville integral operator of the fractional order $0 < \alpha < 1$ for a function $f : (0, \infty) \to \mathbb{R}$ is defined as [5]:

$$J_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\,ds, \tag{1}$$

where $\Gamma$ is the well-known Gamma function.

Riemann–Liouville fractional integral operator has some properties which are stated as follows: for any $\alpha, \beta \geq 0$ and $\gamma > -1$,

$$J^\alpha J^\beta f(t) = J^{\alpha+\beta} f(t),$$
$$J^\alpha J^\beta f(t) = J^\beta J^\alpha f(t),$$
$$J^\alpha t^\beta = \frac{\Gamma(\gamma+1}{\Gamma(\gamma+\alpha+1)} t^{\alpha+\beta}.$$

**Definition 2** The Riemann–Liouville derivative of the fractional order $0 < \alpha < 1$ for a function $f : (0, \infty) \to \mathbb{R}$ is defined as [5]:

$$^{\mathrm{RL}}D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{f(s)}{(t-s)^\alpha}\,ds. \tag{2}$$

**Definition 3** The Liouville–Caputo-type fractional derivative of order $0 < \alpha \leq 1$ for a function $f : (0, \infty) \to \mathbb{R}$ is defined as [19]:

$$D_\tau^\alpha f(\tau) = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \int_0^\tau \frac{f'(s)}{(\tau-\theta)^\alpha}\,d\theta, & 0 < \alpha < 1, \\ \frac{df}{d\tau}, & \alpha = 1. \end{cases}$$

In this paper, we use Liouville–Caputo derivative as the time-fractional derivative because the initial condition for the fractional order derivation is similar to the traditional derivative [49].

**Definition 4** The Mittag-Leffler function is defined as [50]:

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}, \quad \alpha \in \mathbb{C}, \mathrm{Re}(\alpha) > 0. \tag{3}$$

**Definition 5** The generalized Mittag-Leffler function is defined as [50]:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, \quad \alpha, \beta \in \mathbb{C}, \mathrm{Re}(\alpha) > 0, \mathrm{Re}(\beta) > 0. \tag{4}$$

## 3 Generalized time fractional-order Black–Scholes model

The standard two-dimensional Black–Scholes equation for a European put option was presented in Sect. 1. Throughout this paper, we assume that $\sigma_1$, $\sigma_2$, $\rho$, and $r$ are constants.

By transforming coordinates and time to a forward time coordinate $t = T - \tau$ and applying the Liouville–Caputo fractional derivative to the two-dimensional Black–Scholes equation, we obtain the following two-dimension time-fractional order Black–Scholes equation with $\alpha \in (0, 1]$ and the initial and boundary conditions as follows:

$$D_t^\alpha \omega = \frac{1}{2}\sigma_1^2 \frac{\partial^2 \omega}{\partial x^2} + \frac{1}{2}\sigma_2^2 \frac{\partial^2 \omega}{\partial y^2} + \rho\sigma_1\sigma_2 \frac{\partial^2 \omega}{\partial x\,\partial y}, \quad (x, y, t) \in \mathbb{R} \times \mathbb{R} \times [0, T], \tag{5}$$

subject to the initial condition:

$$\omega(x, y, 0) = \max\bigl(E - (\beta_1 e^x + \beta_2 e^y), 0\bigr), \tag{6}$$

and the boundary conditions:

$$\begin{aligned} &\omega(x, y, t) = 0 \quad \text{as } (x, y) \to -\infty, \quad \text{and} \\ &\omega(x, y, t) = E - \bigl(\beta_1 e^{x+\frac{1}{2}\sigma_1^2 t} + \beta_2 e^{y+\frac{1}{2}\sigma_2^2 t}\bigr) \quad \text{as } x \to \infty \text{ or } y \to \infty, \end{aligned} \tag{7}$$

where $\beta_1 = \tilde{\beta}_1 e^{(r-\frac{1}{2}\sigma_1^2)T}$ and $\beta_2 = \tilde{\beta}_2 e^{(r-\frac{1}{2}\sigma_2^2)T}$.

## 4 Solving two-dimensional time-fractional Black–Scholes equation by LHPM

In this section, we apply LHPM techniques for finding the solution of two-dimensional time-fractional Black–Scholes model (5) subject to the initial condition (6) and the boundary conditions (7).

**Theorem 1** *The analytical solution of two-dimensional time-fractional Black–Scholes model* (5) *is given by*

$$\begin{aligned} \omega(x, y, t) = {} & \max\bigl(E - (\beta_1 e^x + \beta_2 e^y), 0\bigr) + e^{x+y} t^\alpha \\ & + \max(\beta_1 e^x, 0) \frac{t^\alpha \sigma_1^2}{2} E_{\alpha,\alpha+1}\left(\frac{t^\alpha \sigma_1^2}{2}\right) + \max(\beta_2 e^y, 0) \frac{t^\alpha \sigma_2^2}{2} E_{\alpha,\alpha+1}\left(\frac{t^\alpha \sigma_2^2}{2}\right) \\ & + \left[e^{x+y}\left(\frac{\sigma_1^2}{2} + \frac{\sigma_2^2}{2} + \rho\sigma_1\sigma_2\right)\Gamma(\alpha+1) t^{2\alpha} E_{\alpha,2\alpha+1}\left(t^\alpha\left(\frac{\sigma_1^2}{2} + \frac{\sigma_2^2}{2} + \rho\sigma_1\sigma_2\right)\right)\right] \\ & - e^{x+y}\Gamma(\alpha+1) t^\alpha E_{\alpha,\alpha+1}\left(t^\alpha\left(\frac{\sigma_1^2}{2} + \frac{\sigma_2^2}{2} + \rho\sigma_1\sigma_2\right)\right), \end{aligned}$$

*where* $E_{\gamma,\eta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\gamma k+\eta)}$ *is the generalized Mittag-Leffler function, in which* $\gamma$ *and* $\eta$ *are constants.*

*Proof* Let

$$\widetilde{N}\bigl(\omega(x, y, t)\bigr) = \frac{1}{2}\sigma_1^2 \frac{\partial^2 \omega}{\partial x^2} + \frac{1}{2}\sigma_2^2 \frac{\partial^2 \omega}{\partial y^2} + \rho\sigma_1\sigma_2 \frac{\partial^2 \omega}{\partial x\,\partial y},$$

and represent Eq. (5) in the general form of

$$D_t^\alpha \omega(x,y,t) = \widetilde{N}\big(\omega(x,y,t)\big).$$

Applying the Laplace transform with respect to $t$, we obtain

$$\mathcal{L}\big\{D_t^\alpha \omega(x,y,t)\big\} = \mathcal{L}\big\{\widetilde{N}\big(\omega(x,y,t)\big)\big\}. \tag{8}$$

By the properties of Laplace transform of the Liouville–Caputo fractional derivative [51], Eq. (8) becomes

$$\begin{aligned}\mathcal{L}\big\{\omega(x,y,t)\big\} &= \frac{1}{s}\max\big(E-\big(\beta_1 e^x+\beta_2 e^y\big),0\big)\\ &\quad+\frac{1}{s^\alpha}\mathcal{L}\left\{\frac{1}{2}\sigma_1^2\frac{\partial^2\omega}{\partial x^2}+\frac{1}{2}\sigma_2^2\frac{\partial^2\omega}{\partial y^2}+\rho\sigma_1\sigma_2\frac{\partial^2\omega}{\partial x\,\partial y}\right\}.\end{aligned} \tag{9}$$

Taking the inverse Laplace transform of (9), we get

$$\begin{aligned}\omega(x,y,t) &= \max\big(E-\big(\beta_1 e^x+\beta_2 e^y\big),0\big)\\ &\quad+\mathcal{L}^{-1}\left\{\frac{1}{s^\alpha}\mathcal{L}\left\{\frac{1}{2}\sigma_1^2\frac{\partial^2\omega}{\partial x^2}+\frac{1}{2}\sigma_2^2\frac{\partial^2\omega}{\partial y^2}+\rho\sigma_1\sigma_2\frac{\partial^2\omega}{\partial x\,\partial y}\right\}\right\}.\end{aligned}$$

Applying techniques of HPM, function $\omega$ can be constructed

$$\omega(x,y,t;q):\mathbb{R}\times\mathbb{R}\times[0,T]\times[0,1]\to\mathbb{R}$$

which satisfies the following equation:

$$\begin{aligned}&(1-q)\big(\omega(x,y,t;q)-\widetilde{\omega}_0(x,y,t)\big)+q\bigg[\omega(x,y,t;q)-\max\big(E-\big(\beta_1 e^x+\beta_2 e^y\big),0\big)\\ &\quad-\mathcal{L}^{-1}\left\{\frac{1}{s^\alpha}\mathcal{L}\left\{\frac{1}{2}\sigma_1^2\frac{\partial^2\omega}{\partial x^2}+\frac{1}{2}\sigma_2^2\frac{\partial^2\omega}{\partial y^2}+\rho\sigma_1\sigma_2\frac{\partial^2\omega}{\partial x\,\partial y}\right\}\right\}\bigg]=0,\end{aligned}$$

or

$$\begin{aligned}\omega(x,y,t;q) &= \widetilde{\omega}_0(x,y,t)-q\widetilde{\omega}_0(x,y,t)+q\max\big(E-\big(\beta_1 e^x+\beta_2 e^y\big),0\big)\\ &\quad+q\mathcal{L}^{-1}\left\{\frac{1}{s^\alpha}\mathcal{L}\left\{\frac{1}{2}\sigma_1^2\frac{\partial^2\omega}{\partial x^2}+\frac{1}{2}\sigma_2^2\frac{\partial^2\omega}{\partial y^2}+\rho\sigma_1\sigma_2\frac{\partial^2\omega}{\partial x\,\partial y}\right\}\right\},\end{aligned} \tag{10}$$

where $q\in[0,1]$ is an embedded parameter and $\widetilde{\omega}_0(x,y,t)$ is an initial approximation of Eq. (10) which can freely be chosen [52]. For this case, we choose $\widetilde{\omega}_0(x,y,t)$ as

$$\widetilde{\omega}_0(x,y,t)=\max\big(E-\big(\beta_1 e^x+\beta_2 e^y\big),0\big)+e^{x+y}t^\alpha.$$

Then, we substitute $\widetilde{\omega}_0(x,y,t)$ into Eq. (10) to get

$$\begin{aligned}\omega(x,y,t;q) &= \max\big(E-\big(\beta_1 e^x+\beta_2 e^y\big),0\big)+e^{x+y}t^\alpha\\ &\quad+q\left(-e^{x+y}t^\alpha+\mathcal{L}^{-1}\left\{\frac{1}{s^\alpha}\mathcal{L}\left\{\frac{1}{2}\sigma_1^2\frac{\partial^2\omega}{\partial x^2}+\frac{1}{2}\sigma_2^2\frac{\partial^2\omega}{\partial y^2}+\rho\sigma_1\sigma_2\frac{\partial^2\omega}{\partial x\,\partial y}\right\}\right\}\right).\end{aligned} \tag{11}$$

For HPM, the solution of Eq. (5) can be assumed to be of the form

$$\omega(x,y,t;q) = \sum_{i=0}^{\infty} q^n \phi_n(x,y,t). \tag{12}$$

Substituting Eq. (12) into Eq. (11), Eq. (11) becomes

$$\begin{aligned}\sum_{n=0}^{\infty} q^n \phi_n(x,y,t) = {}& \max\bigl(E-(\beta_1 e^x + \beta_2 e^y),0\bigr) + e^{x+y}t^\alpha \\ &+ q\Biggl(-e^{x+y}t^\alpha + \mathcal{L}^{-1}\Biggl\{\frac{1}{s^\alpha}\mathcal{L}\Biggl\{\frac{1}{2}\sigma_1^2\sum_{n=0}^{\infty} q^n \frac{\partial^2\phi_n}{\partial x^2} \\ &+ \frac{1}{2}\sigma_2^2\sum_{n=0}^{\infty} q^n \frac{\partial^2\phi_n}{\partial y^2} + \rho\sigma_1\sigma_2\sum_{n=0}^{\infty} q^n \frac{\partial^2\phi_n}{\partial x\,\partial y}\Biggr\}\Biggr\}\Biggr).\end{aligned} \tag{13}$$

When we equate the corresponding powers of $q$ of Eq. (13), we have

$$\begin{aligned}\phi_0(x,y,t) &= \max\bigl(E-(\beta_1 e^x + \beta_2 e^y),0\bigr) + e^{x+y}t^\alpha, \\ \phi_1(x,y,t) &= -e^{x+y}t^\alpha + \mathcal{L}^{-1}\biggl\{\frac{1}{s^\alpha}\mathcal{L}\biggl\{\frac{1}{2}\sigma_1^2\frac{\partial^2\phi_0}{\partial x^2} + \frac{1}{2}\sigma_2^2\frac{\partial^2\phi_0}{\partial y^2} + \rho\sigma_1\sigma_2\frac{\partial^2\phi_0}{\partial x\,\partial y}\biggr\}\biggr\}, \\ \phi_i(x,y,t) &= -e^{x+y}t^\alpha \\ &\quad + \mathcal{L}^{-1}\biggl\{\frac{1}{s^\alpha}\mathcal{L}\biggl\{\frac{1}{2}\sigma_1^2\frac{\partial^2\phi_{i-1}}{\partial x^2} + \frac{1}{2}\sigma_2^2\frac{\partial^2\phi_{i-1}}{\partial y^2} + \rho\sigma_1\sigma_2\frac{\partial^2\phi_{i-1}}{\partial x\,\partial y}\biggr\}\biggr\} \quad \text{for } i \geq 2.\end{aligned}$$

Then, we can write $\phi_0, \phi_1, \phi_2, \phi_3, \ldots$ in the general form, i.e.,

$$\begin{aligned}\phi_0(x,y,t) &= \max\bigl(E-(\beta_1 e^x + \beta_2 e^y),0\bigr) + e^{x+y}t^\alpha, \\ \phi_n(x,y,t) &= \frac{t^{n\alpha}}{\Gamma(n\alpha+1)}\biggl(\frac{1}{2^n}\sigma_1^{2n}\max\bigl(\tilde{\beta}_1 e^x,0\bigr) + \frac{1}{2^n}\sigma_2^{2n}\max\bigl(\tilde{\beta}_2 e^y,0\bigr)\biggr) \\ &\quad + e^{x+y}\frac{t^{(n+1)\alpha}\Gamma(\alpha+1)}{\Gamma((n+1)\alpha+1)}\biggl(\frac{\sigma_1^2}{2} + \frac{\sigma_2^2}{2} + \rho\sigma_1\sigma_2\biggr)^n \\ &\quad - e^{x+y}\frac{t^{n\alpha}\Gamma(\alpha+1)}{\Gamma(n\alpha+1)}\biggl(\frac{\sigma_1^2}{2} + \frac{\sigma_2^2}{2} + \rho\sigma_1\sigma_2\biggr)^{(n-1)} \quad \text{when } n \geq 1.\end{aligned}$$

From Eq. (12), the solution $\omega(x,y,t)$ of the fractional order Black–Scholes equation (5) can be written as

$$\begin{aligned}\omega(x,y,t;q) = {}& \max\bigl(E-(\beta_1 e^x + \beta_2 e^y),0\bigr) + e^{x+y}t^\alpha \\ &+ \sum_{n=0}^{\infty} q^{n+1}\biggl\{\frac{t^{(n+1)\alpha}}{\Gamma((n+1)\alpha+1)}\biggl(\frac{1}{2^{(n+1)}}\sigma_1^{2(n+1)}\max\bigl(\beta_1 e^x,0\bigr) \\ &+ \frac{1}{2^{(n+1)}}\sigma_2^{2(n+1)}\max\bigl(\beta_2 e^y,0\bigr)\biggr) \\ &+ e^{x+y}\frac{t^{(n+2)\alpha}\Gamma(\alpha+1)}{\Gamma((n+2)\alpha+1)}\biggl(\frac{\sigma_1^2}{2} + \frac{\sigma_2^2}{2} + \rho\sigma_1\sigma_2\biggr)^{(n+1)} \\ &- e^{x+y}\frac{t^{(n+1)\alpha}\Gamma(\alpha+1)}{\Gamma((n+1)\alpha+1)}\biggl(\frac{\sigma_1^2}{2} + \frac{\sigma_2^2}{2} + \rho\sigma_1\sigma_2\biggr)^n\biggr\}.\end{aligned}$$

The solution can be obtained by letting $q \to 1$, and then we have

$$\begin{aligned}\omega(x,y,t;1) = {} & \max\bigl(E-\bigl(\beta_1 e^x+\beta_2 e^y\bigr),0\bigr)+e^{x+y}t^\alpha+\sum_{n=0}^{\infty}\Biggl\{\frac{t^{(n+1)\alpha}}{\Gamma((n+1)\alpha+1)} \\ & \times\left(\frac{1}{2^{(n+1)}}\sigma_1^{2(n+1)}\max\bigl(\beta_1 e^x,0\bigr)+\frac{1}{2^{(n+1)}}\sigma_2^{2(n+1)}\max\bigl(\beta_2 e^y,0\bigr)\right) \\ & + e^{x+y}\frac{t^{(n+2)\alpha}\Gamma(\alpha+1)}{\Gamma((n+2)\alpha+1)}\left(\frac{\sigma_1^2}{2}+\frac{\sigma_2^2}{2}+\rho\sigma_1\sigma_2\right)^{(n+1)} \\ & - e^{x+y}\frac{t^{(n+1)\alpha}\Gamma(\alpha+1)}{\Gamma((n+1)\alpha+1)}\left(\frac{\sigma_1^2}{2}+\frac{\sigma_2^2}{2}+\rho\sigma_1\sigma_2\right)^{n}\Biggr\}.\end{aligned}$$

Therefore, the solution of Eq. (5) will be

$$\begin{aligned}\omega(x,y,t) = {} & \max\bigl(E-\bigl(\beta_1 e^x+\beta_2 e^y\bigr),0\bigr)+e^{x+y}t^\alpha \\ & + \max\bigl(\beta_1 e^x,0\bigr)\frac{t^\alpha\sigma_1^2}{2}E_{\alpha,\alpha+1}\left(\frac{t^\alpha\sigma_1^2}{2}\right)+\max\bigl(\beta_2 e^y,0\bigr)\frac{t^\alpha\sigma_2^2}{2}E_{\alpha,\alpha+1}\left(\frac{t^\alpha\sigma_2^2}{2}\right) \\ & + \left[e^{x+y}\left(\frac{\sigma_1^2}{2}+\frac{\sigma_2^2}{2}+\rho\sigma_1\sigma_2\right)\Gamma(\alpha+1)t^{2\alpha}E_{\alpha,2\alpha+1}\left(t^\alpha\left(\frac{\sigma_1^2}{2}+\frac{\sigma_2^2}{2}+\rho\sigma_1\sigma_2\right)\right)\right] \\ & - e^{x+y}\Gamma(\alpha+1)t^\alpha E_{\alpha,\alpha+1}\left(t^\alpha\left(\frac{\sigma_1^2}{2}+\frac{\sigma_2^2}{2}+\rho\sigma_1\sigma_2\right)\right),\end{aligned} \tag{14}$$

where $E_{\gamma,\eta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\gamma k+\eta)}$ is the generalized Mittag-Leffler function, in which $\gamma$ and $\eta$ are constants. □

## 5 Conclusions

In this paper, we have found the analytical solution of the two-dimensional time-fractional Black–Scholes equation for a European put option using the Liouville–Caputo derivative. The Laplace homotopy perturbation method has been used to obtain the solution which can be written in the form of the generalized Mittag-Leffer function. The main feature of the analytical solution is that it is convenient for finding the option price.

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